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COHERENCE FOR PSEUDODISTRIBUTIVE LAWS REVISITED

F. MARMOLEJO AND R. J. WOOD

ABSTRACT. In this paper we show that eight coherence conditions suffice for the definition of a pseudodistributive law between pseudomonads.

1. Introduction

The definition of a pseudodistributive law of one pseudomonad over another is given in [Marmolejo, 1999]. We find there the four familiar diagrams given in the original definiton of a distributive law of [Beck, 1969], but commutativity is replaced by invertible 2-cells. These are required to satisfy coherence conditions; precisely the issue that [Marmolejo, 1999] addressed. There we find nine coherence conditions and a justification of why they should suffice. In the thesis [Tanaka, 2005] it is suggested that the nine coherence axioms given in that paper are *incomplete, in the sense that one of the coherence axioms is missing*, a concern echoed in [Tanaka, Power, 2006], now in the form *one axiom* may *be missing*. This kind of criticism casts doubt on the whole integrity of [Marmolejo, 1999] without pointing out where the mistake might be. We find this unacceptable. It is our contention that [Marmolejo, 1999] is fundamentally correct, if a little conservative in its efforts to provide a complete set of axioms. For here we show that in fact *eight* of the nine axioms of [Marmolejo, 1999] suffice.

We show here that the contentious condition (H-2) of [Tanaka, 2005] and, by duality, (coh 8) of [Marmolejo, 1999] are redundant. Our technique involves showing that given pseudomonads $\mathbb{D}$ on $\mathcal{A}$ and $\mathbb{U}$ on $\mathcal{B}$, a lifting of a 2-functor $F : \mathcal{A} \to \mathcal{B}$ to the corresponding 2-categories of pseudoalgebras is classified by a strong transformation $r : UF \to FD$ together with two invertible modifications subject to *two* coherence conditions. Readers familiar with [Street, 1972] will at once recognize $(F, r, \cdots)$ as a pseudo version of the notion of *morphism of monads*. When we apply this to the particular case $\mathcal{A} = \mathcal{B}$ and $\mathbb{D} = \mathbb{U}$, the corresponding classifying transformation $r : UF \to FU$ is what was defined in [Tanaka, 2005] as a pseudo-distributive law of $\mathbb{U}$ over F — except that the condition (H-2) in that paper turns out to be redundant.

Incidentally, while it is true that [Marmolejo, 1999] does not give the definition of

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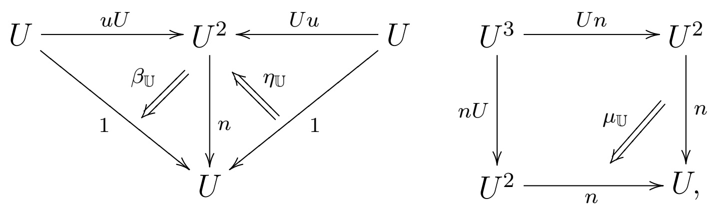
pseudoalgebra for a pseudomonad, such a definition does appear in [Marmolejo, 1997]. Perhaps this reference should have been more prominent. We also point out [Marmolejo, 2004], where we find a study of the algebras for the composite pseudomonad resulting from a distributive law.

For simplicity, we work in the context of 2-categories, 2-functors, strong transformations and modifications, but the results are true in the general context of bicategories, homomorphisms of bicategories, etc. Also for simplicity, we refrain in this paper from producing higher dimensional structures as in [Marmolejo, 1999] or [Tanaka, 2005], but we do feel that the paper would be incomplete without a proof of the equivalence between liftings and the transformations that classify them, so we include this in the paper.

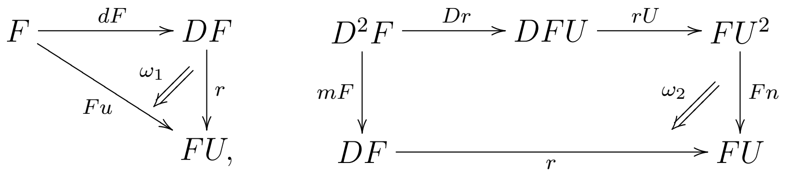
2. From transitions to liftings to algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be 2-categories, $\mathbb{U} = (U, u, n, \beta_{\mathbb{U}}, \eta_{\mathbb{U}}, \mu_{\mathbb{U}})$ a pseudomonad on $\mathcal{A}$, and $\mathbb{D} = (D, d, m, \beta_{\mathbb{D}}, \eta_{\mathbb{D}}, \mu_{\mathbb{D}})$ a pseudomonad on $\mathcal{B}$. Thus



