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CLOSEDNESS PROPERTIES OF INTERNAL RELATIONS II: BOURN LOCALIZATION

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Abstract. We say that a class $\mathbb{D}$ of categories is the *Bourn localization* of a class $\mathbb{C}$ of categories, and we write $\mathbb{D} = \mathrm{Loc}(\mathbb{C})$, if $\mathbb{D}$ is the class of all (finitely complete) categories $\mathcal{D}$ such that for each object A in $\mathcal{D}$, $\mathrm{Pt}(\mathcal{D} \downarrow A) \in \mathbb{C}$, where $\mathrm{Pt}(\mathcal{D} \downarrow A)$ denotes the category of all pointed objects in the comma-category $(\mathcal{D} \downarrow A)$. As D. Bourn showed, if we take $\mathbb{D}$ to be the class of Mal'tsev categories in the sense of A. Carboni, J. Lambek, and M. C. Pedicchio, and $\mathbb{C}$ to be the class of unital categories in the sense of D. Bourn, which generalize pointed Jónsson-Tarski varieties, then $\mathbb{D} = \mathrm{Loc}(\mathbb{C})$. A similar result was obtained by the author: if $\mathbb{D}$ is as above and $\mathbb{C}$ is the class of subtractive categories, which generalize pointed subtractive varieties in the sense of A. Ursini, then $\mathbb{D} = \mathrm{Loc}(\mathbb{C})$. In the present paper we extend these results to abstract classes of categories obtained from classes of varieties. We also show that the Bourn localization of the union of the classes of unital and subtractive categories is still the class of Mal'tsev categories.

Introduction

In this paper's predecessor [11], I gave a general method of finding categorical conditions on varieties of universal algebras, equivalent to term conditions of a particular kind. Each categorical condition arising in this way is determined by an extended term matrix

$$M = \left(\begin{array}{ccc|c} t_{11} & \cdots & t_{1m} & u_1 \\ \vdots & & \vdots & \vdots \\ t_{n1} & \cdots & t_{nm} & u_n \end{array} \right).$$

Here, the terms t_{ij}, u_i belong to some fixed algebraic theory $\mathcal{T}$, which, in the examples, is either the algebraic theory of the variety of sets, $\mathcal{T} = \mathrm{Th}[\textit{sets}]$, or that of the variety of pointed sets, $\mathcal{T} = \mathrm{Th}[\textit{pointed sets}]$. The class of categories determined by the categorical condition corresponding to M (which says that every internal relation is M*-closed*, see Section 1) will be also called the class of categories determined by M.

In [11] it was shown that

- the class of Mal'tsev categories in the sense of A. Carboni, M. C. Pedicchio, and N. Pirovano [7] (see also [6] and [5]), which generalize Mal'tsev varieties of universal

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algebras [13], is the same as the class of categories determined by the extended matrix

$$\left(\begin{array}{ccc|c} x & y & y & x \\ x & x & y & y \end{array}\right) \tag{1}$$

of terms in Th[*sets*];

- the class of unital categories in the sense of D. Bourn [2], which generalize pointed Jónsson-Tarski varieties [12], is the same as the class of categories determined by the matrix

$$\left(\begin{array}{cc|c} x & 0 & x \\ 0 & x & x \end{array}\right)$$

of terms in Th[*pointed sets*], where 0 denotes the unique constant of Th[*pointed sets*];

- the class of strongly unital categories in the sense of D. Bourn [2] is the same as the class of categories determined by the matrix

$$\left(\begin{array}{ccc|c} x & 0 & 0 & x \\ x & x & y & y \end{array}\right)$$

of terms in Th[*pointed sets*];

- the class of subtractive categories in the sense of [9], which generalize pointed subtractive varieties in the sense of A. Ursini [14], is the same as the class of categories determined by the matrix

$$\left(\begin{array}{cc|c} x & 0 & x \\ x & x & 0 \end{array}\right)$$

of terms in Th[*pointed sets*].

Each of the above matrices naturally correspond to the systems of term equations that determine the corresponding classes of varieties of universal algebras. For instance, the extended matrix (1) corresponds to the system

$$\begin{cases} p(x,y,y) = x, \\ p(x,x,y) = y \end{cases}$$

of Mal'tsev equations [13], that determines the class of Mal'tsev varieties (recall that a Mal'tsev variety is a variety whose algebraic theory contains a ternary term p for which the above equations are satisfied).

In [2] D. Bourn gave a characterization of Mal'tsev categories via unital and strongly unital categories, using *fibration of points*. Let $\mathcal{C}$ be a category and let X be an object in $\mathcal{C}$. The X-fibre of the fibration of points over $\mathcal{C}$ is the category $\mathrm{Pt}(\mathcal{C} \downarrow X)$ of pointed objects of the comma category $(\mathcal{C} \downarrow X)$. It was shown in [2] that a finitely complete category $\mathcal{C}$ is a Mal'tsev category if and only if each fibre of the fibration of points over $\mathcal{C}$ is unital, and it was shown also that this is the case precisely when each fibre of the fibration of points is strongly unital (for other similar results due to D. Bourn, see [3] and [4]; see also [1] and the references there). In [9] a similar result was obtained with subtractive categories in the place of unital ones: a finitely complete category $\mathcal{C}$ is a Mal'tsev category if and only if each fibre of the fibration of points over $\mathcal{C}$ is subtractive. In the present paper, which is based on the last chapter of the author's M.Sc. Thesis [8] (see also [10]), we will unify these three results by considering abstract classes of categories determined by term matrices in the place of the classes of Mal'tsev, unital, strongly unital, and subtractive categories (see Theorem 3.9).

Convention. Throughout the paper by a category we will always mean a category having finite limits.

1. Preliminaries

In this section we recall some material from [9], and we also give some new results, which will be used in the following sections (the proofs are given only for the new results).

