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FUNDAMENTAL PUSHOUT TOPOSES

MARTA BUNGE

ABSTRACT. The author [2, 5] introduced and employed certain 'fundamental pushout toposes' in the construction of the coverings fundamental groupoid of a locally connected topos. Our main purpose in this paper is to generalize this construction without the local connectedness assumption. In the spirit of [16, 10, 8] we replace connected components by constructively complemented, or definable, monomorphisms [1]. Unlike the locally connected case, where the fundamental groupoid is localic prodiscrete and its classifying topos is a Galois topos, in the general case our version of the fundamental groupoid is a locally discrete progroupoid and there is no intrinsic Galois theory in the sense of [19]. We also discuss covering projections, locally trivial, and branched coverings without local connectedness by analogy with, but also necessarily departing from, the locally connected case [13, 11, 7]. Throughout, we work abstractly in a setting given axiomatically by a category **V** of locally discrete locales that has as examples the categories **D** of discrete locales, and **Z** of zero-dimensional locales [9]. In this fashion we are led to give unified and often simpler proofs of old theorems in the locally connected case, as well as new ones without that assumption.

Introduction

The author [2, 5] introduced certain 'fundamental pushout toposes' in the construction and study of the prodiscrete fundamental groupoid of a locally connected topos. The main purpose of this paper is to generalize this construction without the local connectedness assumption.

The key to this program is the theory of spreads in topos theory [7], originating in [16, 25], and motivated therein by branched coverings. Our main tools are the factorization theorems [10] and [8] for geometric morphisms whose domains are definable dominances, a notion which states that the constructively complemented subobjects are well-behaved.

In connection with fundamental pushout toposes we also discuss covering projections and branched coverings without local connectedness by analogy with, but also departing from, the locally connected case [7].

The passage from the discrete to the zero-dimensional is not perfect, as the category of zero-dimensional locales does not have good enough closure properties. For this reason, we work with the larger category of locally discrete locales, a category that is also suitable

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for discussing localic reflections.

The setting in which we work is that of a category **V** of (locally discrete) locales, given axiomatically to model the category of zero-dimensional locales. Since also the category of discrete locales is an instance of such a **V**, we are able to discuss the differences between the discrete and the non-discrete (e.g., zero-dimensional) cases in a unified way.

An outline of the paper follows.

In § 1 we review and update definitions and results from [9] in order to include geometric morphisms. A category **V** of locally discrete locales is introduced axiomatically. Its main examples are the categories **D** of discrete, and **Z** of zero-dimensional locales. The key notion in this section is that of a **V**-localic geometric morphism in $\mathbf{Top}_{\mathscr{S}}$. The motivating example for it are the spreads [10]. In the discrete case we prove special properties of the **V**-localic reflection of use in connection with the fundamental progroupoid.

In § 2 we generalize definitions and results of [2, 5] to the **V**-context. For a **V**-determined topos $\mathscr{E}$, and a cover U in $\mathscr{E}$, the localic **V**-reflection of $\mathscr{E}/U$ determines a 'fundamental **V**-pushout' topos $\mathscr{G}_U(\mathscr{E})$. We use results of [21, 15] in order to obtain, from the system of toposes $\mathscr{G}_U(\mathscr{E})$ indexed by a generating category of covers in $\mathscr{E}$, a limit topos. Our results show that this topos (or its corresponding progroupoid) represents first ordered cohomology with coefficients in discrete groups. In the discrete case, we obtain the familiar result that the progroupoid may be replaced by a prodiscrete localic groupoid, using results from § 1.

In § 3 we characterize the fundamental **V**-pushout toposes $\mathscr{G}_U(\mathscr{E})$ of § 2 in terms of what we call **V**-covering projections. In turn, this notion is given alternative equivalent versions, analogue to the result of [11] for locally constant coverings. In connection with the locally constant coverings, we comment on an alternative construction of the fundamental progroupoid of a (Grothendieck) topos given in [14] and inspired by shape theory [17, 18].

In § 4 we first review the **V**-comprehensive factorization [9] of a geometric morphism. with a **V**-determined domain into a **V**-initial geometric morphism followed by a **V**-fibration. The **V**-fibrations are an abstraction of the complete spreads [8], and are the geometric counterpart of the **V**-valued Lawvere distributions [23]. We also establish new closure and stability theorems for the factors of the comprehensive **V**-factorization.

In § 5 we introduce notions of locally trivial **V**-coverings and branched **V**-coverings suitably generalizing the corresponding notions in the locally connected case [7]. The key for the passages from one to the other is given by the analogue of a 'pullback lemma' from [13]. Beyond that point, we are faced with the two 'inconvenient truths' that affect the non-discrete (or non locally connected) case. In particular, and unlike the locally connected case [11], there are no good topological invariants. Also unlike the locally connected case [7], the 'ideal knot' need not exist in the general case.

We remark that this paper does not attempt to be self contained, a task which would have been more suitable for a monograph. A reason for this is that the paper involves a great deal of material from two different lines of research in which the author has been involved, to wit, the fundamental groupoid of a locally connected topos, and complete topos spreads in the non-locally connected case, up until now unrelated. On the other hand, all references that are needed for providing the background that is necessary for a full understanding of this paper are given in the references.

A justification for the work done here is not simply that it provides a construction of the fundamental groupoid of a topos in the non-locally connected case, but primarily that this is done in a conceptually simple manner (to replace connected components by clopen subsets), thereby providing a better understanding of the locally connected case itself. In addition, the systematic use of the universal property of fundamental pushout construction renders this work into a purely categorical one.

Rather than collecting remarks and questions in a separate section at the end of the paper, these are scattered throughout the paper. Some of these remarks constitute ideas for further research, whereas others are just lose ends which might prove interesting to certain readers.

1. **V**-localic reflections

Let $\mathscr{S}$ be an elementary topos. Denote by $\mathbf{Top}_{\mathscr{S}}$ the 2-category of $\mathscr{S}$-bounded toposes, geometric morphisms over $\mathscr{S}$, and iso 2-cells.

Our motivation for the notion of a **V**-localic geometric morphism in $\mathbf{Top}_{\mathscr{S}}$ is the notion of a spread (with a definable dominance domain) [10, 7], originally due to R. H. Fox [16] in topology.

A topos $f : \mathscr{F} \longrightarrow \mathscr{S}$ is a definable dominance [10] if the class of definable monos in $\mathscr{F}$ [1] is well behaved — that is, classifiable and closed under composition. A geometric morphism over a base topos $\mathscr{S}$ is said to be a spread if it has a definable generating family [7]. Spreads are localic geometric morphisms. The defining locales of spreads are said to be zero-dimensional. It is shown in [10] that any geometric morphism whose domain is a definable dominance has a unique pure surjection, spread factorization.

Every Grothendieck topos is a definable dominance since, over *Set*, the definable subobjects of an object are its complemented subobjects. Every locally connected topos is a definable dominance by the characterization theorem in [1]. For a geometric morphism whose domain is locally connected, the notion of a spread may be stated in terms of connected components [7].

We shall work with a category **V** of locales in the base topos $\mathscr{S}$, modeled on the category **Z** of zero-dimensional locales. This generality suits the interplay between the discrete and the zero-dimensional, that is, between the locally connected and the 'general' cases, since the category **D**, of discrete locales, will also be regarded as a model of such a **V** (with special properties).

Denote by **Loc** the category of locales in $\mathscr{E}$, a partial ordering of its morphisms given by

$$m \leq l : W \longrightarrow X$$

if $m^*U \leq l^*U$, for any $U \in \mathscr{O}(X)$. It is $\mathscr{E}$-indexed, with Σ satisfying the BCC.

We shall introduce certain functors $\mathbf{V} \longrightarrow \mathscr{F}$ from a category $\mathbf{V}$ of locales to a topos $\mathscr{F}$ in $\mathbf{Top}_{\mathscr{S}}$. Such functors necessarily would have to forget about 2-cells in $\mathbf{V}$. This suggests that we only consider such $\mathbf{V}$ in which the partial ordering of the morphisms is trivial. This feature justifies restricting our attention to locally discrete locales.

1.1. Definition. We shall say that a locale Z is *locally discrete* if for every locale X the *partial ordering* in $\mathbf{Loc}(X, Z)$ is discrete. Likewise, a map $p : Z \longrightarrow B$ is *locally discrete* if for every $q : X \longrightarrow B$, $\mathbf{Loc}/B(q, p)$ is discrete.

Let $\mathbf{L}$ denote the category of *locally discrete locales* in $\mathscr{S}$. It is an $\mathscr{S}$-indexed category, with Σ satisfying the BCC, and small hom-objects. $\mathbf{L}$ is closed under limits, which are created in $\mathbf{Loc}$, the 2-category of locales in $\mathscr{S}$. The category $\mathbf{D}$, of *discrete locales* is included in $\mathbf{LD}$.

1.2. Remark. $\mathbf{L}$ has the following additional properties:

(i) If $Y \longrightarrow Z$ is a locally discrete map, and Z is locally discrete, then Y is locally discrete.

(ii) If Y is locally discrete, then any locale morphism $Y \longrightarrow Z$ is locally discrete.

(iii) The pullback of a locally discrete map along another locally discrete map is again locally discrete.

(iv) If Z is locally discrete, then any sublocale $S \rightarrowtail Z$ is also locally discrete.

1.3. Assumption. In what follows, $\mathbf{V}$ denotes a full $\mathscr{S}$-indexed subcategory of $\mathbf{Loc}$ such that, in addition, it satisfies

1. $\mathbf{D} \subseteq \mathbf{V} \subseteq \mathbf{L}$.
2. $\mathbf{V}$ is closed under open sublocales.
3. Let

$$\begin{array}{ccc} W & \xrightarrow{\;q\;} & Z \\ {\scriptstyle n}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle m} \\ Y & \xrightarrow{\;p\;} & X \end{array}$$

be a pullback in $\mathbf{Loc}$ in which p is etale. If $m \in \mathbf{V}$ then $n \in \mathbf{V}$.

The *interior* of a localic geometric morphism $Sh_{\mathscr{F}}(Y) \longrightarrow \mathscr{F}$ is an object $d(Y)$ of $\mathscr{F}$ such that

$$\begin{array}{ccc} \mathscr{F}/d(Y) & \longrightarrow & Sh_{\mathscr{F}}(Y) \\ & \searrow & \big\downarrow \\ & & \mathscr{F} \end{array}$$

commutes, and any $\mathscr{F}/D \longrightarrow Sh_{\mathscr{F}}(Y)$ over $\mathscr{F}$ factors uniquely through $\mathscr{F}/d(Y)$. The interior of a localic geometric morphism always exists.

