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GENERATING FAMILIES IN A TOPOS

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ABSTRACT. A *generating family* in a category $\mathcal{C}$ is a collection of objects $\{A_i | i \in I\}$ such that if for any subobject $Y \overset{m}{\rightarrowtail} X$, every $A_i \xrightarrow{f} X$ factors through m, then m is an isomorphism – i.e. the functors $\mathcal{C}(A_i, \_)$ are collectively conservative.

In this paper, we examine some circumstances under which subobjects of 1 form a generating family. Objects for which subobjects of 1 do form a generating family are called *partially well-pointed*. For a Grothendieck topos, it is well known that subobjects of 1 form a generating family if and only if the topos is localic. For the elementary case, little more is known. The problem is studied in [1], where it is shown that the result is internally true, an equivalent condition is found in the boolean case, and certain preservation properties are shown. We look at two different approaches to the problem, one based on a generalization of projectivity, and the other based on looking at the most extreme sorts of counterexamples.

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1. Introduction & Background

The study of generating families in elementary topoi, and in particular of the cases when subterminal objects generate, has been neglected in much of the literature. The question of when subterminal objects form a generating family in a topos is related in some way to the connection between internal and external properties of the topos, since it is always internally true that subterminal objects generate (indeed, it is internally true that 1 is a generator).

In the case of Grothendieck topoi, the answer is well known – the topoi in which subterminal objects generate are exactly the localic topoi – an important subclass of the class of all Grothendieck topoi. In the general case, the problem is studied in [1], where it is shown that subterminals generating implies that Ω is a cogenerator, and it is shown that in a boolean topos, subterminals generate if and only if there are no completely pointless objects (see Section 3). However, very little is known about the non-boolean

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case for elementary topoi. In this paper, we develop two different approaches, and get some results about when subterminal objects form a generating family.

In both sections of the paper, we will use the following definitions relating to whether subterminal objects generate:

1.1. DEFINITION. *For an object X of $\mathcal{E}$, the* support $\sigma(X)$ *of X is the image of the unique morphism from X to 1. An object X of $\mathcal{E}$ is* well-supported *if $\sigma(X) = 1$, i.e. if the morphism from X to 1 is a cover.*

1.2. DEFINITION. *X is* well-pointed *if the only subobject of X through which all morphisms from 1 to X factor is the whole of X. It is* partially well-pointed *if the only subobject of X through which all morphisms from subobjects of 1 to X factor is the whole of X.*

1.3. DEFINITION. *A category is* capital *if all its well-supported objects are well-pointed.*

In Section 2, we weaken the notion of a projective object. Recall that an object X is projective (with respect to epimorphisms) if given any epimorphism $A \xrightarrow{c} B$, and any morphism $X \xrightarrow{f} B$, there is a factorization $X \xrightarrow{f'} A$ of f through c.

Recall also that in a topos, because all pullbacks exist and preserve epis, X is projective if and only if any epi $Y \xrightarrow{x} X$ is split epi, since given any epi $A \xrightarrow{c} B$, and any morphism $X \xrightarrow{f} B$, we can take the pullback of c along f, and compose its splitting with the pullback of f along c, to get a factorization of f through c.

We weaken the concept by instead of requiring f to have a factorization through c, allowing f to have a jointly epi family of partial factorizations through c. This weaker notion is more in the spirit of generating families that are closed under subobjects, and turns out to be exactly what we need for some theorems about when subterminals generate.

Recall that one interpretation of the axiom of choice in a topos is the assertion that all objects are projective. This suggests that in our study of semiprojectivity, various results about this notion of choice will be needed. We give the results we will need here without proof. All of the proofs can be found in [4].

An alternative way of approaching this form of the axiom of choice in a topos is via choice objects, which are defined as follows:

1.4. DEFINITION. *An object X in a topos $\mathcal{E}$ is* choice *if there is a choice function $P^+X \xrightarrow{c} X$ satisfying $(\forall X' : P^+X)(c(X') \in X')$, where P^+X is the object of inhabited subobjects of X, i.e. $P^+X = \{y : PX | (\exists x : X)(x \in y)\}$.*

The assertion that all objects are projective is equivalent to the assertion that all objects are choice. We will show that the assertion that all objects are semiprojective is equivalent to the assertion that choice objects form a generating family.

There is an alternative characterisation of choice objects in terms of entire relations.

1.5. DEFINITION. *A relation $R \rightarrowtail A \times B$, viewed as a relation from A to B is* entire *if the morphism $R \longrightarrow A$ is a cover.*

1.6. Lemma. *An object A in a topos is choice if and only if any entire relation from some object B to A contains a morphism from B to A.*

The way that we will show that choice objects generate in any topos where all objects are semiprojective is to use a particular choice object called the Higgs object, whose relationship to choice objects is studied in [2].

1.7. Definition. *An element x of a distributive lattice L is* widespread *if the sublattice $\{y \in L | y \geqslant x\}$ is boolean. A subterminal object U in a topos $\mathcal{E}$ is* (internally) widespread *if for every object X of $\mathcal{E}$, the subobject $X \times U \rightarrowtail X$ is widespread in the lattice Sub(X). A subobject $X' \rightarrowtail X$ is widespread if it is widespread as a subterminal object in $\mathcal{E}/X$.*

1.8. Definition. *The* Higgs object *W is the classifier of widespread subobjects, i.e. the classifying morphism of a subobject factors through $W \rightarrowtail \Omega$ if and only if the subobject is widespread, i.e. $W = \{u : \Omega | (\forall v : \Omega)(v \vee (v \Rightarrow u))\}$.*

We will show that subobjects of the Higgs object generate. It will then follow that choice objects generate from the following two facts:

1.9. Proposition. [P.Freyd,[2]] *In any topos, the Higgs object is choice.*

1.10. Proposition. *The class of choice objects is closed under subobjects, products and quotients.*

From this it will follow that if Ω is well-pointed and all objects are semiprojective, then subterminal objects generate. We will also show that if all objects are semiprojective then Ω is a cogenerator.

In Section 3, we look at the most extreme case of objects not being partially well-pointed – objects that do not admit any partial points. We call these completely pointless objects. In [1], F. Borceux shows that in a boolean topos, the existence of completely pointless objects is the only way in which subterminal objects can fail to generate.

We classify completely pointless objects in presheaf topoi. For this classification, we need:

1.11. Definition. *A subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is a* cosieve *if there is a subterminal object U, in $[\mathcal{C}, \underline{\mathcal{S}et}]$, for which $\mathcal{D}$ is the full subcategory on objects X with $U(X) = 1$. For such a $\mathcal{D}$, and a functor $F : \mathcal{C} \longrightarrow \underline{\mathcal{S}et}$, I shall denote the restriction of F to $\mathcal{D}$ by $F|_{\mathcal{D}}$.*

We then study when we can deduce that subterminals generate from the non-existence of completely pointless objects. Unfortunately, I do not make much progress in this respect. I conjecture that if there are no completely pointless objects in any closed subtopos of a topos, and if Ω is well-pointed, then subterminal objects generate. However, we only prove this in one fairly limited case.

We also use completely pointless objects to extend the result in [1] that if subterminal objects generate, then Ω is a cogenerator. We show that in fact, if subobjects of Ω generate, then Ω is a cogenerator.

2. Semiprojective Objects

Recall that an object X is projective (with respect to epimorphisms) if given any epimorphism $A \xrightarrow{e} \!\!\!\!\rightarrow B$ and any morphism $X \xrightarrow{f} B$, f factors through e. In a regular category, this is equivalent to all epimorphisms with codomain X being split epi, because the pullback of e along f will be split epi.

In this section, we weaken this by allowing jointly epi families of partial factorizations, rather than one total factorization. We look at a few examples of semiprojective objects. The case where all objects are semiprojective seems of particular interest – recall that the assertion that all objects are projective is one interpretation of the axiom of choice in a topos.

It therefore seems plausible that the assertion that all objects are semiprojective should be some kind of weaker choice principal. We classify the presheaf topoi in which all objects are semiprojective. We then prove that the assertion that all objects in a topos are semiprojective turns out to be equivalent to the assertion that subobjects of the Higgs object form a generating family, and thus that if all objects are semiprojective and Ω is well-pointed, then subobjects of 1 form a generating family.

2.1. Definition. *An object X, in a category $\mathcal{E}$, is* semiprojective *if given any epi $Y \xrightarrow{f} \!\!\!\!\rightarrow Z$, and any morphism $X \xrightarrow{g} Z$, the collection of partial factorizations (i.e. partial morphisms for which the following square commutes) is jointly total (i.e. the left hand monos jointly cover X).*

$$\begin{array}{ccc} X' & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ X & \xrightarrow{g} & Z \end{array}$$

This is clearly implied by projectivity, since if X is projective, the family of partial maps in the definition includes a total map, and so is jointly total. It is however, significantly weaker. For example, in Set, the assertion that all objects are projective is equivalent to the axiom of choice, while all sets are semiprojective without any form of choice, since for any element of a set A, there is a partial factorization for the singleton of that element, and any set is the union of its singletons.

If the category in question has pullbacks, and epis are stable under pullback, then the definition of semiprojectivity can be simplified in the same way as that of projectivity, i.e. we need only consider epimorphisms with codomain X, and X is semiprojective if and only if the partial splittings of any such epi cover X. This is because, given $Y \xrightarrow{f} \!\!\!\!\rightarrow Z$ and $X \xrightarrow{g} Z$, we can take the pullback of f along g, and its partial splittings extend to partial factorizations of g through f.

2.2. Examples. (i) In the category LH, whose objects are topological spaces, and whose morphisms are local homeomorphisms between them, i.e. maps $f : X \longrightarrow Y$ such that

$$(\forall x \in X)(\exists U \subset Y \text{ open})(x \in U \wedge f|_U \text{ is a homeomorphism})$$

all objects are semiprojective. Given a surjective local homeomorphism $f : X \longrightarrow Y$, for any $y \in Y$, take $x \in X$ with $f(x) = y$, and $U \subset X$ with $x \in U$ and $f|_U$ a homeomorphism. Then $f|_U$ has an inverse, which is a partial splitting of f containing y.

(ii) Generalising slightly, all objects are semiprojective in the category whose objects are locales and whose morphisms are local homeomorphisms.

(iii) In the category of sheaves on a locale, all objects are semiprojective. This is obvious from the equivalence $\mathcal{S}h(X) \cong LH/X$.

(iv) If M_2 is the 2-element monoid generated by an element a with $a = a^2$, then in $[M_2, \underline{\mathcal{S}et}]$, all objects are semiprojective. Given $X \overset{f}{\twoheadrightarrow} Y$, and given $y \in Y$, $a(y)$ is certainly fixed, so $\{y, a(y)\}$ is a subobject of Y. Pick $z \in X$ with $f(z) = y$. If $a(y) = y$ then take $x = a(z)$, so that $a(x) = x$. Otherwise just take $x = z$. Then $g(y) = x, g(a(y)) = a(x)$ is a partial splitting containing y.

