Theory and Applications of Categories, Vol. 22, No. 1, 2009, pp. 1–23.

DUALITY FOR CCD LATTICES

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ABSTRACT. The 2-category of constructively completely distributive lattices is shown to be bidual to a 2-category of generalized orders that admits a monadic schizophrenic object biadjunction over the 2-category of ordered sets.

1. Introduction

1.1. Constructively completely distributive (CCD) lattices were introduced in [F&W] and further studied in the series of papers [RW2], [RW3], [RW4], [P&W], [MRW], [RWb], and [CMW] . For background not explicitly given here, we refer the reader to [RW4]. For an ordered set $(A, \leq)$, a subset S of A is said to be a *downset* if $b \leq a \in S$ implies $b \in S$. The downsets of the form $\downarrow a = \{b \mid b \leq a\}$ are said to be *principal*. Denote by DA the lattice of downsets ordered by inclusion. Recall that an ordered set $(A, \leq)$ is said to be *complete* if the embedding of principal downsets $\downarrow : A \to DA$ has a left adjoint $\bigvee : DA \to A$. If $\bigvee$ has a left adjoint $\Downarrow : A \to DA$ then A is said to be *constructively completely distributive*. Recall too that, for any complete ordered set $(A, \leq)$, the *totally below* relation is defined by $b \ll a$ if and only if $(\forall S \in DA)(a \leq \bigvee S \Longrightarrow b \in S)$. To say that A is CCD is equivalent to

$$(\forall a \in A)(a = \bigvee\{b \mid b \ll a\})$$

A restricted class of CCD lattices, called *totally algebraic* in [RW4], are those complete A for which

$$(\forall a \in A)(a = \bigvee\{b \mid b \ll b \leq a\})$$

1.2. At the International Category Theory Conference held in Vancouver in 1997, Paul Taylor raised two questions about the adjunction

$$\mathbf{ord}(-, \Omega)^{\mathrm{op}} \dashv \mathbf{ord}(-, \Omega) : \mathbf{ord}^{\mathrm{op}} \to \mathbf{ord}$$

Here **ord** is the 2-category of ordered sets, order-preserving functions, and inequalities and Ω is the subobject classifier of the base topos $\mathcal{S}$, which is not assumed to be Boolean.

The authors gratefully acknowledge financial support from the Canadian NSERC. Diagrams typeset using M. Barr's diagxy package

Received by the editors 2008-10-06 and, in revised form, 2009-01-25.

Transmitted by Susan Niefield. Published on 2009-01-28.

2000 Mathematics Subject Classification: 18A25.

Key words and phrases: adjunction, completely distributive, idempotent, monadic, proarrow equipment, cauchy complete.

i) What are the algebras for the generated monad?

ii) Is $\mathbf{ord}(-,\Omega)$ monadic?

Taylor had conjectured that the answer to his first question is **ccd**, the 2-category of CCD lattices, order preserving functions that have both left and right adjoints and inequalities. In [MRW] this was affirmed and the second question was answered negatively by recalling from [RW4] that $\mathbf{ord}^{\mathrm{op}}$ is biequivalent to the full subcategory of **ccd** determined by the totally algebraic lattices described in 1.1. (More precisely, we should say that ${\mathbf{ord}_{cc}}^{\mathrm{op}}$, the **ord**-category of *antisymmetric* ordered sets, is biequivalent to that of totally algebraic lattices. The $(-)_{cc}$ terminology will be explained in the sequel.)

1.3. Nevertheless, there is value in expanding on the second question, which sought to generalize Paré's celebrated result in [PAR] that $\Omega^{(-)} : \mathcal{S}^{\mathrm{op}} \to \mathcal{S}$ is monadic, for $\mathcal{S}$ any elementary topos. However, since **ord** is a 2-category and earlier results on CCD lattices fully exploited the extra dimension we asked:

iii) Is there a bimonadic schizophrenic object biadjunction whose domain is biequivalent to **ccd**?

The answer we give here was anticipated in [RW4] but is somewhat subtle. Write **idm** for a certain 2-category in which the objects $(X,<)$ consist of a set together with a transitive, interpolative relation and the arrows are functions that preserve the relation. The 2-category **ccd** is biequivalent to ${\mathbf{idm}_{cc}}^{\mathrm{op}}$, where $\mathbf{idm}_{cc}$ is the full sub-2-category of **idm** consisting of the *Cauchy complete* objects. The emphasized term refers to a slight generalization of the concept given by F.W. Lawvere in [LAW]. What is truly new here is that $\mathbf{idm}_{cc}(-,i\Omega) : {\mathbf{idm}_{cc}}^{\mathrm{op}} \to \mathbf{ord}$, where $i\Omega$ is the subobject classifier seen as an object of **idm**, is bimonadic.

1.4. Lawvere's definition of Cauchy completeness was given for categories enriched over a symmetric monoidal category $\mathcal{V}$, in terms of the pseudofunctor $(-)_* : \mathcal{V}\text{-}\mathbf{cat} \to \mathcal{V}\text{-}\mathbf{prof}$ which sends a $\mathcal{V}$-functor $f : A \to B$ to the $\mathcal{V}$-profunctor $f_* = B(-,f-) : A \to B$. The 2-category **idm**, is not of the form $\mathcal{V}\text{-}\mathbf{cat}$ for some $\mathcal{V}$. However, there is a a pseudofunctor $(-)_\# : \mathbf{idm} \to \mathbf{krl}$, where **krl** is the idempotent splitting completion of **rel**, the 2-category of sets, relations, and containments, such that $(-)_\#$ shares important properties with the $(-)_* : \mathcal{V}\text{-}\mathbf{cat} \to \mathcal{V}\text{-}\mathbf{prof}$. Such pseudofunctors were called proarrow equipments in [Wd1] and [Wd2] and already in [RW4] they were used to study CCD lattices. Moreover, it was recognized in [RW4] that Cauchy completeness could be defined in the generality of proarrow equipment and that applied to $(-)_\# : \mathbf{idm} \to \mathbf{krl}$ it expressed results about left adjoints in **krl** in terms of arrows in **idm**. We refer here to Remark 22 and the last paragraph of Section 4 in [RW4]. What was *not* understood in [RW4] was that the ordered set of (strong) downsets DX of an object of **idm** is *equivalent* to the ordered set $\mathbf{idm}(X^{\mathrm{op}}, i\Omega)$ and that iD provides a very special right biadjoint to $(-)_\# : \mathbf{idm} \to \mathbf{krl}$. The use of equivalences, as opposed to isomorphisms, of ordered sets is critical throughout this paper.

1.5. In Section 2 we review proarrow equipment in a general way and give examples, before introducing the idea of a KZ-adjunction. These are to KZ-doctrines what ordinary adjunctions are to ordinary monads. We define existence of *power objects* relative to a proarrow equipment $(-)_*:\mathcal{K} \to \mathcal{M}$ to mean the existence of a right KZ-adjoint for $(-)_*:\mathcal{K} \to \mathcal{M}$. The general KZ-adjunction simplifies considerably in this case and we explore precisely what is needed to provide power objects. Any proarrow equipment $(-)_*:\mathcal{K} \to \mathcal{M}$ factors to give $(-)_*:\mathcal{K} \to \mathrm{map}\mathcal{M}$, where $\mathrm{map}\mathcal{M}$ is the locally full subbicategory of $\mathcal{M}$ determined by the left adjoints. In Section 3 we define existence of Cauchy completions relative to $(-)_*$ to mean the existence of a *strong* right KZ-adjoint for $(-)_*:\mathcal{K} \to \mathrm{map}\mathcal{M}$. Strong KZ-adjunctions are to KZ-adjunctions what pseudo idempotent pseudomonads are to KZ-doctrines. We prove some basic results about the relationship between power objects and Cauchy completions, noting in particular that power objects are Cauchy complete. In Section 4 we introduce $(-)_\#:\mathbf{idm} \to \mathbf{krl}$ formally and provide a full context for it, especially relative to ordered sets and order ideals. The most important result of this section is Proposition 4.12 which, together with power objects for $(-)_\#:\mathbf{idm} \to \mathbf{krl}$, provides the essential ingredients setting this work apart from earlier CCD papers. In Section 5 we apply the results of Sections 2 and 3 to $(-)_\#:\mathbf{idm} \to \mathbf{krl}$, identifying some additional features present in this example, notably that the dual, Y^{op}, of a Cauchy complete object Y is also Cauchy complete. Finally, in Section 6 we assemble what we have developed so as to state and prove the main bimonadicity result.

2. Proarrow Equipments Redux and Power Objects

2.1. For $\mathcal{K}$ and $\mathcal{M}$ bicategories, a pseudofunctor $(-)_*:\mathcal{K} \to \mathcal{M}$ is said to be *proarrow equipment* in the sense of [Wd1] and [Wd2] if

i) $(-)_*$ is the identity on objects;

ii) $(-)_*$ is locally fully faithful;

iii) (for all $f:A \to B$ in $\mathcal{K}$) (there is an adjunction $f_* \dashv f^*:B \to A$ in $\mathcal{M}$).

We can regard $\mathcal{K}$ via $(-)_*$ as a locally-full subbicategory of $\mathcal{M}$. In the context of proarrow equipment, an arrow in $\mathcal{K}$ is representably fully faithful if and only if its given adjunction in $\mathcal{M}$ has an invertible unit. It is in fact unambiguous in this context to call such an arrow in $\mathcal{K}$ *fully faithful*.