For any topos $f : \mathscr{F} \longrightarrow \mathscr{S}$ in $\mathbf{Top}_{\mathscr{S}}$, then there is a functor

$$F^* : \mathbf{Loc} \longrightarrow \mathscr{F}$$

such that for $X \in \mathbf{Loc}$, $F^*X = d(f^\# X)$ is the *interior* of the locale $f^\# X$, where the latter is defined by the bipullback

$$\begin{array}{ccc} Sh_{\mathscr{F}}(f^\# X) & \longrightarrow & Sh_{\mathscr{S}}(X) \\ \downarrow & \lrcorner & \downarrow \\ \mathscr{F} & \xrightarrow{\quad f \quad} & \mathscr{S} \end{array}$$

in $\mathbf{Top}_{\mathscr{S}}$. In particular, for any locale X in $\mathscr{S}$, there is a canonical geometric morphism

$$\mathscr{F}/F^*X \longrightarrow Sh_{\mathscr{S}}(X)$$

over $\mathscr{S}$, which we refer to as 'the projection'.

1.4. Definition. A topos $f : \mathscr{F} \longrightarrow \mathscr{E}$ over $\mathscr{S}$ is said to be ***V**-determined* if there is an $\mathscr{S}$-indexed left adjoint $F_! \dashv F^* : \mathbf{V} \longrightarrow \mathscr{F}$, such that 'the BCC for etales holds, in the sense that for any etale map $p : Y \longrightarrow X$ in $\mathbf{V}$, the transpose (below, right) of a pullback square (below, left) is again a pullback.

$$\begin{array}{ccc} P & \longrightarrow & F^*Y \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle F^*p} \\ E & \xrightarrow{\quad m \quad} & F^*X \end{array} \qquad\qquad \begin{array}{ccc} F_!P & \longrightarrow & Y \\ {\scriptstyle F_!q}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ F_!E & \xrightarrow{\quad \hat{m} \quad} & X \end{array} .$$

1.5. Remark. The definition of a $\mathbf{V}$-determined topos can be relativized to any geometric morphism. Let $\mathscr{E}$ be any topos in $\mathbf{Top}_{\mathscr{S}}$. Denote by $\mathbf{Loc}(\mathscr{E})$ the category of locales in $\mathscr{E}$ and, similarly, define $\mathbf{D}(\mathscr{E}) \cong \mathscr{E}$, $\mathbf{L}(\mathscr{E})$, and $\mathbf{V}(\mathscr{E})$, the latter axiomatically by analogy with Assumption 1.3. Let $\psi : \mathscr{F} \longrightarrow \mathscr{E}$ be a morphism in $\mathbf{Top}_{\mathscr{S}}$. We say that ψ is a ***V**-determined geometric morphism* if there is an $\mathscr{E}$-indexed left adjoint $\Psi_! \dashv \Psi^* : \mathbf{V}(\mathscr{E}) \longrightarrow \mathscr{F}$, such that 'the BCC for etales in $\mathbf{V}(\mathscr{E})$ holds. (For example, a $\mathbf{D}$-determined geometric morphism in $\mathbf{Top}_{\mathscr{S}}$ is precisely a *locally connected* geometric morphism.)

Let $\psi : \mathscr{F} \longrightarrow \mathscr{E}$ be a $\mathbf{V}$-determined geometric morphism in $\mathbf{Top}_{\mathscr{S}}$. For any object D of $\mathscr{F}$, let

$$\rho_D : \mathscr{F}/D \longrightarrow Sh_{\mathscr{E}}(\Psi_! D) \tag{1}$$

be the composite of the projection $\mathscr{F}/\Psi^*(\Psi_! D) \longrightarrow Sh_{\mathscr{E}}(\Psi_! D)$ with the unit of adjointness $\Psi_! \dashv \Psi^*$.

1.6. Lemma. *The inverse image part of* $\rho_D : \mathscr{F}/D \longrightarrow Sh_{\mathscr{E}}(\Psi_! D)$ *is given as follows. If* $p : Y \longrightarrow \Psi_! D$ *is an etale map in* **Loc** $(\mathscr{E})$*, then* $(\rho_D)^*(p)$ *is the left vertical leg of the pullback*

$$\begin{array}{ccc} (\rho_D)^*(Y) & \longrightarrow & \Psi^* Y \\ \Big\downarrow & \lrcorner & \Big\downarrow{\scriptstyle \Psi^* p} \\ D & \xrightarrow{\ \eta_D\ } & \Psi^*\Psi_! D \end{array} \tag{2}$$

in $\mathscr{F}$.

1.7. Lemma. *Let* $\psi : \mathscr{F} \longrightarrow \mathscr{E}$ *be a* $\boldsymbol{V}$*-determined geometric morphism in* $\boldsymbol{Top}_{\mathscr{S}}$.

1. *Then, for every object* D *of* $\mathscr{F}$, ρ_D *in (1) is a surjection.*
2. *If* $\boldsymbol{V}$ *is such that the BCC holds for all etale morphisms* $Y \longrightarrow X$ *with* $X \in \boldsymbol{V}(\mathscr{E})$*, then* ρ_D *in (1) is connected and* $\boldsymbol{V}$*-determined in the sense of Remark 1.5.*

Proof.

1. If $Y \longrightarrow \Psi_! D$ is etale, then Y is locally in $\mathbf{V}(\mathscr{E})$ by Assumption 1.3(2). For each open inclusion $p : U \hookrightarrow \Psi_! D$, the BCC for etales in $\mathbf{V}(\mathscr{E})$ implies that the transpose $\Psi_! {\rho_D}^*(U) \longrightarrow U$ of the top horizontal in the diagram (2) is an isomorphism. This implies that ρ_D is a surjection.
2. For an arbitrary etale map $Y \longrightarrow \Psi_! D$, the BCC implies that $\Psi_! \rho^*(Y) \cong Y$, so that $\rho_D : \mathscr{F}/D \longrightarrow Sh_{\mathscr{E}}(\Psi_! D)$ has a fully faithful inverse image part, or is connected. That ρ_D is $\mathbf{V}$-determined follows from the same hypothesis, since in that case $\Psi_!$ trivially preserves etaleness.

□

Consider a commutative triangle

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\qquad \pi \qquad} & \mathscr{E} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle e} & \\ & \mathscr{S} & \end{array} \tag{3}$$

in $\mathbf{Top}_{\mathscr{S}}$. If Z is a locale in $\mathbf{V}$, then there is a geometric morphism $\mathscr{F}/\pi^*(E^* Z) \longrightarrow Sh_{\mathscr{F}}(Z)$, which factors through $\mathscr{F}/F^*(Z)$ by a morphism $\pi^*(E^* Z) \longrightarrow \Psi^*(Z)$ in $\mathscr{F}$, since $\Psi^*(Z)$ is the interior of $Sh_{\mathscr{F}}(\psi^{\#} Z)$. Thus, there is a natural transformation

$$\alpha : (\pi^* \cdot E^*) \Rightarrow F^* : \mathbf{V} \longrightarrow \mathscr{F} \tag{4}$$

1.8. Remark. The 2-cell α in (4) is an isomorphism when restricted to discrete locales, or else when π is a local homeomorphism.

1.9. Definition. Consider the commutative triangle (3).

1. We shall say that $\rho : \mathscr{F} \longrightarrow \mathscr{E}$ is ***V**–initial* if the transpose $\hat{\alpha} : E^* \Rightarrow (\rho_* \cdot F^*) : \mathbf{V} \longrightarrow \mathscr{E}$ of (4) under $\rho^* \dashv \rho_*$ is an isomorphism.

2. We shall say that $\rho : \mathscr{F} \longrightarrow \mathscr{E}$ is ***V**-localic* if there is given $Z \in \mathbf{V}(\mathscr{E})$ where $\varphi : Sh_{\mathscr{E}}(Z) \longrightarrow \mathscr{E}$ has a **V**-determined domain, and an equivalence σ in the commutative triangle:

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\quad\sigma\quad} & Sh_{\mathscr{E}}(Z) \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle\varphi} & \\ & \mathscr{E} & \end{array} \tag{5}$$

1.10. Remark. If $f : \mathscr{F} \longrightarrow \mathscr{X}$ is a geometric morphism between **V**-determined toposes, then the canonical geometric morphism ρ in (1) is **V**-initial iff the canonical 2-cell $\beta : (F_! \cdot \rho^*) \Rightarrow X_!$ corresponding to $\hat{\alpha}$ by taking left adjoints, is an isomorphism.

We now state a factorization theorem for **V**-determined geometric morphisms (see Remark 1.5).

1.11. Theorem. *Let $\psi : \mathscr{F} \longrightarrow \mathscr{E}$ in $\boldsymbol{Top}_{\mathscr{S}}$ be a **V**-determined geometric morphism. For any object D of $\mathscr{F}$, let*

$$\mathscr{F}/D \xrightarrow{\rho_D} Sh_{\mathscr{E}}(\Psi_! D) \xrightarrow{\varphi_D} \mathscr{E}$$

be the canonical factorization.

1. *For a general **V**, the first factor is surjective **V**-initial over $\mathscr{E}$, and the second factor is **V**-localic. Such a factorization is unique up to equivalence.*

2. *If **V** is closed under all etale $p : Y \longrightarrow X$ with $X \in \boldsymbol{V}(\mathscr{E})$, then the **V**-initial first factor $\rho_D : \mathscr{F}/D \longrightarrow Sh_{\mathscr{E}}(\Psi_! D)$ is (furthermore) connected and **V**-determined, and the second factor is **V**-localic. Such a factorization is unique up to equivalence.*

1.12. Corollary. *Any locally connected geometric morphism in $\boldsymbol{Top}_{\mathscr{S}}$ factors uniquely into a connected locally connected geometric morphism followed by a local homeomorphism.*

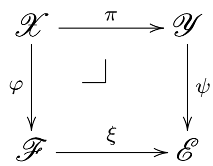
1.13. Example. The main examples of a category **V** satisfying the clauses in Assumption 1.3 are :

1. $\mathbf{V} = \mathbf{D}$. Any discrete locale is locally discrete. Notice that any open inclusion $U \hookrightarrow X$ in **Loc** is etale. In this example, (2) may be strengthened to any etale $p : Y \longrightarrow X$ in **Loc**.

2. $\mathbf{V} = \mathbf{Z}$. Any zero-dimensional locale X is locally discrete since $\mathscr{O}(X)$ has a basis consisting of (constructibly) complemented opens. In this example, (2) holds since any open sublocale of a spread is a spread. However, (2) cannot be strengthened to any etale $p : Y \longrightarrow X$ in $\mathbf{Loc}(\mathscr{E})$, since there exists a zero-dimensional space X and an etale map $p : Y \longrightarrow X$ of locales, where Y is not zero-dimensional [9].

   As for (3), indeed, in $\mathbf{Top}_{\mathscr{S}}$, the pullback of a spread with a definable dominance domain along a local homeomorphism is again one such. We give a proof of (3) below in Proposition 1.14.