(v) If M_3 is the 3-element monoid generated by an element a with $a^3 = a^2$, then in $[M_3, \underline{\mathcal{S}et}]$, 1 is semiprojective (indeed projective), but 2, with the action of a sending both 1 and 0 to 0 is not semiprojective, since the map $3 \overset{f}{\twoheadrightarrow} 2$, where in 3, $a(2) = 1, a(1) = a(0) = 0$, given by $f(0) = 0, f(1) = f(2) = 1$ has no splitting (and therefore no partial splitting containing 1).

(vi) In the topos $[\mathcal{C}, \underline{\mathcal{S}et}]$, where $\mathcal{C}$ is the category with 2 objects and a parallel pair of morphisms between them (i.e. the category for which equalizers and coequalizers are limits and colimits of shape $\mathcal{C}$ respectively), 1 is not semiprojective, since the object $X = 1 \underset{g}{\overset{f}{\rightrightarrows}} 2$ with $f(0) = 0, g(0) = 1$ is well-supported (i.e. admits an epi to 1), but its support has no splitting, and therefore no partial splitting containing the first set of 1. This X is however semiprojective (indeed projective) because it is the representable functor $\mathcal{C}(A, \_)$, where A is the domain of the non-identity morphisms in $\mathcal{C}$.

(vii) If G is a group, then in $[G, \underline{\mathcal{S}et}]$, 1 is not semiprojective, as there are no non-zero partial maps at all from 1 to G with the left multiplication action on itself. We will consider this more extreme sort of behaviour in more detail in the next section.

(viii) In $[\mathbb{Z}, \underline{\mathcal{S}et}_f]$, there are no non-zero semiprojectives. For any finite $\mathbb{Z}$-set A, pick a prime p larger than $|A|$, with a $\mathbb{Z}$-action for which p has only one orbit. Now take $A \times p$. The projection onto A is clearly epi, but it has no partial splittings, since each orbit of $A \times p$ has size at least p, so it can't be the image of an orbit in A.

Semiprojective objects in presheaf topoi

In the case of a presheaf topos, we can see exactly what the semiprojective objects are. Since the representables generate the topos, we only need to check semiprojectivity with respect to representable objects. Recall that in a presheaf topos, the projective objects are free functors on a set of generators. It turns out that semiprojective objects are free for each generator individually, but two different generators might satisfy some relations. Basically, semiprojective objects are a union of free objects (i.e. representables).

2.3. THEOREM. *In $[\mathcal{C}, \underline{\mathcal{S}et}]$ a functor F is semiprojective if and only if:*

$$(\forall X \in \text{ob } \mathcal{C})(\forall x \in F(X))(\exists Z \xrightarrow{h} X \in \text{mor } \mathcal{C})(\exists y \in F(Z))$$

$$(x = F(h)(y) \wedge (\forall X \overset{f}{\underset{g}{\rightrightarrows}} Y \in \mathcal{C})(F(f)(x) = F(g)(x) \Leftrightarrow fh = gh))$$

PROOF. If F is semiprojective, then given the obvious natural transformation

$$\coprod_{Z \in \text{ob } \mathcal{C}} F(Z) \times \mathcal{C}(Z, \_) \xrightarrow{\alpha} F$$

α is surjective, so for any $x \in F(X)$, there will be a partial splitting $F' \overset{\beta}{\rightarrowtail} \mathcal{C}(Z, \_)$, for some $Z \in \text{ob } \mathcal{C}$, with $x \in F'(X)$. Let $\beta_X(x) = (y, h) \in F(Z) \times \mathcal{C}(Z, X)$, where $F(h)(y) = x$. We have $\beta_Y(F(f)(x)) = \beta_Y(F(g)(x))$ if and only if $F(f)(x) = F(g)(x)$ as β is monic. But $\beta_Y(F(f)(x)) = (y, fh)$, and $\beta_Y(F(g)(x)) = (y, gh)$, by naturality, so $fh = gh$ if and only if $F(f)(x) = F(g)(x)$.

It remains to show that any F satisfying the above condition is semiprojective, but given $G \overset{\alpha}{\twoheadrightarrow} F$ an epi natural transformation, and $x \in F(X)$, let Z, h and y be as in the statement of the theorem. There is $y' \in G(Z)$ with $\alpha_Z(y') = y$. Let F' be the subfunctor of F given by

$$F'(Y) = \{y \in F(Y) | (\exists X \xrightarrow{a} Y \in \text{mor } \mathcal{C})(y = F(a)(x))\}$$

There is a partial splitting $F' \xrightarrow{\beta} G$ given by $\beta_Y(F(a)(x)) = G(ah)(y)$. Therefore F is semiprojective. ∎

For a subterminal object, this says that F is semiprojective if and only if for any $X \in \text{ob } \mathcal{C}$ with $F(X) = 1$, there is a $Y \xrightarrow{a} X \in \text{mor } \mathcal{C}$ with $F(Y) = 1$, such that for any parallel pair $X \overset{f}{\underset{g}{\rightrightarrows}} Z$, $ga = fa$. In particular, if F is 1 only on the full subcategory on objects that admit a morphism from X, this says that X has an endomorphism making all parallel pairs with domain X equal. So in $[\mathcal{C}, \underline{\mathcal{S}et}]$, all subterminal objects are semiprojective if and only if every object X of $\mathcal{C}$ has this property.

Recall that in a topos, one interpretation of the axiom of choice is the assertion that all objects are projective. This leads to the question of whether the assertion that all objects are semiprojective corresponds to some sensible weaker notion of choice. Having characterised the semiprojectives in $[\mathcal{C}, \underline{\mathcal{S}et}]$ for small categories $\mathcal{C}$, it is straightforward to check when $[\mathcal{C}, \underline{\mathcal{S}et}]$ has all objects semiprojective.

2.4. LEMMA. *In $[\mathcal{C}, \underline{\mathcal{S}et}]$ for $\mathcal{C}$ a small category, if F is a quotient of a representable functor, then any family of morphisms $G_i \xrightarrow{f_i} F$ that is jointly covering contains a cover.*

PROOF. Let $\mathcal{C}(X, \_) \xrightarrow{\alpha} F$ be a cover. Since the f_i are jointly covering, one of their images must contain $\alpha_X(1_X)$. Let f_i be this morphism. Then the image of f_i contains $\alpha_Y(g)$ for every $g \in \text{mor } \mathcal{C}$. But α is a cover, so the image of f_i is the whole of F, and thus f_i is a cover. ∎

2.5. Lemma. *Let $\mathcal{C}$ be a small Cauchy complete category. Then any semiprojective quotient F of a representable in $[\mathcal{C}, \underline{\mathcal{S}et}]$ is a representable.*

Proof. Since F is semiprojective, it is covered by the partial splittings of the epi from a representable to F, so by Lemma 2.4, one of the partial splittings must be total, making F a retract of a representable, and hence a representable, because $\mathcal{C}$ is Cauchy complete, and thus in $[\mathcal{C}, \underline{\mathcal{S}et}]$, any retract of a representable is representable. ∎

Lemma 2.5 is obvious because quotients of a representable are generated by a single element, and semiprojective objects are free on each element, so if they are generated by one element, then they are free on one element, and thus representable.

If all objects in $[\mathcal{C}, \underline{\mathcal{S}et}]$ are semiprojective, then all representables will be choice, because any quotient of a representable is representable, so it is projective, and the cover is split. Given an entire relation $R \rightarrowtail F \times \mathcal{C}(X, \_)$, take the pushout

$$\begin{array}{ccc} R & \longrightarrow & \mathcal{C}(X,\_) \\ \downarrow & & \downarrow \\ F & \longrightarrow & F' \end{array}$$

The cover $\mathcal{C}(X, \_) \longrightarrow F'$ is split, and composing the splitting with $F \longrightarrow F'$ gives a morphism $F \longrightarrow \mathcal{C}(X, \_)$ contained in R.

2.6. Lemma. *Let $\mathcal{C}$ be a small Cauchy complete category such that $[\mathcal{C}, \underline{\mathcal{S}et}]$ has all objects semiprojective. Let R be an equivalence relation on $\mathcal{C}(X, \_)$. Then there is a morphism $X' \xrightarrow{e} X$ such R is the kernel pair of $\mathcal{C}(e, \_)$.*

Proof. Take the quotient of $\mathcal{C}(X, \_)$ corresponding to R. This is a quotient of a representable, and therefore a representable $\mathcal{C}(X', \_)$. The morphism $\mathcal{C}(X, \_) \longrightarrow \mathcal{C}(X', \_)$ is of the form $\mathcal{C}(e, \_)$ for some $X' \overset{e}{\rightarrowtail} X$, and R is its kernel pair. ∎

2.7. Theorem. *Let $\mathcal{C}$ be as in Lemma 2.6. Then every parallel pair of distinct morphisms in $\mathcal{C}$ must contain exactly one isomorphism.*

Proof. Let R be the equivalence relation on $\mathcal{C}(X, \_)$ generated by the parallel pair $(\mathcal{C}(f, \_), \mathcal{C}(g, \_))$ where f and g are morphisms from X to Y in $\mathcal{C}$. Let $X' \overset{e}{\rightarrowtail} X$ be the morphism obtained from R as in Lemma 2.6. Then e will be the equalizer of f and g, since if $fh = gh$, then the collection $\mathcal{P}'$ of parallel pairs (f', g') with $f'h = g'h$ corresponds to an equivalence relation that contains R, so $\mathcal{C}(H, \_)$, where H is the domain of h, is a retract of $C(X', \_)$, and h factors through e.

$\mathcal{C}(X', \_)$ is a quotient of $\mathcal{C}(X, \_)$, so it is a retract, making e a split mono. If e' is its splitting, then $ee'e = e$, so $(ee', 1) \in R(X)$, meaning (w.l.o.g.)

$$(\exists Y \xrightarrow{f'} X)(f'f = 1, f'g = ee')$$

Thus, given any distinct parallel pair in $\mathcal{C}$, one is split mono. Apply this to ff' and gf', which are distinct because f' is epi, to get that f' is mono, and hence that f is iso.

Now suppose that f and g are both iso. Then $f'g$ is mono, so e' is mono, and thus e is iso, meaning that $f = g$. ∎

This gives us

2.8. THEOREM. *All objects are semiprojective in* $[\mathcal{C}, \underline{\mathcal{S}et}]$ *if and only if* $\mathcal{C}$ *is equivalent to a category in which all hom-sets have at most one non-identity morphism.*

PROOF. One way is obvious by the preceding theorem. For the other direction, if $\mathcal{C}$ is such a category, all its objects will be semiprojective, since given $A \xrightarrow{f} B$ in $[\mathcal{C}, \underline{\mathcal{S}et}]$, any X in $\mathcal{C}$, and any $x \in B(X)$, we can choose $y \in A(X)$ with $f_X(y) = x$, and if x is fixed under the endomorphism g of X (assuming one exists), then $f_X(g(y)) = g(f_X(y)) = x$, so we can find a partial inverse to f, sending x to $g(y)$. If x is not fixed by g, or if X has no non-identity endomorphism, then there is a partial splitting sending x to y. ∎

General topoi

Now we consider the question in a more general topos. In particular, we shall see that if all objects are semiprojective, then the choice objects form a generating family, and indeed the subobjects of the Higgs object form a generating family. The idea is to show that if every partial morphism from W to X factors through $Y \overset{m}{\rightarrowtail} X$, then the image of the classifying map χ_m of m will be a subobject of W, so the partial splittings of the cover part of χ_m all factor through m, meaning that m is an isomorphism. The following lemma will be necessary in the proof; intuitively, it holds because f is the identity everywhere except on m, so its partial splittings cover everything except m.

