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# ACTION ACCESSIBILITY FOR CATEGORIES OF INTEREST

*Dedicated to Dominique Bourn on the occasion of his 60th birthday*

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Abstract. We prove that every category of interest (in the sense of [17]) is action accessible in the sense of [8]. This fact allows us to give an intrinsic description of centers and centralizers in this class of categories. We give also some new examples of categories of interest, mainly arising from Loday's paper [11], [12] and [14].

## 1. Introduction

In a recent paper [8] D. Bourn and G. Janelidze introduced the notion of action accessible category, which allows to construct intrinsically the centralizers of subobjects and of equivalence relations, where the centralizer is defined in the usual sense: if $X$ is a normal subobject of an object $A$ in a semi-abelian category, the centralizer $Z(X, A)$ is the largest subobject $K$ of $A$ such that $[X, K] = 0$, where $[,]$ denotes the commutator of two subobjects, as in [6]. Another important result which is true in action accessible categories is that two equivalence relations $R$ and $S$ on a same object commutes if and only if $[I_R, I_S] = 0$, where $I_R$ and $I_S$ are the normal subobjects associated with $R$ and $S$, respectively. The main examples of action accessible categories given in [8], beyond those of action representative ones [5], are the categories *Rng* of (not necessarily unitary) rings and *Ass* of associative algebras.

The main goal of this paper is to show that every category of interest is action accessible. Categories of interest were introduced in [17] and [18] in order to generalize Barr's results on obstruction theory for commutative algebras [2]. In the context of such categories it is possible to study the existence and the nature of extensions with a fixed kernel. What is particularly interesting is that it is possible to apply this results to some algebras, such as Leibniz algebras and associative dialgebras and trialgebras, introduced by J.L. Loday in [11], [12] and in [14], respectively, as useful tools in algebraic $K$-theory.

The paper is organized as follows.

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In section 2 we recall the definition and the main properties of action accessible categories.

In section 3 we recall the definition and the main properties of categories of interest and we give some examples, with particular attention to Loday's algebras.

In section 4 we first observe that faithful split extensions, in the context of action accessible categories, correspond to strict actions in the varietal context of categories of interest and we use this fact to prove the main result.

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## 2. Action accessible categories

Let $\mathbb{C}$ be a pointed protomodular category. Fixed an object $X \in \mathbb{C}$, a split extension with kernel $X$ is a diagram

$$0 \longrightarrow X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} B \longrightarrow 0$$

such that $ps = 1_B$ and $k = \ker p$. We will denote such a split extension by $(B, A, p, s, k)$. Given another split extension $(D, C, q, t, l)$ with the same kernel $X$, a morphism of split extensions

$$(g, f) : (B, A, p, s, k) \longrightarrow (D, C, q, t, l) \tag{1}$$

is a pair $(g, f)$ of morphisms:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{k} & A & \underset{s}{\overset{p}{\rightleftarrows}} & B & \longrightarrow & 0 \\ & & {\scriptstyle 1_X}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & X & \xrightarrow{l} & C & \underset{t}{\overset{q}{\rightleftarrows}} & D & \longrightarrow & 0 \end{array} \tag{2}$$

such that $l = fk$, $qf = gp$ and $fs = tg$.

Split extensions with fixed kernel $X$ form a category, which we will denote by $SplExt_{\mathbb{C}}(X)$, or simply by $SplExt(X)$.

REMARK. Being the category $\mathbb{C}$ protomodular, the pair $(k, s)$ is jointly strongly epimorphic, and then the morphism $f$ in (2) is uniquely determined by $g$.

2.1. DEFINITION. *An object in $SplExt(X)$ is said to be faithful if any object in $SplExt(X)$ admits at most one morphism into it.*

We have the following description of faithful split extensions:

2.2. PROPOSITION. *([8], Proposition 1.4) For a morphism $(g, f) : (B, A, p, s, k) \longrightarrow (D, C, q, t, l)$ with faithful codomain, the following conditions are equivalent:*

*(a) $(B, A, p, s, k)$ is faithful;*

*(b)* $(g, f)$ *is a monomorphism in* $SplExt(X)$*;*

*(c)* $g$ *is a monomorphism in* $\mathbb{C}$*.*

2.3. Definition. *An object in* $SplExt(X)$ *is said to be accessible if it admits a morphism into a faithful object. If, for any* $X \in \mathbb{C}$*, every object in* $SplExt(X)$ *is accessible, we say that* $\mathbb{C}$ *is action accessible.*

In particular, if $\mathbb{C}$ is action representative in the sense of [5] (see also [3]), then it is action accessible. In fact, in an action representative category every category $SplExt(X)$ has a terminal object. Examples of this situation are the categories $Gp$ of groups and $R - Lie$ of $R$-Lie algebras, where $R$ is a commutative ring. The converse is not true: indeed, the category $Rng$ of (not necessarily unitary) rings is action accessible [8] but not action representative, as shown in [5].

We recall now the construction of the centralizer of an equivalence relation in an action accessible category. In order to do that, we need to introduce the notions of $X$-groupoid and groupoid accessible category.