2.2. As examples of proarrow equipment we mention:

i) $(-)_*:\mathcal{V}\text{-}\mathbf{cat} \to \mathcal{V}\text{-}\mathbf{prof}$, for a cocomplete symmetric monoidal category $\mathcal{V}$. Here for a $\mathcal{V}$-functor $f:B \to A$, the $\mathcal{V}$-profunctor $f_*:B \to A$ is given by $f_*(a,b) = A(a,fb)$.

ii) $(-)_\#:\mathcal{V}\text{-}\mathbf{tax} \to \mathcal{V}\text{-}\mathbf{dist}$, for a cocomplete symmetric monoidal category $\mathcal{V}$. The objects of $\mathcal{V}\text{-}\mathbf{tax}$ are $\mathcal{V}$-taxons (called $\mathcal{V}$-taxonomies in [KOS]). Taxons are similar to categories except that the existence of identity data is replaced by the requirement that the associativity of composition diagram (suitably expressed) be a coequalizer. The arrows are what might be called $\mathcal{V}$-semi-functors. The arrows of $\mathcal{V}$-**dist**, that we call $\mathcal{V}$-distributors, are the i-modules of [KOS]. The 2-cells of $\mathcal{V}$-**tax** are defined by generalizing Mac Lane's "arrows only" description of natural transformations. (See [MAC], page 19.) This paper will involve a very special case of this example and we forego further mention of the general case, referring the interested reader to [KOS].

iii) $(-)_* : \mathbf{TOP_{geo}} \to \mathbf{TOP_{lex}}^{co}$, in which a geometric morphism between toposes is sent to its direct image. This example was studied extensively in [RWp] and [RWc].

iv) The previous example is a special case of map$\mathcal{M} \to \mathcal{M}$ which is proarrow equipment, for $\mathcal{M}$ an arbitrary bicategory. Here map$\mathcal{M}$ is the locally full subbicategory of $\mathcal{M}$ determined by the left adjoint arrows. We follow the convention of calling a left adjoint arrow a *map*.

v) $(-)_* : \mathbf{set} \to \mathbf{rel}$, which sends a function $f : B \to A$ to its graph $f_* : B \to A$ so that af_*b if and only if $a = fb$.

2.3. In [LAW] an object B in $\mathcal{V}$-**cat** was said to be *Cauchy complete* if, for all A, every adjunction $L \dashv R : B \to A$ in $\mathcal{V}$-**prof** arises as one of the form $f_* \dashv f^* : B \to A$ for an $f : A \to B$ in $\mathcal{V}$-**cat**. It is clear immediately that for any proarrow equipment $(-)_* : \mathcal{K} \to \mathcal{M}$ one can define an object Y in $\mathcal{K}$ to be *Cauchy complete* if, for every map $L : X \to Y$ in $\mathcal{M}$, there is an arrow $f : X \to Y$ in $\mathcal{K}$ and an isomorphism $f_* \cong L$ in $\mathcal{M}$. If $\mathcal{V}$ is complete, the question of Cauchy completeness of an object B in $\mathcal{V}$-**cat** can be concentrated in a single $\mathcal{V}$-functor, the inclusion of B in its Cauchy completion $B \to \mathcal{Q}B$, making the condition far more tractable. We turn shortly to the possibility of doing this for a general proarrow equipment $(-)_* : \mathcal{K} \to \mathcal{M}$.

2.4. It will be convenient to write $S : \mathcal{K} \to \mathcal{M}$ for a general pseudofunctor and recall the concept of a right biadjoint $\mathcal{P} : \mathcal{M} \to \mathcal{K}$ for S. In addition to the pseudofunctors S and $\mathcal{P}$, a biadjunction $(S \dashv \mathcal{P}; y, e; \eta, \epsilon)$ consists of a unit $y : 1_{\mathcal{K}} \to \mathcal{P}S$, a counit $e : S\mathcal{P} \to 1_{\mathcal{M}}$, and invertible constraints ϵ and η as shown below.

The constraints $\epsilon : eS.Sy \to 1_S$ and $\eta : 1_{\mathcal{P}} \to \mathcal{P}e.y\mathcal{P}$ for a biadjunction are required to satisfy coherence equations, for which [S&W] is a convenient reference. But since our application will be to locally ordered $\mathcal{M}$, the coherence equations are not relevant for this paper. In [M&W] it was shown that to give an adjunction $Sy \dashv eS$, with unit ϵ^{-1} (and components in $\mathcal{M}$), is to give an adjunction $\mathcal{P}e \dashv y\mathcal{P}$, with counit η^{-1} (and components in $\mathcal{K}$). Moreover, $Sy \dashv eS$ is an adjoint equivalence if and only if $\mathcal{P}e \dashv y\mathcal{P}$ is an adjoint equivalence. A biadjunction $(S \dashv \mathcal{P}; y, e; \eta, \epsilon)$ satisfying either of the equivalent conditions

$$Sy \dashv eS \quad \text{with unit} \quad \epsilon^{-1} \tag{1}$$

or

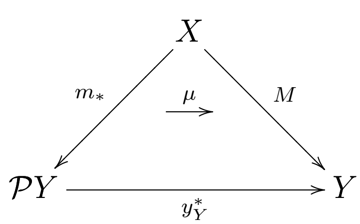
$$\mathcal{P}e \dashv y\mathcal{P} \quad \text{with counit} \quad \eta^{-1} \tag{2}$$

is called a *KZ-adjunction*. It should be noted that for such a biadjunction, for each object X in $\mathcal{K}$, the component SyX is necessarily fully faithful in $\mathcal{M}$ and, for each object Y in $\mathcal{M}$, the component $y\mathcal{P}Y$ is necessarily fully faithful in $\mathcal{K}$. A KZ-adjunction will be said to be *strong* if the counit for the adjunction $Sy \dashv eS$ is also invertible, equivalently the unit for the adjunction $\mathcal{P}e \dashv y\mathcal{P}$ is also invertible. It was shown in [M&W] that a KZ-adjunction generates a KZ-doctrine on $\mathcal{K}$ and that a strong KZ-adjunction generates a pseudo-idempotent pseudomonad on $\mathcal{K}$.

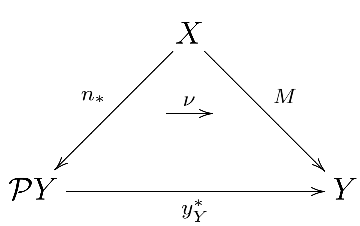
2.5. A proarrow equipment $(-)_* : \mathcal{K} \to \mathcal{M}$ is said to have *power objects* if it has a right KZ-adjoint. Since $(-)_*$ is the identity on objects, the notion of KZ-adjunction simplifies in this case. We will show in the next four subsections that to give a right KZ-adjoint for a proarrow equipment $(-)_* : \mathcal{K} \to \mathcal{M}$ is to give, for each object Y, an arrow $y_Y : Y \to \mathcal{P}Y$ in $\mathcal{K}$ with invertible unit $1_Y \to {y_Y}^* {y_Y}_*$ in $\mathcal{M}$ so that, for each object X, composition with ${y_Y}^* : \mathcal{P}Y \to Y$ preceded by $(-)_*$ provides an equivalence of categories

$$\mathcal{K}(X, \mathcal{P}Y) \xrightarrow{y_Y^*(-)_*} \mathcal{M}(X, Y) \tag{3}$$

2.6. LEMMA. *The equivalence (3) provides, for each $M : X \to Y$ in $\mathcal{M}$, a diagram*



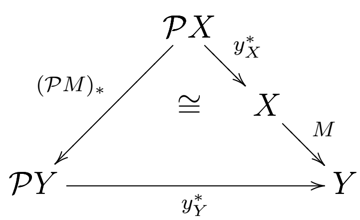
with m in $\mathcal{K}$ and invertible μ, universal among diagrams of the form



for n in $\mathcal{K}$ and arbitrary ν. Moreover if ν is invertible then the induced $n \to m$ is invertible.

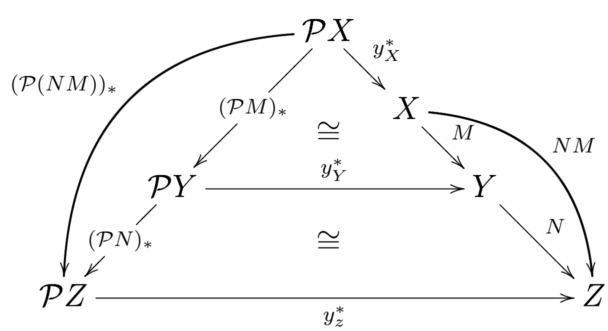
PROOF. Given $M : X \to Y$ in $\mathcal{M}$ we find $m : X \to \mathcal{P}Y$ in $\mathcal{K}$ and invertible μ as shown using the essential surjectivity of $y_Y^*(-)_*$. For the universal property, assume that we have $n : X \to \mathcal{P}Y$ in $\mathcal{K}$ and arbitrary $\nu : y_Y^* n_* \to M$. In this event we have $\mu^{-1}.\nu : y_Y^* n_* \to y_Y^* m_*$. By fully faithfulness of $y_Y^*(-)_*$ there is a unique 2-cell $\tau : n \to m$ in $\mathcal{K}$ so that the whisker composite $y_Y^* \tau_*$ is equal to the pasting of μ^{-1} to ν along M. Of course this immediately says that τ is unique with the property that τ_* pasted to μ along m^* is equal to ν. If ν is invertible then invertibility of τ follows from fully faithfulness of $y_Y^*(-)_*$. ∎

2.7. It is now standard that $\mathcal{P} : \mathcal{M} \to \mathcal{K}$ is given on arrows by requiring, for $M : X \to Y$ in $\mathcal{M}$, that $\mathcal{P}M$ be obtained from the procedure of Lemma 2.6 applied to My_X^* as in the diagram



(4)

(with the universal property of that in Lemma 2.6). From the universal property, we deduce the effect of $\mathcal{P}$ on 2-cells. Moreover, pseudofunctoriality of $\mathcal{P}$ with respect to composition follows from consideration of



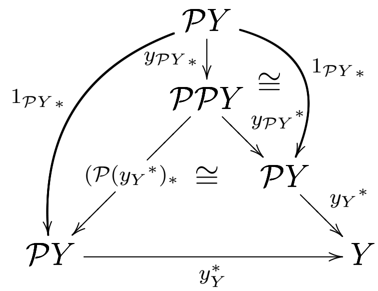
Here each of the smaller triangle, the trapezoid, and the entire region are as provided by Lemma 2.6. It follows from the universality of $\mathcal{P}(NM)$ that we have an isomorphism $(\mathcal{P}N)(\mathcal{P}M) \to \mathcal{P}(NM)$ in $\mathcal{K}$. Pseudofunctoriality with respect to identities is even easier and left for the reader.