1.14. Proposition. *Consider a bipullback in* $\boldsymbol{Top}_{\mathscr{S}}$*, in which* ξ *is a local homeomorphism:*

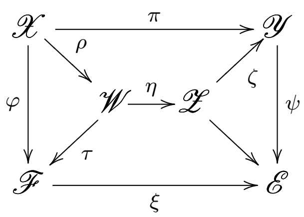


If $\psi : \mathscr{Y} \longrightarrow \mathscr{E}$ *is a spread then* φ *is spread .*

Proof.

In the bipullback, π is a local homeomorphism since ξ is one. Thus, $\mathscr{X} \cong \mathscr{Y}/B$ and π is identified via this equivalence with the canonical local homeomorphism $\Sigma_B \dashv B^* \dashv \Pi_B : \mathscr{Y}/B \longrightarrow \mathscr{Y}$. A monomorphism m in $\mathscr{Y}/B$ is $\mathscr{S}$-definable iff $\Sigma_B(m)$ is $\mathscr{S}$-definable in $\mathscr{Y}$. This implies that $\mathscr{Y}/B$ is a definable dominance since $\mathscr{Y}$ is one.

Since $\mathscr{X}$ is a definable dominance, we may factor φ into its pure surjection $\rho : \mathscr{X} \longrightarrow \mathscr{W}$ and a spread $\tau : \mathscr{W} \longrightarrow \mathscr{F}$ parts: $\varphi \cong \tau \cdot \rho$. Then, since $\mathscr{V}$ is a definable dominance, we may factor the composite $\xi \cdot \tau$ into its pure surjection and spread parts. This is represented in the diagram below, where ζ is uniquely defined using orthogonality, since ψ is a spread and $\eta \cdot \rho$ is the composite of two pure surjections hence a pure surjection.



Since the outer square is a pullback, there is $\mathscr{V} \xrightarrow{\theta} \mathscr{X}$ such that $\pi \cdot \theta \cong \zeta \cdot \eta$ and $\varphi \cdot \theta \cong \tau$. The universal property of the pullback implies that $\theta \cdot \rho \cong id_{\mathscr{X}}$. Hence, $\rho \cdot \theta \cdot \rho \cong \rho$. Therefore, the two geometric morphisms $\rho \cdot \theta$ and $id_{\mathscr{W}}$ from the spread

τ to itself agree when precomposed with the pure surjection ρ. They must therefore agree: $\rho \cdot \theta \cong id_{\mathscr{W}}$. We have shown that ρ is an equivalence, so that φ is a spread.

□

3. $\mathbf{V} = \mathbf{L}$. In this case, (2) and (3) may both be strengthened to any $p : Y \longrightarrow X \in \mathbf{L}$.

1.15. Example. We recall the following example (due to Peter Johnstone and included in [9]). Let X be the subspace $\{0\} \cup \{\frac{1}{n} \mid n \geq 1\}$ of $\mathbb{R}$, and let $Y = X + X/\sim$, obtained by identifying the two $\frac{1}{n}$'s, for every n. The topology on Y is T_1, but not Hausdorff. The map $Y \longrightarrow X$ identifying the two 0's is etale, X is 0-dimensional, but Y is not 0-dimensional.

The converse for spaces (due to Gábor Lukács) holds in the following form. Let X be a zero-dimensional space that has the property: if for all etale maps $Y \longrightarrow X$ of spaces, with Y regular, Y is zero-dimensional, then X is discrete. The argument is by contradiction. Assume that X is zero-dimensional but not discrete. Let $p \in X$ be an isolated point. Construct $Y \longrightarrow X$ etale as above, letting p take the role of 0. The space Y is not zero-dimensional, but (by etaleness) it is locally zero-dimensional. The proof clearly reduces to showing that Y locally zero-dimensional and regular implies that Y is globally zero-dimensional, thus contradicting the assumption. This is shown to be the case. It is an open question whether the analogous statement for arbitrary locales is true and if so in what form.

2. The fundamental V-progroupoid

In this section we abstract the fundamental pushouts of [2, 5] in the $\mathbf{V}$-context of § 1.

We recall that for $\mathscr{E}$ locally connected in $\mathbf{Top}_{\mathscr{S}}$, the terminology 'fundamental pushout' was applied to refer to any pushout

$$\begin{array}{ccc} \mathscr{E}/U & \xrightarrow{\varphi_U} & \mathscr{E} \\ {\scriptstyle \rho_U}\downarrow & & \downarrow{\scriptstyle \sigma_U} \\ \mathscr{E}/e_!U & \xrightarrow[p_U]{} & \mathscr{G}_U(\mathscr{E}) \end{array} \tag{6}$$

where $U \twoheadrightarrow 1$ in $\mathscr{E}$, $\varphi_U : \mathscr{E}/U \longrightarrow \mathscr{E}$ is the canonical local homeomorphism, and ρ_U is the first factor in the canonical factorization of the locally connected geometric morphism

$$\mathscr{E}/U \xrightarrow{\varphi_U} \mathscr{E} \xrightarrow{e} \mathscr{S}$$

into a connected locally connected geometric morphism followed by a local homeomorphism.

This factorization is an instance of Theorem 1.11(2). This suggests that we give a more general notion of 'fundamental pushout' in the $\mathbf{V}$-context, by using Theorem 1.11(1).

2.1. Definition. Let $\mathscr{E}$ be a **V**-determined topos over $\mathscr{S}$. For an epimorphism $U \twoheadrightarrow 1$ in $\mathscr{E}$, the following pushout in $\mathbf{Top}_{\mathscr{S}}$

$$\begin{array}{ccc} \mathscr{E}/U & \xrightarrow{\varphi_U} & \mathscr{E} \\ {\scriptstyle \rho_U}\downarrow & & \downarrow{\scriptstyle \sigma_U} \\ Sh_{\mathscr{E}}(E_!U) & \xrightarrow[p_U]{} & \mathscr{G}_U(\mathscr{E}) \end{array} \tag{7}$$

where $\psi_U : \mathscr{E}/U \longrightarrow \mathscr{E}$ is the canonical local homeomorphism, and where ρ_U was defined in (1) applied to the **V**-determined topos $\mathscr{E}/U \xrightarrow{\varphi_U} \mathscr{E} \xrightarrow{e} \mathscr{S}$, is said to be a *fundamental* ***V****-pushout* .

2.2. Theorem. *Let $\mathscr{E}$ be a* ***V****-determined topos, and let U be a cover in $\mathscr{E}$. Then,*

1. *The topos $\mathscr{G}_U(\mathscr{E})$ in the fundamental* ***V****-pushout is the classifying topos $\mathscr{B}(\mathbb{G}_U)$ of an etale complete groupoid $\mathbb{G}_U$, by an equivalence*

$$\mathscr{G}_U(\mathscr{E}) \simeq \mathscr{B}(\mathbb{G}_U)$$

   which identifies p_U with the canonical localic point of $\mathscr{B}(\mathbb{G}_U)$.

2. *The localic groupoid $\mathbb{G}_U$ is locally discrete.*

3. *The locally discrete groupoid $\mathbb{G}_U$ classifies U-split K-torsors in $\mathscr{E}$ for discrete groups K.*

Proof.

For the morphism $\varphi_U : \mathscr{E}/U \longrightarrow \mathscr{E}$ in $\mathbf{Top}_{\mathscr{S}}$, there is an induced truncated simplicial diagram, namely the kernel of φ_U, denoted $Ker(\varphi_U)$:

$$\mathscr{E}/U \times_{\mathscr{E}} \mathscr{E}/U \times_{\mathscr{E}} \mathscr{E}/U \;\overset{\pi_{01}}{\underset{\pi_{12}}{\overset{\pi_{02}}{\rightrightarrows\!\!\!\rightarrow}}}\; \mathscr{E}/U \times_{\mathscr{E}} \mathscr{E}/U \;\overset{\pi_0}{\underset{\pi_1}{\overset{i}{\leftleftarrows\!\!\!\rightarrow}}}\; \mathscr{E}/U$$

where, for example, $\mathscr{E}/U \times_{\mathscr{E}} \mathscr{E}/U$ denotes the bipullback of φ_U along itself.

There is a category $\mathscr{D}(\varphi_U)$, an object of which consists of an object y of $\mathscr{E}/U$ equipped with an isomorphism $\eta : {\pi_0}^*(y) \cong {\pi_1}^*(y)$ satisfying the cocycle and normalization conditions. A morphism in $\mathscr{D}(\varphi_U)$ from (y, η) to (z, ζ) is a morphism $f : y \longrightarrow z$ in $\mathscr{E}/U$ compatible in the obvious sense with η and ζ.

Let $\Phi_U : \mathscr{D}(\varphi_U) \longrightarrow \mathscr{E}/U$ be the canonical geometric morphism Since φ_U is a locally connected surjection, it is of effective descent. In other words, Φ_U is an equivalence.

Consider now a similar diagram for the morphism $p_U : Sh_{\mathscr{S}}(E_!U) \longrightarrow \mathscr{G}_U(\mathscr{E})$. By methods of classifying toposes [24], we extract from it an etale complete groupoid $\mathbb{G}_U$ in $\mathscr{G}_U$:

$$G_2 \overset{p}{\underset{q}{\overset{m}{\rightrightarrows}}} G_1 \overset{d}{\underset{c}{\overset{i}{\leftrightarrows}}} G_0$$

Denote by $\mathscr{D}(p_U)$the descent category for $Ker(p_U)$ and by $P_U : \mathscr{D}(p_U) \longrightarrow \mathscr{G}_U(\mathscr{E})$ the canonical geometric morphism. We wish to prove any of the following equivalent statements:

(i) $\mathscr{B}(\mathbb{G}_U) \cong \mathscr{G}_U(\mathscr{E})$

(ii) p_U is of effective descent.

(iii) $P_U : \mathscr{D}(p_U) \longrightarrow \mathscr{G}_U(\mathscr{E})$ is an equivalence.

First, notice that $\rho_U : \mathscr{E}/U \longrightarrow Sh_{\mathscr{S}}(E_! U)$, alternatively denoted $\rho_0 : \mathscr{E}/U \longrightarrow Sh(G_0)$ induces a morphism of truncated simplicial objects in $\mathbf{Top}_{\mathscr{S}}$ (in simplified notation) depicted together with the pushout diagram.

$$\begin{array}{ccccccc}
(\mathscr{E}/U)^3 & \longrightarrow & (\mathscr{E}/U)^2 & \longrightarrow & \mathscr{E}/U & \xrightarrow{\varphi_U} & \mathscr{E} \\
\downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_0} & & \downarrow{\scriptstyle \sigma_U} \\
Sh(G_2) & \longrightarrow & Sh(G_1) & \longrightarrow & Sh(G_0) & \xrightarrow{p_U} & \mathscr{G}_U(\mathscr{E})
\end{array}$$

Let $\lambda : Sh(G_0) \longrightarrow \mathscr{K}$ satisfy the descent conditions for $Ker(p_U)$. Then, the composite

$$\mathscr{E}/U \xrightarrow{\rho_0} Sh(G_0) \xrightarrow{\lambda} \mathscr{K}$$

satisfies the descent conditions for $Ker(\varphi_U)$. (For instance, for y an object of $\mathscr{K}$, and $\eta : d_0^*\lambda^*(y) \cong c^*\lambda^*(y)$ in $Sh(G_1)$, then $\rho_0{}^*\eta : \rho_1{}^*d_0^*\lambda^*(y) \cong \rho_1{}^*d_1^*\lambda^*(y)$ and, in turn, $d_0^*\rho_0{}^*\lambda^*(y) \cong d_1^*\rho_0{}^*\lambda^*(y)$.) Since φ_U is of effective descent, there exists a unique $\tau : \mathscr{E} \longrightarrow \mathscr{K}$, such that the square

$$\begin{array}{ccc}
\mathscr{E}/U & \xrightarrow{\varphi_U} & \mathscr{E} \\
\downarrow{\scriptstyle \rho_U} & & \downarrow{\scriptstyle \tau} \\
Sh(G_0) & \xrightarrow[\lambda]{} & \mathscr{K}
\end{array} \tag{8}$$

commutes up to an iso 2-cell.