Fixed $X \in \mathbb{C}$, a reflexive graph structure on an object $(B, A, p, s, k)$ in $SplExt(X)$ is a morphism $u : A \longrightarrow B$ such that $us = 1_B$; we will say also that $(B, A, p, s, u)$ is the underlying reflexive graph of $((B, A, p, s, k), u)$. Conversely, given a reflexive graph $(B, A, d_0, s_0, d_1)$, the morphism $d_1$ gives a reflexive graph structure on the object $(B, A, d_0, s_0, k)$ in $SplExt(X)$, where $(X, k)$ is a kernel of $d_0$.

In a protomodular category $\mathbb{C}$, being an internal groupoid in $\mathbb{C}$ is the same as being an internal reflexive graph satisfying certain property. Specifically, a reflexive graph $(B, A, d_0, s_0, d_1)$ is a groupoid if and only if $[R[d_0], R[d_1]] = 0$, where $R[f]$ denotes, as in [4], the equivalence relation determined by the kernel pair of any morphism $f$, and where, given any pair $(R, S)$ of equivalence relations on the same object $X$, we write $[R, S] = 0$ if the pair $(R, S)$ has a connector (see [7]).

Accordingly, by a groupoid structure on an object $(B, A, p, s, k)$ in $SplExt(X)$ we mean a morphism $u : A \longrightarrow B$ such that $us = 1_B$ and $[R[p], R[u]] = 0$; the system $(B, A, p, s, k, u)$ is then called an $X$-groupoid. $X$-groupoids form a category $Grpd(X)$, in which a morphism

$$(g, f) : (B, A, p, s, k, u) \longrightarrow (D, C, q, t, l, v)$$

is a morphism $(g, f) : (B, A, p, s, k) \longrightarrow (D, C, q, t, l)$ in $SplExt(X)$ such that $vf = gu$.

Similarly to definitions 2.1 and 2.3, we introduce

2.4. Definition. *An* $X$*-groupoid is said to be faithful if any* $X$*-groupoid admits at most one morphism into it. An* $X$*-groupoid is said to be accessible if it admits a morphism into a faithful* $X$*-groupoid. If, for any* $X \in \mathbb{C}$*, every* $X$*-groupoid is accessible, we say that* $\mathbb{C}$ *is groupoid accessible.*

2.5. Proposition. *([8], Corollary 3.5) If* $\mathbb{C}$ *is homological, then it is action accessible if and only if it is groupoid accessible.*

2.6. Theorem. *([8], Theorem 4.1) Let $R$ be an equivalence relation on an object $B$ in a homological action accessible category $\mathbb{C}$, $X$ an object in $\mathbb{C}$ and*

$$(g,f):(B,A,p,s,k,u) \longrightarrow (D,C,q,t,l,v)$$

*a morphism in $Grpd(X)$, in which the domain is the relation $R$ seen as a groupoid and the codomain is faithful. Then the kernel pair $R[g]$ of $g$ is the centralizer of $R$.*

## 3. Categories of interest

The definition of a category of interest was introduced by Orzech in [17]. Now we recall it (in the form expressed in [10]).

3.1. Definition. *A category of interest is a category $\mathbb{C}$ whose objects are groups with a set of operation $\Omega$ and with a set of identities $\mathbb{E}$, such that $\mathbb{E}$ includes the group laws and the following conditions hold. If $\Omega_i$ is the set of i-ary operations in $\Omega$, then:*

- *(a)* $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;
- *(b) the group operations (written additively: 0, -, +, even if the group is not necessarily abelian) are elements of $\Omega_0$, $\Omega_1$ and $\Omega_2$ respectively. Let $\Omega_2' = \Omega_2 \backslash \{+\}$, $\Omega_1' = \Omega_1 \backslash \{-\}$ and assume that if $* \in \Omega_2'$, then $\Omega_2'$ contains $*^\circ$ defined by $x *^\circ y = y * x$. Assume further that $\Omega_0 = \{0\}$;*
- *(c) for any $* \in \Omega_2'$, $\mathbb{E}$ includes the identity $x * (y+z) = x*y + x*z$;*
- *(d) for any $\omega \in \Omega_1'$ and $* \in \Omega_2'$, $\mathbb{E}$ includes the identities $\omega(x+y) = \omega(x) + \omega(y)$ and $\omega(x) * y = \omega(x*y)$;*
- *(e)* ***Axiom 1*** $x_1 + (x_2 * x_3) = (x_2 * x_3) + x_1$ *for any* $* \in \Omega_2'$;
- *(f)* ***Axiom 2*** *for any ordered pair $(*, \bar{*}) \in \Omega_2' \times \Omega_2'$ there is a word $W$ such that*

$$(x_1 * x_2)\bar{*}x_3 = W(x_1(x_2x_3), x_1(x_3x_2), (x_2x_3)x_1, (x_3x_2)x_1,$$
$$x_2(x_1x_3), x_2(x_3x_1), (x_1x_3)x_2, (x_3x_1)x_2),$$

  *where each juxtaposition represents an operation in $\Omega_2'$. This means that $(x_1 * x_2)\bar{*}x_3$ belongs to the subalgebra generated by the elements:*

$$x_1(x_2x_3), x_1(x_3x_2), (x_2x_3)x_1, (x_3x_2)x_1, x_2(x_1x_3), x_2(x_3x_1), (x_1x_3)x_2, (x_3x_1)x_2.$$

We describe now some examples (and counterexamples) of categories of interest.