2.9. LEMMA. *Let the following diagram be a pushout, and let C be semiprojective. Then m and the partial splittings of f cover B:*

$$\begin{array}{ccc} A & \longrightarrow & 1 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle a} \\ B & \xrightarrow{f} & C \end{array}$$

PROOF. Let $C \overset{m_i}{\leftarrowtail} C_i \overset{g_i}{\rightarrowtail} B$ be a partial splitting, so that $m_i = fg_i$, and let $m_i \cap a = U_i$. Consider the following diagram:

$$\begin{array}{ccccc} U_i & \rightarrowtail & A & \longrightarrow & 1 \\ \downarrow & & {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle a} \\ C_i & \overset{g_i}{\rightarrowtail} & B & \xrightarrow{f} & C \end{array}$$

The right hand square and the outer rectangle are pullbacks, so the left hand square is also a pullback, i.e. $m \cap g_i = U_i$. Thus, the squares in the following diagram are pushouts:

$$\begin{array}{ccccc} U_i & \rightarrowtail & A & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow \\ C_i & \rightarrowtail & C_i \cup A & \longrightarrow & C_i \cup 1 \\ & & \downarrow & & \downarrow \\ & & B & \longrightarrow & C \end{array}$$

which means that the bottom right square is also a pullback, as in a topos, pushout squares of monos are also pullback squares. Now suppose that the $C_i \cup A$ all factor through $B' \rightarrowtail B$. Consider the pushouts:

$$\begin{array}{ccc} C_i \cup A & \longrightarrow & C_i \cup 1 \\ \downarrow & & \downarrow \\ B' & \longrightarrow & C_i' \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & C \end{array}$$

In each such pushout, C_i' is the image of $B' \rightarrowtail B \xrightarrow{f} C$, so it does not depend on i, and hence each $C_i \overset{m_i}{\rightarrowtail} C$ factors through it. Therefore, $C_i' \rightarrowtail C$ is an iso, and so is $B' \rightarrowtail B$, being a pullback of it. ■

2.10. Lemma. *Let $\mathcal{E}$ be a topos whose objects are all semiprojective, and let $1 \overset{a}{\rightarrowtail} Z$ have a monic classifying map and contain all subterminal subobjects of Z. Then a is widespread, and hence Z is choice, and indeed a subobject of the Higgs object.*

Proof. We need to show that for any X and any factorization $X \overset{r}{\rightarrowtail} R \overset{m}{\rightarrowtail} Z \times X$ of $X \overset{a\times 1_X}{\rightarrowtail} Z \times X$, there is an $R' \overset{m'}{\rightarrowtail} Z \times X$, such that $m \cap m' = a \times 1_X$ and $m \cup m' = 1_{Z\times X}$. If X is not well-supported, work in the slice of $\mathcal{E}$ over the support $\sigma(X)$. If there is a value of m' for which this holds, then it must hold for $m' = \forall_m(r)$, so we will show that this works. By definition, $m' \cap m \leqslant a \times 1_X$, and $m' \cap m \geqslant a \times 1_X$ as $a \times 1_X \cap m = a \times 1_X$, so we need only show that $m' \cup m = 1_{Z\times X}$. Consider the pushout:

$$\begin{array}{ccc} R & \longrightarrow & 1 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle b} \\ Z \times X & \xrightarrow{f} & Y \end{array}$$

By Lemma 2.9, m and the partial splittings of f cover $Z \times X$. Furthermore, for any partial splitting $Y \leftarrowtail Y_i \overset{f_i}{\rightarrowtail} Z \times X$, $f_i \cap m$ is a subterminal object, and so its composite with $Z \times X \xrightarrow{\pi_1} Z$ factors through a. This means that $f_i \cap m \leqslant a \times 1_X$, so that f_i factors through m'. Therefore, all the f_i and m factor through $m' \cup m$, so $m' \cup m = 1_{Z\times X}$.

Thus a is widespread, and therefore its classifying map factors through the Higgs object, W. But the classifying map of a is mono, and therefore Z is a subobject of W, which is choice (shown in [2]). Therefore, Z is choice. ■

2.11. Theorem. *In a topos $\mathcal{E}$, the following are equivalent:*
(i) All objects are semiprojective.
(ii) The choice objects form a generating family.
(iii) Subobjects of the Higgs object form a generating family.

Proof. (i)⇒(iii): Given a subobject $Y \overset{y}{\rightarrowtail} X$, suppose it contains all partial maps from W to X, and thus all subterminal subobjects of X. Let its classifying map have cover-mono factorization $X \overset{c}{\twoheadrightarrow} Z \overset{a}{\rightarrowtail} \Omega$, and let $1 \overset{m}{\rightarrowtail} Z$ be the subobject classified by a. (If X is not well-supported, slice over its support, so that the domain of m is 1. X and Y have the same support because all partial splittings of the support of X factor through y.)

Let $U \overset{n}{\rightarrowtail} Z$ be a subterminal subobject of Z, and pull back the partial splittings of c along n. These are partial maps from U to X, so they factor through y, and their composites with c therefore factor through m. But their domains cover U, so $n \cap m = n$, i.e. n factors through m. However, n was an arbitrary subterminal subobject of Z, so Z is a subobject of W by Lemma 2.10.

Z is semiprojective, so the partial splittings of c cover Z. Any partial splitting of c is a partial map from W to X, and thus factors through y. Therefore, any partial splitting of c factors through m, making m, and thus y, an isomorphism.

(iii)⇒(ii): It is shown in [2] that W is choice, and it is well known that subobjects of choice objects are choice, whence this is obvious.

(ii)⇒(i): If the choice objects form a generating family, then given any $A \overset{c}{\twoheadrightarrow} B$, the maps from choice objects to A cover A. Their images therefore also cover A, and quotients of choice objects are choice, so A is covered by the monos from choice objects to it. The composites of these monos with c cover B. But the maps from these composites to their images are covers with choice domains, and therefore split epis. The splittings of these epis give partial splittings of c that cover B. ∎

The conditions of Theorem 2.11 also imply that Ω is a cogenerator in $\mathcal{E}$. This result will be extended in the next section, but the methods used there will be different, so we include the proof here.

The idea is that if $X \overset{f}{\twoheadrightarrow} Y$ is a cover through which every map $X \longrightarrow \Omega$ factors, then $\exists_f$ will be an equivalence of categories between $\mathrm{Sub}(X)$ and $\mathrm{Sub}(Y)$, and so conservative, so we just need to prove:

2.12. Lemma. *Let $\mathcal{E}$ be a topos in which the choice objects form a generating family, and let $X \overset{f}{\twoheadrightarrow} Y$ be a cover such that $\exists_f$ is a conservative functor:* $\mathrm{Sub}(X) \longrightarrow \mathrm{Sub}(Y)$. *Then f is an isomorphism.*

Proof. Let $R \underset{b}{\overset{a}{\rightrightarrows}} X$ be the kernel pair of f. We will show that any choice subobject of R is contained in the diagonal $X \overset{\Delta}{\longrightarrow} X \times X$. This will mean that $R \rightarrowtail X \times X$ is the diagonal, and therefore that f is an isomorphism.

To do this, we will show that if $C \xrightarrow{r} X$ is a morphism from a choice object to R, and $C_1 \xrightarrow{c_1} X$ and $C_2 \xrightarrow{c_2} X$ are the images of ar and br respectively, then the restriction of f to $c_1 \cup c_2$ is mono. Therefore, the kernel pair of this restriction is the diagonal. But ar and br both factor through this kernel pair, so $ar = br$ as required.

In fact, we will show that the restriction of f to the union of any two choice subobjects is mono. Let $C \xrightarrow{c} X$ be a choice subobject of X. Let $C \xrightarrow{g} C' \xrightarrow{h} X'$ be the cover-mono factorization of fc. As g is a cover with choice domain, it is split epi. Let g' be a splitting of g. Then $\exists_f(g') = 1_{C'}$, so g' is iso, and therefore fc is mono.

Now let $C_1 \xrightarrow{c_1} X$ and $C_2 \xrightarrow{c_2} X$ be two choice subobjects of X. Let

$$\begin{array}{ccc} C' & \xrightarrow{d_1} & C_1 \\ {\scriptstyle d_2}\downarrow & & \downarrow{\scriptstyle fc_1} \\ C_2 & \xrightarrow{fc_2} & X' \end{array}$$

be a pullback (fc_1 and fc_2 are monic because C_1 and C_2 are choice). Now let $D \xrightarrow{d} X$ be the union of c_1d_1 and c_2d_2 in $\mathrm{Sub}(X)$. Since $fc_1d_1 = fc_2d_2$, they must both equal $\exists_f(d)$, because $\exists_f$ preserves unions. But then $\exists_f$ sends the factorizations of c_1d_1 and c_2d_2 through d to the identity on D. Therefore, the factorizations are isomorphisms as $\exists_f$ is conservative, meaning that $c_1d_1 = c_2d_2$ as subobjects of X. This means that the union of c_1 and c_2 has monic composite with f, because the union of c_1 and c_2 and the union of fc_1 and fc_2 are both given by the pushout

$$\begin{array}{ccc} C' & \xrightarrow{d_1} & C_1 \\ {\scriptstyle d_2}\downarrow & & \downarrow \\ C_2 & \longrightarrow & C \end{array}$$

■

2.13. THEOREM. *If in the topos $\mathcal{E}$, the choice objects form a generating family, then Ω is a cogenerator.*

PROOF. Let $X \xrightarrow{f} Y$ be a cover through which every map $X \longrightarrow \Omega$ factors. Let $X' \xrightarrow{m} X$ be a subobject of X. The classifying map χ_m factors through f, so m is the pullback of a subobject of Y along f. This means that $f^* : \mathrm{Sub}(Y) \longrightarrow \mathrm{Sub}(X)$ is full and faithful and surjective on objects. Therefore, the adjunction $\exists_f \dashv f^*$ is an equivalence, so $\exists_f$ is conservative, and by Lemma 2.12, f is an isomorphism. ■

This extends a result in [1] that states that if subobjects of 1 form a generating family then Ω is a cogenerator. This result already has some extensions, for example, in [5], it is shown that any in any graphic topos, Ω is a cogenerator. This result implies Theorem 2.13 in the case of a presheaf topos. It is also known that Ω is a cogenerator in the simplicial topos, but I have not yet found a reference for this fact.