By the universal property of the pushout (7), there exists a unique $\kappa : \mathscr{G}_U \longrightarrow \mathscr{K}$ such that $\kappa \cdot p_U \cong \lambda$ and $\kappa \cdot \sigma_U \cong \tau$. Since both φ_U and ρ_U are surjections, we have the following

$$(\kappa\sigma_U \cong \tau) \Leftrightarrow (\kappa\sigma_U\varphi_U \cong \tau\varphi_U) \Leftrightarrow (\kappa p_U \rho_U \cong \lambda\rho_U) \Leftrightarrow (\kappa p_U \cong \lambda)$$

Thus, we may rephrase the comment above as follows. There exists a unique $\kappa : \mathscr{G}_U(\mathscr{E}) \longrightarrow \mathscr{K}$ such that $\kappa \cdot p_U \cong \lambda$. This shows that p_U is of effective descent and it establishes the first claim.

We observe that since $Sh_{\mathscr{G}_U}(G_0) \cong Sh_{\mathscr{S}}(E!U) \longrightarrow \mathscr{S}$ is localic defined by a (zero-dimensional, hence) locally discrete locale $E_!U$, then also $p_U : Sh(E_!U) \longrightarrow \mathscr{G}_U(\mathscr{E})$ is localic defined by a locally discrete locale G_0 in $\mathscr{G}_U(\mathscr{E})$, by Remark 1.2.

It also follows from Remark 1.2 that all objects and morphisms defining $\mathbb{G}_U$ are locally discrete, that is, $\mathbb{G}_U$ is a locally discrete groupoid in $\mathscr{G}_U(\mathscr{E})$. (For instance, in the pullback

$$\begin{array}{ccc} Sh(G_1) & \xrightarrow{\pi_0} & Sh(G_0) \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle p_U} \\ Sh(G_0) & \xrightarrow{p_U} & \mathscr{G}_U(\mathscr{E}) \end{array}$$

G_1 is locally discrete.) This establishes the second claim.

We now argue that $\mathbb{G}_U$ (or, equivalently, $\mathscr{B}(\mathbb{G}_U)$) classifies U-split K torsors for discrete groups K. Crucial for this is that $\mathbf{D} \subseteq \mathbf{V}$ in Assumption 1.3 and the universal property of the $\mathbf{V}$-reflection.

For any *cover* $\mathscr{U} = \langle U \twoheadrightarrow 1, I, U \xrightarrow{\zeta} e^*I \rangle$ in a topos $\mathscr{E} \xrightarrow{e} \mathscr{S}$, the topos $\mathscr{P}_{\mathscr{U}}(\mathscr{E})$ in the following pushout is equivalent to the category *Split*$(\mathscr{U})$ [11]:

$$\begin{array}{ccc} \mathscr{E}/U & \xrightarrow{\varphi_U} & \mathscr{E} \\ {\scriptstyle \lambda_{\mathscr{U}}}\downarrow & & \downarrow{\scriptstyle \sigma_{\mathscr{U}}} \\ \mathscr{S}/I & \xrightarrow[p_{\mathscr{U}}]{} & \mathscr{P}_{\mathscr{U}}(\mathscr{E}) \end{array} \tag{9}$$

For a group K in $\mathscr{S}$, we consider the K-torsors in $\mathscr{E}$ split by a cover $\mathscr{U} = \langle U, \zeta \rangle$, where $U \twoheadrightarrow 1$ and $\zeta : U \longrightarrow e^*I$ in $\mathscr{E}$. By Diaconescu's theorem, a K-torsor in $\mathscr{E}$ is equivalently given by a geometric morphism $\tau : \mathscr{E} \longrightarrow \mathscr{B}(K)$. A morphism of K-torsors in $\mathscr{E}$ corresponds to a natural transformation. There is a category

$$\mathrm{Tors}(\mathscr{E}; K)^{\mathscr{U}}$$

of K-torsors in $\mathscr{E}$ split by $\mathscr{U}$.

To say that the K-torsor (represented by) τ is $\mathscr{U}$-split is to say that there is a factorization

$$\begin{array}{ccc} \mathscr{E} & & \\ {\scriptstyle \sigma_{\mathscr{U}}}\downarrow & \searrow^{\tau} & \\ \mathscr{P}_{\mathscr{U}}(\mathscr{E}) & \xrightarrow[\lambda]{} & \mathscr{B}(K) \end{array}$$

where the 'point' $p_{\mathscr{U}} : \mathscr{S} \longrightarrow \mathscr{P}_{\mathscr{U}}(\mathscr{E})$ plays no role.

There is induced a double pushout diagram

$$\begin{array}{ccc} \mathscr{E}/U & \xrightarrow{\varphi_U} & \mathscr{E} \\ \downarrow{\scriptstyle \rho_U} & & \downarrow{\scriptstyle \sigma_U} \\ Sh(E_!U) & \xrightarrow{q_U} & \mathscr{G}_U(\mathscr{E}) \\ \downarrow{\scriptstyle \kappa_{\mathscr{U}}} & & \downarrow{\scriptstyle \chi_{\mathscr{U}}} \\ \mathscr{S}/I & \xrightarrow{p_{\mathscr{U}}} & \mathscr{P}_{\mathscr{U}}(\mathscr{E}) \end{array} \qquad \lambda_{\mathscr{U}} : \mathscr{E}/U \to \mathscr{S}/I, \quad \tau_{\mathscr{U}} : \mathscr{E} \to \mathscr{P}_{\mathscr{U}}(\mathscr{E}) \tag{10}$$

where $\kappa_{\mathscr{U}}$ is induced the universal property of the **V**-localic reflection, and $\chi_{\mathscr{U}}$ by the pushout property.

It follows from the commutative square

$$\begin{array}{ccc} \mathscr{E}/U & \xrightarrow{\varphi_U} & \mathscr{E} \\ {\scriptstyle \rho_U}\downarrow & & \downarrow{\scriptstyle \tau} \\ Sh(E_!U) & \xrightarrow[\gamma]{} & \mathscr{B}(K) \end{array} \tag{11}$$

where $\gamma = \lambda \cdot p_{\mathscr{U}} \cdot \kappa_{\mathscr{U}}$ and the pushout property of (7), that there exists a unique $\delta : \mathscr{G}_U(\mathscr{E}) \longrightarrow \mathscr{B}(K)$ such that the diagram

$$\begin{array}{ccc} \mathscr{E} & & \\ {\scriptstyle \sigma_U}\downarrow & \searrow^{\tau} & \\ \mathscr{G}_U(\mathscr{E}) & \xrightarrow[\delta]{} & \mathscr{B}(K) \end{array}$$

commutes up to an iso 2-cell. This shows the third claim. □

2.3. Corollary. *For any group K and any cover $U \twoheadrightarrow 1$ in a **V**-determined topos $\mathscr{E}$, there is an equivalence*

$$\boldsymbol{Top}_{\mathscr{S}}(\mathscr{B}(\mathbb{G}_U), \mathscr{B}(K)) \simeq \mathrm{Tors}(\mathscr{E}; K)^U$$

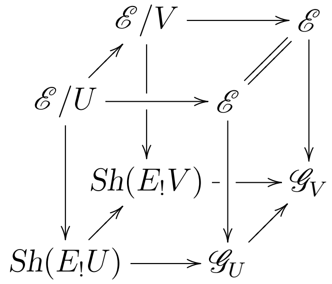
natural in K, where we write simply U for the canonical cover $\mathscr{U} = \langle U \twoheadrightarrow 1, \eta : E_!U, U \longrightarrow E^*E_!U \rangle$.

We proceed to define, for $\mathrm{Cov}(\mathscr{E})$ a small generating category of covers in $\mathscr{E}$ and morphisms $\alpha : U \longrightarrow V$ witnessing (not uniquely) that $U \leq V$, a functor

$$\mathscr{G} : \mathrm{Cov}(\mathscr{E}) \longrightarrow \mathbf{Top}_{\mathscr{S}}.$$

We let $\mathscr{G}(\mathscr{U}) = \mathscr{G}_{\mathscr{U}}(\mathscr{E})$. Let $U \leq V$, where U and V are covers in $\mathscr{E}$, and let $\alpha : U \longrightarrow V$ be a given morphism in $\mathrm{Cov}(\mathscr{E})$ witnessing the relation. Then there is an induced geometric morphism $\varphi_\alpha : \mathscr{G}_U(\mathscr{E}) \longrightarrow \mathscr{G}_V(\mathscr{E})$ over $\mathscr{S}$ which commutes with the canonical localic points p_U and p_V.

This is clear from the cube below (in simplified notation), using the pushout property (7) of the back face:



Recall that the geometric morphisms $\rho_U : \mathscr{E}/U \longrightarrow Sh(E_!U)$ are (in general) surjective by Lemma 1.7. It follows from the pushout that each $\sigma_U : \mathscr{E} \longrightarrow \mathscr{G}_U(\mathscr{E})$ is also surjective and, in particular, for each α as above, also the geometric morphisms $\varphi_\alpha : \mathscr{G}_U(\mathscr{E}) \longrightarrow \mathscr{G}_V(\mathscr{E})$ are surjective.

Since each induced geometric morphism $\varphi_\alpha : \mathscr{G}_U(\mathscr{E}) \longrightarrow \mathscr{G}_V(\mathscr{E})$ over $\mathscr{S}$ commutes with the localic points p_U and p_V, it may therefore be interpreted as an object of the full subcategory

$$\mathbf{Top}_{\mathscr{S}}[\mathscr{B}(\mathbb{G}_U), \mathscr{B}(\mathbb{G}_V)]_+$$

of $\mathbf{Top}_{\mathscr{S}}[\mathscr{B}(\mathbb{G}_U, \mathscr{B}(\mathbb{G}_V)]$ whose objects are geometric morphisms commuting with the canonical localic points [3]. The square brackets in both cases indicate that the morphisms in those Hom-categories are to be taken to be iso 2-cells in $\mathbf{Top}_{\mathscr{S}}$.