(1) The categories $Gp$ of groups and $R - Lie$ of $R$-Lie algebras (where $R$ is a commutative ring) are categories of interest.

(2) The category *Rng* of (not necessarily unitary) rings is of interest.

(3) Let $K$ be a field. A $K$-Poisson algebra is a vector space over $K$ with two bilinear operations (let us denote them $\cdot$ and $\{,\}$) satisfying the following conditions:

- the operation $\cdot$ gives rise to an associative $K$-algebra;
- the operation $\{,\}$ gives rise to a $K$-Lie algebra;
- $\{x, yz\} = \{x, y\}z + y\{x, z\}$ for all $x, y, z$.

The category of Poisson algebras is of interest.

(4) Let $R$ be a commutative ring. A $R$-Leibniz algebra [11] is a $R$-module $X$ with a bilinear map

$$[,] : X \times X \longrightarrow X$$

such that

$$[[x, y], z] = [[x, z], y] + [x, [y, z]] \qquad \text{for all } x, y, z \in X.$$

It is easy to verify that any Lie algebra is a Leibniz algebra. The category of Leibniz algebras is of interest.

(5) Again, let $R$ be a commutative ring. An associative dialgebra (or simply dialgebra) [12], [13] over $R$ is an $R$-module $X$ equipped with two bilinear maps

$$\dashv : X \times X \longrightarrow X, \qquad \vdash : X \times X \longrightarrow X$$

satisfying, for any $x, y, z \in X$, the following axioms:

(i) $(x \dashv y) \dashv z = x \dashv (y \vdash z)$;
(ii) $(x \dashv y) \dashv z = x \dashv (y \dashv z)$;
(iii) $(x \vdash y) \dashv z = x \vdash (y \dashv z)$;
(iv) $(x \dashv y) \vdash z = x \vdash (y \vdash z)$;
(v) $(x \vdash y) \vdash z = x \vdash (y \vdash z)$.

The category of associative dialgebras is a category of interest.

(6) Another example of category of interest is those of associative trialgebras. An associative trialgebra [14] over a field $K$ is a vector space $X$ equipped with three binary bilinear operations, denoted by $\dashv$, $\vdash$ and $\perp$, which are associative and satisfy, for any $x, y, z \in X$, the following axioms:

(i) $(x \dashv y) \dashv z = x \dashv (y \vdash z)$;

(ii) $(x \vdash y) \dashv z = x \vdash (y \dashv z)$;

(iii) $(x \dashv y) \vdash z = x \vdash (y \vdash z)$;

(iv) $(x \dashv y) \dashv z = x \dashv (y \perp z)$;

(v) $(x \perp y) \dashv z = x \perp (y \dashv z)$;

(vi) $(x \dashv y) \perp z = x \perp (y \vdash z)$;

(vii) $(x \vdash y) \perp z = x \vdash (y \perp z)$;

(viii) $(x \perp y) \vdash z = x \vdash (y \vdash z)$.

(7) In [17] the author showed that Jordan algebras are not categories of interest, since Axiom 2 fails. We recall that, given a field $K$, a $K$-Jordan algebra is a vector space over $K$ equipped with a binary bilinear operation which is commutative and satisfies the following axiom:

$$(xy)(xx) = x(y(xx)) \quad \text{for all } x, y.$$

Orzech provided also an example of category for which Axiom 2 is valid, but Axiom 1 fails: see [17], example 1.12.

(8) Any variety of groups (in the sense of [16]) is a category of interest. More generally, any subvariety of a category of interest is of interest. This fact, according to Theorem 4.7 below, should be compared with Proposition 2.3 in [8]. Beyond the well known examples of abelian, $n$-nilpotent, $n$-solvable groups, groups with exponent $m$ and polynilpotent groups of class $(c_1, \ldots, c_r)$ (where $n$, $m$ and the $c_i$'s are positive integers), other interesting examples of varieties of groups (see, for instance, [15] and [9]) are (we simply write the axiom satisfied by the groups belonging to the corresponding variety):

(i) $n$-abelian groups: $(xy)^n = x^n y^n$;

(ii) $n$-central groups: $[x^n, y] = 1$;

(iii) 2-Engel groups: $[[x, y], y] = 1$;

(iv) $n$-Levi groups: $[x^n, y] = [x, y]^n$;

(v) $n$-Bell groups: $[x^n, y] = [x, y^n]$,

where $n$ is a fixed positive integer.

## 4. Action accessibility of categories of interest

In this section we prove that any category of interest is action accessible. From now on we will denote by $\mathbb{C}$ a category of interest.