Theorem 2.11 also gives a characterisation of when subobjects of the terminal objects form a generating family, and thence of when a topos is capital.

2.14. Lemma. *Any subobject of a partially well-pointed object is also partially well-pointed.*

Proof. Morally, this is obvious, because the intersection of a partial point with a subobject is a partial point of the subobject, and if the partial points cover an object, they must also cover any of its subobjects.

More formally, let $X \stackrel{m}{\rightarrowtail} Y$ be a subobject, and let Y be partially well-pointed. Suppose any partial point of X, and in particular the pullback of a partial point $U \stackrel{u}{\rightarrowtail} Y$ along $X \stackrel{m}{\rightarrowtail} Y$, factors through $X' \stackrel{x}{\rightarrowtail} X$. Then u must factor through $\forall_m(x)$. Therefore, $\forall_m(x)$ will be the identity on Y, meaning that x must be the identity on X. Therefore X is partially well-pointed. ∎

2.15. Lemma. *If X is partially well-pointed and injective, then it is well-pointed.*

Proof. Let $\widetilde{X}$ be the partial map classifier of X. As X is injective, the mono $X \rightarrowtail \widetilde{X}$ is split. Any partial map from 1 to X corresponds to a morphism from 1 to $\widetilde{X}$, and therefore extends to a morphism from 1 to X. Therefore, if every morphism from 1 to X factors through $X' \rightarrowtail X$, then so does every partial map from 1 to X, so $X' \cong X$. ∎

2.16. Corollary. *The following are equivalent:*

(i) Subterminal objects form a generating family.

(ii) All objects are semiprojective and W is partially well-pointed.

(iii) All objects are semiprojective and Ω is well-pointed.

(iv) All subterminal objects are semiprojective and Ω is well-pointed.

Proof. (iv)⇒(i): Let $Y \stackrel{m}{\rightarrowtail} X$ be an arbitrary subobject in $\mathcal{E}$, through which every partial map from 1 to X factors, and let $X \stackrel{g}{\twoheadrightarrow} Z \stackrel{h}{\rightarrowtail} \Omega$ be the cover-mono factorization of the classifying map of m. Let $R \rightarrowtail \Omega = \forall_{hg}(m)$, and let $Z' \rightarrowtail Z$ be the pullback of $R \rightarrowtail \Omega$ along h. We will show that any morphism $1 \stackrel{n}{\rightarrowtail} \Omega$ is contained in R, and therefore, that R is isomorphic to Ω because Ω is well-pointed.

Given a map $1 \stackrel{n}{\rightarrowtail} \Omega$, the pullback of n along $Z \rightarrowtail \Omega$ is a subterminal subobject $U \rightarrowtail Z$. Any partial map from U to X factors through Y, and therefore, its composite with g factors through $Z' \rightarrowtail Z$. U is semiprojective, and if its pullback along $Z' \rightarrowtail Z$ is U', then any partial splitting of $X \times U \stackrel{\pi_2}{\twoheadrightarrow} U$ factors through U', so $U' \cong U$. Thus $U \rightarrowtail Z$ factors through $Z' \rightarrowtail Z$, so n factors through $\forall_h(Z') = R$.

Since n was an arbitrary point of Ω, and Ω is well-pointed, $R \cong \Omega$. Thus, $Y \cong X$, as m is a pullback of $R \rightarrowtail \Omega$.

(i)⇒(iii): 1 is a subobject of W, so the subterminal objects are a subcollection of the subobjects of W, so this follows from Theorem 2.11 and Lemma 2.15.

(iii)⇒(ii): W is a subobject of Ω, so this follows from Lemma 2.14.

(iii)⇒(iv) is obvious.

(ii)⇒(i) is immediate from Theorem 2.11. ∎

There is a more direct proof of (i)⇒(iv): Suppose all partial splittings of $A \xrightarrow{c} U$ factor through $U' \rightarrowtail U$, for some subterminal U. Then any partial map from U to A, and therefore from 1 to A must factor through $A \times U' \rightarrowtail A$, so that $A \times U' \cong A$, and thus, $U' \cong U$.

In particular, Corollary 2.16 gives a characterisation of capital topoi.

2.17. Corollary. *A topos $\mathcal{E}$ is capital if and only if the following three conditions hold:*
(i) 1 is projective.
(ii) Ω is well-pointed.
(iii) Any subterminal object is the sup of a family of complemented subterminal objects.

Proof. Only if: (i) and (ii) are obvious. For (iii), note that for any subterminal U, $1 \amalg U$ is well-supported and hence well-pointed, so U is the sup of the complemented subobjects of 1 contained in it.

If: Suppose that conditions (i)-(iii) hold. 1 is projective. Therefore, so is any complemented subterminal object; so any subterminal object is the sup of a family of projective subobjects, and therefore semiprojective. Therefore, subterminal objects form a generating family by Corollary 2.16.

Let A be a well-supported object in $\mathcal{E}$, and $A' \rightarrowtail A$ a subobject through which all maps from 1 to A factor. For any map $U \overset{m}{\rightarrowtail} A$, for U a subterminal object, let $U' \rightarrowtail U$ be the pullback of $A' \rightarrowtail A$ along m. Any map from a complemented subterminal to A extends to a map from 1 to A (A admits a map from 1 because 1 is projective, and if we fix a map a from 1 to A, we can extend our original map by making it equal to a wherever it wasn't defined originally) and therefore factors through A'. Therefore, if V is a complemented subterminal object contained in U, then $V \rightarrowtail U \overset{m}{\rightarrowtail} A$ factors through A', so $V \rightarrowtail U$ factors through $U' \rightarrowtail U$. U is the sup of the complemented subterminal objects contained in it, so $U' \cong U$, and m must factor through $A' \rightarrowtail A$. Since this m was arbitrary, $A' \cong A$. ∎

Another situation that has been studied for projective objects is the case where a category has enough projectives, i.e. every object admits an epimorphism from a projective object. If we say that a category has enough semiprojectives if every object admits an epimorphism from a semiprojective object, then we get the following:

2.18. Lemma. *Any étendue has enough semiprojectives.*

Proof. Let $\mathcal{E}$ be an étendue, and let $\mathcal{E}/X$ be localic, with X well-supported. Given an object A of $\mathcal{E}$, A is covered by $A \times X$, which is semiprojective in $\mathcal{E}/X$ (because all objects are semiprojective in $\mathcal{E}/X$) and therefore also semiprojective in $\mathcal{E}$. ∎

If we assume not only that there are enough semiprojectives, but also that we can choose a semiprojective object covering each object (if the axiom of choice holds for all collections of objects, then having enough semiprojective objects will imply this) then we get:

2.19. LEMMA. *A locally small topos with a chosen semiprojective cover* $X' \xrightarrow{c} X$ *for every object* X *has a generating set.*

PROOF. Let $\mathcal{S}$ be the set of subobjects of chosen covers of subobjects of Ω. ($\mathcal{S}$ is a set by local smallness.) Given any $Y \overset{m}{\rightarrowtail} X$ through which all morphisms from a member of $\mathcal{S}$ to X factor, let the cover-mono factorization of the classifying map of m be $X \xrightarrow{\phi_m} Z \overset{\psi_m}{\rightarrowtail} \Omega$, and let $W \overset{a}{\rightarrowtail} Z$ be $\forall_{\phi_m}(m)$. Then m is the pullback of a along ϕ_m. Let $Z' \xrightarrow{c} Z$ be the chosen semiprojective cover of Z and let

$$\begin{array}{ccc} X' & \xrightarrow{\theta} & Z' \\ \downarrow{\scriptstyle c'} & & \downarrow{\scriptstyle c} \\ X & \xrightarrow{\phi_m} & Z \end{array}$$

be a pullback. Any subobject of Z' is in $\mathcal{S}$, so any partial splitting of θ, when composed with c' factors through m. Therefore, the domains of partial splittings of θ, when composed with c, factor through a. But partial splittings of θ cover Z', so their composites with c cover Z. Thus a is an isomorphism, and therefore, so is m. ∎

It is not the case that all Grothendieck topoi have enough semiprojectives. As we saw, the example of finite $\mathbb{Z}$-sets has no semiprojectives. Similarly, if we make $\mathbb{Z}$ into a topological group by making a subset open if it is a union of non-zero ideals (viewing $\mathbb{Z}$ as a ring) then the topos of continuous $\mathbb{Z}$-sets is a Grothendieck topos with no semiprojectives.

Recall from our classification of semiprojective objects in presheaf topoi that if all objects are semiprojective in $[\mathcal{C}, \underline{\mathcal{S}et}]$ then representable functors are choice, and in fact subobjects of W. We now give two more proofs of this fact, to show which properties of representable objects make them choice in topoi where all objects are semiprojective.

2.20. DEFINITION. *A proper subobject* $X' \rightarrowtail X$ *(a subobject other than the identity) is a* maximum proper subobject *if any subobject of* X *either factors through* X' *or is the identity.*

Any representable functor in a presheaf topos has a maximum proper subobject. In $[\mathcal{C}, \underline{\mathcal{S}et}]$, the functor $\mathcal{C}(X, \_)$ has the maximum proper subfunctor F given by

$$F(Y) = \{f : X \longrightarrow Y | f \text{ is not split mono}\}$$

In a topos in which all objects are semiprojective, the property of having a maximum proper subobject is sufficient to make an object choice, indeed a subobject of W.

2.21. LEMMA. *In a topos* $\mathcal{E}$*, in which all objects are semiprojective, if* X *has a maximum proper subobject* $X' \overset{m}{\rightarrowtail} X$ *then the classifying map* χ_m *of* m *is monic, and therefore* X' *is subterminal, and if* X *is not subterminal then* X' *is its support.*

PROOF. Let $X \xrightarrow{f} Y \overset{g}{\rightarrowtail} \Omega$ be the cover-mono factorization of χ_m. Let $Y' \overset{m'}{\rightarrowtail} Y$ be the image of fm. m is the pullback of m' along f. Y' is not isomorphic to Y, so there is a partial splitting $X \longleftarrow Y_i \rightarrowtail Y$ of f which does not factor through m'. Therefore,

$Y_i \rightarrowtail X$ cannot factor through m, so it must be an isomorphism, making f monic, and thus also an isomorphism. This proves that χ_m must be monic.

Therefore, the pullback of χ_m along $\top$ is monic, making X' subterminal. The support of X is semiprojective, but if X is not subterminal, then all its partial splittings factor through X', as they are not isomorphisms, so X' must be the support of X. ∎

2.22. Corollary. *In a topos $\mathcal{E}$ in which all objects are semiprojective, if X has a maximum proper subobject $X' \rightarrowtail X$, then it is choice.*

Proof. By Lemma 2.21, either X is subterminal, in which case it is choice, or X' is the support of X, in which case slicing over X' reduces to Theorem 2.10. ∎

Another way of seeing that representable functors are choice if all objects are semiprojective is given by the following definition, which will also be useful in the next section.