Let $A_1, ..., A_n$ be arbitrary sets. We will consider extended matrices

$$M = \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1m} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nm} & b_n \end{array}\right),$$

where $a_{i1}, ..., a_{im}, b_i \in A_i$ for each $i \in \{1, ..., n\}$; we then say that M is an $n \times (m+1)$ extended matrix with columns from $A_1 \times ... \times A_n$. Here we assume $n \geq 1$ and $m \geq 0$. If $m = 0$ then M becomes

$$\left(\begin{array}{c|c} & b_1 \\ & \vdots \\ & b_n \end{array}\right).$$

The columns of a's will be called *left columns* of M, and the column of b's will be called *right column* of M.

Let $\mathcal{T}$ be an arbitrary algebraic theory and let

$$M = \left(\begin{array}{ccc|c} t_{11} & \cdots & t_{1m} & u_1 \\ \vdots & & \vdots & \vdots \\ t_{n1} & \cdots & t_{nm} & u_n \end{array}\right)$$

be an extended matrix of terms of $\mathcal{T}$. We identify a term of $\mathcal{T}$ with the corresponding element of the free $\mathcal{T}$-algebra $\mathrm{Fr}_{\mathcal{T}}\mathcal{X}$ over the alphabet $\mathcal{X}$ of $\mathcal{T}$. We also assume to be given a fixed sequence $x_1, ..., x_k$ of distinct variables, such that each term in M depends only on those variables that belong to this sequence; this allows to regard each term w in M as a k-ary term $w = w(x_1, ..., x_k)$.

1.1. DEFINITION. *Let $A_1, ..., A_n$ be $\mathcal{T}$-algebras and let M' be an $n \times (m+1)$ extended matrix with columns from $A_1 \times ... \times A_n$.*

(a) M' is said to be a row-wise interpretation of M if

$$M' = \left(\begin{array}{ccc|c} t_{11}(c_{11}, ..., c_{1k}) & \cdots & t_{1m}(c_{11}, ..., c_{1k}) & u_1(c_{11}, ..., c_{1k}) \\ \vdots & & \vdots & \vdots \\ t_{n1}(c_{n1}, ..., c_{nk}) & \cdots & t_{nm}(c_{n1}, ..., c_{nk}) & u_n(c_{n1}, ..., c_{nk}) \end{array} \right)$$

for some $c_{11}, ..., c_{1k} \in A_1, ..., c_{n1}, ..., c_{nk} \in A_n$. Equivalently, M' is a row-wise interpretation of M if there there exist $\mathcal{T}$-algebra homomorphisms

$$f_1 : \mathrm{Fr}_{\mathcal{T}}\mathcal{X} \to A_1, ..., f_n : \mathrm{Fr}_{\mathcal{T}}\mathcal{X} \to A_n$$

such that

$$M' = \left(\begin{array}{ccc|c} f_1(t_{11}) & \cdots & f_1(t_{1m}) & f_1(u_1) \\ \vdots & & \vdots & \vdots \\ f_n(t_{n1}) & \cdots & f_n(t_{nm}) & f_n(u_n) \end{array} \right).$$

(b) Suppose $A_1 = ... = A_n = A$. Then, M' is said to be a regular interpretation of M if

$$M' = \left(\begin{array}{ccc|c} t_{11}(c_1, ..., c_k) & \cdots & t_{1m}(c_1, ..., c_k) & u_1(c_1, ..., c_k) \\ \vdots & & \vdots & \vdots \\ t_{n1}(c_1, ..., c_k) & \cdots & t_{nm}(c_1, ..., c_k) & u_n(c_1, ..., c_k) \end{array} \right)$$

for some $c_1, ..., c_k \in A$. Equivalently, M' is said to be a regular interpretation of M if there exists a $\mathcal{T}$-algebra homomorphism $f : \mathrm{Fr}_{\mathcal{T}}\mathcal{X} \to A$ such that

$$M' = \left(\begin{array}{ccc|c} f(t_{11}) & \cdots & f(t_{1m}) & f(u_1) \\ \vdots & & \vdots & \vdots \\ f(t_{n1}) & \cdots & f(t_{nm}) & f(u_n) \end{array} \right).$$

(c) Two $n \times (m+1)$ extended term matrices M and M' are said to be row-wise similar to each other if each one of them is, regarded as an extended matrix of elements of $\mathrm{Fr}_{\mathcal{T}}\mathcal{X}$, a row-wise interpretation of the other.

Let $R \longrightarrow A_1 \times ... \times A_n$ be an internal relation in a category $\mathcal{C}$, and X an object in $\mathcal{C}$. Consider an extended matrix

$$M' = \left(\begin{array}{ccc|c} f_{11} & \cdots & f_{1m} & g_1 \\ \vdots & & \vdots & \vdots \\ f_{n1} & \cdots & f_{nm} & g_n \end{array} \right)$$

whose each i-th row consists of morphisms $f_{i1}, ..., f_{im}, g_i : X \longrightarrow A_i$ of $\mathcal{C}$. We will say that the relation R is *compatible* with M' if whenever the morphisms $(f_{1j}, ..., f_{nj}) : X \longrightarrow A_1 \times ... \times A_n$ factor through r, so does the morphism $(g_1, ..., g_n) : X \longrightarrow A_1 \times ... \times A_n$.

1.2. Definition. *Let $\mathcal{C}$ be a category, and let $A_1, ..., A_n$ be objects in $\mathcal{C}$, each A_i equipped with an internal $\mathcal{T}$-algebra structure. Then the $\mathcal{T}$-algebra structure on A_i induces a $\mathcal{T}$-algebra structure on* $\hom(X, A_i)$*, for every object X in $\mathcal{C}$. Let $R \longrightarrow A_1 \times ... \times A_n$ be an internal relation in $\mathcal{C}$.*

- *(a) The relation R is said to be strictly M-closed, if for every object X, R is compatible with every interpretation M' of M, whose each i-th row consists of morphisms from the $\mathcal{T}$-algebra* $\hom(X, A_i)$.
- *(b) Suppose $A_1 = ... = A_n = A$, and suppose the corresponding $\mathcal{T}$-algebra structures also coincide. Then, the relation R is said to be M-closed, if for every object X, R is compatible with every interpretation M' of M, consisting of morphisms from the $\mathcal{T}$-algebra* $\hom(X, A)$.