As shown in [5] there is a strong equivalence of categories:

$$\mathrm{Hom}(\mathbb{G}_U, \mathbb{G}_V) \simeq \mathbf{Top}_{\mathscr{S}}[\mathscr{B}(\mathbb{G}_U), \mathscr{B}(\mathbb{G}_V)]_+.$$

We derive from it that there are induced groupoid homomorphisms $g_\alpha : \mathbb{G}_U \longrightarrow \mathbb{G}_V$, for each α. Furthermore, since the geometric morphisms φ_α are surjective, the homomorphisms g_α are also surjective.

The system consisting of the locally discrete groupoids $\mathbb{G}_U$ and the surjective homomorphisms g_α is not filtered in general. Nevertheless, the system is a bifiltered 2-category in the sense of [21] or, equivalently, a 2-filtered 2-category in the sense of [15]. As in [15], the bicolimit of the 2-functor exists.

Let $\mathbb{G} = \lim\{G_U \mid U \in \mathrm{Cov}(\mathscr{E})\}$. This is a locally discrete groupoid. There is a canonical geometric morphism

$$\mathscr{B}(\lim\{G_U \mid U \in \mathscr{U}\}) \longrightarrow \lim\{\mathscr{B}(\mathbb{G}_U) \mid U \in \mathscr{U}\}$$

Since the transition geometric morphisms $\varphi_\alpha : \mathscr{B}(\mathbb{G}_U) \longrightarrow \mathscr{B}(\mathbb{G}_V)$ for $U \leq V$ are only surjective, this canonical geometric morphism need not be an equivalence.

In spite of this, the locally discrete *progroupoid* $\mathbb{P} = \{\mathbb{G}_U \mid U \in \mathrm{Cov}(\mathscr{E})\}$ represents cohomology of $\mathscr{E}$ with coefficients in discrete groups. Explicitly, the claim is that for a discrete group K, there is an isomorphism

$$H^1(\mathscr{E};K) \simeq [\mathbb{P},K].$$

This is shown as in [14].

2.4. Remark. There is a case of special interest, namely when $\mathbf{V}=\mathbf{D}$, the category of discrete locales. A $\mathbf{D}$-determined topos $\mathscr{E}$ is precisely a locally connected topos. In this case, there are additional properties of the fundamental progroupoid $\mathbb{P}=\langle \mathbb{G}_U \mid U \in \mathrm{Cov}(\mathscr{E})\rangle$ of $\mathscr{E}$, which we proceed to indicate.

1. Firstly, the fundamental progroupoid $\mathbb{P}$ may in this case be equivalently replaced by a prodiscrete localic groupoid G. That the localic groupoids G_U are discrete follows from the fact that the $p_U : \mathscr{S}/e_!U \longrightarrow \mathscr{G}_U(\mathscr{E})$ are local homeomorphisms, in turn since the $\mathscr{S}/e_!U \longrightarrow \mathscr{S}$ are local homeomorphisms. Further, it follows from Lemma 1.7(2) and properties of the pushouts (7) that each σ_U, hence also the transition geometric morphisms $\varphi_\alpha : \mathscr{B}(G_U) \longrightarrow \mathscr{B}(G_V)$, are connected. This in turn implies that the canonical geometric morphism

$$\mathscr{B}(\lim\{G_U \mid U \in \mathscr{U}\}) \longrightarrow \lim\{\mathscr{B}(G_U) \mid U \in \mathscr{U}\}$$

   is an equivalence. In particular, $\mathbb{G}=\lim\{G_U \mid U \in \mathscr{U}\}$ is a prodiscrete localic groupoid which represents cohomology of $\mathscr{E}$ with coefficients in discrete groups.

2. Secondly, in addition to the fundamental theorem of Galois theory, in the form $\mathscr{G}(\mathscr{E}) \cong \mathscr{B}(\mathscr{G})$, there is implicit a Galois theory in the sense of [19]. From Lemma 1.7(2) we deduce that the $\rho_U : \mathscr{E}/U \longrightarrow \mathscr{S}/e_!U$ are $\mathbf{D}$-determined, that is, locally connected. This fact, applied to the fundamental pushout (7) gives, using a result from [2], that $p_U : \mathscr{S}/e_!U \longrightarrow \mathscr{G}_U(\mathscr{E})$ (and $\sigma_U : \mathscr{E} \longrightarrow \mathscr{G}_U(\mathscr{E})$) is locally connected and a surjection, hence an $\mathscr{S}$-essential surjection, therefore with a representable inverse image part. The representor is a locally constant object $\langle A,\sigma\rangle$, which is a 'normal' object in the sense of [19]. In particular, each $\mathscr{G}_U(\mathscr{E}) \cong \mathscr{B}(G_U)$ is a Galois topos, from which it follows that also the limit $\lim\{\mathscr{B}(G_U) \mid U \in \mathscr{U}\}$ is a Galois topos. The reader is referred to [5] for details.

2.5. Remark. The fundamental pushout definition of the coverings fundamental groupoid of a locally connected topos [2] became a useful tool in the comparison theorems between the coverings and the paths versions of the fundamental groupoid [4, 12]. Such a comparison could not be attempted in this context without a prior generalization of the paths fundamental groupoid to the non-locally connected case. The latter might prove to be an interesting project.

3. V-covering projections

In this section we introduce and discuss a notion of $\mathbf{V}$-covering projection and use it to analyze the fundamental $\mathbf{V}$-pushout toposes of Definition 2.1.

3.1. Definition. Let $\mathscr{E}$ be a $\mathbf{V}$-determined topos. An object A of $\mathscr{E}$ is said to be a *$\boldsymbol{V}$-covering projection object* split by a cover $\mathscr{U} = \langle U \twoheadrightarrow 1, X, \eta : U \longrightarrow E^*U\rangle$ in $\mathscr{E}$ if A is part of a 3-tuple $\langle A, \alpha, \theta\rangle$ where α is an etale morphism $Y \longrightarrow X$ in $\mathbf{Loc}$, with $X \in \mathbf{V}$, and θ is an isomorphism

$$E^*Y \times_{E^*X} U \xrightarrow{\theta} A \times U$$

over U. We call the corresponding geometric morphism $\mathscr{E}/A \longrightarrow \mathscr{E}$ *a $\boldsymbol{V}$-covering projection* split by $\mathscr{U}$.

A *morphism* $\langle A, \alpha, \theta\rangle \longrightarrow \langle B, \beta, \kappa\rangle$ between $\mathbf{V}$-covering projection objects split by $\mathscr{U} = \langle U \twoheadrightarrow 1, X, \eta : U \longrightarrow E^*X\rangle$ is given by a pair $(f\ ,\ \gamma)$ with $f : A \longrightarrow B$ a morphism in $\mathscr{E}$ and $\gamma : Y \longrightarrow Z$ a morphism in $\mathbf{Loc}$ over X, such that

$$\begin{array}{ccc} E^*Y \times_{E^*X} U & \xrightarrow{\theta} & A \times U \\ \Big\downarrow{\scriptstyle E^*\gamma\times U} & & \Big\downarrow{\scriptstyle f\times U} \\ E^*Z \times_{E^*X} U & \xrightarrow[\kappa]{} & B \times U \end{array}$$

commutes.

3.2. Proposition. *Let $\varphi : \mathscr{F} \longrightarrow \mathscr{E}$ be a local homeomorphism in $\boldsymbol{Top}_{\mathscr{S}}$. Let $\mathscr{U} = \langle U \twoheadrightarrow 1, X, \eta : U \longrightarrow E^*U\rangle$ be a cover in $\mathscr{E}$. Then*

(i) $\varphi^*\mathscr{U} = \langle \varphi^*U \longrightarrow \varphi^*1 \cong 1, X, \varphi^*\eta : (\varphi^* \cdot E^*)X \cong F^*X\rangle$ is a cover in $\mathscr{F}$.

(ii) If A is $\boldsymbol{V}$-covering projection object in $\mathscr{E}$ split by $\mathscr{U}$, then φ^*A is a $\boldsymbol{V}$-covering projection object in $\mathscr{F}$, split by $\varphi^*\mathscr{U}$.

Proof. The conclusion is easily checked using Definition 3.1, and the fact that the 2-cell $\varphi^*E^* \Rightarrow F^*$ is an isomorphism when φ is a local homeomorphism (see Remark 1.8). □

The following statement is similar to one shown in [11] for locally constant coverings (or covering projections). Due to the restricted nature of the BCC in the $\mathbf{V}$-case the proof given therein is not directly applicable. We give one which relies directly on the $\mathbf{V}$-reflection universal property.

3.3. Proposition. *If $e : \mathscr{E} \longrightarrow \mathscr{S}$ is $\boldsymbol{V}$-determined, A an object of $\mathscr{E}$, and $U \twoheadrightarrow 1$ (a 'cover') in $\mathscr{E}$, the following are equivalent:*

(i) A is a $\boldsymbol{V}$-covering projection object split by U.

(ii) There is an etale morphism $Y \xrightarrow{\alpha} E_!U$ in $\mathbf{Loc}(\mathscr{S})$, and a pullback

$$\begin{array}{ccc} A \times U & \xrightarrow{\pi_2} & U \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \eta_U} \\ E^*Y & \xrightarrow[E^*\alpha]{} & E^*E_!U \end{array}$$

where η_U is the unit of adjointness $E_! \dashv E^*$ evaluated at U.

(iii) There is an etale morphism $Y \xrightarrow{\beta} X$ *in* $\mathbf{Loc}(\mathscr{S})$ *with* $X \in \boldsymbol{V}(\mathscr{S})$*, a morphism* $\eta : U \longrightarrow E^*X$*, and a pullback*

$$\begin{array}{ccc} A \times U & \xrightarrow{\pi_2} & U \\ \downarrow & & \downarrow \eta \\ E^*Y & \xrightarrow[E^*\beta]{} & E^*X \end{array}$$

Proof. The items (i) and (iii) are equivalent. Trivially, $(ii) \Rightarrow (iii)$ As for $(iii) \Rightarrow (ii)$, suppose that A is U-split via $\beta : Y \longrightarrow X$, $\zeta : U \longrightarrow E^*X$ and an iso $\theta : E^*Y \times_{E^*X} U \longrightarrow A \times U$ over U. Then the composite rectangle

$$\begin{array}{ccc} A \times U & \xrightarrow{\pi_2} & U \\ \downarrow & & \downarrow \eta_U \\ E^*Y & \xrightarrow{E^*\alpha} & E^*E_!U \\ = \downarrow & & \downarrow E^*\zeta' \\ E^*Y & \xrightarrow[E^*\beta]{} & E^*X \end{array}$$

is a pullback and, in the lower 'square', $\zeta' : E_!U \longrightarrow X$ exists and is the unique such that $\zeta' \cdot \alpha = \beta$ is obtained from the universal property of the **V**-localic reflection. it follows easily that the top square is a pullback. □

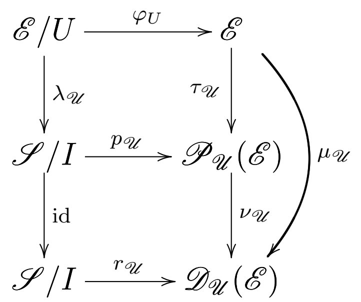
Let $\mathscr{E}$ be an **V**-determined topos. Denote by $\mathscr{C}_{\mathbf{V}}(\mathscr{E})(\mathscr{U})$ the category of the **V**-covering projections objects split by $\mathscr{U} = \langle U \twoheadrightarrow 1, \eta : U \longrightarrow E^*E_!U \rangle$ and their morphisms. There is a forgetful functor $\Psi_U : \mathscr{C}_{\mathbf{V}}(\mathscr{U}) \longrightarrow \mathscr{E}$ defined by $\langle A, \alpha, \theta \rangle \mapsto A$ and $(f, \gamma) \mapsto f$. This functor is in general neither full nor faithful.