2.23. Definition. *An object X is* irreducible *if any jointly covering family of subobjects of X contains a jointly covering family of subobjects of the form $X \times U$ for a subterminal object U.*

Morally, we want to say that X is irreducible if, whenever we have a covering family of subobjects of X, one of them is the whole of X. However, this doesn't quite give what we want in topoi that are not presheaf topoi (in presheaf topoi, representable functors are irreducible in this stronger sense) since for example, in $\mathcal{S}\mathbf{h}(\mathbb{R})$, 1 can be covered by a family of subterminal objects none of which is 1. To ensure that subterminal objects are irreducible, we allow objects to be covered by families of subobjects of the form $X \times U_i$.

In a presheaf topos, representable functors are irreducible because if they are covered by a family of subfunctors, one of those subfunctors will contain the identity on the representing object, and will therefore have to be the whole representable functor.

2.24. Lemma. *If all objects in the topos $\mathcal{E}$ are semiprojective, X is irreducible, and all unions exist in* Sub(1)*, then any subobject of X is the union of a subterminal object and a subobject of the form $X \times U$ for some subterminal U. Therefore, given any $X' \overset{m_1}{\underset{m_2}{\rightrightarrows}} X$, there are three subterminals U, V, W that cover 1, such that $m_1 \cap U = m_2 \cap U$, $m_1 \times V \cong 1_{X \times V}$, and $m_2 \times W \cong 1_{X \times W}$.*

Proof. Let $X'' \overset{m}{\rightarrowtail} X$ be a subobject of X. Form the pushout:

$$\begin{array}{ccc} X'' & \longrightarrow & \sigma(X'') \\ {\scriptstyle m}\downarrow & & \downarrow \\ X & \xrightarrow{\;g\;} & Z \end{array}$$

By Lemma 2.9, m and the partial splittings of g jointly cover X. Therefore, we can find a family of subterminal objects U_i whose union is 1, such that for each U_i, either $m \times U_i$ is iso, or one of the partial splittings of g is iso. If we let U be the union of the U_i for which $m \times U_i$ is iso, and V be the union of the U_i for which one of the partial splittings of g is iso, then $m \times U$ is iso, while if $g \times U_i$ has a partial splitting which is iso, then it is iso, so $g \times V$ is iso, as a union of isomorphisms (any two partial inverses must agree on their intersection, since they are both inverse to the restriction of g).

Since $g \times V$ is an isomorphism, so is $X'' \times V \longrightarrow \sigma(X'') \times V$, i.e. $X'' \times V$ is a subterminal object. Therefore, $X'' = X'' \times V \cup X'' \times U$ is a union of a subterminal object and a subobject of the form $X \times U$.

For the second part, we consider the union m of m_1 and m_2. There are subobjects U and U' such that $m \times U = U$, while $m \times U' = 1_{X \times U'}$. Now, $X = (X \times U) \cup m_1 \cup m_2$, so by irreducibility of X, there are subterminal objects V and W such that $X \times V \subseteq m_1$ and $X \times W \subseteq m_2$, and $U \cup V \cup W = 1$. ■

The use of Lemma 2.24 is that if we have two partial splittings of $X \longrightarrow Y$, then these partial splittings must agree on their intersection as subobjects of Y, because any two monomorphisms from this intersection to X must be equal. Therefore, we get a partial splitting from their union to X. This allows us to show:

2.25. Proposition. *If all objects in the topos $\mathcal{E}$ are semiprojective, $\mathcal{E}$ has arbitrary unions of subterminal objects, and X is irreducible in $\mathcal{E}$, then X is choice.*

Proof. By Theorem 2.11, choice objects generate in $\mathcal{E}$. Therefore, X is covered by its choice subobjects (recall that quotients of choice objects are choice, so we can take just the images of morphisms from choice objects to X). As X is irreducible, the cover by its choice subobjects contains a cover by choice objects of the form $X \times U_i$ for subterminal objects U_i.

Given a cover $X \xrightarrow{c} Y$, there is a family of partial splittings $Y \times U_i \overset{m_i}{\rightarrowtail} X \times U_i$. By Lemma 2.24, these partial splittings must agree on their intersections as subobjects of Y. Therefore, by taking their union, we get a total splitting of c. Now, if we have any entire relation $R \rightarrowtail Z \times X$, we can find a splitting of the pushout of $R \rightarrowtail Z \times X \longrightarrow Z$ along $R \rightarrowtail Z \times X \longrightarrow X$. This gives a morphism contained in R. Therefore, X is choice. ■

3. Completely Pointless Objects

In this section, we consider an extreme case where subterminal objects do not generate – the case of objects which do not have any non-zero partial points, which I call completely pointless objects. In [1], F. Borceux observes that the existence of such objects is the only reason why subterminal objects might not generate in a boolean topos.

I believe that in the non-boolean case, the only reasons why subterminal objects might fail to generate are the existence of non-zero completely pointless objects, and the failure of Ω to be well-pointed. In other words, I believe that if Ω is well-pointed, and the only completely pointless object is 0 (in a slightly more general sense, which will be explained shortly) then subterminal objects will generate. Unfortunately, I have not yet been able to prove this.

In this section, we study completely pointless objects, paying particular attention to the connections with generating families. We classify completely pointless objects in presheaf topoi. We use them to show that if subobjects of Ω form a generating family, then Ω is a cogenerator, extending Theorem 2.13. We also prove the conjecture in the previous paragraph (in fact, we prove a stronger result) in the case where the topos is generated by the irreducible objects (Definition 2.23).

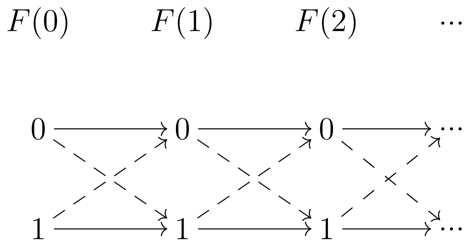
3.1. Definition. *An object X, in a category $\mathcal{C}$, is* completely pointless *if the only partial map from 1 to X is the zero partial map.*

3.2. Examples. (i) In the topos $[G, \underline{\mathcal{S}et}]$, for a group G, the left action of G on itself (the representable functor) is completely pointless, since any morphism from 1 to an object corresponds to a fixed point under the action of G.

(ii) Consider the topos $[\mathcal{C}, \underline{\mathcal{S}et}]$, where $\mathcal{C}$ is the category with objects the natural numbers and two morphisms $f_{m,n}$ and $g_{m,n}$ from m to n whenever $m < n$, and no other non-identity morphisms, satisfying $f_{m,n}f_{l,m} = g_{m,n}g_{l,m} = f_{l,n}$ and $f_{m,n}g_{l,m} = g_{m,n}f_{l,m} = g_{l,n}$.

There is a completely pointless functor F, defined as follows:

$F(n) = 2$ for all $n \in \mathbb{N}$, and F sends $f_{m,n}$ to the identity on 2, and $g_{m,n}$ to the non-identity automorphism. As in the following diagram:



Complete pointlessness is one of the more extreme ways in which an object can fail to be partially well-pointed. However, for studying partial well-pointedness, it is not as useful as we might hope. This is perhaps best seen in the following example.

3.3. Example. Consider the topos $[\mathcal{C}, \underline{\mathcal{S}et}]$, where $\mathcal{C}$ is the category with two objects A and B, and non-identity morphisms $f : A \longrightarrow A$ and $g : A \longrightarrow B$ satisfying $f^2 = 1_A$ and $gf = g$. Let F be the representable functor $\mathcal{C}(A, \_)$, so $F(A) = 2$, $F(B) = 1$, and $F(f)$ is the non-identity automorphism of 2. Then F is restrictively pointed, as the topos of closed sheaves complementary to the subterminal object U given by $U(A) = 0, U(B) = 1$ is equivalent to $[C_2, \underline{\mathcal{S}et}]$, and the associated sheaf of F is the two-object set with the non-trivial action of C_2 (An instance of example 3.2(i)).

Here, F fails to be partially well-pointed because all its partial points factor through U. In general, what we want to study is the following generalization:

3.4. Definition. *An object X, in a topos $\mathcal{E}$, is* restrictively pointed *if either $X = 0$ or there is a subterminal object U, strictly less than the support of X, such that in the topos $\mathcal{S}\mathbf{h}_{c(U)}(\mathcal{E})$, the associated sheaf of X is completely pointless.*

3.5. Remark. Since any geometric inclusion can be factored as a dense inclusion (one whose direct image preserves 0) followed by a closed one, and the inverse image of a dense inclusion clearly reflects complete pointlessness, it would be equivalent to consider arbitrary subtopoi of $\mathcal{E}$ in which the associated sheaf of X is not completely pointless unless it is 0.

If X is a non-zero restrictively pointed object in the topos $\mathcal{E}$, then its support $\sigma(X)$ will not be semiprojective, as all partial splittings of the morphism $X \twoheadrightarrow \sigma(X)$ factor through the associated sheaf of 0. However, subterminal objects can fail to be semiprojective even if there are no restrictively pointed objects in $\mathcal{E}$ (see example2.2(vi)).

Recall that in the previous section, we observed that $[\mathbb{Z}, \underline{\mathcal{S}et}_f]$ has no non-zero semiprojectives. This is explained by the fact that any slice of it contains a completely pointless object – let (A, f) be a finite $\mathbb{Z}$-set i.e. A is a finite set, and f is any endomorphism of A, which we think of as the action of 1 on A. f has finite order, n say, and the map $(2n, s) \xrightarrow{g} (A, f)$ of $\mathbb{Z}$-sets, where s is the function sending k to $k+1$, for $k < 2n-1$, and $2n-1$ to 0, given by $g(0) = a$ for some $a \in A$ is completely pointless. This means that in any slice of $[\mathbb{Z}, \underline{\mathcal{S}et}_f]$, 1 is not semiprojective, so no object can be semiprojective in $[\mathbb{Z}, \underline{\mathcal{S}et}_f]$.

3.6. Proposition. *In a capital topos $\mathcal{E}$, the only restrictively pointed object is 0.*

Proof. The point here is that all subterminal objects are semiprojective (see the proof of Corollary 2.17). Therefore, if X has support U, then X admits a family of partial points whose domains cover U. Therefore, for any subterminal $V < U$, X admits a partial point that does not factor through V. This becomes a non-zero partial point in $\mathcal{S}\mathbf{h}_{c(U)}(\mathcal{E})$, meaning that X is only restrictively pointed if $U = 0$. ∎

3.7. Lemma. *In any topos $\mathcal{E}$, the only restrictively pointed subobject of Ω is 0.*

Proof. Let $X \overset{m}{\rightarrowtail} \Omega$ be a completely pointless subobject. It classifies a subterminal subobject of X, which must be 0. Therefore, m must factor through $1 \xrightarrow{\perp} \Omega$, so X must be subterminal, and hence 0.