Observe:

- If an extended term matrix M' is an interpretation of M, then M-closedness always implies M'-closedness.
- If an extended term matrix M' is a row-wise interpretation of M, then strict M-closedness always implies strict M'-closedness. In particular, this gives that if M and M' are row-wise similar to each other, then strict M-closedness is the same as strict M'-closedness.
- A strictly M-closed relation $R \longrightarrow A^n$ is always M-closed.

Let $\mathcal{C}$ be a category and let $r = (r_1, ..., r_n) : R \longrightarrow A^n$ be an n-ary relation in $\mathcal{C}$, where A is an object of $\mathcal{C}$ equipped with an internal $\mathcal{T}$-algebra structure. Consider the relation

$$r^M = (r_1^M, ..., r_k^M) : R^M \longrightarrow A^k$$

where R^M is the object obtained as the limit of the diagram

$$\begin{array}{ccccccc}
A^k & = & A^k & = & \cdots & = & A^k \\
\Big\downarrow \begin{pmatrix} t_{11} \\ \vdots \\ t_{n1} \end{pmatrix} & & \Big\downarrow \begin{pmatrix} t_{12} \\ \vdots \\ t_{n2} \end{pmatrix} & & \cdots & & \Big\downarrow \begin{pmatrix} t_{1m} \\ \vdots \\ t_{nm} \end{pmatrix} \\
A^n & & A^n & & \cdots & & A^n \\
\Big\uparrow r & & \Big\uparrow r & & \cdots & & \Big\uparrow r \\
R & & R & & \cdots & & R
\end{array}$$

(here t_{ij} denotes the operation $t_{ij} : A^k \longrightarrow A$ of the $\mathcal{T}$-algebra structure of A, corresponding to the term t_{ij}) and r^M is the limit projection $R^M \longrightarrow A^k$.

1.3. Lemma. *The following conditions are equivalent:*

(a) The relation R is M-closed.

(b) The morphism

$$(u_1(r_1^M, ..., r_k^M), ..., u_n(r_1^M, ..., r_k^M)) : R^M \longrightarrow A^n$$

factors through r.

(c) The relation R is compatible with

$$\left(\begin{array}{ccc|c} t_{11}(r_1^M, ..., r_k^M) & \cdots & t_{1m}(r_1^M, ..., r_k^M) & u_1(r_1^M, ..., r_k^M) \\ \vdots & & \vdots & \vdots \\ t_{n1}(r_1^M, ..., r_k^M) & \cdots & t_{nm}(r_1^M, ..., r_k^M) & u_n(r_1^M, ..., r_k^M) \end{array}\right).$$

Proof. This Lemma is the same as Lemma 2.1 of [11], but with the additional equivalent condition (c) added to the conditions (a) and (b) in Lemma 2.1 of [11]. The equivalence of (b) and (c) follows immediately from the fact that the morphisms $R^M \longrightarrow A^n$, induced by the left columns of the matrix in (c), factor through r. ■

1.4. Definition. *A category $\mathcal{C}$ is said to have M-closed relations (or, it is said to be a category with M-closed relations) if any internal relation $R \longrightarrow A^n$ in $\mathcal{C}$ is M-closed with respect to any internal $\mathcal{T}$-algebra structure on A.*

1.5. Theorem. *For any category $\mathcal{C}$, the following conditions are equivalent:*

(a) Every relation $R \longrightarrow A_1 \times ... \times A_n$ in $\mathcal{C}$ is strictly M-closed, with respect to any choice of a $\mathcal{T}$-algebra structure on each A_i.

(b) $\mathcal{C}$ has M-closed relations.

Proof. (a)⇒(b) is trivial. Let $\mathcal{C}$ be a category and let $R \longrightarrow A_1 \times ... \times A_n$ be an internal relation in $\mathcal{C}$, where $A_1, ..., A_n$ are objects in $\mathcal{C}$, each equipped with an internal $\mathcal{T}$-algebra structure. Construct from R a relation $R' \longrightarrow (A_1, ..., A_n)^n$ via the pullback

$$\begin{array}{ccc} R' & \longrightarrow & R \\ \downarrow & & \downarrow \\ (A_1 \times ... \times A_n)^n & \xrightarrow[\pi_1 \times ... \times \pi_n]{} & A_1 \times ... \times A_n \end{array}$$

where each π_i denotes the i-th product projection $\pi_i : A_1 \times ... \times A_n \longrightarrow A_i$. If $\mathcal{C}$ has M-closed relations, then the relation R' is M-closed, with respect to any $\mathcal{T}$-algebra structure of $A_1 \times ... \times A_n$, and in particular, with respect to its product $\mathcal{T}$-algebra structure, obtained from the $\mathcal{T}$-algebra structures of $A_1, ..., A_n$. It is easy to show that this implies that R is strictly M-closed (see [11]) (with respect to the given $\mathcal{T}$-algebra structures of $A_1, ..., A_n$). This shows (b)⇒(a), concluding the proof. ■

1.6. Corollary. *If an extended term matrix M' is a row-wise interpretation of an extended term matrix M, then any category with M-closed relations has M'-closed relations. If M and M' are row-wise similar to each other, then categories with M-closed relations are the same as categories with M'-closed relations.*

1.7. Proposition. *Suppose M' is a (nonempty) extended matrix of terms obtained from an extended term matrix M by applying to it any one of the following operations:*

(a) permutation of left columns,

(b) addition of a left column,

(c) deletion of one of two identical left columns,

(d) permutation of rows,

(e) deletion of a row,

(f) addition of a row that is identical to an existing row.

Then, any category with M-closed relations has M'-closed relations.

Proof. The cases of (a),(b),(c) are obvious. Suppose M' is obtained from M by permuting the rows of M. That is, suppose there is a permutation $\varphi : \{1, ..., n\} \longrightarrow \{1, ..., n\}$, such that each i-th row of M is identical to the $\varphi(i)$-th row of M'. Consider a relation $(r_1, ..., r_n) : R \longrightarrow A^n$ in a category $\mathcal{C}$, where A is an object of $\mathcal{C}$ equipped with a $\mathcal{T}$-algebra structure. It is obvious that R is M'-closed if and only if the relation $(r_{\varphi^{-1}(1)}, ..., r_{\varphi^{-1}(n)}) : R \longrightarrow A^n$ is M-closed. Hence, if $\mathcal{C}$ has M-closed relations, then it has M'-closed relations.