2.8. The un-named isomorphisms given by (4) provide the pseudonaturalities for a pseudonatural transformation $y^* : (-)_*\mathcal{P} \to 1_{\mathcal{M}}$, playing the role of e in 2.4. Taking mates we have, for each $M : X \to Y$ in $\mathcal{M}$, a 2-cell $y_{Y*}M \to (\mathcal{P}M)_* y_{X*}$ in $\mathcal{M}$. In particular, for each $k : X \to Y$ in $\mathcal{K}$ we have

$$y_{Y*}k_* \to (\mathcal{P}k_*)_* y_{X*} \qquad \text{and} \qquad y_{X*}k^* \to (\mathcal{P}k^*)_* y_{Y*}$$

For each k in $\mathcal{K}$, the first of these gives, by pseudofunctoriality and local fully faithfullness of $(-)_*$, a 2-cell $y_Y k \to (\mathcal{P}k_*) y_X$ in $\mathcal{K}$, the second has a mate $(\mathcal{P}k_*)_* y_{X*} \to y_{Y*}k_*$

in $\mathcal{M}$ (using $(\mathcal{P}k^*)_* \cong (\mathcal{P}k_*)^*$ via pseudofunctoriality of $\mathcal{P}$) and this last gives a 2-cell $(\mathcal{P}k_*)y_X \to y_Y k$ in $\mathcal{K}$, provably inverse to $y_Y k \to (\mathcal{P}k_*)y_X$. Again, it follows that the isomorphisms $y_Y k \to (\mathcal{P}k_*)y_X$ provide the pseudonaturalities for a pseudonatural transformation $y : 1_{\mathcal{K}} \to \mathcal{P}(-)_*$. Turning again to the terminology of subsection 2.4, we can construct the invertible constraint ϵ by using the inverses for the unit of the family of adjunctions ${y_X}_* \dashv {y_X}^*$ since we have y fully faithful. To construct the constraint η, consider



The triangle is as provided by Lemma 2.6 while the other displayed isomorphism is another instance of the fully faithfulness of y. Finally since the entire region can also be seen as an instance of the diagram in Lemma 2.6, universality provides us with a unique isomorphism $\mathcal{P}({y_Y}^*)y_{\mathcal{P}Y} \to 1_{\mathcal{P}Y}$ in $\mathcal{K}$ whose image under $(-)_*$ coheres with the pasting of the other isomorphisms in the diagram. This completes the exhibition of the data needed for a biadjunction $(-)_* \dashv \mathcal{P}$. Again we point out that since our application will be to locally ordered bicategories, verification of 2-cell equations will not be considered here.

2.9. By the very construction of the counit e as y^* we have (1) so that the biadjunction $(-)_* \dashv \mathcal{P}$ has the KZ-property. Finally, we note that the defining fully faithful adjoint string, in the sense of [MAR], showing that $(\mathcal{P}(-)_*, y)$ is a KZ-doctrine on $\mathcal{K}$ is given by

$$\mathcal{P}Y \;\underset{y_{\mathcal{P}Y}}{\overset{\mathcal{P}({y_Y}_*)}{\rightrightarrows}}\; \mathcal{P}\mathcal{P}Y, \qquad \mathcal{P}Y \xleftarrow{\mathcal{P}({y_Y}^*)} \mathcal{P}\mathcal{P}Y, \qquad \mathcal{P}({y_Y}_*) \dashv \mathcal{P}({y_Y}^*) \dashv y_{\mathcal{P}Y} \tag{5}$$

2.10. Remark. It should not be supposed that all proarrow equipments with right biadjoints provide KZ-adjunctions. For example, the proarrow equipment $(-)_* : \mathbf{set} \to \mathbf{rel}$ which sends a function to its graph has the power set construction as right adjoint. The unit is singleton with components $\{-\}_Y : Y \to \mathcal{P}Y$; the counit is membership with components $\in_Y : \mathcal{P}Y \to Y$ and $\in_Y$ is not the right adjoint of the graph of $\{-\}_Y$ in $\mathbf{rel}$.

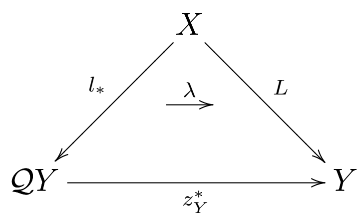
3. Cauchy Completion Relative to Proarrow Equipments

3.1. We now consider the question of existence of a *strong* right KZ-adjoint $\mathcal{Q}$ for $(-)_* : \mathcal{K} \to \mathrm{map}\mathcal{M}$. From the general discussion in Section 2 we see that to give such a biadjunction $(-)_* \dashv \mathcal{Q}$ it suffices to give, for each object Y, an arrow $z_Y : Y \to \mathcal{Q}Y$ in $\mathcal{K}$ with invertible unit $1_Y \to {z_Y}^*{z_Y}_*$ and invertible counit ${z_Y}_*{z_Y}^* \to 1_{\mathcal{Q}Y}$ in $\mathrm{map}\mathcal{M}$ so that, for each object X, composition with $e : \mathcal{Q}Y \to Y$ preceded by $(-)_*$ provides an equivalence of categories

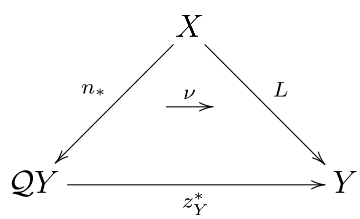
$$\mathcal{K}(X, \mathcal{Q}Y) \xrightarrow{z_Y{}^*(-)_*} \mathrm{map}\mathcal{M}(X, Y) \tag{6}$$

It should be noted that to say z^* is the counit for the biadjunction $(-)_* \dashv \mathcal{Q}$ makes the components $z_Y{}^*$ themselves maps in $\mathcal{M}$. The proof of the following Lemma is easily adapted from that of Lemma 2.6.

3.2. Lemma. *The equivalence (6) provides, for each* map $L : X \to Y$ *in* $\mathcal{M}$, *a diagram*

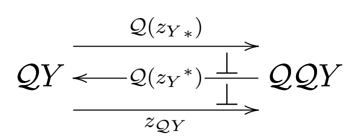


with l *in* $\mathcal{K}$ *and invertible* λ, *universal among diagrams of the form*



for n *in* $\mathcal{K}$ *and arbitrary* ν. ∎

3.3. Just as we did for $(\mathcal{P}, y)$, we can carry out a similar discussion for $(\mathcal{Q}, z)$ to show that our assumptions about $z_Y : Y \to \mathcal{Q}Y$ lead to a *strong* KZ-adjunction $(-)_* \dashv \mathcal{Q} : \mathrm{map}\mathcal{M} \to \mathcal{K}$ which in turn leads to a string of adjoint equivalences



(7)

showing that $(\mathcal{Q}(-)_*, z)$ is a pseudo-idempotent pseudo monad on $\mathcal{K}$. In this generality we refer to $z_Y : Y \to \mathcal{Q}Y$ as the *Cauchy completion* of Y and we say that the proarrow equipment $(-)_* : \mathcal{K} \to \mathcal{M}$ *has Cauchy completions* if $(-)_* : \mathcal{K} \to \mathrm{map}\mathcal{M}$ has a strong right KZ-adjoint. In any bicategory $\mathcal{K}$, an arrow $s : X \to Y$ is a *pseudo section* if there is an arrow $r : Y \to X$ and an isomorphism $1_X \cong rs$. For a pseudo-idempotent pseudo monad $(\mathcal{T}, t)$ on any bicategory $\mathcal{K}$, it is standard that an object Y in $\mathcal{K}$ underlies a $(\mathcal{T}, t)$-algebra if and only if $t_Y : Y \to \mathcal{T}Y$ is an equivalence if and only if $t_Y : Y \to \mathcal{T}Y$ is a pseudo section. We recall from 2.3 that an object Y in $\mathcal{K}$ is Cauchy complete if, for every map $L : X \to Y$ in $\mathcal{M}$, there is an arrow $f : X \to Y$ in $\mathcal{K}$ and an isomorphism $f_* \cong L$ in $\mathcal{M}$. Since $(-)_* : \mathcal{K}(X, Y) \to \mathrm{map}\mathcal{M}(X, Y)$ is in any event fully faithful, we see that Y is Cauchy complete if and only if, for all X, $(-)_* : \mathcal{K}(X, Y) \to \mathrm{map}\mathcal{M}(X, Y)$ is an equivalence of categories.

3.4. Proposition. *For a proarrow equipment $(-)_*:\mathcal{K} \to \mathcal{M}$ such that $(-)_*:\mathcal{K} \to \mathrm{map}\mathcal{M}$ has a strong right KZ-adjoint $\mathcal{Q}$ with unit z, an object Y in $\mathcal{K}$ is Cauchy complete if and only if it is a $(\mathcal{Q}(-)_*, z)$ algebra if and only if $z_Y : Y \to \mathcal{Q}Y$ is a pseudo section in $\mathcal{K}$.*

Proof. Only the first part of the statement requires comment. Consider

$$\begin{array}{ccccc}
\mathcal{K}(X,Y) & & \xrightarrow{\quad (-)_* \quad} & & \mathrm{map}\mathcal{M}(X,Y) \\
\Big\downarrow{\scriptstyle \mathcal{K}(X,z_Y)} & \cong & & & \| \\
\mathcal{K}(X,\mathcal{Q}Y) & \xrightarrow[(-)_*]{} & \mathrm{map}\mathcal{M}(X,\mathcal{Q}Y) & \xrightarrow[\mathrm{map}\mathcal{M}(X,{z_Y}^*)]{} & \mathrm{map}\mathcal{M}(X,Y)
\end{array}$$

which commutes to within isomorphism since z_Y is fully faithful. Our global assumption ensures that the bottom side of the triangle is an equivalence of categories (and we use here that ${z_Y}^*$ is also a map in $\mathcal{M}$). Now Y is Cauchy complete if and only if, for all X, the hypotenuse is an equivalence of categories which is so if and only if, for all X, $\mathcal{K}(X, z_Y)$ is an equivalence of categories if and only if $z_Y : Y \to \mathcal{Q}Y$ is an equivalence in $\mathcal{K}$. ■