4.1. Definition. *[17] Let $B \in \mathbb{C}$. A subobject of $B$ is called an ideal if it is the kernel of some morphism.*

4.2. Theorem. *[17] Let $A$ be a subobject of $B$ in $\mathbb{C}$. Then $A$ is an ideal of $B$ if and only if the following conditions hold:*

*(i) $A$ is a normal subgroup of $B$;*

*(ii) for any $a \in A$, $b \in B$ and $* \in \Omega'_2$, we have $a * b \in A$.*

Given $B, X \in \mathbb{C}$, a family of (set-theoretical) maps $\{f_* : B \times X \longrightarrow X\}_{* \in \Omega_2}$ is called "action of $B$ on $X$". Every split extension $(B, A, p, s, k)$ with kernel $X$ induces an action of $B$ on $X$, given by:

$$f_+(b, x) = s(b) + k(x) - s(b)$$

and

$$f_*(b, x) = s(b) * k(x),$$

where the first equality defines the map $f_+$ relative to the group operation and the second one defines $f_*$ for any $* \in \Omega'_2$. We will denote $f_+(b, x)$ simply by $b \cdot x$ and $f_*(b, x)$ by $b * x$. Actions arising from split extensions are called derived actions in [17].

Given an action of $B$ on $X$, the semidirect product $X \rtimes B$ has $X \times B$ as a set of elements and operations defined by:

$$(x', b') + (x, b) = (x' + b' \cdot x, b' + b),$$

$$(x', b') * (x, b) = (x' * x + b *^{\circ} x' + b' * x, b' * b) \qquad \text{for all } * \in \Omega'_2.$$

In general, the semidirect product $X \rtimes B$ is not an object of $\mathbb{C}$; we have the following characterization:

4.3. Theorem. *[17] An action of $B$ on $X$ in $\mathbb{C}$ is derived if and only if $X \rtimes B$ is an object of $\mathbb{C}$.*

4.4. Definition. *[10] An action of $B$ on $X$ in $\mathbb{C}$ is said to be strict if, for any $b, b' \in B$ such that $b \cdot x = b' \cdot x$, $(\omega b) \cdot x = (\omega b') \cdot x$, $b * x = b' * x$ for any $x \in X$, $\omega \in \Omega'_1$ and $* \in \Omega'_2$, we have $b = b'$.*

It is immediate to observe that an action of $B$ on $X$ is strict if and only if, for any $b \in B$ such that $b \cdot x = x$, $(\omega b) \cdot x = x$, $b * x = 0$ for any $x \in X$, $\omega \in \Omega'_1$ and $* \in \Omega'_2$, we have $b = 0$.

4.5. Definition. *An action of $B$ on $X$ in $\mathbb{C}$ is said to be super-strict if, for any $b, b' \in B$ such that $b \cdot x = b' \cdot x$ and $b * x = b' * x$ for any $x \in X$ and $* \in \Omega'_2$, we have $b = b'$.*

4.6. Proposition. *Given $(D, C, q, t, l) \in SplExt(X)$, then the following conditions are equivalent:*

*(a) the corresponding action of $D$ on $X$ is super-strict;*

*(b) the corresponding action of $D$ on $X$ is strict;*

*(c) $(D, C, q, t, l)$ is faithful.*

Proof. (a) $\Longrightarrow$ (b) Obvious.

(b) $\Longrightarrow$ (c) Let $(g, f)$ and $(g', f')$ be morphisms from $(B, A, p, s, k)$ to $(D, C, q, t, l)$ (see diagram 2). It suffices to show that $g(b) = g'(b)$, for all $b \in B$. In order to do that, since the action is strict, we will prove the following equalities for any $x \in X$:

(i) $tg(b) + l(x) - tg(b) = tg'(b) + l(x) - tg'(b)$;

(ii) $t(\omega g(b)) + l(x) - t(\omega g(b)) = t(\omega g'(b)) + l(x) - t(\omega g'(b))$ for all $\omega \in \Omega_1'$;

(iii) $tg(b) * x = tg'(b) * x$ for all $* \in \Omega_2'$.

The equality (i) is equivalent to:

$$-tg'(b) + tg(b) + l(x) - tg(b) + tg'(b) = l(x);$$

since $tg = fs$, $tg' = f's$ and $l = fk = f'k$, we obtain:

$$-f's(b) + fs(b) + fk(x) - fs(b) + f's(b) = f'k(x),$$

that is:

$$-f's(b) + f(s(b) + k(x) - s(b)) + f's(b) = f'k(x).$$

$s(b) + k(x) - s(b) \in k(X)$, so there exists $y \in X$ such that $s(b) + k(x) - s(b) = k(y)$ and we have:

$$\begin{aligned} & -f's(b) + fk(y) + f's(b) = f'k(x) \\ \Longleftrightarrow \quad & -f's(b) + f'k(y) + f's(b) = f'k(x) \\ \Longleftrightarrow \quad & f'(-s(b) + k(y) + s(b)) = f'k(x). \end{aligned}$$

But

$$-s(b) + k(y) + s(b) = -s(b) + s(b) + k(x) - s(b) + s(b) = k(x)$$

and then the equality (i) holds for any $x \in X$.