Now suppose M' is obtained from M by deleting it's i-th row. Without loss of generality we can assume $i = n$. Consider a relation $(r_1, ..., r_{n-1}) : R \longrightarrow A^{n-1}$ in a category $\mathcal{C}$, where, as above, A is an object of $\mathcal{C}$ equipped with a $\mathcal{T}$-algebra structure. Let T denote the terminal object of $\mathcal{C}$ and let t denote the unique morphism $t : R \longrightarrow T$. It is easy to see that the relation R is strictly M'-closed if and only if the relation $(r_1, ..., r_{n-1}, t) : R \longrightarrow A^{n-1} \times T$ is strictly M-closed, with respect to the given $\mathcal{T}$-algebra structure of A and the unique $\mathcal{T}$-algebra structure of T. This shows that if $\mathcal{C}$ has M-closed relations then $\mathcal{C}$ has M'-closed relations.

Finally, suppose M' is obtained from M by adding to M a new row that is a copy of an existing row. Without loss of generality we can assume that the new row, and its copy, are the last two rows of M. Consider an $(n+1)$-ary relation $(r_1, ..., r_{n+1}) : R \longrightarrow A^{n+1}$ in a category $\mathcal{C}$, where A is again an object equipped with a $\mathcal{T}$-algebra structure, and form a new n-ary relation $(r_1, ..., r_{n-1}, (r_n, r_{n+1})) : R \longrightarrow A^{n-1} \times (A \times A)$. It is easy to see that the first relation is strictly M'-closed if and only if the second relation is strictly M-closed with respect to the given $\mathcal{T}$-algebra structure of A and the induced product $\mathcal{T}$-algebra structure of $A \times A$. We obtain, once again, that if $\mathcal{C}$ has M-closed relations then $\mathcal{C}$ has M'-closed relations. ■

2. Matrices of variables, pointed matrices of variables, and admissible pairs (M, x)

Let $\mathcal{T}$ and M be as in the previous section. In the case when $\mathcal{T}$ is the algebraic theory of sets, M becomes an *extended matrix of variables*. If $\mathcal{T}$ is the algebraic theory of pointed sets, then the entries of M are either variables or a fixed nullary term 0. In this case let us say that M is a *pointed (extended) matrix of variables*.

Let M be any $n \times (m+1)$ extended matrix of variables. An $n \times (m+1)$ extended matrix M' of elements of an arbitrary set is a regular interpretation of M if and only if whenever two entries in M coincide, the corresponding entries in M' also coincide. An $n \times (m+1)$ extended matrix M', whose each i-th row consists of elements of an arbitrary set, is a row-wise interpretation of M if and only if whenever in each row of M two entries coincide, the corresponding entries of the corresponding row of M' also coincide.

Let M be any $n \times (m+1)$ pointed extended matrix of variables. An an $n \times (m+1)$ extended matrix M' of elements of a pointed set A is a regular interpretation of M if and only if whenever two entries in M coincide, the corresponding entries in M' also coincide, and an entry in M is 0 implies that the corresponding entry in M' is the base point of A. An $n \times (m+1)$ extended matrix M', whose each i-th row consists of elements of a pointed set A_i, is a row-wise interpretation of M if and only if whenever in each row of M two entries coincide, the corresponding entries of the corresponding row of M' also coincide, and an entry of some i-th row of M is 0 implies that the corresponding entry of the i-th row of M' is the base point of A_i.

For an arbitrary extended matrix M of variables, and for an arbitrary variable x, the pair (M, x) will denote the pointed matrix of variables obtained from M by writing 0 in the place of each instance of x encountered in M, i.e. (M, x) is the pointed matrix obtained from M by *substituting* 0 in x. Note that if x is not an entry of M, then the pointed matrix (M, x) is equal to M, however, while we regard M as a matrix of terms in the algebraic theory of sets, we regard (M, x) as a matrix of terms in the algebraic theory of pointed sets.

For an extended matrix M of variables (pointed or not) by $\mathbf{Cat}_M$ we denote the class of all categories with M-closed relations, and by $\mathbf{Cat}^*_M$ the class of all pointed categories with M-closed relations.

2.1. Definition. *Let M be an arbitrary extended matrix of variables and let x be an arbitrary variable. The pair (M, x) is said to be admissible if there is a left column*

$$\begin{pmatrix} t_{1j} \\ \vdots \\ t_{nj} \end{pmatrix}$$

in M such that for each $i \in \{1, ..., n\}$, if the i-th row of M contains x then $t_{ij} = x$.

Note that if there exists a variable x such that (M, x) is admissible, then M contains at least one left column.

The following proposition is a special case of (the "internal version" of) Proposition 1.10 in [11].

2.2. Proposition. *Let*

$$(r_1, ..., r_n) : R \longrightarrow A_1 \times ... \times A_n$$

be an arbitrary internal relation in an arbitrary category $\mathcal{C}$. Consider the relation $S \longrightarrow R^n$ obtained from R via the pullback

$$\begin{array}{ccc} S & \longrightarrow & R \\ \downarrow & & \downarrow{\scriptstyle (r_1,...,r_n)} \\ R^n & \xrightarrow[r_1 \times ... \times r_n]{} & A_1 \times ... \times A_n \end{array}$$

For any extended matrix M of variables such that for any variable x the pair (M, x) is admissible, the following conditions are equivalent:

(a) R is strictly M-closed.

(b) S is strictly M-closed.

(c) S is M-closed.

2.3. Remark. Note that the relation $S \longrightarrow R^n$ constructed in Proposition 2.2 is a reflexive relation, i.e. it contains the diagonal $R \longrightarrow R^n$. Hence, from the above proposition we can deduce: for an extended matrix M of variables as in Proposition 2.2, a category $\mathcal{C}$ has M-closed relations if and only if every reflexive relation $R \longrightarrow A^n$ in $\mathcal{C}$ has M-closed relations.