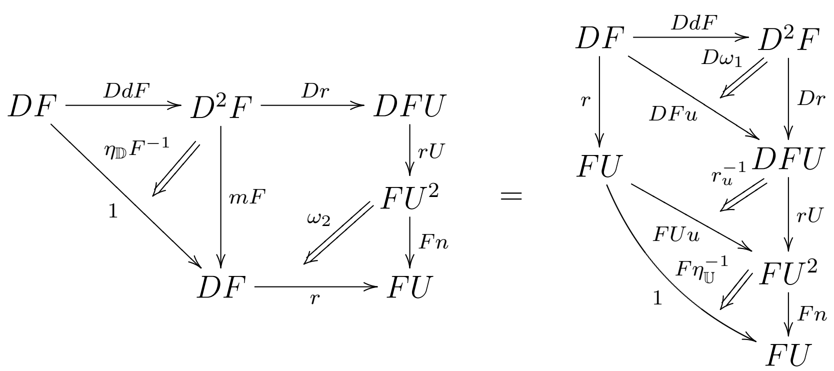
with $\beta_{\mathbb{U}}$, $\eta_{\mathbb{U}}$, and $\mu_{\mathbb{U}}$ invertible modifications satisfying the coherence conditions found in, say, [Marmolejo, 1997] and similary for $\mathbb{D}$. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a 2-functor.

2.1. Definition. *A transition from $\mathbb{U}$ to $\mathbb{D}$ along $F : \mathcal{A} \to \mathcal{B}$ is a strong transformation $r\colon DF \to FU$ together with invertible modifications*



(1)

that satisfy the following coherence conditions:



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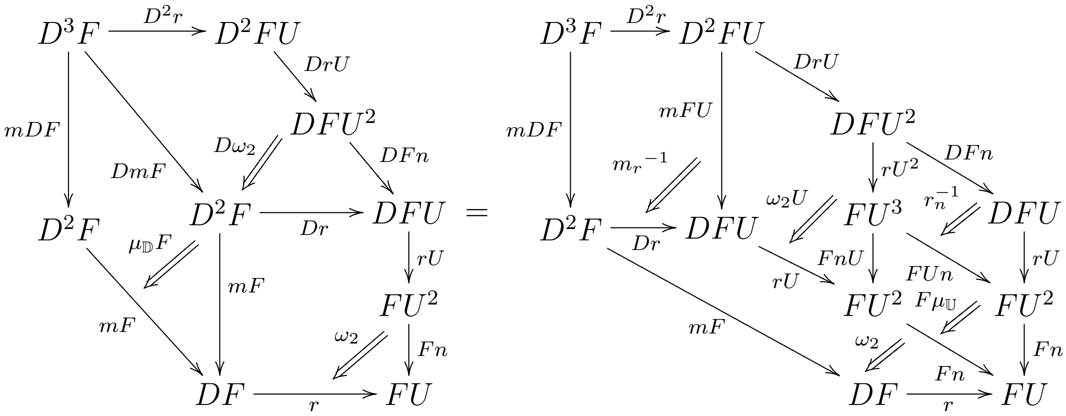
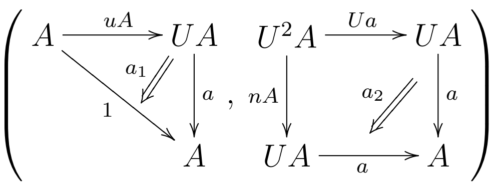

(3)

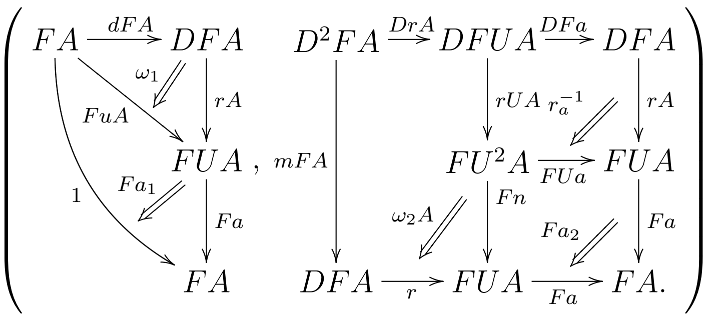
In the case $\mathcal{A} = \mathcal{B}$ a transition from $\mathbb{U}$ to $\mathbb{U}$ is the same thing as a distributive law of $\mathbb{U}$ over F in the sense of Definition 5.1 in [Tanaka, 2005] without the coherence condition (H-2). We will show shortly that this condition follows from the other two.

A transition induces a lifting of F to algebras, $\widehat{F}$, whose definition is given by the following proposition. (See [Marmolejo, 1997] for the definition of algebras.)