Now suppose $X \overset{m}{\rightarrowtail} \Omega$ is a subobject of Ω, and U is a subterminal object for which the associated sheaf of X is completely pointless in $\mathcal{S}\mathbf{h}_{c(U)}(\mathcal{E})$. Let Ω_U be the image of $c(U)$ i.e. the subobject $\{V : \Omega | U \leqslant V\}$. Ω_U is therefore the subobject classifier of $\mathcal{S}\mathbf{h}_{c(U)}(\mathcal{E})$, so the associated sheaf of $X \cap \Omega_U$ is 0 in $\mathcal{S}\mathbf{h}_{c(U)}(\mathcal{E})$. Therefore, the support of $X \cap \Omega_U$ must be a subobject of U. $X \cap \Omega_U$ is a retract of X, so $\sigma(X) \leqslant U$, and X must therefore be 0. ∎

The following alternative characterisation of completely pointless objects is useful for showing their preservation properties.

3.8. Lemma. *An object X in a coherent category is completely pointless if and only if for any morphism $Y \xrightarrow{f} X$ satisfying $f\pi_1 = f\pi_2$, we have $Y = 0$ (where π_1 and π_2 are the projections $Y \times Y \longrightarrow Y$).*

Proof. If: Suppose X admits a morphism f from a subterminal object U. Since U is subterminal, we have $\pi_1 = \pi_2$, so $f\pi_1 = f\pi_2$, forcing $U = 0$, so X is completely pointless.

Only if: Let X be completely pointless, and let $Y \xrightarrow{f} X$ be such that $f\pi_1 = f\pi_2$. Then let $Y \xrightarrow{h} Z \rightarrowtail^{g} X$ be the cover-image factorization of f. Let π_1' and π_2' be the projections from $Z \times Z$ to Z. Now $g\pi_1'(h \times h) = gh\pi_1 = gh\pi_2 = g\pi_2'(h \times h)$, but $h \times h$ is a cover, and in particular is epi, and g is mono, so $\pi_1' = \pi_2'$, which can only occur if Z is a subterminal object. The map from Z to X is therefore a partial map from 1 to X, which must be the zero partial map. Y therefore admits a morphism to 0, so it must be 0. ∎

In particular, this means that cartesian functors that reflect 0 between coherent categories reflect completely pointless objects, since if F is a cartesian functor and $F(X)$ is completely pointless, then given $Y \xrightarrow{f} X$ such that $f\pi_1 = f\pi_2$, $F(f)\pi_1 = F(f)F(\pi_1) = F(f)F(\pi_2) = F(f)\pi_2$, so that $F(Y) = 0$, and hence $Y = 0$.

Therefore, if $\mathcal{C}$ has products, the Yoneda embedding reflects this alternative definition of completely pointless objects. Since any functor in a presheaf category admits a natural transformation from a representable functor, the topos $[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{S}et}]$, where $\mathcal{C}$ has products, cannot contain a non-zero completely pointless object unless it contains a completely pointless representable functor. In this case, $\mathcal{C}$ has an object X such that whenever $Y \xrightarrow{f} X$ in $\mathcal{C}$ satisfies $f\pi_1 = f\pi_2$, Y must be an initial object.

3.9. Lemma. *The direct image of a geometric surjection preserves completely pointless objects.*

Proof. Let $\mathcal{E} \xrightarrow{f} \mathcal{F}$ be a geometric surjection with direct image f_* and inverse image f^*. Now let Z be a completely pointless object in $\mathcal{E}$, and let $Y \xrightarrow{g} f_*(Z)$ satisfy $g\pi_1 = g\pi_2$. Consider the adjunction:

$$\frac{Y \times Y \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} Y \xrightarrow{g} f_*(Z)}{f^*(Y) \times f^*(Y) \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} f^*(Y) \xrightarrow{\bar{g}} Z}$$

Clearly, $\bar{g}\pi_1 = \bar{g}\pi_2$, so $f^*(Y) = 0$. f is a surjection, so f^* reflects isomorphisms. Any morphism in $\mathcal{F}$ with codomain Y is sent to an isomorphism by f^*, and hence is an isomorphism in $\mathcal{F}$. Thus $Y = 0$. ∎

For restrictively pointed objects, consider the following diagram of topoi:

$$\begin{array}{ccc} \mathcal{S}\mathbf{h}_{c(U)}(\mathcal{E}) & \xrightarrow{g} & \mathcal{S}\mathbf{h}_{c(V)}(\mathcal{F}) \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle j} \\ \mathcal{E} & \xrightarrow{f} & \mathcal{F} \end{array}$$

If $i^*(X)$ is completely pointless, then so is $g_*i^*(X) = j^*f_*(X)$, by Lemma 3.9. Therefore, as long as $f_*(U) < \sigma(f_*(X))$, $f_*(X)$ will be restrictively pointed.

3.10. THEOREM. *For a functor F in $[\mathcal{C}, \underline{\mathcal{S}et}]$, the following are equivalent:*

(i) F is completely pointless.

(ii) For any nonempty cosieve $\mathcal{D}$ of $\mathcal{C}$, $\varprojlim F|_{\mathcal{D}} = \emptyset$.

(iii) For any nonzero subterminal object U in $[\mathcal{C}, \underline{\mathcal{S}et}]$, $\varprojlim F^U = \emptyset$

PROOF. (i)⇒(ii): For any category $\mathcal{C}$, there is a geometric surjection $f : [\mathcal{C}, \underline{\mathcal{S}et}] \longrightarrow \underline{\mathcal{S}et}$, given by $f^*(S) = \Delta_S$ (the constantly S functor) and $f_*(F) = \varprojlim F$. The only completely pointless object in $\underline{\mathcal{S}et}$ is $\emptyset$. Thus, if F is completely pointless in $[\mathcal{C}, \underline{\mathcal{S}et}]$ then $f_*(F) = \emptyset$. However, if F is completely pointless in $[\mathcal{C}, \underline{\mathcal{S}et}]$, then for any nonempty cosieve $\mathcal{D}$ of $\mathcal{C}$, $F|_{\mathcal{D}}$ is completely pointless in $[\mathcal{D}, \underline{\mathcal{S}et}]$, since a natural transformation $V \longrightarrow F$ in $[\mathcal{C}, \underline{\mathcal{S}et}]$ is the same thing as a natural transformation $V|_{\mathcal{D}} \longrightarrow F|_{\mathcal{D}}$ in $[\mathcal{D}, \underline{\mathcal{S}et}]$, where V is a subterminal object that is 0 except on $\mathcal{D}$.

(ii)⇒(i): Suppose, for every nonempty cosieve $\mathcal{D}$ of $\mathcal{C}$, $\varprojlim F|_{\mathcal{D}} = \emptyset$. Any partial map from 1 to F corresponds to a natural transformation $1 \longrightarrow F|_{\mathcal{D}}$ in some $[\mathcal{D}, \underline{\mathcal{S}et}]$, where $\mathcal{D}$ is a cosieve of $\mathcal{C}$, which must factor through $\Delta_\emptyset$, as $\emptyset$ is the limit of $F|_{\mathcal{D}}$. for nonempty $\mathcal{D}$, $\Delta_\emptyset$ is strict initial in $[\mathcal{D}, \underline{\mathcal{S}et}]$ so there is no morphism $1 \longrightarrow |Delta_\emptyset$. Hence $\mathcal{D}$ must be empty, so F must be completely pointless.

(ii)⇔(iii): If $\mathcal{D}$ is the subcategory of $\mathcal{C}$ corresponding to the subterminal U, then any natural transformation $X|_{\mathcal{D}} \longrightarrow F|_{\mathcal{D}}$ in $[\mathcal{D}, \underline{\mathcal{S}et}]$ extends to a natural transformation $X \times U \longrightarrow F \times U$ in $[\mathcal{C}, \underline{\mathcal{S}et}]$. This extension then corresponds to a natural transformation $X \longrightarrow (F \times U)^U \cong F^U \times U^U \cong F^U$. Therefore, the right Kan extension of the inclusion of $\mathcal{D}$ into $\mathcal{C}$, being right adjoint to the restriction map, must send F to F^U, so the limit of $F|_{\mathcal{D}}$ must be the limit of F^U. ∎

3.11. LEMMA. *Let $(\mathcal{C}, J)$ be a site. An object F of $\mathcal{S}\mathbf{h}(\mathcal{C}, J)$ is completely pointless if and only if as a functor in $[\mathcal{C}^{\text{op}}, \underline{\mathcal{S}et}]$ it admits no maps from subterminal $(\mathcal{C}, J)$-sheaves, i.e. if and only if for any subterminal J-sheaf U, $\varprojlim F^U = \emptyset$.*

PROOF. This is basically the same argument as in the preceding theorem – the functor $\varprojlim$ sends F to the set of natural transformations from 1 to F, so it will send F^U to $\emptyset$ whenever F admits no map from U. ∎

3.12. LEMMA. *Let $\mathcal{E}$ be a topos, X an object of $\mathcal{E}$, U a subterminal object of $\mathcal{E}$, and f the geometric inclusion of $\mathcal{S}\mathbf{h}_{c(U)}(\mathcal{E})$ into $\mathcal{E}$. The following are equivalent:*

(i) $f^*(X)$ is completely pointless in $\mathcal{S}\mathbf{h}_{c(U)}(\mathcal{E})$.

(ii) For any Y such that $f^*(Y)$ is subterminal, and any subterminal V admitting a morphism to X^Y, $f^*(Y) \times V \leqslant U$.

(iii) For any Y such that $f^*(Y)$ is subterminal and $Y \times U \cong X \times U$, any morphism from a subterminal V to X^Y has $f^*(Y) \times V \leqslant U$.

PROOF. (i)⇒(ii): If $Y \times V$ admits a morphism to X, and $f^*(Y) \leqslant 1$, then $f^*(Y) \times f^*(V)$ admits a morphism to $f^*(X)$, so $f^*(Y) \times f^*(V) = 0$ in $\mathcal{S}\mathbf{h}_{c(U)}(\mathcal{E})$, i.e. $f^*(V) \times f^*(Y) = U$, so $V \times f^*(Y) \leqslant U$.

(ii)⇒(iii) is obvious.