3.5. We assume now that we have both $\mathcal{P}$ and $\mathcal{Q}$ as above. Because $\mathrm{map}\mathcal{M}(X,Y)$ is a full subcategory of $\mathcal{M}(X,Y)$, for all X, we have $\mathcal{K}(X,\mathcal{Q}Y)$ a full subcategory of $\mathcal{K}(X,\mathcal{P}Y)$, for all X, so that by the bicategorical Yoneda lemma we have $i_Y : \mathcal{Q}Y \to \mathcal{P}Y$ in $\mathcal{K}$ representably fully faithful. Hence $1_{\mathcal{Q}Y} \to {i_Y}^* i_{Y*}$. is invertible. From pseudonaturality we have

$$\begin{array}{ccc}
\mathcal{K}(\mathcal{Q}Y,\mathcal{Q}Y) & \xrightarrow{\mathcal{K}(\mathcal{Q}Y,i)} & \mathcal{K}(\mathcal{Q}Y,\mathcal{P}Y) \\
\Big\downarrow{\scriptstyle z^*(-)_*} & \cong & \Big\downarrow{\scriptstyle y^*(-)_*} \\
\mathrm{map}\mathcal{M}(\mathcal{Q}Y,Y) & \hookrightarrow & \mathcal{M}(\mathcal{Q}Y,Y)
\end{array}$$

and

$$\begin{array}{ccc}
\mathcal{K}(Y,\mathcal{Q}Y) & \xrightarrow{\mathcal{K}(Y,i)} & \mathcal{K}(Y,\mathcal{P}Y) \\
\Big\downarrow{\scriptstyle z^*(-)_*} & \cong & \Big\downarrow{\scriptstyle y^*(-)_*} \\
\mathrm{map}\mathcal{M}(Y,Y) & \hookrightarrow & \mathcal{M}(Y,Y)
\end{array}$$

Chasing the identity on $\mathcal{Q}Y$ around the first diagram and z_Y around the second diagram we have isomorphisms

$$\begin{array}{ccc}
Y & \xleftarrow{{z_Y}^*} & \mathcal{Q}Y \\
{\scriptstyle {y_Y}^*}\nwarrow & \cong & \Big\downarrow{\scriptstyle i_{Y*}} \\
 & & \mathcal{P}Y
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
Y & \xrightarrow{z_Y} & \mathcal{Q}Y \\
 & {\scriptstyle y_Y}\searrow \;\cong & \Big\downarrow{\scriptstyle i_Y} \\
 & & \mathcal{P}Y
\end{array}$$

with the first in $\mathcal{M}$ and the second in $\mathcal{K}$ (since it results from $y^*(iz)_* \cong z^*(z)_* \cong 1_Y \cong y^*(y)_*$ and $y^*(-)_*$ being an equivalence is a fully faithful functor). It follows immediately that $i(-)_* : \mathcal{Q}(-)_* \to \mathcal{P}(-)_*$ is a morphism of pseudomonads. It will be a convenient abuse of notation to omit $(-)_*$ in such contexts and speak of $\mathcal{P}$ and $\mathcal{Q}$ as pseudomonads on $\mathcal{K}$ and $i : \mathcal{Q} \to \mathcal{P}$ as a morphism of pseudomonads. An example is our usage in:

3.6. Proposition. *The morphism of pseudofunctors $z\mathcal{P} : \mathcal{P} \to \mathcal{QP}$ is an equivalence.*

Proof. By Proposition 3.4, it suffices to show that each $z\mathcal{P}Y : \mathcal{P}Y \to \mathcal{QP}Y$ is a pseudo section. But from

$$(y^*\mathcal{P}Y.i\mathcal{P}Y).z\mathcal{P}Y \cong y^*\mathcal{P}Y.(i\mathcal{P}Y.z\mathcal{P}Y) \cong y^*\mathcal{P}Y.y\mathcal{P}Y \cong 1_{\mathcal{P}}Y$$

(using the second triangle in 3.5 instantiated at $\mathcal{P}y$) we see this is so. ∎

3.7. Corollary. *Any object of the form $\mathcal{P}Y$ in $\mathcal{K}$ is Cauchy complete.* ∎

3.8. We denote by $\mathcal{K}_{\mathrm{cc}}$ the full sub 2-category of $\mathcal{K}$ determined by the Cauchy complete objects. Any object of the form $\mathcal{Q}Y$ is Cauchy complete (since $\mathcal{Q}$ is a pseudo-idempotent pseudomonad). If, for each Y in $\mathcal{K}$, the Cauchy completion of Y exists then it is a formality that $(-)_* : \mathcal{K}_{cc} \to \mathrm{map}\mathcal{M}$ is a biequivalence with inverse given by $\mathcal{Q}$.

4. Proarrow Equipments for Orders and Idempotents

4.1. As with earlier papers on CCD lattices, we would like to make our work constructive so that in the sequel the word *set* refers to an object of the base topos $\mathcal{S}$. The words *ordered set* refer to a set together with a reflexive and transitive relation, what some authors call a preordered set. Many of the ordered sets $(A, \leq)$ that we consider are definitely not anti-symmetric. We write **ord** for the 2-category of ordered sets, order-preserving functions, and inequalities. We always have in mind that **ord** is a symmetric monoidal 2-category via its finite products and most categories occurring in the rest of this paper are **ord**-categories in the sense of enriched category theory. In fact, **ord** is itself Ω-**cat**. (The subobject classifier Ω is an ordered set with finite products which endow it with symmetric monoidal structure.) For much of this paper it would be possible to replace Ω by a general symmetric monoidal category $\mathcal{V}$. However, we limit further mention of Ω-**cat** to this section while we clarify the relationship of **ord**, especially in the context of proarrow equipment, with a more general **ord**-category of sets equipped with a relation.

4.2. The bicategory of order ideals **idl** has objects those of **ord** and an arrow $R : A \to B$ in **idl** is a relation $R : A \to B$ for which $((b' \leq bRa \leq a') \Longrightarrow (b'Ra'))$. We have a 2-cell from $R : A \to B$ to $S : A \to B$ in **idl** if and only if the relation R is contained in S. It is easy to see that **idl** is Ω-**prof** and that **idl** is an **ord**-category. We have $(-)_* : \mathbf{ord} \to \mathbf{idl}$, given by the identity on objects and, for $f : A \to B$ in **ord**, we have $f_* : A \to B$ in **idl** given by bf_*a if and only if $b \leq fa$. For $f : A \to B$ in **ord**, there is also $f^* : B \to A$ in **idl**, where af^*b if and only if $fa \leq b$. It is standard that $f_* \dashv f^*$ in **idl** and $(-)_* : \mathbf{ord} \to \mathbf{idl}$

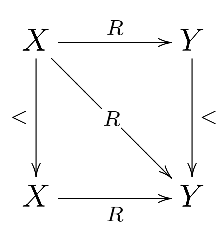
is precisely the proarrow equipment of example i) of subsection 2.2 for the case of $\mathcal{V} = \Omega$. Moreover, $(-)_* : \mathbf{ord} \to \mathbf{idl}$ has a right KZ-adjoint that we call D, with DB being the lattice of downsets of B ordered by inclusion. Clearly

$$DB = \mathbf{idl}(1, B) \cong \mathbf{ord}(B^{\mathrm{op}}, \Omega)$$

4.3. In [C&S] the authors studied the notion of Cauchy completeness of objects of **ord**. In the terminology of Section 3, $(-)_* : \mathbf{ord} \to \mathrm{map}\mathbf{idl}$ has a strong right KZ-adjoint (Cauchy completion) that sends an ordered set B to $\mathcal{Q}B = \{\downarrow b \mid b \in B\}$, the set of principal downsets of B. This amounts to the antisymmetrization of B. In [C&S] it is shown that $z_B : B \to \mathcal{Q}B$ is an equivalence, for each B, if and only if supports split in $\mathcal{S}$.

4.4. An idempotent in the bicategory **rel** of relations consists of a set X together with a relation $< \ : X \to X$ equal to its composite with itself. One containment providing the equality makes $<$ transitive; the other makes it *interpolative*, meaning that if $x < y$ then there exists z in X with $x < z < y$. Any order $\leq \ : X \to X$ provides an example of an idempotent in **rel** since interpolation can be realized trivially via reflexivity. It is convenient to read $x < y$ as "x is below y". If $(X, <)$ and $(Y, <)$ are idempotents then a function $f : X \to Y$ is said to be *below-preserving* if $x < x'$ implies $fx < fx'$. Idempotents and below-preserving functions form a category that we denote $\mathbf{idm}_0$. The category $\mathbf{idm}_0$ becomes an **ord**-category that we call **idm** as follows. For $f, g : X \to Y$ in $\mathbf{idm}_0$ we define $f \preceq g$ to mean $(\forall x, x' \text{ in } X)(x < x' \Longrightarrow fx < gx')$.

4.5. Another **ord**-category whose objects are the idempotents $(X, <)$ is the *idempotent splitting completion* of **rel**. We call it **krl** (a contraction of "the Karoubian envelope of the category of relations"). From the standard description of idempotent splitting completions, an arrow $R : (X, <) \to (Y, <)$ in **krl** is a relation $R : X \to Y$ such that



commutes. It follows that, for all $y \in Y$ and $x \in X$ we have

$$(\exists y')(y < y' R x) \quad \text{iff} \quad yRx \quad \text{iff} \quad (\exists x')(yRx' < x)$$

There is a 2-cell from $R : X \to Y$ to $S : X \to Y$ in **krl** if and only if the relation R is contained in S.