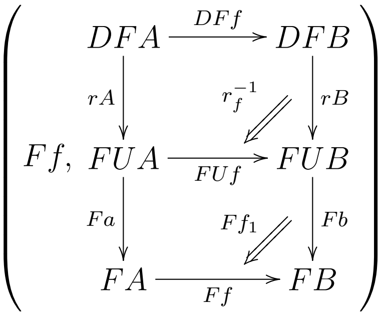
2.2. Proposition. *Let (r, ω_1, ω_2) be a transition from $\mathbb{U}$ to $\mathbb{D}$ along $F : \mathcal{A} \to \mathcal{B}$. We define $\widehat{F} : \mathcal{A}^{\mathbb{U}} \to \mathcal{B}^{\mathbb{D}}$ as follows. For an object*



in $\mathcal{A}^{\mathbb{U}}$, we define $\widehat{F}(A, a, a_1, a_2)$ as



For $(f, f_1) : (A, a, a_1, a_2) \to (B, b, b_1, b_2)$ *in* $\mathcal{A}^{\mathbb{U}}$*, we define* $\widehat{F}(f, f_1)$ *as*



For $\xi : (f, f_1) \to (g, g_1)$ *in* $\mathcal{A}^{\mathbb{U}}$*, define* $\widehat{F}\xi = F\xi$*. Then* $\widehat{F} : \mathcal{A}^{\mathbb{U}} \to \mathcal{B}^{\mathbb{D}}$ *is a 2-functor such that the diagram*

$$\begin{array}{ccc} \mathcal{A}^{\mathbb{U}} & \xrightarrow{\widehat{F}} & \mathcal{B}^{\mathbb{D}} \\ \downarrow & & \downarrow \\ \mathcal{A} & \xrightarrow[F]{} & \mathcal{B} \end{array} \tag{4}$$

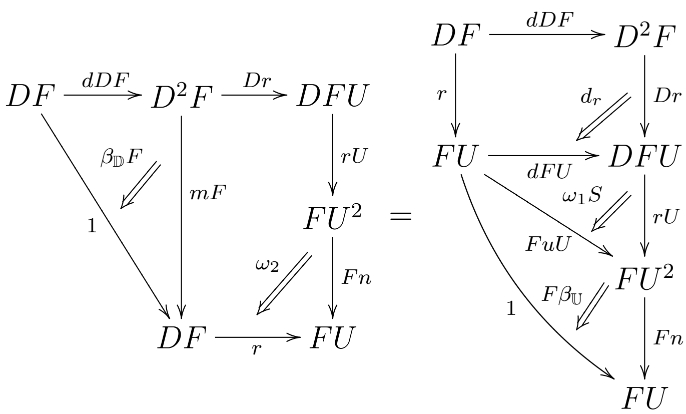
commutes, where the vertical arrows are the usual forgetful functors.

PROOF. We prove that $\widehat{F}$ is well defined on objects and leave the rest to the reader. Thus, we must show that $\widehat{F}(A, a, a_1, a_2)$ is a $\mathbb{D}$-algebra. Paste $\eta_{\mathbb{D}} FA^{-1}$ on the left of the second coordinate of the above pasting. Use (2). Since (A, a, a_1, a_2) is a $\mathbb{U}$-algebra, we can replace the pasting of $F\eta_{\mathbb{U}} A^{-1}$ and Fa_2 by $Fa \circ FUa_1$. Then replace the pasting of ${r_{uA}}^{-1}$, r_a^{-1} and FUa_1 by $rA \circ DFa_1$. This gives us one of the equations.

For the other, paste D of the second coordinate of the above pasting with the second coordinate of the above pasting, and with $\mu_{\mathbb{D}} FA$. Use (3). Using that (A, a, a_1, a_2) is a $\mathbb{U}$-algebra, replace the pasting of DFa_2, r_a^{-1}, r_{nA}^{-1}, Fa_2 and $F\mu_{\mathbb{U}} A$ by the pasting of r_a^{-1}, r_{Ua}^{-1}, Fa_2, Fn_a^{-1} and Fa_2. Conclude the calculation replacing the pasting of Dr_a^{-1}, r_{Ua}^{-1}, Fn_a^{-1} and $\omega_2 UA$ by the pasting of $\omega_2 A$, m_{Fa}^{-1} and r_a^{-1}. ∎

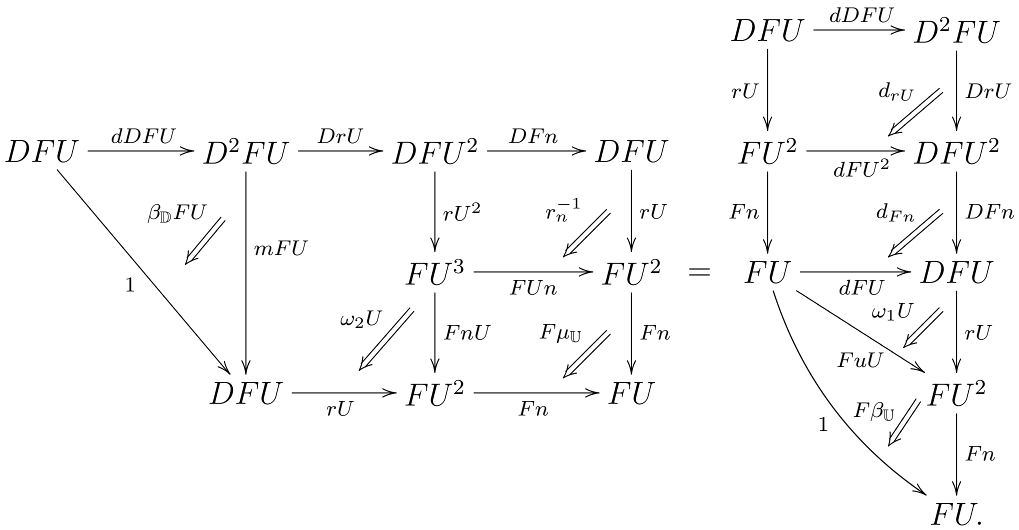
We show now the missing condition.