2.4. Proposition. *Let M be an extended matrix of variables containing a row*

$$\left(\begin{array}{ccc|c} t_{i1} & \cdots & t_{im} & u_i \end{array} \right)$$

such that $u_i \notin \{t_{i1}, ..., t_{im}\}$, and let x be any variable such that $x \neq u_i$. Then:

(a) If $m \neq 0$ then $\mathbf{Cat}_M$ is the class of all preorders and $\mathbf{Cat}^*_{(M,x)}$ is the class of all codiscrete categories.

(b) If $m = 0$ then both $\mathbf{Cat}_M$ and $\mathbf{Cat}^*_{(M,x)}$ are equal to the class of all codiscrete categories.

PROOF. Let $\mathcal{C}$ be a category with M-closed relations. Then, by Proposition 1.7, $\mathcal{C}$ has M'-closed relations, where

$$M' = \left(\begin{array}{ccc|c} t_{i1} & \cdots & t_{im} & u_i \end{array} \right).$$

Suppose $m \neq 0$. To show that $\mathcal{C}$ is a preorder, consider any two parallel morphisms $f, g : X \longrightarrow A$ in $\mathcal{C}$. The diagonal $\Delta : A \longrightarrow A \times A$, regarded as a unary relation on $A \times A$, is compatible with $((f,f) \cdots (f,f)|(f,g))$, where (f,f) and (f,g) denote the induced morphisms $X \longrightarrow A \times A$. This gives $f = g$, which shows that $\mathcal{C}$ is a preorder. Conversely, it is easy to see that any preorder has M-closed relations. If a pointed category $\mathcal{D}$ has (M,x)-closed relations, then $\mathcal{D}$ has M''-closed relations, where $M'' = (0|u_i)$. A similar argument as above shows that any morphism $f : X \longrightarrow A$ in $\mathcal{D}$ coincides with the zero morphism $0 : X \longrightarrow A$. Hence $\mathcal{D}$ is codiscrete. Conversely, any codiscrete category has (M,x)-closed relations. This proves (a).

Now suppose $m = 0$. Then each diagonal $\Delta : A \longrightarrow A \times A$ in $\mathcal{C}$ is compatible with $(|1_{A\times A})$, which gives that Δ is an isomorphism. This implies that $\mathcal{C}$ is a preorder. Since $\mathcal{C}$ is also connected (as it has all binary products), to show that $\mathcal{C}$ is codiscrete, it remains to show that for any morphism $f : X \longrightarrow A$ there exists a morphism backwards $A \longrightarrow X$. Indeed, since $\mathcal{C}$ is a preorder, f is a monomorphism; now, as a unary relation on A, f is compatible with $(|1_A)$, which implies the existence of a morphism $A \longrightarrow X$. Thus, we showed that if a category has M-closed relations then it is codiscrete. Conversely, any codiscrete category has M-closed relations. Next we show that $\mathbf{Cat}^*_{(M,x)}$ is the class of all codiscrete categories. Any codiscrete category is pointed and has (M,x)-closed relations. To show the converse, consider a pointed category $\mathcal{D}$ with (M,x)-closed relations. As in the non-pointed case, we have: by Proposition 1.7, any unary relation in $\mathcal{D}$ is closed with respect to $(|u_i)$, which implies that each diagonal $\Delta : A \longrightarrow A \times A$ in $\mathcal{D}$ is compatible with $(|1_{A\times A})$, and hence $\mathcal{D}$ is a preorder. Since $\mathcal{D}$ is also pointed, $\mathcal{D}$ is automatically codiscrete. ■

For a category $\mathcal{C}$ and an object A in $\mathcal{C}$, by $\mathrm{Pt}(\mathcal{C} \downarrow A)$ we denote the category of pointed objects of the comma category $(\mathcal{C} \downarrow A)$. The objects of $\mathrm{Pt}(\mathcal{C} \downarrow A)$ can be represented as triples (X, g, h), where X is an object in $\mathcal{C}$ and g, h are morphisms

$$g : X \longrightarrow A, \quad h : A \longrightarrow X \quad \text{with} \quad gh = 1_A.$$

A morphism $f : (X, g, h) \longrightarrow (X', g', h')$ in $\mathrm{Pt}(\mathcal{C} \downarrow A)$ is a morphism $f : X \longrightarrow X'$ in $\mathcal{C}$ such that $fh = h'$ and $g'f = g$. The triple $(A, 1_A, 1_A)$ is a zero object in $\mathrm{Pt}(\mathcal{C} \downarrow A)$, and the zero morphism 0 from (X, g, h) to (X', g', h') is the composite $h'g$.

2.5. PROPOSITION. *Let $\mathcal{C}$ be an arbitrary category and let A be an arbitrary object in $\mathcal{C}$. For any extended matrix M of variables and any variable x, if the pointed category* $\mathrm{Pt}(\mathcal{C} \downarrow A)$ *has (M,x)-closed relations, then every n-ary reflexive relation $R \longrightarrow A^n$ in $\mathcal{C}$ is M-closed.*

Proof. Without loss of generality we can assume that x is an entry of some left column of M. Indeed, let M' denote the extended matrix obtained from M by adding to M a left column consisting of just x's:

$$M' = \left(\begin{array}{cccc|c} x & t_{11} & \cdots & t_{1m} & u_1 \\ \vdots & \vdots & & \vdots & \vdots \\ x & t_{n1} & \cdots & t_{nm} & u_n \end{array}\right).$$

Since (M', x) is different from (M, x) only by a left column of 0's, $\mathrm{Pt}(\mathcal{C} \downarrow A)$ has (M', x)-closed relations if and only if it has (M, x)-closed relations. At the same time, M'-closedness of a reflexive relation is the same as M-closedness.