(iii)⇒(i): Let there be a morphism $V \xrightarrow{a} f^*(X)$, where V is a subterminal with $U \leqslant V$. Let Y be the pullback of $X \xrightarrow{\eta_X} f^*(X)$ along a, where η is the unit of the adjunction $f^* \dashv f_*$. $Y \times Y$ is therefore the pullback of the kernel pair $\hat{X}$ of η_X, and the diagonal $\Delta_Y : Y \longrightarrow Y \times Y$ is the pullback of the factorisation $X \rightarrowtail \hat{X}$ of the square:

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ \downarrow{\scriptstyle 1_X} & & \downarrow{\scriptstyle \eta_X} \\ X & \xrightarrow{\eta_X} & f^*(X) \end{array}$$

through the kernel pair. However, $\hat{X} \rightarrowtail X \times X \underset{\eta_X \pi_2}{\overset{\eta_X \pi_1}{\rightrightarrows}} f^*(X)$ is an equalizer, and therefore, so is $f^*(\hat{X}) \rightarrowtail f^*(X) \times f^*(X) \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} f^*(X)$. Therefore, $f^*(\hat{X}) \cong f^*(X)$, and thus $X \rightarrowtail \hat{X}$ is $c(U)$-dense. Thus, Δ_Y is also $c(U)$-dense, so $f^*(Y)$ is subterminal. Since the morphism $Y \longrightarrow V$ is a cover (being the pullback of the cover η_X) $f^*(Y) = V$. $Y \times U$ is the pullback of η_X along $U \rightarrowtail V \rightarrowtail f^*(X)$, which is $X \times U$. Finally, $V \times Y \cong Y$, so there is a morphism from $V \times Y$ to X, and therefore, a corresponding morphism from V to X^Y. Thus, by hypothesis, $f^*(Y) \times V \leqslant U$. But $f^*(Y) = V$, so $V \leqslant U$, and $f^*(V) = U$. Therefore, $f^*(X)$ is completely pointless in $\mathcal{S}\mathbf{h}_{c(U)}(\mathcal{E})$. ■

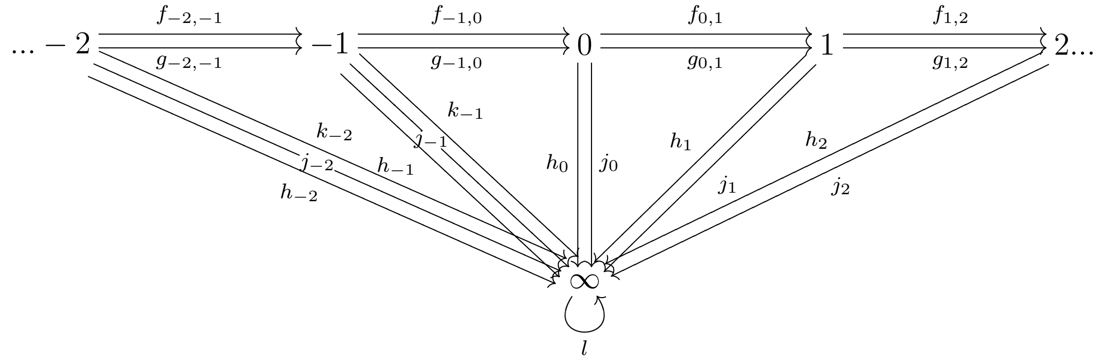
3.13. Corollary. *F is restrictively pointed in $[\mathcal{C}, \underline{\mathcal{S}et}]$ if and only if there is some $U < \sigma(F)$ such that for any subterminal V, and any functor G satisfying $G \times U \cong F \times U$ such that the associated sheaf of G in $\mathcal{S}\mathbf{h}_c(U)([\mathcal{C}, \underline{\mathcal{S}et}])$ is subterminal,* $\varprojlim F^{(G \times V)} = \emptyset$.

Proof. This is obvious from Lemma 3.12 and part of the proof of Theorem 3.10. ■

The question of which objects are restrictively pointed in a general Grothendieck topos is more difficult. It is clear that if a functor is restrictively pointed in a subtopos of $\mathcal{E}$, then it will be restrictively pointed in $\mathcal{E}$, since the composite of two inclusions is an inclusion.

In the other direction, let F be restrictively pointed in $\mathcal{E}$, and a sheaf for a local operator j corresponding to an inclusion f. Let U be a subterminal object for which the associated sheaf of F is completely pointless in $\mathcal{S}\mathbf{h}_{c(U)}(\mathcal{E})$, such that f^*U is less than the support of F. Then the associated sheaf of F in $\mathcal{S}\mathbf{h}_{c(f^*U)}(\mathcal{E})$ will be completely pointless. However, there need not be such a U. For example, if F is the disjoint union of a subterminal object and a completely pointless object that has partial points in any non-trivial $\mathcal{S}\mathbf{h}_{c(U)}(\mathcal{E})$ except when $U \rightarrowtail 1$ is j-dense, then the associated j-sheaf of F will not be restrictively pointed.

3.14. Example. Let $\mathcal{C}$ be a category whose objects are the integers and ∞, with morphisms $f_{m,n}, g_{m,n} : m \longrightarrow n$ for any $m < n$, $h_m, j_m : m \longrightarrow \infty$, for all $m \in \mathbb{Z}$, $k_m : m \longrightarrow \infty$ for all $m < 0$, and $l : \infty \longrightarrow \infty$, as in the following diagram:



satisfying: $l^2 = 1_\infty$, $lh_m = j_m$, $lk_m = k_m$, $f_{m,n}f_{i,m} = f_{i,n}$, $g_{m,n}g_{i,m} = f_{i,n}$, $g_{m,n}f_{i,m} = g_{i,n}$ and $f_{m,n}g_{i,m} = g_{i,n}$. (compare Example 3.2(ii)). Let T be the coverage whose only nontrivial covering is that ∞ is covered by all morphisms from members of $\mathbb{Z}$ to it.

T is subcanonical. Let F be the representable functor $\mathcal{C}(\_, \infty)$. Then F is a sheaf for T. Let U be the subterminal object in $[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{S}et}]$ given by $U(n) = 1$ if $n \in \mathbb{Z}$, and $U(\infty) = 0$. The associated sheaf of F is completely pointless in $\mathcal{S}\mathbf{h}_{c(U)}([\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{S}et}])$, but F is not restrictively pointed in $\mathcal{S}\mathbf{h}(\mathcal{C}, T)$, since the associated T-sheaf of U is 1.

The idea here is that because of the third element of $F(n)$ for n negative, there is a map $U \longrightarrow F$, where U is the subterminal object that is 1 for all negative numbers, and empty for non-negative integers and ∞. Indeed, to make F completely pointless in $\mathcal{S}\mathbf{h}_{c(U)}(\mathcal{E})$, U must contain all integers, and the associated T-sheaf will then be 1.

More on generating families & cogenerators

Now we consider the situation for more general topoi. In particular, the relation between restrictively pointed objects (or the non-existence of such objects) and generating families.

3.15. THEOREM. *In the topos $\mathcal{E}$, if subobjects of Ω form a generating family, then there are no restrictively pointed objects in any slice category $\mathcal{E}/Z$, or more constructively, given a morphism $X \xrightarrow{f} Z$ in $\mathcal{E}$ and a local operator $Z \times \Omega \xrightarrow{j} Z \times \Omega$ in $\mathcal{E}/Z$, where f does not factor through the associated j-sheaf of 0, such that the only partial map from 1 to the associated j-sheaf of X in $\mathcal{S}\mathbf{h}_j(\mathcal{E}/Z)$ is the zero partial map, then X must be 0.*

PROOF. Any morphism with completely pointless codomain must have completely pointless domain, since a partial map from 1 to the domain extends to a partial map to the codomain. Therefore, if $\mathcal{E}$ has a non-zero restrictively pointed object, then some $\mathcal{S}\mathbf{h}_j(\mathcal{E})$ has a completely pointless object, so any generating family for $\mathcal{E}$ contains an object whose associated sheaf in $\mathcal{S}\mathbf{h}_j(\mathcal{E})$ is completely pointless (and non-zero) i.e. a restrictively pointed object. This means that if $\mathcal{E}$ has a restrictively pointed object, then subobjects of Ω cannot form a generating family. Indeed if any slice of $\mathcal{E}$ has a restrictively pointed object, then subobjects of Ω won't be a generating family, because if $X \xrightarrow{f} Z$ is restrictively pointed in $\mathcal{E}/Z$, and in particular the associated sheaf of X in $\mathcal{S}\mathbf{h}_{c(Z')}(\mathcal{E}/Z)$

is non-zero and completely pointless, then for any morphism $Y \xrightarrow{g} X$ where Y is a subobject of Ω, fg is a subobject of $\Omega \times Z \xrightarrow{\pi_2} Z$, which is the subobject classifier in $\mathcal{E}/Z$. Therefore, fg must factor through $Z' \rightarrowtail Z$, so g must factor through its pullback along f, which is not the whole of X. Therefore, subobjects of Ω do not generate. ∎

The converse does not hold. For example, if $\mathcal{C}$ is the category with three objects A, B and C, and morphisms, $f, g : A \longrightarrow B$, $h, i : B \longrightarrow C$ such that $hf = hg$ and $if = ig$, then in $[\mathcal{C}, \underline{\mathcal{S}et}]$, the representable $\mathcal{C}(A, \_)$ is not generated by subobjects of Ω, but there are no restrictively pointed objects in any non-degenerate slice of $[\mathcal{C}, \underline{\mathcal{S}et}]$.

There is still the question of when Ω is a cogenerator. In the previous section we saw that Ω is a cogenerator whenever subobjects of the Higgs object form a generating family. This can be extended.

3.16. Definition. *An object X of the topos $\mathcal{E}$ is* Sub(1)-valued *if its only subobjects are of the form $X \times U$ for U a subterminal object.*

3.17. Lemma. *In the topos $\mathcal{E}$, Ω is a cogenerator if and only if the only* Sub(1)*-valued objects in any slice of $\mathcal{E}$ are subterminal objects.*

Proof. Only If: Let Ω be a cogenerator. Given $A \xrightarrow{f} B$ a Sub(1)-valued object in $\mathcal{E}/B$, any subobject of A in $\mathcal{E}/B$ is the pullback of a subobject of B along f. Any subobject of A in $\mathcal{E}$ is also a subobject in $\mathcal{E}/B$. Therefore, any subobject of A is a pullback of some subobject of B along f, so the classifying map of this subobject factors through f. Therefore, every morphism from A to Ω factors through f, making f monic.

If: Let $A \xrightarrow{f} B$ be a cover through which every morphism from A to Ω factors. Any subobject of A is a pullback of $\top$ along a morphism from A to Ω, which factors through f. Therefore, any subobject of A is a pullback of a subobject of B along f. Therefore, f is Sub(1)-valued in $\mathcal{E}/B$, so it must be a subterminal in this topos, meaning that f is mono, and therefore an iso. ∎

In a presheaf topos, this condition is relatively straightforward to check:

3.18. Lemma. *A natural transformation $F \xrightarrow{\alpha} G$ is* Sub(1)*-valued in $[\mathcal{C}, \underline{\mathcal{S}et}]/G$, if and only if, for any pair $a, b \in F(X)$ for some object $X \in \mathcal{C}$, such that $\alpha_X(a) = \alpha_X(b)$, there are morphisms $X \xrightarrow{f} X$ and $X \xrightarrow{g} X$ in $\mathcal{C}$, such that $F(f)(a) = b$ and $F(g)(b) = a$.*

Proof. For any pair $a, b \in F(X)$, we have morphisms $\mathcal{C}(X, \_) \xrightarrow{f_a} F$ and $\mathcal{C}(X, \_) \xrightarrow{f_b} F$. These morphisms have the same composite with α. Therefore, if α is Sub(1)-valued, then the images of f_a and f_b must be equal. Thus a must be in the image of f_b, so there must be $X \xrightarrow{g} X$ such that $F(g)(b) = a$, and similarly, there must be $X \xrightarrow{f} X$ such that $F(f)(a) = b$.