Condition (ii) is very similar to condition (i); we have:

$$\begin{aligned} & t(\omega g(b)) + l(x) - t(\omega g(b)) = t(\omega g'(b)) + l(x) - t(\omega g'(b)) \\ \Longleftrightarrow \quad & -t\omega g'(b) + t\omega g(b) + l(x) - t\omega g(b) + t\omega g'(b) = l(x); \end{aligned}$$

but $g\omega = \omega g$ and $g'\omega = \omega g'$ and so the equality becomes:

$$-tg'(\omega b) + tg(\omega b) + l(x) - tg(\omega b) + tg'(\omega b) = l(x)$$

$$\iff \quad -f's(\omega b) + fs(\omega b) + fk(x) - fs(\omega b) + f's(\omega b) = f'k(x)$$

$$\iff \quad -f's(\omega b) + f(s(\omega b) + k(x) - s(\omega b)) + f's(\omega b) = f'k(x).$$

Again, $s(\omega b) + k(x) - s(\omega b) \in k(X)$, so there exists $z \in k(X)$ such that $s(\omega b) + k(x) - s(\omega b) = k(z)$, then:

$$-f's(\omega b) + fk(z) + f's(\omega b) = f'k(x)$$

$$\iff \quad -f's(\omega b) + f'k(z) + f's(\omega b) = f'k(x)$$

$$\iff \quad f'(-s(\omega b) + k(z) + s(\omega b)) = f'k(x).$$

Since

$$-s(\omega b) + k(z) + s(\omega b) = -s(\omega b) + s(\omega b) + k(x) - s(\omega b) + s(\omega b) = k(x),$$

also the equality (ii) holds for any $x \in X$.

Concerning (iii), we have that

$$p(s(b) * k(x)) = ps(b) * pk(x) = ps(b) * 0 = 0,$$

so there exists $y \in X$ such that $k(y) = s(b) * k(x)$ and then

$$tg(b) * l(x) = fs(b) * fk(x) = f(s(b) * k(x)) = fk(y)$$

$$= f'k(y) = f'(s(b) * k(x)) = f's(b) * f'k(x) = tg'(b) * l(x).$$

Now, since the action of $B$ on $X$ is strict, we have that $g(b) = g'(b)$, for all $b \in B$, and this completes the proof, since $f$ is uniquely determined by $g$.

(c) $\Longrightarrow$ (a) Fixed $d \in D$, it is possible to construct $(B, A, p, s, k) \in SplExt(X)$ and a morphism

$$(g, f) : (B, A, p, s, k) \longrightarrow (D, C, q, t, l)$$

as follows:

- let $B$ the free object in $\mathbb{C}$ on a one-element set $\{z\}$ (it exists, since, as observed in [17], all categories of interest are monadic over the category $\mathcal{S}et$ of sets; for an explicit description of free objects, see for example [1]);
- $g : B \longrightarrow D$ is the unique morphism in $\mathbb{C}$ such that $g(z) = d$ (whose existence and unicity is given by the universal property of $B$);

- $A = X \rtimes B$, where the action of $B$ on $X$ is induced, via $g$, by the action of $D$ on $X$, that is:
$$b \cdot x = g(b) \cdot x = tg(b) + l(x) - tg(b),$$
$$b * x = g(b) * x = tg(b) * l(x) \qquad \text{for all } b \in B, x \in X \text{ and } * \in \Omega_2',$$
where we are identifying $l(X)$ and $X$. Hence the operations in $A$ are defined as follows:
$$(x, b) + (x', b') = (x + b \cdot x', b + b'),$$
$$(x, b) * (x', b') = (x * x' + b' *^{\circ} x + b * x', b * b') \quad \text{for all } * \in \Omega_2'.$$
We observe that $A \in \mathbb{C}$, since the action of $B$ on $X$ is derived: in fact, it is induced by the action of $D$ on $X$, which is derived (see [10], where some necessary and sufficient conditions for an action in order to be derived are given).