2.3. THEOREM.



(5)

PROOF. Lemma 9.1 of [Marmolejo, 1997], applied to $\widehat{F}(UA, nA, \beta_{\mathbb{U}}A, \mu_{\mathbb{U}}A)$ for every possible A gives us



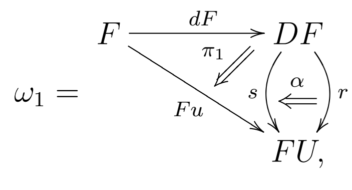
Precompose both sides with DFu. On top of both sides paste first with d_{DFu}, then with Dr_u, and then with $DF\eta_{\mathbb{U}}$. On the left and down of both sides paste first with r_u^{-1} and then with $F\eta_{\mathbb{U}}^{-1}$.

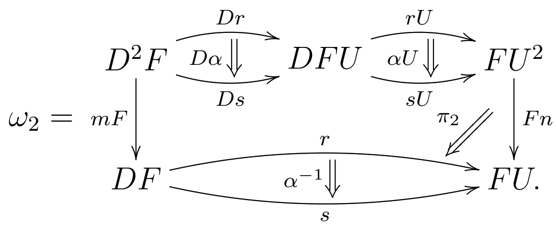
On the left replace the pasting of Dr_u, d_{DFu}, $\omega_2 U$, $\beta_{\mathbb{D}}FU$ and r_u^{-1} by the pasting of r_{Uu}^{-1}, Fn_u^{-1}, ω_2 and $\beta_{\mathbb{D}}F$. Observe that the pasting of all the 2-cells left with the exception of $\beta_{\mathbb{U}}F$ and ω_2 is an identity.

On the right, replace the pasting of Dr_u, d_{DFu}, d_{rU} and r_u^{-1} by the pasting of d_r and d_{FUu}. Now the pasting of $DF\eta_{\mathbb{U}}$, d_{FUu}, d_{Fn} and $F\eta_{\mathbb{U}}^{-1}$ cancels out. ∎

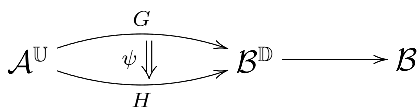
3. Transitions and liftings are essentially the same

3.1. DEFINITION. *Transitions (r, ω_1, ω_2) and (s, π_1, π_2) from $\mathbb{U}$ to $\mathbb{D}$ along F are said to be coherently isomorphic if there is an invertible $\alpha : r \to s$ such that*





3.2. PROPOSITION. *If $\alpha : (r, \omega_1, \omega_2) \to (s, \pi_1, \pi_2)$ is a coherent isomorphism between transitions from $\mathbb{U}$ to $\mathbb{D}$ along F, and $G, H : \mathcal{A}^{\mathbb{U}} \to \mathcal{B}^{\mathbb{D}}$ are the corresponding induced liftings, then there is a 2-isomorphism $\psi : G \to H$ such that*



is the identity, where the rightmost arrow is the usual forgetful functor.

Proof. For any $(a_1, a_2) = (A, a, a_1, a_2) \in \mathcal{A}^{\mathbb{U}}$, define $\psi(a_1, a_2) = (Id_{FA}, Fa \circ \alpha A)$. ∎

We now produce a transition from a lifting. Assume $G : \mathcal{A}^{\mathbb{U}} \to \mathcal{B}^{\mathbb{D}}$ such that

$$\begin{array}{ccc} \mathcal{A}^{\mathbb{U}} & \xrightarrow{G} & \mathcal{B}^{\mathbb{D}} \\ \downarrow & & \downarrow \\ \mathcal{A} & \xrightarrow[F]{} & \mathcal{B} \end{array}$$

commutes. For every $A \in \mathcal{A}$ we have $(\beta_{\mathbb{U}}A, \mu_{\mathbb{U}}A) = (UA, nA, \beta_{\mathbb{U}}A, \mu_{\mathbb{U}}A) \in \mathcal{A}^{\mathbb{U}}$. Thus we have the $\mathbb{D}$-algebra $G(\beta_{\mathbb{U}}A, \mu_{\mathbb{U}}A) = (FUA, G(\beta_{\mathbb{U}}A, \mu_{\mathbb{U}}A)_0, G(\beta_{\mathbb{U}}A, \mu_{\mathbb{U}}A)_1, G(\beta_{\mathbb{U}}A, \mu_{\mathbb{U}}A)_2)$:

$$\left(\begin{array}{ccc} FUA & \xrightarrow{dDUA} & DFUA \\ & \searrow^{1} \; \Downarrow G(\beta_{\mathbb{U}}A,\mu_{\mathbb{U}}A)_1 & \downarrow G(\beta_{\mathbb{U}}A,\mu_{\mathbb{U}}A)_0 \\ & & FUA \end{array} \; , \; \begin{array}{ccc} D^2FUA & \xrightarrow{DG(\beta_{\mathbb{U}}A,\mu_{\mathbb{U}}A)_0} & DFUA \\ {\scriptstyle mFUA}\downarrow & \Downarrow G(\beta_{\mathbb{U}}A,\mu_{\mathbb{U}}A)_2 & \downarrow G(\beta_{\mathbb{U}}A,\mu_{\mathbb{U}}A)_0 \\ DFUA & \xrightarrow[G(\beta_{\mathbb{U}}A,\mu_{\mathbb{U}}A)_0]{} & FUA \end{array} \right).$$

For $f : A \to B$ in $\mathcal{A}$, we have $(Uf, n_f^{-1}) : (\beta_{\mathbb{U}}A, \mu_{\mathbb{U}}A) \to (\beta_{\mathbb{U}}B, \mu_{\mathbb{U}}B)$ in $\mathcal{A}^{\mathbb{U}}$. Applying G we obtain

$$\left(FUf, \; \begin{array}{ccc} DFUA & \xrightarrow{DFUf} & DFUB \\ {\scriptstyle G(\beta_{\mathbb{U}}A,\mu_{\mathbb{U}}A)_0}\downarrow & \Downarrow G(Uf,n_f^{-1}) & \downarrow {\scriptstyle G(\beta_{\mathbb{U}}B,\mu_{\mathbb{U}}B)_0} \\ FUA & \xrightarrow[FUf]{} & FUB \end{array} \right) : G(\beta_{\mathbb{U}}A, \mu_{\mathbb{U}}A) \to G(\beta_{\mathbb{U}}B, \mu_{\mathbb{U}}B)$$

in $\mathcal{B}^{\mathbb{D}}$. Given $A \in \mathcal{A}$, we define

$$rA := (\; DFA \xrightarrow{DFuA} DFUA \xrightarrow{G(\beta_{\mathbb{U}}A,\mu_{\mathbb{U}}A)_0} FUA, \;)$$

and, for $f : A \to B$ a 1-cell in $\mathcal{A}$, we define

$$r_f := \begin{array}{ccccc} DFA & \xrightarrow{DFuA} & DFUA & \xrightarrow{G(\beta_{\mathbb{U}}A,\mu_{\mathbb{U}}A)_0} & FUA \\ {\scriptstyle DFf}\downarrow & \Downarrow DFu_f & \downarrow {\scriptstyle DFUf} & \Downarrow G(Uf,n_f^{-1})^{-1} & \downarrow {\scriptstyle FUf} \\ DFB & \xrightarrow[DFuB]{} & DFUB & \xrightarrow[G(\beta_{\mathbb{U}}B,\mu_{\mathbb{U}}B)_0]{} & FUB \end{array}$$

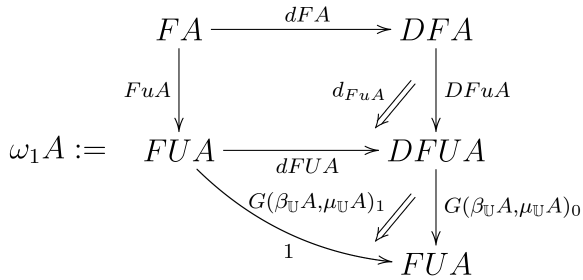
For any $\varphi : f \to g : A \to B$ in $\mathcal{A}$,

$$D\varphi : (Df, Dn_f^{-1}) \to (Dg, Dn_g^{-1}) : (\beta_{\mathbb{U}}A, \mu_{\mathbb{U}}A) \to (\beta_{\mathbb{U}}B, \mu_{\mathbb{U}}B)$$

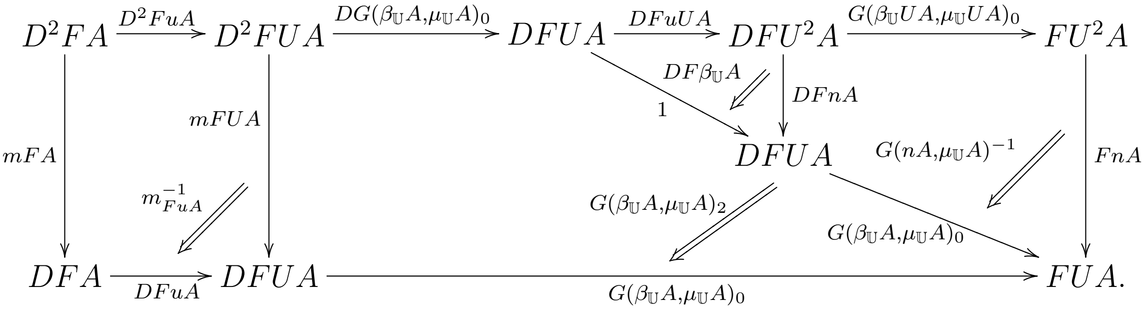
is a 2-cell in $\mathcal{A}^{\mathbb{U}}$. Applying G to $D\varphi$ it follows that $FU\varphi$ is a 2-cell in $\mathcal{B}^{\mathbb{D}}$. It is now easy to see that with the given definitions:

3.3. Lemma. *$r : DF \to FU$ is a strong transformation.* ∎

For $A \in \mathcal{A}$, define



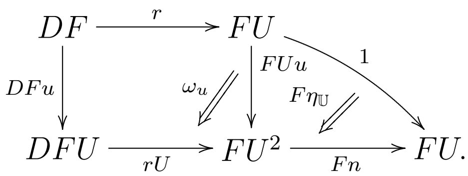
Observe that $(nA, \mu_{\mathbb{U}}A): (\beta_{\mathbb{U}}UA, \mu_{\mathbb{U}}UA) \to (\beta_{\mathbb{U}}A, \mu_{\mathbb{U}}A)$ in $\mathcal{A}^{\mathbb{U}}$. Apply G to $(nA, \mu_{\mathbb{U}}A)$ and define $\omega_2 A$ as the pasting



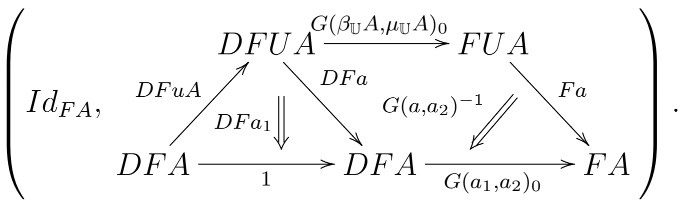
3.4. Proposition. *If $G: \mathcal{A}^{\mathbb{U}} \to \mathcal{B}^{\mathbb{D}}$ is a lifting of $F: \mathcal{A} \to \mathcal{B}$ to algebras and r, ω_1, and ω_2 are defined as above, then (r, ω_1, ω_2) is a transition from $\mathbb{U}$ to $\mathbb{D}$ along F.* ■

3.5. Theorem. *Let (r, ω_1, ω_2) be a transition from $\mathbb{U}$ to $\mathbb{D}$ along F and write $G: \mathcal{A}^{\mathbb{U}} \to \mathcal{B}^{\mathbb{D}}$ for the corresponding lifting. If (s, π_1, π_2) is the transition induced by G, then (r, ω_1, ω_2) and (s, π_1, π_2) are coherently isomorphic. Let $G: \mathcal{A}^{\mathbb{U}} \to \mathcal{B}^{\mathbb{D}}$ be a lifting to algebras of $F: \mathcal{A} \to \mathcal{B}$ and write (r, ω_1, ω_2) for the transition from $\mathbb{U}$ to $\mathbb{D}$ along F induced by G. If H is the lifting induced by (r, ω_1, ω_2), then there is a 2-natural isomorphism $\psi: G \to H$ such that $U^{\mathbb{D}} \circ \psi$ is the identity.*

Proof. In one direction, define $\alpha: r \to s$ as the pasting



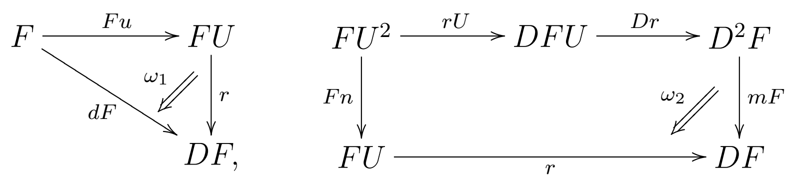
In the other, start with $(A, a, a_1, a_2) \in \mathcal{A}^{\mathbb{U}}$. Now $(a, a_2): (\beta_{\mathbb{U}}A, \mu_{\mathbb{U}}A) \to (a_1, a_2)$ is a 1-cell in $\mathcal{A}^{\mathbb{U}}$. We define $\psi(a_1, a_2)$ as



4. Op-transitions

Dually, we have op-transitions:

4.1. Definition. *An op-transition from* $\mathbb{U}$ *to* $\mathbb{D}$ *along* F *is a strong transformation* $r : FU \to DF$ *together with invertible modifications*