Conversely, if for any $a, b \in F(X)$ such that $\alpha_X(a) = \alpha_X(b)$ we have $X \xrightarrow{f} X$ and $X \xrightarrow{g} X$ such that $F(f)(a) = b$ and $F(g)(b) = a$, then any subobject of F must either contain both a and b, or neither of them. Therefore, for any $x \in G(X)$, any subobject of F must either contain all preimages of x under α, or none of them. It is therefore the pullback of its image under α, along α. Therefore, α is Sub(1)-valued. ∎

In a general topos, expressing the assertion that Ω is a cogenerator in terms of Sub(1)-valued objects allows us to show:

3.19. THEOREM. *In the topos $\mathcal{E}$, if subobjects of Ω form a generating family, then Ω is a cogenerator.*

PROOF. This can be proved by showing that if the only restrictively pointed object in a topos is 0, then the only Sub(1)-valued objects are the subterminal objects. Let $\mathcal{E}$ be a topos in which 0 is the only restrictively pointed object, and let X be a Sub(1)-valued object in $\mathcal{E}$. Let $V = (\forall x, y : X)(x = y)$. Given a morphism $U \rightarrowtail X$, for $U \geqslant V$ a subterminal object, U must be of the form $X \times W$, for some other subterminal object W. This gives a morphism from U to W, so $U \leqslant W$, so $U \cong X \times U$, and hence, $X \times U \cong X \times X \times U$. Therefore, $X \times X \times U \rightarrowtail X \times X$ factors through the diagonal, making $U \leqslant V$. Thus $U \cong V$, and hence X is completely pointless in $\mathcal{S}\mathbf{h}_{c(V)}(\mathcal{E})$ (X is a $c(V)$-sheaf because $X \times V \cong V$ by definition of V). V is therefore 1, since it must contain the support of X. The diagonal $X \rightarrowtail X \times X$ is therefore an iso, so X must be subterminal. ∎

The converse to Theorem 3.19 does not hold – there are topoi in which Ω is a cogenerator but subobjects of Ω do not form a generating family:

3.20. EXAMPLES. (i) Let $\mathcal{C}$ be the category whose objects are rational numbers and such that between two rationals q and r there is only a nonidentity morphism if $q < r$, in which case there is a morphism $f_{n,q,r}$ for every natural number n, with composition given by $f_{n,r,s} \circ f_{m,q,r} = f_{m+n,q,s}$.

In $[\mathcal{C}, \underline{\mathcal{S}et}]$, let $X(q) = \mathbb{N}$ for every $q \in \mathbb{Q}$, and let $X(f_{n,q,r})(m) = m+n$. X is completely pointless, since a partial map from 1 to X corresponds to an element of some $X(q)$ that is mapped to the same point by $X(f_{n,q,r})$ and $X(f_{m,q,r})$ for every m, n, and r. However, in $[\mathcal{C}, \underline{\mathcal{S}et}]$, any Sub(1)-valued object Y must have that $Y(q) \cong 1$ or $Y(q) \cong \emptyset$ for every $q \in \mathbb{Q}$, as if x and y are distinct in $Y(q)$, then $Z_x(r) = \{z \in Y(r) | (\exists n \in \mathbb{N})(z = Y(f_{n,q,r})(x))\}$ and $Z_y(r) = \{z \in Y(r) | (\exists n \in \mathbb{N})(z = Y(f_{n,q,r})(y))\}$ give distinct subobjects of Y that both have the same support. Therefore, a Sub(1)-valued object in $[\mathcal{C}, \underline{\mathcal{S}et}]$ must be subterminal, so Ω is a cogenerator in $[\mathcal{C}, \underline{\mathcal{S}et}]$, but subobjects of Ω do not form a generating family.

(ii) The topos in Example 3.2(ii).

(iii) The example earlier in this section that shows that the converse to Theorem 3.15 does not hold.

(iv) If we consider $\mathbb{N}$ as a monoid under the operation $\vee$, then in the topos $[\mathbb{N}, \underline{\mathcal{S}et}]$, the only Sub(1)-valued objects are subterminal objects, since the image of the object X under the action of 1 is a subobject of X, so must be the whole of X (in $[\mathbb{N}, \underline{\mathcal{S}et}]$, 1 has only two subobjects) but the action of 1 is idempotent, so must be the identity, i.e. X must be well-pointed and therefore subterminal. However, $\mathbb{N}$, with the action $\vee$, is completely pointless, as it has no fixed points.

We now consider how this relates to the case where subobjects of 1 form a generating family. We will show that if the irreducible objects in a topos form a generating family, then subobjects of 1 form a generating family if and only if Ω is a well-pointed cogenerator.

3.21. Lemma. *If X is irreducible and partially well-pointed, then it is a subterminal object.*

Proof. The partial points of X are jointly covering, so there is a sub-covering family of partial points of the form $X \times U_i$ for subterminal objects U_i. This means that $X \times U_i \cong U_i$. Given a pair of morphisms $Y \underset{g}{\overset{f}{\rightrightarrows}} X$, we know that for every i, $f \times U_i = g \times U_i$, so the equalizer of f and g contains every $Y \times U_i$, and is therefore iso. Thus $f = g$, so X is a subterminal object. ∎

3.22. Lemma. *Any quotient of an irreducible object is irreducible.*

Proof. Let X be irreducible, and let $X \xrightarrow{q} Y$ be a quotient. Now suppose $Y_i \xrightarrow{m_i} Y$ is a jointly covering family of subobjects. The pullbacks $X_i \xrightarrow{n_i} X$ of the m_i along q are also jointly covering, since if they all factor through some $X' \xrightarrow{n} X$, then the m_i will all factor through $\forall_q(n)$. Therefore, there is a jointly covering subcollection of the n_i of the form $X \times U_i \rightarrowtail X$. These can only be the pullback of subobjects of the form $Y \times U_i \rightarrowtail Y$, so there is a covering subcollection of the m_i of this form, meaning that Y is irreducible. ∎

3.23. Lemma. *If Ω is well-pointed, then any irreducible object is* Sub(1)*-valued.*

Proof. Let X be irreducible and $X' \xrightarrow{m} X$ any subobject. If we let $X \xrightarrow{q} Z \xrightarrow{z} \Omega$ be the cover-mono factorization of the classifying map χ_m, then Z is irreducible and partially well-pointed, so it is subterminal. This means that z must be the classifying map of some subterminal object U, and so m must be $X \times U$. ∎

This means that if Ω is well-pointed and irreducible objects generate, then Sub(1)-valued objects generate.

3.24. Theorem. *If in the topos $\mathcal{E}$, irreducible objects form a generating family and Ω is a well-pointed cogenerator, then subobjects of 1 form a generating family.*

Proof. This is immediate from Lemma 3.23 and Lemma 3.17. ∎

This is only of use if we know that there are a number of topoi in which the irreducible objects generate.

3.25. Proposition. *In a presheaf topos, irreducible objects generate.*

PROOF. Representable objects are irreducible, since if $\mathcal{C}(\_, X)$ is covered by a family of subobjects, then one of these subobjects must contain 1_X, and must therefore be the whole of $\mathcal{C}(\_, X)$. It is well-known that representable objects form a generating family. ■

However, in a general Grothendieck topos, irreducible objects do not have to generate.

3.26. EXAMPLE. Let $\mathcal{C}$ be the category whose objects are the natural numbers, with 2 morphisms from $n+1$ to n for every n, satisfying no equations (so there are 2^k morphisms from $n+k$ to n).

Define a coverage T on $\mathcal{C}$ by defining a family of morphisms $\mathcal{M}$ with common codomain n, such that if $f \in \mathcal{M}$ then $fg \in \mathcal{M}$ for any g such that the composite fg exists, to be covering if every morphism contains a morphism in $\mathcal{M}$. i.e. for every f, there is a g such that $fg \in \mathcal{M}$.

Now in $\mathcal{S}\mathbf{h}(\mathcal{C}, T)$, the only subterminal objects are 1 and 0. Furthermore, in $[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{S}et}]$, two copies of the representable $\mathcal{C}(\_, n+1)$ are T-dense in $\mathcal{C}(\_, n)$, so in $\mathcal{S}\mathbf{h}(\mathcal{C}, T)$, the associated sheaves of representable functors have no irreducible subobjects. Therefore, irreducible objects do not generate.

3.27. PROPOSITION. *Ω is well-pointed in $\mathcal{S}\mathbf{h}(\mathcal{C}, T)$.*

PROOF. We will show that the points of Ω are T-dense in $[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{S}et}]$. Let $\mathcal{S}$ be an element of $\Omega(0)$ in $[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{S}et}]$. It corresponds to a collection of morphisms with codomain 0 in $\mathcal{C}$, with the property that if $f \in \mathcal{S}$, then for any g such that the composite fg exists, $fg \in \mathcal{S}$. A morphism $n \xrightarrow{f} 0$ sends $\mathcal{S}$ to $\{g | fg \in \mathcal{S}\}$. Therefore, if $f \in \mathcal{S}$, then $f(\mathcal{S})$ will give a partial point of Ω. Similarly, if there is no $fg \in \mathcal{S}$, then $f(\mathcal{S})$ will be the empty collection, and so will give a partial point of Ω.

Thus, we just need to show that $\mathcal{S} \cup \{f | (\nexists g)(fg \in \mathcal{S})\}$ is T-dense. Given $n \xrightarrow{f} 0$, either there is an element of $\mathcal{S}$ factoring through f, or $f \in \{h | (\nexists g)(hg \in \mathcal{S})\}$. Therefore, the partial points of Ω are T-dense in $[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{S}et}]$, so Ω is well-pointed in $\mathcal{S}\mathbf{h}(\mathcal{C}, T)$. ■

However, Ω is not a cogenerator in $\mathcal{S}\mathbf{h}(\mathcal{C}, T)$, since if F is the functor in $[\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{S}et}]$ with $F(n) = 2$ for every n, and if the two morphisms from $n+1$ to n are a_n and b_n, then $F(a_n)(0) = F(b_n)(1) = 0$ and $F(b_n)(0) = F(a_n)(1) = 1$ for every n, then any T-closed subobject of F is either 0 or the whole of F. Therefore, the associated sheaf of F is Sub(1)-valued in $\mathcal{S}\mathbf{h}(\mathcal{C}, T)$, so Ω is not a cogenerator.

I have not yet been able to determine whether Ω being a well-pointed cogenerator implies that subterminal objects generate, even in a Grothendieck topos.

References

[1] F. Borceux. When is Ω a cogenerator in a topos? *Cah. Top. Geo. Diff.*, 16:1–5, 1975.

[2] P. Freyd. Choice and well-ordering. *Ann. Pure Appl. logic*, 35:149–166, 1987.

[3] P. J. Freyd and A. Scedrov. *Categories, allegories.* North-Holland, 1990.

[4] P. T. Johnstone. *Sketches of an Elephant: a Topos Theory Compendium*, volume 1 and 2. Clarendon Press, 2002.

[5] F. W. Lawvere. More on graphic toposes. *Cah. Top. Geo. Diff.*, 32:5–10, 1991.

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