4.6. As explained in more detail in [RW4], for $f : X \to Y$ in **idm**, we define $f_\# : X \to Y$ in **krl** by $y f_\# x$ if and only if $(\exists x')(y < fx' \text{ and } x' < x)$. We have also $f^\# : Y \to X$ with $x f^\# y$ if and only if $(\exists x')(x < x' \text{ and } fx' < y)$. It is easy to verify that both $f_\#$ and $f^\#$ are arrows in **krl** and that $(-)_\# : \mathbf{idm} \to \mathbf{krl}$ is a (strict) pseudofunctor. Moreover, if X

and Y are idempotents in **rel** in virtue of being ordered sets then any such f is in **ord**, any arrow of **krl** is in **idl** and $f_\# = f_*$ in **idl**. The following was stated in [RW4] after Proposition 1 but in that paper $f \preceq g : X \to Y$ was *defined* to mean that $f_\# \subseteq g_\#$.

4.7. Proposition. *The pseudofunctor* $(-)_\# : \mathbf{idm} \to \mathbf{krl}$ *is proarrow equipment.*

Proof. Proposition 1 of [RW4] showed that $f_\# \dashv f^\#$ in **krl**. We have only to show here that $f \preceq g$ if and only if $f_\# \subseteq g_\#$ using our current definition of $f \preceq g$. So assume $f \preceq g$ and for $y \in Y$, $x \in X$, assume that $(\exists x')(y < fx'$ and $x' < x)$. Interpolate $x' < x$ to get $x' < x'' < x$. Since $f \preceq g$ we have $fx' < gx''$ and hence $(\exists x'')(y < gx''$ and $x'' < x)$. Conversely, assume $f_\# \subseteq g_\#$ and $x < x'$. Interpolate $x < x'$ to get $x < x'' < x'$ giving $fx < fx''$ and $x'' < x'$. Since $f_\# \subseteq g_\#$ we have $(\exists \bar{x})(fx < g\bar{x}$ and $\bar{x} < x')$ from which follows $fx < gx'$. ∎

4.8. It should not be supposed that $(-)_\# : \mathbf{idm} \to \mathbf{krl}$ is an ad hoc proarrow equipment. It is precisely the proarrow equipment of example ii) of subsection 2.2 for the case of $\mathcal{V} = \Omega$. In general, there is a 2-functor $i : \mathcal{V}\text{-}\mathbf{cat} \to \mathcal{V}\text{-}\mathbf{tax}$ which interprets a $\mathcal{V}$-category as a $\mathcal{V}$-taxon. We have nothing further to say here about $\mathcal{V}$-taxons and the like for general $\mathcal{V}$. Rather, we will shortly exhibit a right KZ-adjoint to $(-)_\# : \mathbf{idm} \to \mathbf{krl}$ and a strong right KZ-adjoint (Cauchy completion) to $(-)_\# : \mathbf{idm} \to \text{map}\mathbf{krl}$ but we think that these are best understood as generalizations to their counterparts for ordered sets.

4.9. Consider the following diagram of *ordinary* categories and functors.

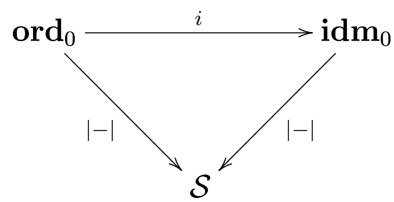
$$\mathbf{ord}_0 \;\begin{array}{c} \xleftarrow{\quad a \quad} \\ \bot \\ \xrightarrow{\quad i \quad} \\ \bot \\ \xleftarrow[\quad c \quad]{} \end{array}\; \mathbf{idm}_0$$

Here i is the literal inclusion of orders as idempotents. For an idempotent $X = (X, <)$, $aX = (X, < \cup 1_X)$, where 1_X is the identity relation on X, and $cX = (\{x \in X | x < x\}, <|)$, where $<|$ is the restricted idempotent. It is clear that we have *ordinary* adjunctions $a \dashv i \dashv c$. It is also clear that i is an **ord**-functor, for if $f \leq g : X \to A$ in **ord** then $if \preceq ig$ in **idm**. Similarly, if $f \preceq g : X \to A$ in **idm** then $cf \leq cg$ in **ord** and it follows that we can regard $i \dashv c$ as an **ord**-adjunction.

$$\mathbf{ord} \;\begin{array}{c} \xrightarrow{\quad i \quad} \\ \bot \\ \xleftarrow[\quad c \quad]{} \end{array}\; \mathbf{idm}$$

But $a : \mathbf{idm}_0 \to \mathbf{ord}_0$ does not admit enrichment to an **ord**-functor. For example, consider the two distinct sum injections $f, g : 1 \to 2$ in $\mathcal{S}$. These give rise to distinct arrows $f, g : (1, \emptyset) \to (2, \emptyset)$ in **idm**, where each idempotent in question has no instances of $<$. We have $f \preceq g$ (and $g \preceq f$) but we do not have $af \leq ag$.

4.10. We have also the commuting diagram of forgetful functors



and note that $\mathbf{ord}_0(1,-) = |-| : \mathbf{ord}_0 \to \mathcal{S}$, where 1 is the one-point ordered set — the terminal object of **ord** and the identity object for the cartesian monoidal structure. Clearly $i1$ is terminal in **idm** and we have $\mathbf{idm}(1,-) \cong c : \mathbf{idm} \to \mathbf{ord}$. By contrast, $\mathbf{idm}_0(N,-) = |-| : \mathbf{idm}_0 \to \mathcal{S}$, where $N = (1, \emptyset)$ is 'the naked one'. Since $a : \mathbf{idm}_0 \to \mathbf{ord}_0$ commutes with the forgetful functors, it follows that the chaotic order construction $k : \mathcal{S} \to \mathbf{ord}_0$, right adjoint to $|-| : \mathbf{ord}_0 \to \mathcal{S}$, yields $ik : \mathcal{S} \to \mathbf{idm}$ right adjoint to $|-| : \mathbf{idm}_0 \to \mathcal{S}$. Penultimately, in regard to this diagram, observe that the discrete order construction $d : \mathcal{S} \to \mathbf{ord}_0$ is left adjoint to $|-| : \mathbf{ord}_0 \to \mathcal{S}$ while $n : \mathcal{S} \to \mathbf{idm}_0$, the left adjoint to $|-| : \mathbf{idm}_0 \to \mathcal{S}$ is given by $nS = (S, \emptyset)$. Finally, we have $\pi_0 \dashv d : \mathcal{S} \to \mathbf{ord}_0$, with $\pi_0 A$ the set of connected components of A, but there is no left adjoint for $n : \mathcal{S} \to \mathbf{idm}_0$ because n does not preserve the terminal object.

4.11. In [RW4] and [MRW] the **ord**-functor $D = \mathbf{krl}(1,-) : \mathbf{krl} \to \mathbf{ord}$ was studied in detail. For X an idempotent, $DX = D(X,<)$ is the set of downsets of $(X,<)$ ordered by inclusion. Recall that a *downset* S of $(X,<)$ is a subset S of X with the property that $x \in S$ if and only if $(\exists y)(x < y \in S)$. Of course as with any subset, such an S has a characteristic function that takes values in Ω. Note that the inclusion ordering of subsets is that given by the pointwise-order for Ω-valued functions. For $(X,<)$ an idempotent, the *set* $\mathbf{idm}_0(X^{\mathrm{op}}, i\Omega)$ can be identified with the set of those subsets T of X with the property that $x < y \in T$ implies $x \in T$. Call such a T a *weak downset* of $(X,<)$. (X^{op} for idempotents is just as for orders, that is $X^{\mathrm{op}} = (X,<)^{\mathrm{op}} = (X,<^{\mathrm{op}})$, where $x <^{\mathrm{op}} y$ if and only if $y < x$.) For the record, observe that, for any idempotent X and any order A, the set $\mathbf{idm}_0(X, iA)$ admits the order $f \leq g$ defined by $(\forall x)(fx \leq gx)$ but this is in general distinct from the order $\preceq$ of $\mathbf{idm}(X, iA)$. In general, $f \leq g$ implies $f \preceq g$ but not conversely. For example, for any parallel pair of the form $f, g : (X, \emptyset) \to (A, \leq)$ we have $f \preceq g$, but not necessarily $f \leq g$. Identifying subsets with their characteristic functions we see that $\mathbf{idm}(X^{\mathrm{op}}, i\Omega)$ is the set of weak downsets of X with $T \preceq U$ if and only $x < y \in T$ implies $x \in U$.

4.12. Proposition. *For any idempotent $(X,<)$, the inclusion $iDX \to \mathbf{idm}(X^{\mathrm{op}}, i\Omega)$ is an equivalence of ordered sets.*

Proof. Since a downset is a weak downset and, by our generalities about orders, $R \subseteq S$ implies $R \preceq S$; it follows that inclusion provides a functor $iDX \to \mathbf{idm}(X^{\mathrm{op}}, i\Omega)$. For T a weak downset, define $T^{\circ} = \{x | (\exists t)(x < t \in T)\}$. Then T° is certainly a downset. If $T \preceq U$ and $x \in T^{\circ}$ then $x < t \in T$ implies $(\exists y)(x < y < t \in T)$ which implies $(\exists y)(x <$

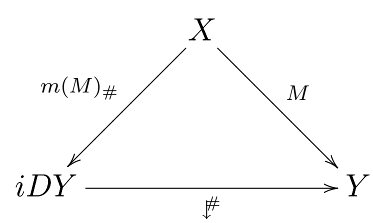
$y \in U$) (by $T \preceq U$) and hence $x \in U^\circ$. Thus $T^\circ \subseteq U^\circ$ and $(-)^\circ : \mathbf{idm}(X^{\mathrm{op}}, i\Omega) \to iDX$ is a functor, right adjoint to the inclusion $iDX \to \mathbf{idm}(X^{\mathrm{op}}, i\Omega)$, since $T^\circ \subseteq T$ and hence $T^\circ \preceq T$ provides a counit. For T a downset we have also $T \subseteq T^\circ$ so that $(-)^\circ : \mathbf{idm}(X^{\mathrm{op}}, i\Omega) \to iDX$ is split by the inclusion. To show that we have an equivalence it suffices to show that $T^\circ \preceq T$ is invertible, meaning here that $T \preceq T^\circ$, for every weak downset T. But if $x < y \in T$ then, by definition, $x \in T^\circ$ which shows $T \preceq T^\circ$. ∎

4.13. Remark. Of course we do *not* have $T^\circ = T$ for every weak downset so that the inclusion $iDX \to \mathbf{idm}(X^{\mathrm{op}}, i\Omega)$ is *not* an order isomorphism. Also, for A an order, weak downsets *are* downsets (and the order relation on $\mathbf{idm}(iA^{\mathrm{op}}, i\Omega)$ is containment, modulo identification of subsets with their characteristic functions). Thus our use of D for both $\mathbf{ord}(1, -)$ and $\mathbf{krl}(1, -)$ is unambiguous. We also generalize $\downarrow$ to objects of $\mathbf{idm}$ so that $\downarrow_X : X \to iDX$ in $\mathbf{idm}$ is the arrow that sends x to $\downarrow x = \{x' \in X \mid x' < x\}$.