Let $r = (r_1, ..., r_n) : R \longrightarrow A^n$ be an n-ary reflexive relation in $\mathcal{C}$. Recall from Section 1 that each entry w of M can be regarded as a k-ary term. Since w is a variable, to regard w as a k-ary term is the same as to regard w as a natural number $w \in \{1, ..., k\}$; this natural number is then the same as i in the formula $w(x_1, ..., x_k) = x_i$, which defines the k-ary term w. So, the extended matrix in Lemma 1.3 can be written as

$$M_r = \left(\begin{array}{ccc|c} r^M_{t_{11}} & \cdots & r^M_{t_{1m}} & r^M_{u_1} \\ \vdots & & \vdots & \vdots \\ r^M_{t_{n1}} & \cdots & r^M_{t_{nm}} & r^M_{u_n} \end{array}\right).$$

By this lemma, R is M-closed if and only if R is compatible with M_r. Thus, we must show that if $\mathrm{Pt}(\mathcal{C} \downarrow A)$ has (M, x)-closed relations, then R is compatible with M_r. Since R is reflexive,

- there exists a morphism $d_0 : A \longrightarrow R$ such that $r_1 d_0 = ... = r_n d_0 = 1_A$, and
- there exists a morphism $d_1 : A \longrightarrow R^M$ such that $r^M_1 d_1 = ... = r^M_k d_1 = 1_A$ (where R^M is the same as in Lemma 1.3).

Next, we observe:

- the triples

$$\begin{aligned} B &= (R^M, r^M_x, d_1), \\ C &= (A \times A, \mathrm{pr}_2 : A \times A \longrightarrow A, (1_A, 1_A) : A \longrightarrow A \times A), \\ S &= (R \times A, \mathrm{pr}_2 : R \times A \longrightarrow A, (d_0, 1_A) : A \longrightarrow R \times A) \end{aligned}$$

  are objects in $\mathrm{Pt}(\mathcal{C} \downarrow A)$;
- for each $i \in \{1, ..., n\}$, the morphism $s_i = r_i \times 1_A : R \times A \longrightarrow A \times A$ is a morphism $s_i : S \longrightarrow C$ in $\mathrm{Pt}(\mathcal{C} \downarrow A)$, and $s = (s_1, ..., s_n) : S \longrightarrow C^n$ is a relation in $\mathrm{Pt}(\mathcal{C} \downarrow A)$;

- for each entry w in M, the morphism
$$(r^M_w, r^M_x) : R^M \longrightarrow A \times A$$
is a morphism $(r^M_w, r^M_x) : B \longrightarrow C$ in $\mathrm{Pt}(\mathcal{C} \downarrow A)$; in the case when $w = x$, this morphism is the zero morphism $0 : B \longrightarrow C$ in $\mathrm{Pt}(\mathcal{C} \downarrow A)$, and so the extended matrix
$$M'_r = \left(\begin{array}{ccc|c} (r^M_{t_{11}}, r^M_x) & \cdots & (r^M_{t_{1m}}, r^M_x) & (r^M_{u_1}, r^M_x) \\ \vdots & & \vdots & \vdots \\ (r^M_{t_{n1}}, r^M_x) & \cdots & (r^M_{t_{nm}}, r^M_x) & (r^M_{u_n}, r^M_x) \end{array}\right)$$
is a regular interpretation of the pointed matrix (M, x).

After these observations it suffices to show that if S is compatible with M'_r, then R is compatible with M_r. Indeed, this follows from the fact that for each column
$$\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$
of M, the following two conditions are equivalent to each other:

- In $\mathrm{Pt}(\mathcal{C} \downarrow A)$, the morphism $((r^M_{w_1}, r^M_x), ..., (r^M_{w_n}, r^M_x)) : B \longrightarrow C^n$ factors through s.
- In $\mathcal{C}$, the morphism $(r^M_{w_1}, ..., r^M_{w_n}) : R^M \longrightarrow A^n$ factors through r.

This concludes the proof. ■

3. Bourn localization

3.1. Definition. *Let $\mathbb{C}$ be an arbitrary class of categories. The Bourn localization of $\mathbb{C}$ is the class* $\mathrm{Loc}(\mathbb{C})$ *of all categories $\mathcal{C}$ such that for every object A in $\mathcal{C}$, the pointed category* $\mathrm{Pt}(\mathcal{C} \downarrow A)$ *belongs to $\mathbb{C}$.*

Note that for any matrix M of variables (pointed or not), $\mathrm{Loc}(\mathbf{Cat}_M) = \mathrm{Loc}(\mathbf{Cat}^*_M)$.

The proof of Theorem 3.3 below will use the following theorem, which follows easily from Lemma 1.3 and the fact that a pullback preserving functor always preserves jointly monomorphic families of morphisms:

3.2. Theorem. *Let M be an extended matrix of variables and let $F : \mathcal{D} \to \mathcal{C}$ be a pullback preserving functor which reflects isomorphisms. Then, if $\mathcal{C}$ has M-closed relations, so does $\mathcal{D}$.*

3.3. Theorem. *For any extended matrix M of variables, and for any variable x such that the pair (M, x) is admissible,*
$$\mathbf{Cat}_M = \mathrm{Loc}(\mathbf{Cat}_{(M,x)}).$$

Proof. Let M and x be as in the theorem. Let $\mathcal{X}$ denote the set of all variables (i.e. $\mathcal{X}$ is the alphabet of the algebraic theory of sets). For each $i \in \{1, ..., n\}$, let f_i be an injective map

$$f_i : \{t_{i1}, ..., t_{im}, u_i\} \longrightarrow \mathcal{X}$$

from the set of all entries of the i-th row of M, to the set $\mathcal{X}$. Further, suppose $f_1, ..., f_n$ is chosen in such a way that the following two conditions are met:

- for each $i \in \{1, ..., n\}$, and for each entry y of the i-th row of M,

$$f_i(y) = x \quad \Leftrightarrow \quad y = x,$$

- for each $i, i' \in \{1, ..., n\}$, $\mathrm{Im}(f_i) \cap \mathrm{Im}(f_i') \subseteq \{x\}$.