3.4. Proposition. *Let* $\mathscr{G}_U(\mathscr{E})$ *be the topos in the fundamental pushout applied to a* $\boldsymbol{V}$*-determined topos* $\mathscr{E}$ *and a cover* U*. Then there exists an equivalence functor* $\Phi_U : \mathscr{G}_U(\mathscr{E}) \longrightarrow \mathscr{C}_{\mathbf{V}}(\mathscr{E})(U)$*. Under this equivalence, the functor* ${\sigma_U}^* : \mathscr{G}_U(\mathscr{E}) \longrightarrow \mathscr{E}$ *corresponds to the forgetful functor* $\Psi_U : \mathscr{C}_{\mathbf{V}}(\mathscr{E})(U) \longrightarrow \mathscr{E}$*, defined by* $\langle A, \alpha, \zeta \rangle \mapsto A$ *and* $\langle a, \gamma \rangle \mapsto a$*. This functor is in general neither full nor faithful.*

Proof. An object of $\mathscr{G}_U(\mathscr{E})$ in the fundamental pushout of toposes is precisely a locally **V**-trivial object split by U in the sense of Definition 3.3. A similar observation applies to morphisms in the category $\mathscr{G}_U(\mathscr{E})$. This is a consequence of the well-known fact that pushouts in $\mathbf{Top}_{\mathscr{S}}$ are calculated as pullbacks in **CAT** of the inverse image parts. □

3.5. Remark. An alternative construction of the fundamental progroupoid of a general Grothendieck topos $\mathscr{E}$ which also uses pushout toposes is that of [14]. However, the locally constant pushouts employed therein (albeit as a first approximation) just give the toposes $\mathscr{P}_{\mathscr{U}}(\mathscr{E})$ for a cover $\mathscr{U} = \langle U \twoheadrightarrow 1, I, U \longrightarrow e^*I \rangle$. That this is not adequate in the non locally connected case is indicated by the fact that the canonical 'point' $p_{\mathscr{U}} : \mathscr{S}/I \longrightarrow \mathscr{P}_{\mathscr{U}}(\mathscr{E})$ need not be of effective descent . A correction is subsequently made in [14] by means of categories $\mathscr{D}_{\mathscr{U}}(\mathscr{E}) \subseteq \mathscr{P}_{\mathscr{U}}(\mathscr{E})$, with $\mathscr{D}_{\mathscr{U}}(\mathscr{E})$ defined as the full subcategory of $\mathscr{P}_{\mathscr{U}}(\mathscr{E})$ whose objects are sums of $\mathscr{U}$-split 'covering projections', a notion that is a generalization to Grothendieck toposes of the 'covering projections' off [18] for localic toposes, and which is itself inspired by the overlays of [17].

In pictures, the locally split pushout (top) can be completed to a commutative diagram



(12)

The category $\mathscr{D}_{\mathscr{U}}(\mathscr{E}) \subseteq \mathscr{P}_{\mathscr{U}}(\mathscr{E})$ is then shown to be a topos, that it is atomic, and that there exists an etale complete localic groupoid $\mathbb{D}_{\mathscr{U}}$ such that $\mathscr{D}_{\mathscr{U}}(\mathscr{E}) \cong \mathscr{B}(\mathbb{D}_{\mathscr{U}})$, and such that it classifies $\mathscr{U}$-split K-torsors in $\mathscr{E}$ for discrete groups K.

The construction of the $\mathscr{D}_{\mathscr{U}}(\mathscr{E})$ is different from our construction of the $\mathscr{G}_U(\mathscr{E})$. For instance, in our case (first exposed in [4]), the use of the fundamental pushouts requires no correction.

We also point out that (without further information) we cannot conclude that there is an equivalence between the toposes $\mathscr{D}_{\mathscr{U}}(\mathscr{E})$ and $\mathscr{G}_{\mathscr{U}}(\mathscr{E})$. Since both constructions generalize the locally connected case, these toposes are certainly equivalent when $\mathscr{E}$ is locally connected. For a general $\mathscr{E}$, a property that $\mathscr{D}_{\mathscr{U}}(\mathscr{E})$ and $\mathscr{G}_{\mathscr{U}}(\mathscr{E})$ both share is the classification of torsors in $\mathscr{E}$ for *discrete* groups. However, the localic groupoids that these toposes respectively classify are not in general discrete, so no conclusion about their possible equivalence can be drawn from this observation alone.

4. The comprehensive $\mathbf{V}$-factorization

We now review and update the $\mathbf{V}$-comprehensive factorization theorem [9], modeled on the pure, complete spread factorization in case of geometric morphisms with a locally connected domain [7], and on the hyperpure, complete spread factorization of [8] for geometric morphisms with a definable dominance domain.

The following is a generalization of the notion of a Lawvere distribution on a topos [23].

4.1. Definition. A ***V**–distribution* on a topos $\mathscr{E}$ is an $\mathscr{S}$-indexed functor $\mu : \mathscr{E} \longrightarrow \mathbf{V}$ with an $\mathscr{S}$-indexed right adjoint $\mu \dashv \mu_*$. Denote by $\mathbf{E}_{\mathbf{V}}(\mathscr{E})$ the category of $\mathbf{V}$-distributions on $\mathscr{E}$.

Let μ be a $\mathbf{V}$-distribution on $\mathscr{E} \simeq Sh(\mathbb{C}, J)$. Let $\mathbb{M}$ be the category in $\mathscr{S}$ with objects (C, U) with $U \in \mathscr{O}(\mu(C))$, and morphisms $(C, U) \longrightarrow (D, V)$ given by $C \xrightarrow{m} D$ in $\mathbb{C}$ such that $U \leq \mu(m)^*(V)$. For $U \in \mathscr{O}(\mu(C))$, denote by $U \rightarrowtail \mu(C)$ the corresponding open sublocale.

Let $\mathscr{Z}$ be the topos of sheaves for the topology on $\mathbb{M}$ generated by the following families, which we call *weak μ-covers*: a family

$$\{(C, U_a) \xrightarrow{1_C} (C, U) \mid a \in A\}$$

is a weak μ-cover if $\bigvee U_a = U$ in $\mathscr{O}(\mu(C))$. As usual there is a functor $\mathbb{M} \longrightarrow \mathbb{C}$ that induces a geometric morphism $P(\mathbb{M}) \longrightarrow P(\mathbb{C})$, and hence one $\mathscr{Z} \longrightarrow P(\mathbb{C})$.

4.2. Definition. Let μ be a $\mathbf{V}$-distribution on $\mathscr{E}$. The geometric morphism φ in the topos pullback

$$\begin{array}{ccc} \mathscr{X} & \rightarrowtail & \mathscr{Z} \\ {\scriptstyle\varphi}\downarrow & \lrcorner & \downarrow \\ \mathscr{E} & \rightarrowtail & P(\mathbb{C}) \end{array}$$

is said to be *the support of* μ, denoted $\{\mu\}_{\mathscr{E}}$. (A priori, this construction depends on a site presentation of $\mathscr{E}$.)

4.3. Proposition. *Consider a topos $\mathscr{E} \simeq Sh(\mathbb{C}, J)$. Suppose that $\mathscr{F}$ is a $\boldsymbol{V}$-determined topos and that $\psi : \mathscr{F} \longrightarrow \mathscr{E}$ is a geometric morphism. Let $\mu = F_! \psi^*$. As in Def. 4.2, we may consider the category $\mathbb{M}$, and the support $\varphi : \mathscr{X} \longrightarrow \mathscr{E}$ associated with μ as in Def 4.2. Then, $\mathscr{X}$ is $\boldsymbol{V}$-determined.*

4.4. Definition. *A geometric morphism $\varphi : \mathscr{X} \longrightarrow \mathscr{E}$ is said to be a* $\mathbf{V}$-fibration *if its domain $\mathscr{F}$ is a $\boldsymbol{V}$-determined topos and furthermore φ equals the support of the $\boldsymbol{V}$-distribution $\mu = F_! \psi^*$.*

The following is the analogue of the comprehensive factorization [22, 29, 7] in the $\mathbf{V}$-setting. It was established in [9].

4.5. Theorem. *Any geometric morphism $\psi : \mathscr{F} \longrightarrow \mathscr{E}$ with a $\boldsymbol{V}$-determined domain $\mathscr{F}$ admits a factorization*

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\eta} & \mathscr{X} \\ & {\scriptstyle\psi}\searrow \quad \swarrow{\scriptstyle\varphi} & \\ & \mathscr{E} & \end{array} \tag{13}$$

into a first factor η that is $\boldsymbol{V}$-initial followed by a $\boldsymbol{V}$-fibration φ. This factorization, said to be comprehensive*, is unique up to equivalence.*

In the rest of this section we obtain several new results about the factors in the $\mathbf{V}$-comprehensive factorization, needed in the rest of the paper.

4.6. Lemma. *Let $\pi : \mathscr{F} \longrightarrow \mathscr{E}$ be a local homeomorphism. If $\mathscr{E}$ is $\boldsymbol{V}$-determined then also $\mathscr{F}$ is $\boldsymbol{V}$-determined.*

Proof. We observed (Remark 1.8) that since π is a local homeomorphism, the 2-cell α in (4) is an isomorphism. Hence, a left adjoint $F_!$ to $F^* \cong (\pi^* \cdot E^*)$ is given by the composite $(E_! \cdot \pi_!)$. The BCC for etales holds for $F_! \dashv F^*$ since it does for $E_! \dashv E^*$ and since π is in particular locally connected, $\pi_! \dashv \pi^*$ is an $\mathscr{E}$-indexed adjointness.