- the morphisms $p, s, k$ and $f$ are given by:
$$p(x, b) = b, \quad s(b) = (0, b), \quad k(x) = (x, 0), \quad f(x, b) = l(x) + tg(b).$$
Clearly, $p, s$ and $k$ are morphisms in $\mathbb{C}$ such that $(B, A, p, s, k)$ is a split extension and the following diagram commutes:
$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{k} & A & \underset{s}{\overset{p}{\rightleftarrows}} & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_X} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & X & \xrightarrow{l} & C & \underset{t}{\overset{q}{\rightleftarrows}} & D & \longrightarrow & 0. \end{array}$$
It remains to prove that $f$ is a morphism in $\mathbb{C}$:
$$f[(x_1, b_1) + (x_2, b_2)] = f(x_1 + b_1 \cdot x_2, b_1 + b_2) = l(x_1 + b_1 \cdot x_2) + tg(b_1 + b_2)$$
$$= l(x_1) + tg(b_1) + l(x_2) - tg(b_1) + tg(b_1) + tg(b_2)$$
$$= l(x_1) + tg(b_1) + l(x_2) + tg(b_2) = f(x_1, b_1) + f(x_2, b_2)$$
and
$$f[(x_1, b_1) * (x_2, b_2)] = f(x_1 * x_2 + b_2 *^{\circ} x_1 + b_1 * x_2, b_1 * b_2)$$
$$= l(x_1 * x_2 + b_2 *^{\circ} x_1 + b_1 * x_2) + tg(b_1 * b_2)$$
$$= l(x_1) * l(x_2) + tg(b_2) *^{\circ} l(x_1) + tg(b_1) * l(x_2) + tg(b_1) * tg(b_2)$$
$$= l(x_1) * l(x_2) + l(x_1) * tg(b_2) + tg(b_1) * l(x_2) + tg(b_1) * tg(b_2)$$
$$= l(x_1) * [l(x_2) + tg(b_2)] + tg(b_1) * [l(x_2) + tg(b_2)]$$
$$= [l(x_1) + tg(b_1)] * [l(x_2) + tg(b_2)] = f(x_1, b_1) * f(x_2, b_2).$$
Hence $(g, f)$ is a morphism in $SplExt(X)$.

Let us now construct $(B', A', p', s', k') \in SplExt(X)$ and a morphism

$$(g', f') : (B', A', p', s', k') \longrightarrow (D, C, q, t, l)$$

in the same way as above, but choosing another element $d' \in D$ instead of $d$. We will prove that, if $d$ and $d'$ act in the same way on $X$, then $(B', A', p', s', k') = (B, A, p, s, k)$. Of course $B' = B$ as objects in $\mathbb{C}$, $p' = p$, $s' = s$, $k' = k$ and $A' = A$ as sets; it remains to prove that the operations defined on $A$ and $A'$ are the same. In order to do that, it suffices to prove that:

(i) $tg(b) + l(x) - tg(b) = tg'(b) + l(x) - tg'(b)$

(ii) $tg(b) * l(x) = tg'(b) * l(x) \qquad$ for all $x \in X,\ b \in B$ and $* \in \Omega_2'$.

Since, by hypothesis on $d$ and $d'$, (i) and (ii) hold for $b = z$, it suffices to show that the set $Z$ of those elements $b \in B$ for which (i) and (ii) hold for every $x \in X$ is a subobject of $B$, i.e. it is closed under the operations in $B$.

Let $b_1, b_2 \in Z$; then, for all $x \in X$:

$$tg(b_1 - b_2) + l(x) - tg(b_1 - b_2) = tg(b_1) - tg(b_2) + l(x) + tg(b_2) - tg(b_1)$$

$$= tg(b_1) - tg'(b_2) + l(x) + tg'(b_2) - tg(b_1) = tg'(b_1) - tg'(b_2) + l(x) + tg'(b_2) - tg'(b_1)$$
$$= tg'(b_1 - b_2) + l(x) - tg'(b_1 - b_2),$$

where the third equality holds because $-tg'(b_2) + l(x) + tg'(b_2) \in l(X)$, and

$$tg(b_1 - b_2) * l(x) = [tg(b_1) - tg(b_2)] * l(x)$$

$$= tg(b_1) * l(x) - tg(b_2) * l(x) = tg'(b_1) * l(x) - tg'(b_2) * l(x)$$
$$= [tg'(b_1) - tg'(b_2)] * l(x) = tg'(b_1 - b_2) * l(x).$$

Then $b_1 - b_2 \in Z$ and $Z$ is a subgroup of $B$. Moreover, if $b_1, b_2 \in Z$, $x \in X$ and $* \in \Omega_2'$, then:

$$tg(b_1 * b_2) + l(x) - tg(b_1 * b_2) = [tg(b_1) * tg(b_2)] + l(x) - [tg(b_1) * tg(b_2)]$$

$$= l(x) + [tg(b_1) * tg(b_2)] - [tg(b_1) * tg(b_2)] = l(x)$$
$$= l(x) + [tg'(b_1) * tg'(b_2)] - [tg'(b_1) * tg'(b_2)]$$
$$= [tg'(b_1) * tg'(b_2)] + l(x) - [tg'(b_1) * tg'(b_2)] = tg'(b_1 * b_2) + l(x) - tg'(b_1 * b_2),$$

where the second and the fifth equalities follow from Axiom 1, and

$$tg(b_1 * b_2) * l(x) = [tg(b_1) * tg(b_2)] * l(x)$$

$$= W(tg(b_1)(tg(b_2)l(x)), tg(b_1)(l(x)tg(b_2)), (tg(b_2)l(x))tg(b_1), (l(x)tg(b_2))tg(b_1),$$
$$tg(b_2)(tg(b_1)l(x)), tg(b_2)(l(x)tg(b_1)), (tg(b_1)l(x))tg(b_2), (l(x)tg(b_1))tg(b_2))$$

$$= W(tg'(b_1)(tg'(b_2)l(x)), tg'(b_1)(l(x)tg'(b_2)), (tg'(b_2)l(x))tg'(b_1), (l(x)tg'(b_2))tg'(b_1),$$
$$tg'(b_2)(tg'(b_1)l(x)), tg'(b_2)(l(x)tg'(b_1)), (tg'(b_1)l(x))tg'(b_2), (l(x)tg'(b_1))tg'(b_2))$$
$$= [tg'(b_1) * tg'(b_2)] * l(x) = tg'(b_1 * b_2) * l(x),$$

where we are using Axiom 2 and the fact that $l(X)$ is an ideal of $C$. Hence $Z$ is a subobject of $B$ and then $Z = B$. Since $(D, C, q, t, l)$ is faithful, from $(B', A', p', s', k') = (B, A, p, s, k)$ it follows that $g = g'$ and so $d = d'$ and the action of $D$ on $X$ is super-strict. ∎