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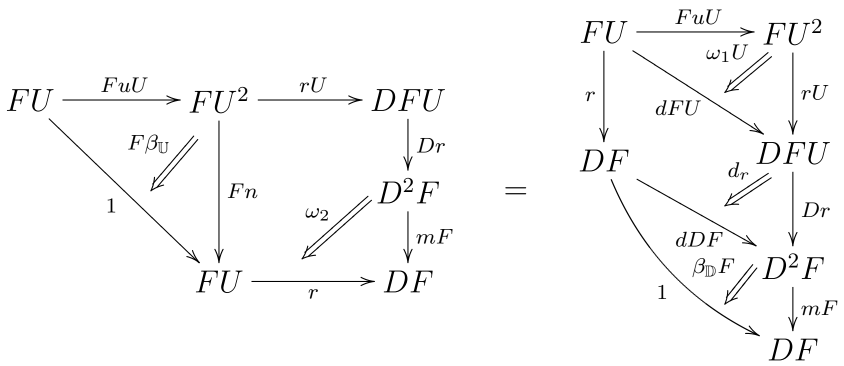
that satisfy the following coherence conditions:



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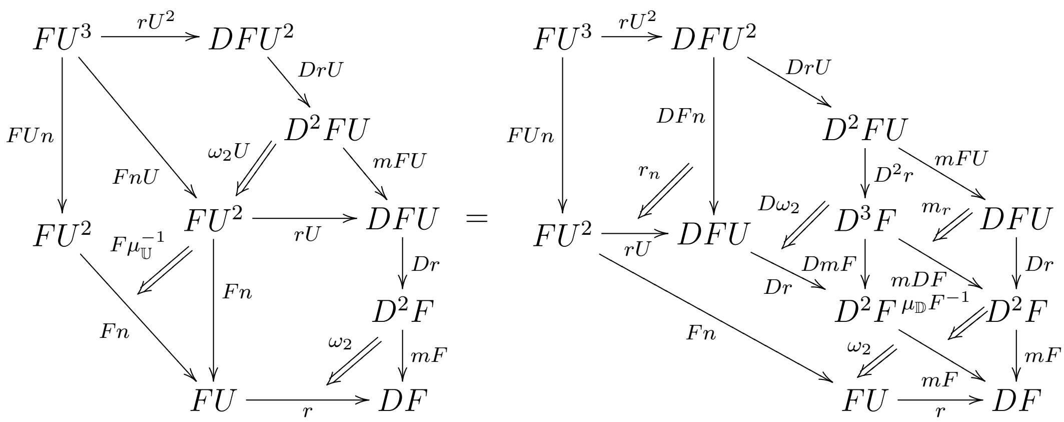

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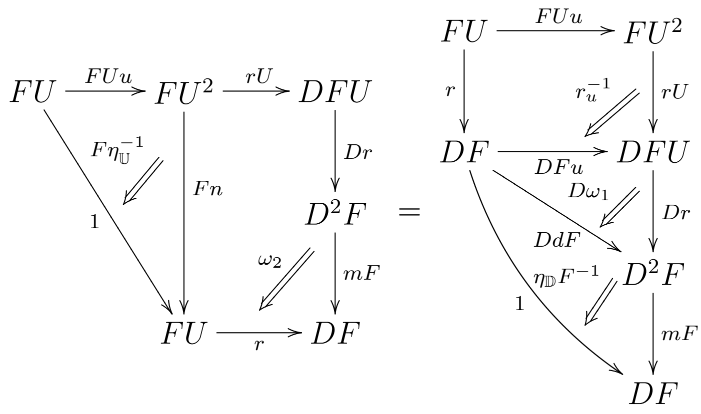
We obtain the dual of Theorem 2.3

4.2. Proposition.

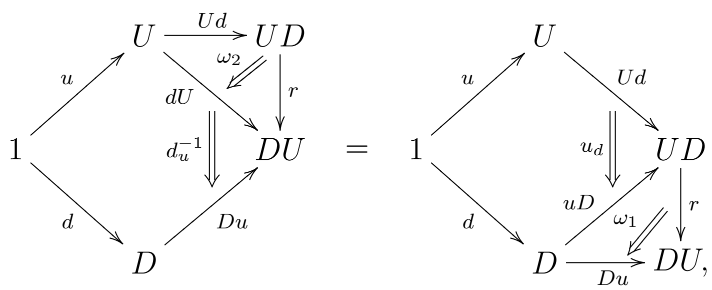


(9)

5. Distributive laws

We are now in a position to explain why there are eight coherence conditions for a pseudodistributive law.

5.1. Proposition. *A distributive law of* $\mathbb{U}$ *over* $\mathbb{D}$ *consists of a transition* $(r : UD \to DU, \omega_1, \omega_3)$ *from* $\mathbb{U}$ *to* $\mathbb{U}$ *along* D*, together with an op-transition* (r, ω_2, ω_4) *from* $\mathbb{D}$ *to* $\mathbb{D}$ *along* U *that satisfy the following coherence conditions:*