5. Power Objects and Cauchy Completions for $(-)_\# : \mathbf{idm} \to \mathbf{krl}$

5.1. We first show that the proarrow equipment $(-)_\# : \mathbf{idm} \to \mathbf{krl}$ has a right KZ-adjoint. We recall that for general $f : X \to Y$ in $\mathbf{idm}$, we have $f^\# : Y \to X$ in $\mathbf{krl}$ given by $x f^\# y$ if and only if $(\exists x')(x < x'$ and $fx' < y)$. It is easy to see that $y \downarrow_Y^\# S$ (if and only if $(\exists y')(y < y'$ and $\downarrow_Y y' \subseteq S)$) simplifies to $y \in S$. For all $M : X \to Y$ in $\mathbf{krl}$, define $m = m(M) : X \to iDY$ by $m(x) = \{y \mid yMx\}$. Since $y \in m(x)$ if and only if yMx it is at once clear that each $m(x)$ is a downset and that $m : X \to iDY$ is in $\mathbf{idm}$.

5.2. Lemma. *For all $M : X \to Y$ in* $\mathbf{krl}$*,*



commutes.

Proof. We have $y(\downarrow^\# m_\#)x$ if and only if $(\exists S)(y \in S$ and $S m_\# x)$ if and only if $(\exists S, x')(y \in S \subseteq m(x')$ and $x' < x)$ which holds if and only if yMx. ∎

5.3. Lemma. *The diagram in Lemma 5.2 exhibits $m(M)_\#$ as a right lifting in* $\mathbf{krl}$ *of M through $\downarrow^\# : iDY \to Y$.*

Proof. We recall from [RW4] that $\mathbf{krl}$ has all right liftings and, for any $L : A \to Y \leftarrow X : M$, the right lifting of M through L is given by the relational composite $<_A .(L \Rightarrow M). <_X$, where $L \Rightarrow M$ is the right lifting in the bicategory $\mathbf{rel}$ of all relations. When A is an order it is easy to see that the description simplifies to $(L \Rightarrow M). <_X$. Now $S((\downarrow^\# \Rightarrow M). <)x$ holds if and only if $(\exists x')((\forall y)(y \in S$ implies $yMx')$ and $x' < x)$ which is the case if and only if $(\exists x')(S \subseteq m(x')$ and $x' < x)$ which means precisely $S m_\# x$. ∎

5.4. Proposition. *The proarrow equipment* $(-)_{\#} : \mathbf{krl} \to \mathbf{idm}$ *has power objects, given by* iD *as in 4.11 with unit* $\downarrow$.

Proof. As we have seen in Section 2, it suffices to show that

$$\mathbf{idm}(X, iDY) \xrightarrow{\downarrow^{\#}.(-)_{\#}} \mathbf{krl}(X, Y)$$

provides an equivalence of categories. Since $\downarrow^{\#}.(-)_{\#}$ is surjective by Lemma 5.2 it suffices to show that $\downarrow^{\#}.(-)_{\#}$ is fully faithful. So assume that we have $f, g : X \to iDY$ in $\mathbf{idm}$ with $\downarrow^{\#}.f_{\#} \subseteq \downarrow^{\#}.g_{\#}$. We must show that $f \preceq g$. For this we must show that if $x' < x$ then $fx' \subseteq gx$. So along with $x' < x$ assume that we have $y \in fx'$ Since $y \downarrow^{\#}.f_{\#}x$ translates to $(\exists x')(y \in fx'$ and $x' < x)$ and we are assuming $\downarrow^{\#}.f_{\#} \subseteq \downarrow^{\#}.g_{\#}$ we have $(\exists x'')(y \in gx''$ and $x'' < x)$ and hence $y \in gx$. ∎

5.5. Remark. We know from our general study of power objects for proarrow equipments in Section 2, that from fully faithfulness of $\downarrow^{\#}.(-)_{\#}$ we have the universal property for $m(M)$ given in Lemma 2.6. For $(-)_{\#} \dashv iD : \mathbf{krl} \to \mathbf{idm}$ we have the stronger right lifting property given by Lemma 5.3.

5.6. As shown in [RW4], there is an important idempotent relation, denoted $\subset\!\subset$, on the set of downsets of an idempotent $(Y, <)$, with $S \subset\!\subset T$ if and only if $(\exists t)(S \subseteq \downarrow t$ and $t \in T)$. We write $\mathbb{D}Y = (|DY|, \subset\!\subset)$ for the set of downsets of $(Y, <)$ together with $\subset\!\subset$. Since $S \subset\!\subset T$ implies $S \subseteq T$, the identity function provides a functor $k : \mathbb{D}Y \to iDY$ in $\mathbf{idm}$. We showed in [RW4] that $\downarrow_Y$ factors through k. We will let $\downarrow = \downarrow_Y$ serve double duty and write also $\downarrow = \downarrow_Y : Y \to \mathbb{D}Y$. Observe that for this interpretation of $\downarrow$ we still have $y \downarrow^{\#} S$ if and only if $y \in S$. The idempotent relation $\subset\!\subset$ on $|DY|$ gives rise to a new idempotent relation on $\mathbf{idm}_0(X^{\mathrm{op}}, i\Omega)$. For weak downsets U and V we define $U \prec\!\prec V$ to mean $U^{\circ} \subset\!\subset V^{\circ}$ where $(-)^{\circ} : (\mathbf{idm}_0(X^{\mathrm{op}}, i\Omega), \prec\!\prec) \to \mathbb{D}Y$ is as in Proposition 4.12. Note that $U^{\circ} \subset\!\subset V^{\circ}$ is equivalent to saying $(\exists y, v)((x < u \in U$ implies $x < y)$ and $(y < v \in V))$.

5.7. Proposition. *The inclusion* $\mathbb{D}Y \to (\mathbf{idm}_0(Y^{\mathrm{op}}, i\Omega), \prec\!\prec)$ *provides an equivalence of objects in* $\mathbf{idm}$ *with inverse given by* $(-)^{\circ} : (\mathbf{idm}_0(Y^{\mathrm{op}}, i\Omega), \prec\!\prec) \to \mathbb{D}Y$ *as defined in Proposition 4.12.*

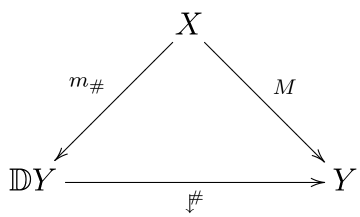
Proof. Assume that $S \subset\!\subset T$ in $\mathbb{D}Y$. Since the inclusion of downsets in weak downsets is split by $(-)^{\circ}$ we have $S^{\circ} \subset\!\subset T^{\circ}$ which means $S \prec\!\prec T$ in weak downsets. This shows that the inclusion provides an arrow $\mathbb{D}Y \to (\mathbf{idm}_0(Y^{\mathrm{op}}, i\Omega), \prec\!\prec)$ in $\mathbf{idm}$. If, for weak downsets U and V, $U \prec\!\prec V$ then $U^{\circ} \subset\!\subset V^{\circ}$ by definition so that $(-)^{\circ} : (\mathbf{idm}_0(Y^{\mathrm{op}}, i\Omega), \prec\!\prec) \to \mathbb{D}Y$ is an arrow in $\mathbf{idm}$, split by the inclusion, call it j. To show that j and $(-)^{\circ}$ constitute inverse equivalences in $\mathbf{idm}$ it suffices to show the inequalities

$$1_{(\mathbf{idm}_0(Y^{\mathrm{op}}, i\Omega), \prec\!\prec)} \preceq j(-)^{\circ} \quad \text{and} \quad j(-)^{\circ} \preceq 1_{(\mathbf{idm}_0(Y^{\mathrm{op}}, i\Omega), \prec\!\prec)}$$

For the first, assume $U \prec\!\prec V$. We must show $U \prec\!\prec V^{\circ}$. But this just means showing that $U^{\circ} \subset\!\subset V^{\circ}$ implies $U^{\circ} \subset\!\subset V^{\circ\circ}$, which is trivial since $(-)^{\circ}$ is idempotent. For the second, we must show that $U \prec\!\prec V$ implies $U^{\circ} \prec\!\prec V$, which is as trivial as the first. ∎

5.8. It follows that we can replace $\mathbb{D}Y$ by $(\mathbf{idm}_0(Y^{\mathrm{op}}, i\Omega), \prec\!\prec)$ in the next results. For $M : X \to Y$ in **krl** we again use $m = m(M) : X \to iDX$ in **idm**, with $m(x) = \{y \mid yMx\}$, as we did in 5.1. Recall that $M \dashv R$ in **krl** if and only if $1_X \subseteq RM$ and $MR \subseteq 1_Y$. The first of these, the unit condition, is equivalent to $x < x'$ in X implies $(\exists y)(xRyMx')$ and the second, the counit condition, is equivalent to $yMxRy'$ implies $y < y'$.

5.9. Lemma. *For any* map $M : X \to Y$ *in* **krl**, $m = m(M) : X \to iDY$ *factors through* $k : \mathbb{D}Y \to iDY$ *and*



commutes.