4.7. Theorem. *Every category of interest $\mathbb{C}$ is action accessible.*

Proof. Let $(B, A, p, s, k)$ be in $SplExt(X)$; we have to construct a morphism

$$(g, f) : (B, A, p, s, k) \longrightarrow (D, C, q, t, l)$$

with a faithful codomain. Let us consider the set

$$I = \{i \in B | s(i) + k(x) - s(i) = k(x), \quad s(i) * k(x) = 0 \quad \text{for all } x \in X, \text{ for all } * \in \Omega_2'\};$$

we will show that $I$ is an ideal of $B$. Let $i_1, i_2 \in I$; then

$$s(i_1 - i_2) + k(x) - s(i_1 - i_2) = s(i_1) - s(i_2) + k(x) + s(i_2) - s(i_1) = k(x),$$

since $s(i_j) + k(x) - s(i_j) = k(x)$, $j = 1, 2$, and

$$s(i_1 - i_2) * k(x) = (s(i_1) - s(i_2)) * k(x) = s(i_1) * k(x) - s(i_2) * k(x) = 0 - 0 = 0,$$

then $I$ is a subgroup of $B$. If $i \in I$ and $b \in B$, then

$$s(b + i - b) + k(x) - s(b + i - b) = s(b) + s(i) - s(b) + k(x) + s(b) - s(i) - s(b);$$

$-s(b)+k(x)+s(b) \in k(X)$ (since $k(X)$ is a normal subgroup of $A$), so $-s(b)+k(x)+s(b) = k(y)$ for some $y \in X$; then

$$s(b) + s(i) - s(b) + k(x) + s(b) - s(i) - s(b) = s(b) + k(y) - s(b) = k(x),$$

where the first equality holds because $i \in I$; moreover

$$s(b+i-b) * k(x) = (s(b) + s(i) - s(b)) * k(x) = s(b) * k(x) + s(i) * k(x) - s(b) * k(x) = 0$$

since $i \in I$. Then $I$ is a normal subgroup of $B$. Finally, let $i \in I$, $b \in B$ and $* \in \Omega_2'$; then

$$s(i * b) + k(x) - s(i * b) = (s(i) * s(b)) + k(x) - (s(i) * s(b))$$
$$= k(x) + (s(i) * s(b)) - (s(i) * s(b)) = k(x),$$

where the second equality follows from Axiom 1 of the definition of category of interest. Furthermore, for any $\bullet \in \Omega_2'$, $s(i * b) \bullet k(x) = 0$ since, by Axiom 2,

$$s(i * b) \bullet k(x) = (s(i) * s(b)) \bullet k(x) = W(s(i)(s(b)k(x)), s(i)(k(x)s(b)), (s(b)k(x))s(i),$$

$$(k(x)s(b))s(i), s(b)(s(i)k(x)), s(b)(k(x)s(i)), (s(i)k(x))s(b), (k(x)s(i))s(b)),$$

and, in any component of the word $W$, $s(i)$ operates (w.r.t. the operations in $\Omega_2'$) on elements of $k(X)$, then any component is 0, so that $W = 0$. So, by theorem 4.2, $I$ is an ideal of $B$. We define $D = B/I$.

Let us define now an action of $B/I$ on $X$ in the following way:

$$(b + I) \cdot x = s(b) + k(x) - s(b);$$

$$(b + I) * x = s(b) * k(x) \quad \text{for all } * \in \Omega_2'.$$

This action is well-defined; indeed, if $b_1 + I = b_2 + I$, i.e. $b_1 - b_2 \in I$, then

$$(s(b_1) + k(x) - s(b_1)) - (s(b_2) + k(x) - s(b_2)) = s(b_1) + k(x) - s(b_1) + s(b_2) - k(x) - s(b_2)$$

$$= s(b_2) - s(b_2) + s(b_1) + k(x) - s(b_1) + s(b_2) - k(x) - s(b_2)$$

$$= s(b_2) + s(-b_2 + b_1) + k(x) - s(-b_2 + b_1) - k(x) - s(b_2) = s(b_2) + k(x) - k(x) - s(b_2) = 0$$