Then, the extended matrix

$$M_x = \left(\begin{array}{ccc|c} f_1(t_{11}) & \cdots & f_1(t_{1m}) & f_1(u_1) \\ \vdots & & \vdots & \vdots \\ f_n(t_{n1}) & \cdots & f_n(t_{nm}) & f_n(u_n) \end{array} \right)$$

has the following properties:

- if a variable y is repeated in two different rows of M_x, then $y = x$,
- the pair (M_x, x) is admissible,
- $\mathbf{Cat}_{(M,x)} = \mathbf{Cat}_{(M_x,x)}$ and $\mathbf{Cat}_M = \mathbf{Cat}_{M_x}$.

The two equalities above follow from the fact that M and M_x are each other's row-wise interpretations, and also the pointed matrices (M, x) and (M_x, x) are each other's row-wise interpretations.

Thus, to prove the theorem it suffices to show $\mathbf{Cat}_{M_x} = \mathrm{Loc}(\mathbf{Cat}^*_{(M_x,x)})$. First suppose there exists a variable y such that the pair (M_x, y) is not admissible. Since (M_x, x) is admissible, M_x must contain at least one left column, which implies that y must be an entry of M_x. Next, we claim that y is not an entry of any of the left columns of M_x. Indeed, since $y \neq x$, there is only one row

$$\left(\begin{array}{ccc|c} t_{i1} & \cdots & t_{im} & u_i \end{array} \right)$$

in M_x which contains y, and if $y \in \{t_{i1}, ..., t_{im}\}$, then we would get that the pair (M_x, y) is admissible, contrary to our assumption. So $y = u_i \notin \{t_{i1}, ..., t_{im}\}$. By Proposition 2.4, $\mathbf{Cat}_{M_x}$ is the class of all preorders, and $\mathbf{Cat}^*_{(M_x,x)}$ is the class of all codiscrete categories. Then the inclusion $\mathbf{Cat}_{M_x} \subseteq \mathrm{Loc}(\mathbf{Cat}^*_{(M_x,x)})$ is obvious. We now show the converse inclusion, $\mathrm{Loc}(\mathbf{Cat}^*_{(M_x,x)}) \subseteq \mathbf{Cat}_{M_x}$. Let $\mathcal{C}$ be a category such that for each object A in $\mathcal{C}$ the category $\mathrm{Pt}(\mathcal{C} \downarrow A)$ is a codiscrete category. To show that $\mathcal{C}$ is a preorder, it suffices to show that for each object A in $\mathcal{C}$, the diagonal $\Delta : A \longrightarrow A \times A$ is an isomorphism.

Indeed, since $\mathrm{Pt}(\mathcal{C} \downarrow A)$ is codiscrete, the morphism $\Delta : (A, 1_A, 1_A) \longrightarrow (A, \mathrm{pr}_2, \Delta)$ in $\mathrm{Pt}(\mathcal{C} \downarrow A)$ is an isomorphism, which implies that Δ is an isomorphism in $\mathcal{C}$ as well.

Now suppose for any variable y, the pair (M_x, y) is admissible. From Proposition 2.5 we obtain that if $\mathcal{C} \in \mathrm{Loc}(\mathbf{Cat}^*_{(M_x,x)})$, then every reflexive relation in $\mathcal{C}$ is M_x-closed. By Remark 2.3, this implies that $\mathcal{C}$ has M_x-closed relations. Hence $\mathrm{Loc}(\mathbf{Cat}^*_{(M_x,x)}) \subseteq \mathbf{Cat}_{M_x}$. It remains to show $\mathbf{Cat}_{M_x} \subseteq \mathrm{Loc}(\mathbf{Cat}^*_{(M_x,x)})$. Let $\mathcal{C}$ be a category with M_x-closed relations. Take an arbitrary object A in $\mathcal{C}$. The forgetful functor $\mathrm{Pt}(\mathcal{C} \downarrow A) \longrightarrow \mathcal{C}$, which takes each triple (X, f, g) to the object X, preserves pullbacks and reflects isomorphisms. By Theorem 3.2, $\mathrm{Pt}(\mathcal{C} \downarrow A)$ has M_x-closed relations. This obviously implies that $\mathrm{Pt}(\mathcal{C} \downarrow A)$ has (M_x, x)-closed relations. We obtain $\mathbf{Cat}_{M_x} \subseteq \mathrm{Loc}(\mathbf{Cat}^*_{(M_x,x)})$, completing the proof. ■

Theorem 3.3 unifies the following three results:

(i) The class of all Mal'tsev categories is the Bourn localization of the class of all unital categories [2].

(ii) The class of all Mal'tsev categories is the Bourn localization of the class of all strongly unital categories [2].

(iii) The class of all Mal'tsev categories is the Bourn localization of the class of all subtractive categories [11].

To see how the above three statements follow from Theorem 3.3, we just have to make the following observations, which are based on the results of the last section of [11]:

(i) A Mal'tsev category is the same as a category with M-closed relations, where

$$M = \left(\begin{array}{ccc|c} y & x & x & y \\ x & x & y & y \end{array} \right).$$

In this case

$$(M, x) = \left(\begin{array}{cc|c} y & 0 & 0 & y \\ 0 & 0 & y & y \end{array} \right).$$

A pointed category with (M, x)-closed relations is the same as a pointed category with M'-closed relations, where

$$M' = \left(\begin{array}{cc|c} y & 0 & y \\ 0 & y & y \end{array} \right),$$

which is the same as a unital category.

(ii) A Mal'tsev category is the same as a category with M-closed relations, where

$$M = \left(\begin{array}{ccc|c} y & x & x & y \\ y & y & z & z \end{array} \right).$$

In this case

$$(M,x)=\left(\begin{array}{ccc|c} y & 0 & 0 & y \\ y & y & z & z \end{array}\right).$$

A pointed category with (M,x)-closed relations is the same as a strongly unital category.

(iii) A Mal'tsev category is the same as a category with M-closed relations, where

$$M=\left(\begin{array}{ccc|c} y & x & x & y \\ y & y & x & x \end{array}\right).$$

In this case

$$(M,x)=\left(\begin{array}{ccc|c} y & 0 & 0 & y \\ y & y & 0 & 0 \end{array}\right).$$

A pointed category with (M,x)-closed relations is the same as a pointed category with M'-closed relations, where

$$M'=\left(\begin{array}{cc|c} y & 0 & y \\ y & y & 0 \end{array}\right),$$

which is the same as a subtractive category.