□

4.7. Proposition. *The pullback of a $\boldsymbol{V}$-initial geometric morphism along a local homeomorphism is $\boldsymbol{V}$-initial.*

Proof. This is a straightforward diagram chase, using the fact that (4) is an isomorphism when ψ is a local homeomorphism. □

4.8. Lemma. *Consider a triangle of geometric morphisms*

$$\begin{array}{ccc} \mathscr{F} & & \\ {\scriptstyle \eta}\downarrow & \searrow^{p} & \\ \mathscr{X} & \underset{\tau}{\rightarrowtail} & \mathscr{Z} \end{array}$$

in which τ is an inclusion.

1. *If p and τ are both $\boldsymbol{V}$-initial, then so is η.*

2. *If p is $\boldsymbol{V}$-initial and $\tau^* Z^* \Rightarrow X^*$ is an isomorphism, then η is $\boldsymbol{V}$-initial.*

Proof. 1. Consider

$$\begin{array}{ccc} Z^* & & \\ \Downarrow & \searrow\!\!\searrow & \\ \tau_* X^* & \Longrightarrow & \tau_* \eta_* F^* \end{array}$$

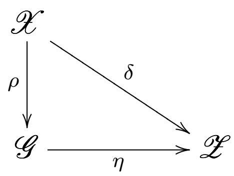
If $p = \tau\eta$ is $\mathbf{V}$-initial, then the hypotenuse is an isomorphism. If τ is $\mathbf{V}$-initial, then the vertical is an isomorphism. Therefore, the horizontal is an isomorphism, and therefore η is $\mathbf{V}$-initial since τ is an inclusion.

2. Applying τ^* to the isomorphism $Z^* \cong p_* F^* \cong \tau_* \eta_* F^*$ gives the top horizontal in the following diagram, which is an isomorphism.

$$\begin{array}{ccc} \tau^* Z^* & \Longrightarrow & \tau^* \tau_* \eta_* F^* \\ \Downarrow & & \Downarrow \\ X^* & \Longrightarrow & \eta_* F^* \end{array}$$

The right vertical is the counit of $\tau^* \dashv \tau_*$, which is an isomorphism since τ is an inclusion. The left vertical is an isomorphism by assumption. We conclude the bottom horizontal is an isomorphism, which says that η is $\mathbf{V}$-initial. □

4.9. Proposition. *Consider a commutative triangle*



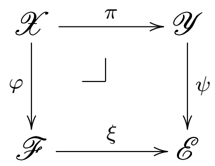
where $\mathscr{X}$, $\mathscr{G}$, and $\mathscr{Z}$ are $\mathbf{V}$-determined toposes. If ρ and η are $\mathbf{V}$-initial, so is the composite δ.

Proof. From Remark 1.8, we have the canonical isomorphisms $X_! \cdot \rho^* \cong G_!$ and $G_! \cdot \eta^* \cong Z_!$. It follows that

$$X_! \cdot \delta^* \cong X_! \cdot (\rho^* \cdot \eta^*) \cong (X_! \cdot \rho^*) \cdot \eta^* \cong G_! \cdot \eta^* \cong Z_!$$

and this is also given by a canonical (iso) 2-cell. Hence the result. □

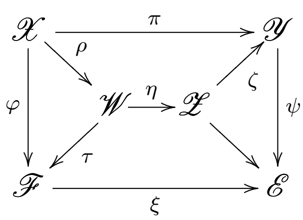
4.10. Proposition. *Consider a bipullback in $\mathbf{Top}_{\mathscr{S}}$, in which $\psi : \mathscr{Y} \longrightarrow \mathscr{E}$ is a $\mathbf{V}$-fibration and ξ is a local homeomorphism:*



Then φ is a $\mathbf{V}$-fibration.

Proof. First factor the $\mathbf{V}$-determined φ into its $\mathbf{V}$-initial and $\mathbf{V}$-fibration parts: $\varphi \cong \tau \cdot \rho$. Then factor the $\mathbf{V}$-determined $\xi \cdot \tau$ into its $\mathbf{V}$-initial and $\mathbf{V}$-fibration parts.

We now have a diagram as follows.



There is such a ζ because ψ is a $\mathbf{V}$-fibration and since $\eta \cdot \rho$ is the composite of two $\mathbf{V}$-initial hence a $\mathbf{V}$-initial (by Proposition 4.9).

Since the outer square is a pullback, there is $\mathscr{W} \xrightarrow{\theta} \mathscr{X}$ such that $\pi \cdot \theta \cong \zeta \cdot \eta$ and $\varphi \cdot \theta \cong \tau$. The universal property of the pullback implies that $\theta \cdot \rho \cong id_{\mathscr{X}}$. Hence, $\rho \cdot \theta \cdot \rho \cong \rho$. Therefore, the two geometric morphisms $\rho \cdot \theta$ and $id_{\mathscr{W}}$ from the $\mathbf{V}$-fibration τ to itself agree when precomposed with the $\mathbf{V}$-initial ρ. They must therefore agree: $\rho \cdot \theta \cong id_{\mathscr{W}}$. We have shown that ρ is an equivalence, so that φ is a $\mathbf{V}$-fibration. □

We have already considered the transpose $\hat{\alpha} : F^* \Rightarrow \rho_* G^*$ of $\alpha : \rho^* F^* \Rightarrow G^*$ under $\rho^* \dashv \rho_*$.

$$\begin{array}{ccc} F^* & & \\ {\scriptstyle \eta F^*}\Downarrow & \overset{\hat{\alpha}}{\searrow\!\!\searrow} & \\ \rho_*\rho^* F^* & \underset{\rho_*\alpha}{\Longrightarrow} & \rho_* G^* \end{array}$$

Recall that ageometric morphism $\mathscr{G} \xrightarrow{\rho} \mathscr{F}$ is ***V**-initial* if $\hat{\alpha}$ is an isomorphism .

4.11. Definition. A geometric morphism $\mathscr{G} \xrightarrow{\rho} \mathscr{F}$ is such that its direct image part ρ_* *preserves **V**-coproducts* if ηF^* is an isomorphism.

4.12. Proposition. *Assume that the 2-cell α of (4) is an isomorphism. Then, the following are equivalent:*

1. *ρ is **V**-initial.*
2. *ρ_* preserves **V**-coproducts.*

Proof.
Consider

$$\begin{array}{ccc} E^* & & \\ \Downarrow & \searrow\!\!\searrow & \\ \rho_*\rho^* E^* & \Longrightarrow & \rho_* F^* \end{array}$$

where the bottom 2-cell is $\rho_*(\alpha)$, hence an isomorphism by assumption. Then, the vertical 2-cell ηE^* is an isomorphism iff the hypothenuse $\hat{\alpha}$ is an isomorphism. $\square$

5. Branched **V**-coverings

The following 'pullback lemma' follows an argument similar to that employed in [13] (Proposition 8.2), shown therein in the locally connected case.

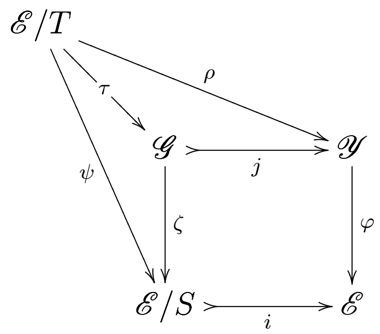
5.1. Lemma. *Let $\mathscr{E}/S \rightarrowtail \mathscr{E}$ be a local homeomorphism which is also a **V**-initial subtopos, where $\mathscr{E}$ is **V**-determined. Let $\psi : \mathscr{E}/T \longrightarrow \mathscr{E}/S$ be induced by a morphism $p : T \longrightarrow S$ in $\mathscr{E}$. Assume that ψ is a **V**-fibration. Then the associated **V**-fibration φ of $i \cdot \psi$ (exists and) forms a topos pullback square.*

$$\begin{array}{ccc} \mathscr{E}/T & \xrightarrow{\rho} & \mathscr{Y} \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathscr{E}/S & \underset{i}{\rightarrowtail} & \mathscr{E} \end{array}$$

*Consequently, the **V**-initial factor ρ is an inclusion.*

Proof.

We claim that in the diagram below, where the bottom square is a bipullback, the induced geometric morphism τ is an equivalence.



First, observe that, since $\mathscr{E}$ is **V**-determined, so is the slice topos $\mathscr{E}/T$, by Lemma 4.6. Hence the comprehensive **V**-factorization of the composite

$$\mathscr{E}/T \xrightarrow{\psi} \mathscr{E}/S \xrightarrow{i} \mathscr{E}$$

exists as indicated, with ρ a **V**-initial geometric morphism, and φ a **V**-fibration.

That τ is a **V**-fibration follows from the equation $\psi \cong \zeta \cdot \tau$ and the facts that ψ is a **V**-fibration by assumption, and that ζ is a **V**-fibration as it is obtained from the **V**-fibration φ by pullback along a local homeomorphism geometric morphism. For this, quote Proposition 4.10.

That τ is **V**-initial follows from the fact that $j \cdot \tau \cong \rho$, ρ **V**-initial, and j an inclusion, using Lemma 4.8. Therefore, τ is both **V**-initial and a **V**-fibration hence an equivalence. This completes the proof. $\square$

5.2. Definition. Let $\mathscr{E}$ be a **V**-determined topos. An object A of $\mathscr{E}$ is said to be a *locally **V**-trivial object* split by a cover $\mathscr{U} = \langle U \twoheadrightarrow 1, X, \eta : U \longrightarrow E^*X\rangle$ if A is part of a 3-tuple $\langle A, \alpha, \theta\rangle$ where α is an etale morphism $Y \longrightarrow X$ in $\mathscr{E}$, with $X \in \mathbf{V}$ and $Y \in \mathbf{V}$, and θ is an isomorphism

$$E^*Y \times_{E^*X} U \xrightarrow{\theta} A \times U$$

over U. We call the corresponding geometric morphism $\mathscr{E}/A \longrightarrow \mathscr{E}$ *a locally **V**-trivial covering* split by $\mathscr{U}$.

A *morphism* $\langle A, \alpha, \theta\rangle \longrightarrow \langle B, \beta, \kappa\rangle$ between locally **V**-trivial objects split by $\mathscr{U} = \langle U \twoheadrightarrow 1, X, \eta : U \longrightarrow E^*X\rangle$ is given by a pair $(f\ ,\ \gamma)$ with $f : A \longrightarrow B$ a morphism in $\mathscr{E}$ and $\gamma : Y \longrightarrow Z$ a morphism in **Loc** over X, such that

$$\begin{array}{ccc} E^*Y \times_{E^*X} U & \xrightarrow{\theta} & A \times U \\ \Big\downarrow{\scriptstyle E^*\gamma \times U} & & \Big\downarrow{\scriptstyle f \times U} \\ E^*Z \times_{E^*X} U & \xrightarrow[\kappa]{} & B \times U \end{array}$$

commutes.