and then

$$s(b_1) + k(x) - s(b_1) = s(b_2) + k(x) - s(b_2).$$

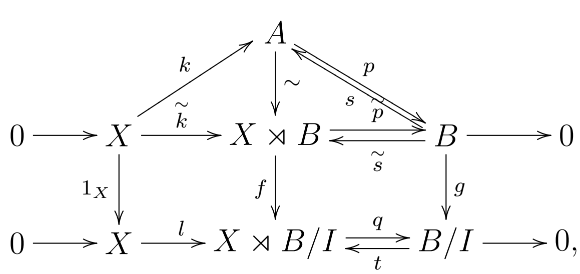
Analogously

$$s(b_1) * k(x) - s(b_2) * k(x) = (s(b_1) - s(b_2)) * k(x) = s(b_1 - b_2) * k(x) = 0$$

and then

$$s(b_1) * k(x) = s(b_2) * k(x) \qquad \text{for all} * \in \Omega_2'.$$

The action of $B/I$ on $X$ is derived, in fact the conditions needed to be satisfied (again, see [10]) are the same already satisfied by the action of $B$ on $X$. We define then $C = X \rtimes B/I$ with respect to the action defined above. Since $A \simeq X \rtimes B$, we can consider the following diagram:




where $q, t$, and $l$ are the usual semidirect product projection and injections, $g = \pi_B$ is the canonical projection and $f = (1_X, g)$. $f$ is a morphism in $\mathbb{C}$ (since it is equivariant with respect to the action) and the diagram commutes, so $(g, f)$ is a morphism in $SplExt(X)$.

It remains to show that $(B/I, X \rtimes B/I, q, t, l)$ is faithful. Thanks to Proposition 4.6, it suffices to prove that the corresponding action is super-strict, that is, given $b + I \in B/I$ with

(i) $(b+I)\cdot x = x$;

(ii) $(b+I) * x = 0$

for any $x \in X$ and $* \in \Omega_2'$, then $b+I=0$. Conditions (i) and (ii) can be then rewritten in the following form:

(i') $s(b)+k(x)-s(b)=k(x)$ for all $x \in X$;

(ii') $s(b) * k(x) = 0$ for all $x \in X$, $* \in \Omega_2'$.

But, if $b$ satisfies (i') and (ii'), then $b \in I$ and $b+I=0$. So $(B/I, X \rtimes B/I, q, t, l)$ is faithful and this concludes the proof. ∎

## References


[1] J. Adámek, J. Rosicky, *Locally presentable and accessible categories*, London Mathematical Society lecture notes series, vol. 189 (1994), Cambridge University Press.

[2] M. Barr, *Cohomology and obstructions: commutative algebras*, in Sem. Triples and Categorical homology theory, Lecture Notes in Mathematics, vol. 80 (1969), Springer-Verlag, 357-375.

[3] F. Borceux, D. Bourn, *Split extension classifier and centrality*, Contemporary Mathematics, vol. 431 (2007), 85-104.

[4] F. Borceux, D. Bourn, *Mal'cev, protomodular, homological and semi-abelian categories*, Mathematics and its applications, vol. 566 (2004), Kluwer.

[5] F. Borceux, G. Janelidze, G.M. Kelly, *On the representability of actions in a semi-abelian category*, Theory and Applications of Categories 14 (2005), 244-286.

[6] D. Bourn, *Commutator theory in regular Malcev categories*, in Hopf algebras and Semi-Abelian Categories, the Fields Institute Communications, vol 43 (2004), Amer. Math. Soc., 61-75.

[7] D. Bourn, M. Gran *Centrality and connectors in Maltsev categories*, Algebra Universalis 48 (2002), 309-331.

[8] D. Bourn, G. Janelidze, *Centralizers in action accessible categories*, preprint Cahiers LMPA n. 344 (2007).

[9] R. Brandl, L.C. Kappe, *On n-Bell groups*, Comm. Algebra 17 n.4 (1989), 787-807.

[10] J.M. Casas, T. Datuashvili and M. Ladra, *Universal strict general actors and actors in categories of interest*, Appl. Cat. Structures, in press, DOI 10.1007/510485-008-9166-z.

[11] J.L. Loday, *Une version non commutative des algèbres de Lie: les algèbres de Leibniz*, Enseign. Math 39 n.2 (1993), 269-293.

[12] J.L. Loday, *Algèbres ayant deux opérations associatives (digèbres)*, C.R. Acad. Sci. Paris 321 (1995), 141-146.

[13] J.L. Loday, *Dialgebras*, in Dialgebras and Related Operads, Lecture Notes in Mathematics, vol. 1763 (2001), Springer-Verlag.

[14] J.L. Loday, M.O. Ronco, *Trialgebras and families of polytopes*, Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-theory, Contemporary Mathematics 346 (2004), 369-398.

[15] L.C. Kappe, *On n-Levi groups*, Arch. Math. 47 (1986), 198-210.

[16] H. Neumann, *Varieties of groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 37 (1967), Springer-Verlag.

[17] G. Orzech, *Obstruction theory in algebraic categories, I*, J. Pure and Appl. Algebra 2 (1972), 287-314.

[18] G. Orzech, *Obstruction theory in algebraic categories, II*, J. Pure and Appl. Algebra 2 (1972), 315-340.

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