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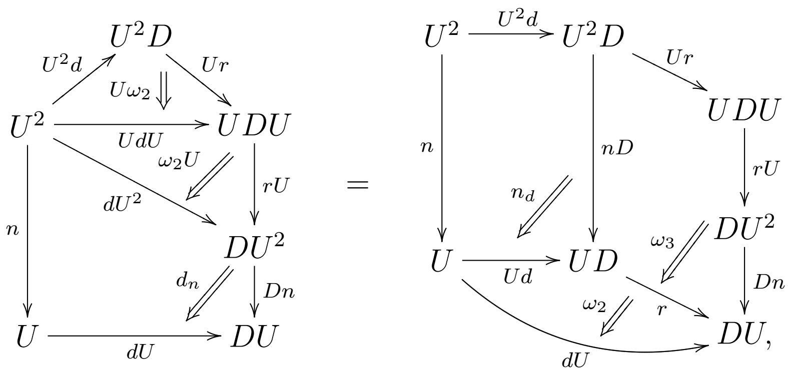

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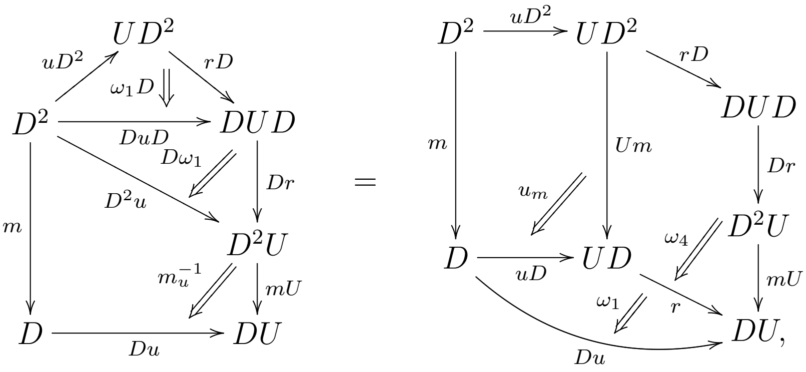
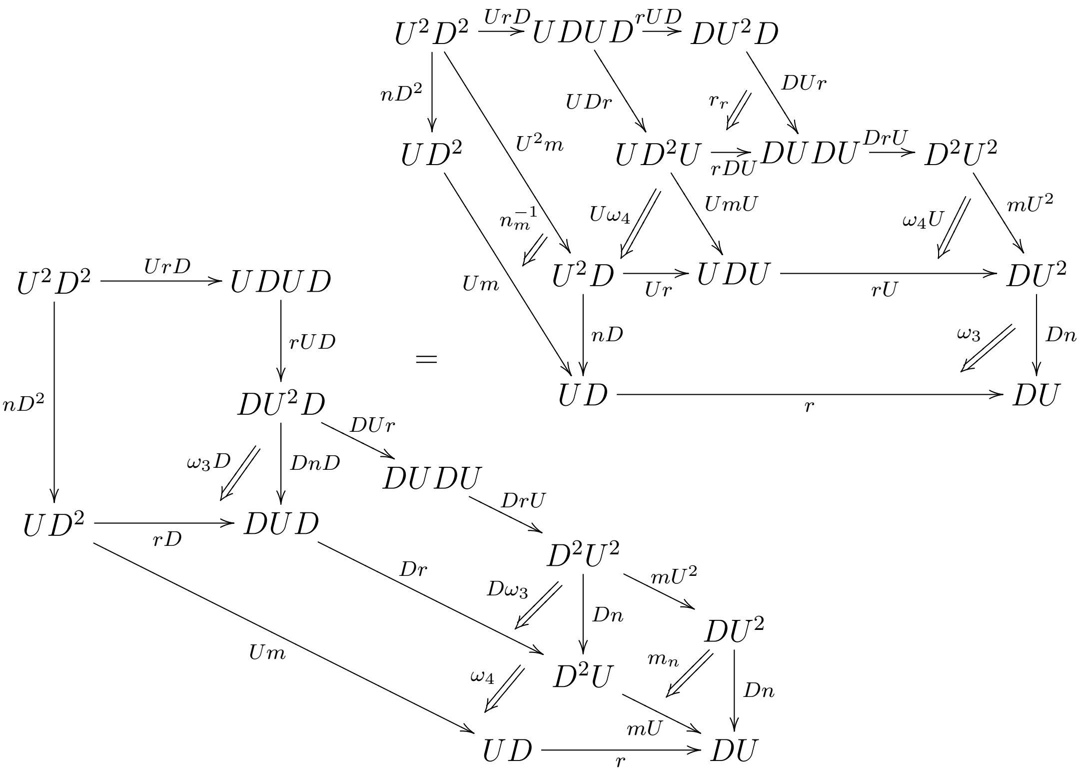


(12)

and



(13)

PROOF. The references (coh1), ..., (coh9) are to the paper [Marmolejo, 1999]. The coherence conditions for the transition of $\mathbb{U}$ to $\mathbb{U}$ along D correspond to the coherence conditions (coh2) and (coh4). The coherence conditions for the op-transition from $\mathbb{D}$ to $\mathbb{D}$ along U correspond to the coherence conditions (coh7) and (coh9). The coherence condition (coh8) follows from these last two according to Proposition 4.2. (10), (11), (12) and (13) are (coh1), (coh3), (coh5) and (coh6) rewritten. ■

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