Proof. Let $M \dashv R$. We have seen in Lemma 5.2 that each $m(x)$ is a downset of X. Under the assumption that M is a map we want to show that m factors through $k : \mathbb{D}X \to iDX$ in **idm**. Assume that $x < x'$ in X. To show $m(x) \subset\!\subset m(x')$ is to show

$$(\exists y')(m(x) \subseteq \downarrow y' \text{ and } y' \in m(x'))$$

which is to show

$$(\exists y')((yMx \text{ implies } y < y') \text{ and } y'Mx')$$

From the unit condition for $M \dashv R$ we have $(\exists y')(xRy'Mx')$. So if yMx then xRy' gives $y < y'$, using the counit condition for $M \dashv R$, and this together with $y'Mx'$ shows that y' witnesses $m(x) \subset\!\subset m(x')$. Now $y(\downarrow^\# m_\#)x$ if and only if $(\exists S)(y \downarrow^\# S m_\# x)$ which is equivalent to

$$(\exists S)(y \in S \text{ and } (\exists x')(S \subset\!\subset m(x') \text{ and } x' < x))$$

which holds if and only if

$$(\exists S)(y \in S \text{ and } (\exists x')((\exists y')(S \subseteq \downarrow y' \text{ and } y' \in m(x')) \text{ and } x' < x))$$

which holds if and only if

$$(\exists y', x')(y < y' \text{ and } y'Mx' \text{ and } x' < x)$$

which is yMx. ∎

5.10. Lemma. *For any* map $M : X \to Y$ *in* **krl**, *the diagram in Lemma 5.9 exhibits* $m_\#$ *as a right lifting in* **krl** *of* M *though* $\downarrow^\# : \mathbb{D}Y \to Y$.

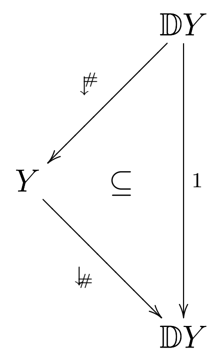
Proof. Here $S(\subset\!\subset.(\downarrow^\# \Rightarrow M).<_X)x$ holds if and only if $(\exists T, x')(S \subset\!\subset T \text{ and } (\forall y)(y \in T \text{ implies } yMx') \text{ and } x' < x)$ if and only if $(\exists T, x')(S \subset\!\subset T \subseteq m(x'))$ and $x' < x)$ if and only if $(\exists x')(S \subset\!\subset m(x') \text{ and } x' < x)$ which here means $Sm_\# x$. ∎

5.11. PROPOSITION. *The proarrow equipment* $(-)_\# : \mathbf{krl} \to \mathbf{idm}$ *has Cauchy completions, given by* $\mathbb{D}$ *as in 5.6 with unit* $\downarrow : 1_{\mathbf{idm}} \to \mathbb{D}(-)_*$.

PROOF. It suffices to show that

$$\mathbf{idm}(X, \mathbb{D}Y) \xrightarrow{\downarrow^\#.(-)_\#} \mathrm{map}\mathbf{krl}(X, Y)$$

provides an equivalence of categories and that the counit



is an isomorphism in **krl**. It is convenient to deal first with the latter, by noting that it is part of Proposition 11 in [RW4] — which says that $\downarrow_\#$ and $\downarrow^\#$ are inverse isomorphisms in **krl**. In particular, $\downarrow^\#$ is also a *map* in **krl**. For the first requirement, since $\downarrow^\#.(-)_\#$ is surjective by Lemma 5.9, it suffices to show that $\downarrow^\#.(-)_\#$ is fully faithful. So assume that we have $f, g : X \to \mathbb{D}Y$ in **idm** with $\downarrow^\#.f_\# \subseteq \downarrow^\#.g_\#$. We must show that $f \preceq g$. Since $\downarrow^\#$ is a map, $\downarrow^\#.g_\#$ is a map whose right lifting through $\downarrow^\#$ is easily seen to be $g_\#$. It follows that $f_\# \subseteq g_\#$ and hence $f \preceq g$ since $(-)_\#$ is locally fully faithful.

5.12. REMARK. From Corollary 3.7 it follows that any object of the form iDY is Cauchy complete. Note too that, for B an order, we have $DiB = DB$ where the second D is the power object construction for **ord**.

5.13. We will also need to know that Cauchy completeness is preserved by application of $(-)^{\mathrm{op}}$. While this topic can also be treated for general proarrow equipments that admit a duality, we content ourselves here with the specific case of $(-)_\# : \mathbf{krl} \to \mathbf{idm}$. For $Y = (Y, <)$ in **idm** we define $UY = \mathbf{krl}(Y, 1)^{\mathrm{op}}$. It is easy to see that $UY = D(Y^{\mathrm{op}})^{\mathrm{op}}$ (where we recall $(Y, <)^{\mathrm{op}} = (Y, <^{\mathrm{op}})$) and that for $T \in UY$ we have

$$t \in T \text{ iff } (\exists t')(T \ni t' < t)$$

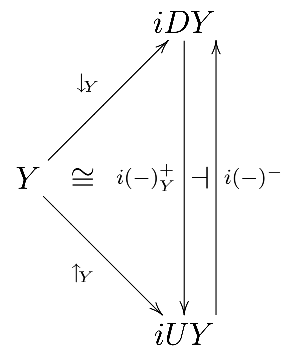
We refer to the elements of UY as *upsets* of Y. We also write $\uparrow = \uparrow_Y = (\downarrow_{Y^{\mathrm{op}}})^{\mathrm{op}}$, so that $\uparrow y = \{y' \mid y < y'\}$. We define $\mathbb{U}Y = (\mathbb{D}Y^{\mathrm{op}})^{\mathrm{op}}$ and write $\supset\!\supset$ for the dual of $\subset\!\subset$ for DY^{op}. Thus $\mathbb{U}Y = (|UY|, \supset\!\supset)$ where $T' \supset\!\supset T$ if and only if $(\exists t')(T' \ni t'$ and $\uparrow t' \supseteq T)$. We define $(-)^+_Y : DY \to UY$ by

$$S^+ = \{u \mid (\exists v)(S \subseteq \downarrow v \text{ and } v < u\} = \{u \mid S \subset\!\subset \downarrow u\}$$

and $(-)^-_Y : UY \to DY$ by

$$T^- = \{l \mid \uparrow l \supset\!\supset T\}$$

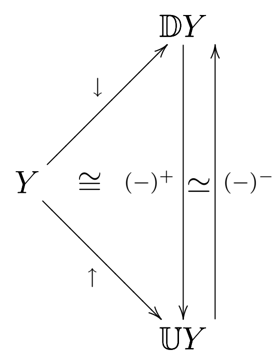
5.14. Lemma. *In* **idm**



meaning that both triangles commute to within isomorphism *and we have the displayed adjunction (in* **idm***).*

Proof. The proof is routine but some care needs to be exercised in applying the definitions pertaining to **idm**. Thus, for example, to show $i(-)^+_Y.\downarrow_Y\ \cong\ \uparrow_Y$ we must show that if $y_1 < y_2$ then both $(\downarrow y_1)^+ \supseteq \uparrow y_2$ and $\uparrow y_1 \supseteq (\downarrow y_2)^+$. The details are left to the reader. ∎

5.15. Lemma. *The functions* $(-)^+_Y$ *and* $(-)^-_Y$ *also provide arrows* $(-)^+_Y : \mathbb{D}Y \twoheadrightarrow \mathbb{U}Y$ *and* $(-)^-_Y : \mathbb{U}Y \twoheadrightarrow \mathbb{D}Y$ *in* **idm** *which are inverse equivalences and in*



both triangles commute to within isomorphism.

Proof. To see that $(-)^+$ defines an arrow $(-)^+ : \mathbb{D}Y \twoheadrightarrow \mathbb{U}Y$, suppose that, for downsets S and S', we have $S \subset\!\!\subset S'$. We can assume this to be witnessed by $S \subset\!\!\subset \downarrow s'$ with $s' \in S'$ (for the asumption that $S \subseteq \downarrow s''$ with $s'' \in S'$ immediately gives us $s'' < s' \in S'$). It follows that we have $S^+ \supset\!\!\supset S'^+$ witnessed by $S^+ \ni s'$ and $\uparrow s' \supseteq S'^+$. The first conjunct is clear and for the second, if we have $u \in S'^+$ then we have $s' \in S' \subset\!\!\subset \downarrow u$ giving $s' < u$, which can be read as $\uparrow s' \ni u$. The case of $(-)^-$ is similar.

To show that $(-)^+_Y : \mathbb{D}Y \twoheadrightarrow \mathbb{U}Y$ and $(-)^-_Y : \mathbb{U}Y \twoheadrightarrow \mathbb{D}Y$ are inverse equivalences consider first the task of showing that $1_{\mathbb{D}Y} \cong (-)^{+-}$. We assume that $R \subset\!\!\subset S$ in $\mathbb{D}X$ and show that both $R \subset\!\!\subset S^{+-}$ and $R^{+-} \subset\!\!\subset S$. If we have $R \subset\!\!\subset \downarrow s$ with $s \in S$ then $R \subset\!\!\subset S^{+-}$ is also witnessed by s since we have $S \subseteq S^{+-}$ (the unit of the adjunction in Lemma 5.14). To show that $R^{+-} \subset\!\!\subset S$ it suffices to show that $R^{+-} \subseteq \downarrow s$. If $l \in R^{+-}$ then we have $\uparrow l \supset\!\!\supset \{u \mid R \subset\!\!\subset \downarrow u\} \ni s$ which gives $\uparrow l \ni s$ and hence $l < s$. The calculation to show $1_{\mathbb{U}Y} \cong (-)^{-+}$ is dual.