5.3. REMARK. The analogue of Proposition 3.3 holds for locally $\mathbf{V}$-trivial objects in a $\mathbf{V}$-determined topos. Moreover, in this case, the proof is entirely analogous to the corresponding one in [11] for locally constant objects. This is so on account of the BCC holding for etale maps $Y \longrightarrow X$ where both X and Y are in $\mathbf{V}$.

Denote by $\mathscr{Z}_{\mathbf{V}}(\mathscr{E})(U)$ the category of locally $\mathbf{V}$-trivial objects of $\mathscr{E}$ split by $U = \langle U \twoheadrightarrow 1, \eta : U \longrightarrow E^* E_! U\rangle$ and their morphisms.

5.4. REMARK. There is an inclusion

$$\mathscr{Z}_{\mathbf{V}}(\mathscr{E})(U) \subseteq \mathscr{C}_{\mathbf{V}}(\mathscr{E})(U).$$

It follows from Example 1.13 that this inclusion is in general proper. For instance, not every $\mathbf{Z}$-covering projection is locally $\mathbf{Z}$-trivial. On the other hand, every $\mathbf{D}$-covering projection is locally $\mathbf{D}$-trivial (or locally constant).

5.5. DEFINITION. A geometric morphism $\psi : \mathscr{W} \longrightarrow \mathscr{E}$ of a $\mathbf{V}$-determined topos $\mathscr{E}$ is said to be *an unramified* $\mathbf{V}$*-covering* if it is both a local homeomorphism and a $\mathbf{V}$-fibration.

5.6. PROPOSITION. *A locally* $\mathbf{V}$*-trivial covering is an unramified* $\mathbf{V}$*-covering.*

Proof. Let $\langle A, \alpha, \theta\rangle$ be a locally $\mathbf{V}$-trivial object of a $\mathbf{V}$-determined topos $\mathscr{E}$, trivialized by $U \twoheadrightarrow 1$, in the sense of Definition 3.1. Recall that $\alpha : Y \longrightarrow X$, $\eta : U \longrightarrow E^*U$, and that

$$E^*Y \times_{E^*X} U \xrightarrow{\theta} A \times U$$

is an isomorphism over U.

For any etale morphism $p : Y \longrightarrow X$ and corresponding geometric morphism $Sh(Y) \longrightarrow Sh(X)$, with X in $\mathbf{V}$, we have that for each open inclusion $p_a : W_a \hookrightarrow X$, where $\lim\{W_a\} \cong Y$, there is induced an unramified $\mathbf{V}$-covering $Sh(W_a) \cong Sh(X)/(p_a) \longrightarrow Sh(X)$. For the local homeomorphism $Sh(Y) \longrightarrow Sh(X)$ to be a $\mathbf{V}$-fibration, hence an unramified $\mathbf{V}$-covering, we must have that $Y \in \mathbf{V}$, by Proposition 3.3. □

5.7. REMARK. A $\mathbf{V}$-covering projection of $\mathscr{E}$ is not necessarily an unramified $\mathbf{V}$-covering. This follows from the observation, already employed in the proof of Proposition 5.6, that for a local homeomorphism that $Sh(Y) \longrightarrow Sh(X)$ to be a $\mathbf{V}$-fibration, hence an unramified $\mathbf{V}$-covering, we must have, by Proposition 3.3, that $Y \in \mathbf{V}$. By Example 1.15 this need not be the case, for instance for $\mathbf{V} = \mathbf{Z}$.

5.8. DEFINITION. Let $\mathscr{E}$ be a $\mathbf{V}$-determined topos. A *branched* $\mathbf{V}$*-covering of* $\mathscr{E}$ (with 'non-singular part' $i : \mathscr{E}/S/ \rightarrowtail \mathscr{E}$) is given by the following data:

(i) A $\mathbf{V}$-fibration $\psi : \mathscr{F} \longrightarrow \mathscr{E}$.

(ii) A $\mathbf{V}$-initial inclusion $i : \mathscr{E}/S \rightarrowtail \mathscr{E}$.

(iii) An locally trivial $\mathbf{V}$-covering $\varphi : \mathscr{E}/T \longrightarrow \mathscr{E}/S$ induced by a morphism $p : T \longrightarrow S$.

(iv) A **V**-initial geometric morphism $\pi : \mathscr{E}/T \longrightarrow \mathscr{F}$ for which the square

$$\begin{array}{ccc} \mathscr{E}/T & \xrightarrow{\pi} & \mathscr{F} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \mathscr{E}/S & \underset{i}{\rightarrowtail} & \mathscr{E} \end{array} \tag{14}$$

commutes.

5.9. Corollary. *The square (14) is a pullback.*

Proof. This is a consequence of Lemma 5.1 and Proposition 5.6. □

In view of Remark 5.4, Proposition 5.6, and Corollary 5.9, the definition we have given of a branched **V**-covering not only generalizes that of [7] in the locally connected case, but it is also the only possible such. An alternative is to require directly unramified **V**-coverings in both the discrete and the general **V**-case, but unless special assumptions are made on $\mathscr{E}$ (locally paths simply connected) there is no connection with the fundamental groupoid.

Let $\mathscr{E}$ be a **V**-determined topos in $\mathbf{Top}_{\mathscr{S}}$. Denote by $\mathscr{B}_{\mathbf{V},S}(\mathscr{E})$ the full sub 2-category of $\mathbf{Top}_{\mathscr{S}}$ whose objects are the branched **V**-coverings of $\mathscr{E}$ with non-singular part $i : \mathscr{E}/S \rightarrowtail \mathscr{E}$.

5.10. Theorem. *The category $\mathscr{B}_{\mathbf{V},S}(\mathscr{E})$ is canonically equivalent to the category $\mathscr{Z}_{\mathbf{V}}(\mathscr{E}/S)$.*

Proof. We pass from a branched **V**-covering ψ of $\mathscr{E}$ as given in Definition 5.8 to a locally **V**-trivial covering of $\mathscr{E}/S$ by pullback.

Conversely, we pass from a locally **V**-trivial covering φ of $\mathscr{E}/S$ to a branched covering of $\mathscr{E}$ with non-singular part $i : \mathscr{E}/S \rightarrowtail \mathscr{E}$ by the comprehensive **V**-factorization of the composite

$$\mathscr{X} \xrightarrow{\varphi} \mathscr{E}/S \rightarrowtail \mathscr{E}.$$

By Lemma 5.9, the square

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{\rho} & \mathscr{Y} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \mathscr{E}/S & \underset{i}{\rightarrowtail} & \mathscr{E} \end{array} \tag{15}$$

is a pullback so that the **V**-initial ρ is an inclusion. The passages indicated give an equivalence of categories.

□

5.11. COROLLARY. *Let $\mathscr{E}$ be a $\boldsymbol{V}$-determined topos. Then, for each cover U in $\mathscr{E}$, there is an inclusion*

$$\mathscr{B}_{\mathbf{V},S}(\mathscr{E})(U) \subseteq \mathscr{C}_{\mathbf{V}}(\mathscr{E}/S)(U).$$

This inclusion is in general proper.

5.12. REMARK. In the locally connected case, the results above reduce to the known ones [7]. Moreover, If $\mathbf{V}$ is the category $\mathbf{D}$, of discrete locales, then there is an equivalence $\mathscr{B}_{\mathbf{V},S}(\mathscr{E})(U) \cong \mathscr{C}_{\mathbf{V}}(\mathscr{E}/S)(U)$ so that the branched fundamental groupoid topos with respect to a given 'non-singular part' $i : \mathscr{E}/S \rightarrowtail \mathscr{E}$ may be identified with the fundamental groupoid topos $\Pi_1^{(c)}(\mathscr{E}/S) = \lim\{\mathscr{G}_U(\mathscr{E}/S) \mid U \in \mathrm{Cov}(\mathscr{E})\}$, for the pushout toposes $\mathscr{G}_U(\mathscr{E}/S)$.

We may always consider the largest topology in any Grothendieck topos $\mathscr{E}$ for which certain objects of the topos are sheaves [27]. A monomorphism $m : A \rightarrowtail B$ is dense for this largest topology iff $B^*A \longrightarrow B^*A^m$ (transpose of the projection) is an isomorphism.

We apply this idea, as in [7], to the class of objects of the form E^*X for $X \in \mathbf{V}$. Clearly, a monomorphism $m : S \rightarrowtail T$ in $\mathscr{E}$ is dense for such a topology iff the induced geometric morphism $\sigma : \mathscr{E}/S \longrightarrow \mathscr{E}/T$ is such that σ_* preserves $\mathbf{V}$-coproducts. Let us call this the $\mathbf{V}$-topology. Consider the topos of sheaves for the $\mathbf{V}$-topology on $\mathscr{E}$, with inclusion $i : \mathscr{E}_{\mathbf{V}} \rightarrowtail \mathscr{E}$.

5.13. THEOREM. *$\mathscr{E}_{\mathbf{V}}$ is the smallest subtopos of $\mathscr{E}$ whose inclusion preserves $\boldsymbol{V}$-coproducts.*

Proof.

The unit $E^*(\Omega_{\mathscr{S}}) \longrightarrow i_*i^*E^*(\Omega_{\mathscr{S}})$ for the inclusion $i : \mathscr{E}_{\mathbf{V}} \rightarrowtail \mathscr{E}$ is an isomorphism because $E^*(\Omega_{\mathscr{S}})$ is a $\mathbf{V}$-initial-sheaf. But this says that i_* preserves $\mathbf{V}$-coproducts.

We now show that it is the smallest such subtopos. Let $Sh_j(\mathscr{E}) \rightarrowtail \mathscr{E}$ be an inclusion for which j_* preserves $\mathbf{V}$-coproducts. Every $E^*(\Omega_{\mathscr{S}})$ is a j-sheaf. Hence, every j-dense monomorphism is dense for the $\mathbf{V}$-topology. Therefore, $\mathscr{E}_{\mathbf{V}}$ is included in $Sh_j(\mathscr{E})$. □

Since for each local homeomorphism $i : \mathscr{E}/S \hookrightarrow \mathscr{E}$ is $\mathbf{V}$-initial iff i_* preserves $\mathbf{V}$-coproducts, $\mathscr{E}_{\mathbf{V}} \hookrightarrow \mathscr{E}$ factors through every $\mathbf{V}$-initial local homeomorphism $i : \mathscr{E}/S \hookrightarrow \mathscr{E}$. However, $\mathscr{E}_{\mathbf{V}} \hookrightarrow \mathscr{E}$ itself can 'never' be a local homeomorphism.

It follows that the smallest $\mathbf{V}$-initial subtopos of $\mathscr{E}$ does not exist in general and that therefore the intrinsic characterization of branched coverings [7] does not hold in general either. In particular, the 'ideal knot' does not exist in the general case. We conclude that the locally connected assumption leads to miracles which we cannot expect without it.

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Department of Mathematics and Statistics, McGill University
805 Sherbrooke St. W, Montreal, QC, Canada H3A 2K6

Email: marta.bunge@mcgill.ca

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