Finally, we show $(-)^+.\!\downarrow\,\cong\,\uparrow$ in **idm** and leave $(-)^-.\!\uparrow\,\cong\,\downarrow$ for the reader. Assume $y_1 < y_2$ in Y. We will show $(\downarrow y_1)^+ \supset\!\supset \uparrow y_2$ and $\uparrow y_1 \supset\!\supset (\downarrow y_2)^+$. From $y_1 < y_2$ there exists y such that $y_1 < y < y_2$ from which we have $\downarrow y_1 \subset\!\subset \downarrow y$ and also $\uparrow y \supseteq \uparrow y_2$. From these we see that y witnesses $(\downarrow y_1)^+ \supset\!\supset \uparrow y_2$. The same y also witnesses $\uparrow y_1 \supset\!\supset (\downarrow y_2)^+$. ∎

5.16. Proposition. *If $Y = (Y, <)$ is Cauchy complete then Y^{op} is Cauchy complete.*

Proof. Since $\downarrow_Y : Y \to \mathbb{D}Y$ has a pseudo section and $(-)^+_Y : \mathbb{D}Y \to \mathbb{U}Y$ is an equivalence by Lemma 5.15, $(-)^+_Y.\downarrow_Y : Y \to \mathbb{U}Y$ has a pseudo section. Thus its isomorph $\uparrow_Y : Y \to \mathbb{U}Y$, also using Lemma 5.15, has a pseudo section so that $(\uparrow_Y)^{\mathrm{op}} : Y^{\mathrm{op}} \to (\mathbb{U}Y)^{\mathrm{op}}$ has a pseudo section. But this last is $\downarrow_{Y^{\mathrm{op}}} : Y^{\mathrm{op}} \to \mathbb{D}Y^{\mathrm{op}}$ showing that Y^{op} is Cauchy complete. ∎

5.17. Remark. We remark that the display in Lemma 5.14 shows that we have what is called the "Isbell conjugation" adjunction in the context of enriched category theory. In [K&S] it is shown, for enriched categories, that the Isbell conjugation adjunction restricted from power objects to Cauchy completions gives an equivalence and hence the self dual property for Cauchy completeness.

6. The Main Results

6.1. Proposition. *The **ord**-functors*

$$\begin{array}{c}\mathbf{idm}^{\mathrm{op}}\\ i\mathbf{ord}(-,\Omega)\uparrow\ \dashv\ \downarrow\mathbf{idm}(-,i\Omega)\\ \mathbf{ord}\end{array}$$

*are biadjoint in the sense indicated and give rise to an **ord**-monad $\mathbf{idm}(i\mathbf{ord}(-,\Omega), i\Omega)$ on **ord** whose **ord**-category of algebras is **ccd**, the **ord** category of CCD lattices and functors having both right and left adjoints.*

Proof. To establish the biadjunction we must show that, for all X in **idm** and A in **ord**, we have a pseudonatural equivalence of categories (here ordered sets)

$$\mathbf{idm}(X, i\mathbf{ord}(A,\Omega)) \simeq \mathbf{ord}(A, \mathbf{idm}(X, i\Omega))$$

For $(X, <)$ an idempotent, there is an equivalence $\mathbf{idm}(X, i\Omega) \overset{(vii)}{\simeq} DX^{\mathrm{op}}$ by Proposition 4.12 and, for $(A, \le)$ an order, the isomorphism $\mathbf{ord}(A,\Omega) \overset{(i)}{\cong} DA^{\mathrm{op}}$ is standard, as reviewed in 4.2. In Remark 5.12 we disambiguated the use of D in both **ord** and **idm** power object contexts with the equation $DiB \overset{(ii)}{=} DB$. In the string of pseudonatural equivalences below, we have also explicitly used Proposition 5.4 that iD is right biadjoint to $(-)_\#$ in (iii) and in (v), while (iv) arises from the fact that $(-)^{\mathrm{op}} : \mathbf{krl}^{\mathrm{op}} \to \mathbf{krl}$ is an involutory isomorphism. Finally, we have (vi) because $i : \mathbf{ord} \to \mathbf{idm}$ is fully faithful. (In anticipation of possible future developments we note that *pseudo fully faithfulness* of i would have sufficed.)

$$\begin{array}{rcl}\mathbf{idm}(X, i\mathbf{ord}(A,\Omega)) & \overset{(i)}{\cong} & \mathbf{idm}(X, iDA^{\mathrm{op}}) \\ & \overset{(ii)}{=} & \mathbf{idm}(X, iDiA^{\mathrm{op}}) \\ & \overset{(iii)}{\simeq} & \mathbf{krl}(X, iA^{\mathrm{op}}) \\ & \overset{(iv)}{\cong} & \mathbf{krl}(iA, X^{\mathrm{op}}) \\ & \overset{(v)}{\simeq} & \mathbf{idm}(iA, iDX^{\mathrm{op}}) \\ & \overset{(vi)}{\cong} & \mathbf{ord}(A, DX^{\mathrm{op}}) \\ & \overset{(vii)}{\simeq} & \mathbf{ord}(A, \mathbf{idm}(X, i\Omega))\end{array}$$

The involutory isomorphism $(-)^{\mathrm{op}}$ mentioned above for **krl** leads to an involutory isomorphism $(-)^{\mathrm{op}} : \mathbf{idm}^{\mathrm{co}} \to \mathbf{idm}$ (just as taking opposites has variance 'co' for categories and functors). If the now established biadjunction of the statement is composed with

$$\mathbf{idm}^{\mathrm{coop}} \underset{(-)^{\mathrm{op}}}{\overset{(-)^{\mathrm{op}}}{\rightleftarrows}} \mathbf{idm}^{\mathrm{op}}$$

seen as a trivial biadjunction, then the composite monad is unchaged (because $(-)^{\mathrm{op}}$ is involutory) but in this form it is immediately recognizable as the composite monad $DU : \mathbf{ord} \to \mathbf{ord}$ arising from the distributive law $UD \to DU$ of the upset monad over the downset monad (both on **ord**) studied in [MRW]. The 2-category of algebras for DU was shown in [MRW] to be **ccd**. ∎

6.2. Theorem. *The* **ord**-*functors*

$$\begin{array}{c}{\mathbf{idm}_{cc}}^{\mathrm{op}} \\ i\mathbf{ord}(-,\Omega)\ \big\uparrow \dashv \big\downarrow\ \mathbf{idm}_{cc}(-,i\Omega) \\ \mathbf{ord}\end{array}$$

are biadjoint in the sense indicated, $\mathbf{idm}_{cc}(-, i\Omega) : \mathbf{idm}^{\mathrm{op}} \to \mathbf{ord}$ *is bimonadic and the* **ord**-*category of algebras is* **ccd**.

Proof. To establish this biadjunction we refer to that of Proposition 6.1 and first note that the left biadjoint applied to an order A produces an isomorph of iDA^{op} and we have seen in 5.12 that idempotents of the form iDB are Cauchy complete so that it factors through ${\mathbf{idm}_{cc}}^{\mathrm{op}}$. On the other hand, $\Omega = D1$ so that 5.12 also shows that $i\Omega$ is Cauchy complete. It follows that the restriction of the right biadjoint of Proposition 6.1

to ${\mathbf{idm}_{cc}}^{\mathrm{op}}$ is $\mathbf{idm}_{cc}(-, i\Omega)$. Now with X an arbitrary Cauchy complete idempotent we have the following string of pseudonatural equivalences adapted from those in the proof of Proposition 6.1

$$\begin{aligned}
\mathbf{idm}_{cc}(X, i\mathbf{ord}(A,\Omega)) &\overset{(a)}{\cong} \mathbf{idm}_{cc}(X, iDA^{\mathrm{op}}) \\
&\overset{(b)}{=} \mathbf{idm}_{cc}(X, iDiA^{\mathrm{op}}) \\
&\overset{(c)}{\simeq} \mathbf{krl}(X, iA^{\mathrm{op}}) \\
&\cong \mathbf{krl}(iA, X^{\mathrm{op}}) \\
&\simeq \mathbf{idm}(iA, iDX^{\mathrm{op}}) \\
&\cong \mathbf{ord}(A, DX^{\mathrm{op}}) \\
&\overset{(d)}{\simeq} \mathbf{ord}(A, \mathbf{idm}_{cc}(X, i\Omega))
\end{aligned}$$

Their justifications are as before except that at (a), (b), (c), and (d) we need also to note that $\mathbf{idm}_{cc}$ is the *full* sub-**ord**-category of **idm** determined by the Cauchy complete objects. There is the trivial biadjunction

$$ {\mathbf{idm}_{cc}}^{\mathrm{coop}} \underset{(-)^{\mathrm{op}}}{\overset{(-)^{\mathrm{op}}}{\rightleftarrows}} {\mathbf{idm}_{cc}}^{\mathrm{op}} $$

because we have seen in Proposition 5.16 that the dual of a Cauchy complete idempotent is Cauchy complete. If this biadjunction is composed with that in the statement it is clear that the biadjunction of the statement gives rise to the **ord**-monad DU on **ord** whose **ord**-category of algebras was shown in [MRW] to be **ccd**. The comparison **ord**-functor ${\mathbf{idm}_{cc}}^{\mathrm{op}} \to \mathbf{ccd}$ is evidently equivalent to $D(-)^{\mathrm{op}} : {\mathbf{idm}_{cc}}^{\mathrm{op}} \to \mathbf{ccd}$ by Proposition 4.12. The latter can seen as the composite

$$ {\mathbf{idm}_{cc}}^{\mathrm{op}} \xrightarrow{(-)^{\mathrm{op}}} {\mathbf{idm}_{cc}}^{\mathrm{coop}} \longrightarrow \mathrm{map}\mathbf{krl}^{\mathrm{coop}} \xrightarrow{\mathrm{map}D^{\mathrm{coop}}} \mathrm{map}{\mathbf{ccd}_{\mathrm{sup}}}^{\mathrm{coop}} = \mathbf{ccd} $$

where $\mathbf{ccd}_{\mathrm{sup}}$ is the **ord**-category of CCD lattices and functors having a right adjoint. The first displayed arrow is an isomorphism. The second is an instance of the formal biequivalence given in subsection 3.8 since $(-)_{\#} : \mathbf{idm} \to \mathbf{krl}$ has Cauchy completions. The third is obtained from the biequivalence $D : \mathbf{krl} \to \mathbf{ccd}_{\mathrm{sup}}$ of Theorem 17 in [RW4] by application of $\mathrm{map}(-)^{\mathrm{coop}}$. It follows that ${\mathbf{idm}_{cc}}^{\mathrm{op}} \to \mathbf{ccd}$ is a biequivalence so that $\mathbf{idm}_{cc}(-, i\Omega) : {\mathbf{idm}_{cc}}^{\mathrm{op}} \to \mathbf{ord}$ is bimonadic. ∎

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