3.4. Remark. Note that the three M's in (i), (ii) and (iii) are all row-wise similar, and they are also row-wise similar to the matrix (1). In [11], any such matrix, i.e. any matrix which is row-wise similar to (1), was called a *Mal'tsev matrix*. Thus, the classes of unital, strongly unital, and subtractive categories are all of the form $\mathbf{Cat}^*_{(M,x)}$, where M is some Mal'tsev matrix and x is a variable such that the pair (M,x) is admissible. It can be shown that no other class of categories is of that form, apart from also the class of all pointed Mal'tsev categories.

Let M be an arbitrary extended matrix of variables. From Theorem 3.3 we obtain, that if M has at least one left column, then $\mathbf{Cat}_M = \mathrm{Loc}(\mathbf{Cat}_M)$. What happens when M does not have any left columns? By Proposition 2.4, in this case $\mathbf{Cat}_M$ is the class of all codiscrete categories.

3.5. Proposition. *The class of all codiscrete categories is not a Bourn localization of any class of categories.*

Proof. Let $\mathbb{C}$ be a class of categories. Suppose $\mathrm{Loc}(\mathbb{C})$ contains all codiscrete categories. Consider a category $\mathcal{C}$ which consists of exactly two objects, X and Y, and exactly three morphisms: the two identity morphisms 1_X and 1_Y of X and Y, respectively, and one morphism $f : X \longrightarrow Y$. And, consider its subcategories $\mathcal{D}_1$ and $\mathcal{D}_2$, where $\mathcal{D}_1$ consists of just the object X (and its identity morphism), and $\mathcal{D}_2$ consists of just the object Y. Then $\mathrm{Pt}(\mathcal{C} \downarrow X) = \mathrm{Pt}(\mathcal{D}_1 \downarrow X)$ and $\mathrm{Pt}(\mathcal{C} \downarrow Y) = \mathrm{Pt}(\mathcal{D}_2 \downarrow Y)$. Since $\mathcal{D}_1$ and $\mathcal{D}_2$ are codiscrete, they belong to $\mathrm{Loc}(\mathbb{C})$. From this we obtain that $\mathcal{C}$ belongs to $\mathrm{Loc}(\mathbb{C})$ as well, but $\mathcal{C}$ is not codiscrete. We showed that if the Bourn localization of a class of categories contains all codiscrete categories, then it also contains a category which is not codiscrete. Hence, the class of all codiscrete categories is not a Bourn localization of any class of categories. ∎

Thus, if M does not have any left columns, then $\mathbf{Cat}_M$ is not a Bourn localization of any class of categories.

Suppose M has at least one left column. For any variable x such that the pair (M, x) is admissible, we have

$$\mathbf{Cat}^*_M \subseteq \mathbf{Cat}^*_{(M,x)}.$$

Since $\mathrm{Loc}(\mathbf{Cat}^*_M) = \mathrm{Loc}(\mathbf{Cat}^*_{(M,x)})$, for any class $\mathbb{C}$ of categories such that

$$\mathbf{Cat}^*_M \subseteq \mathbb{C} \subseteq \mathbf{Cat}^*_{(M,x)},$$

we would have $\mathbf{Cat}_M = \mathrm{Loc}(\mathbb{C})$. This gives that the fact that the Bourn localization of the class of all strongly unital categories is the class of all Mal'tsev categories follows from the fact that the Bourn localization of the class of all unital categories is the class of all Mal'tsev categories, and the fact that the Bourn localization of the class of all (pointed) Mal'tsev categories is the class of all Mal'tsev categories (since any pointed Mal'tsev category is strongly unital, and any strongly unital category is unital). At the same time the class of all strongly unital categories is the intersection of the classes of all unital and subtractive categories. Observe that, more generally, Bourn localization always preserves intersections of classes of categories, that is, for any family $(\mathbb{C}_i)_{i \in I}$ of classes of categories, we have:

$$\mathrm{Loc}\left(\bigcap_{i\in I} \mathbb{C}_i\right) = \bigcap_{i \in I} \mathrm{Loc}(\mathbb{C}_i).$$

However, in general, for unions we only have

$$\mathrm{Loc}\left(\bigcup_{i\in I} \mathbb{C}_i\right) \supseteq \bigcup_{i \in I} \mathrm{Loc}(\mathbb{C}_i).$$

So the following result is somewhat unexpected:

3.6. Theorem. *For any extended matrix M of variables with at least one left column, one has*

$$\mathbf{Cat}_M = \mathrm{Loc}\left(\bigcup_{x \in \mathbf{X}} \mathbf{Cat}_{(M,x)}\right)$$

where $\mathbf{X}$ denotes the set of all variables x such that the pair (M, x) is admissible.

Proof. Since M has at least one left column, $\mathbf{X}$ is not empty (any variable that is not in M belongs to $\mathbf{X}$). By Theorem 3.3, $\mathbf{Cat}_M = \mathrm{Loc}(\mathbf{Cat}_{(M,x)})$ for all $x \in \mathbf{X}$, so we have

$$\mathbf{Cat}_M \subseteq \mathrm{Loc}\left(\bigcup_{x \in \mathbf{X}} \mathbf{Cat}_{(M,x)}\right).$$

Suppose first there exists a row

$$\left(\begin{array}{ccc|c} t_{i1} & \cdots & t_{im} & u_i \end{array}\right)$$

in M such that $u_i \notin \{t_{i1}, ..., t_{im}\}$. Note $u_i \notin \mathbf{X}$. Hence, by Proposition 2.4, for each $x \in \mathbf{X}$, $\mathbf{Cat}^*_{(M,x)}$ is the class of all codiscrete categories. Therefore, for any $y \in \mathbf{X}$ we have

$$\mathbf{Cat}^*_{(M,y)} = \bigcup_{x \in \mathbf{X}} \mathbf{Cat}^*_{(M,x)},$$

which means that the equality in Theorem 3.6 becomes the same as the equality in Theorem 3.3 (these equalities state that the class of all preorders is the Bourn localization of the class of all codiscrete categories).

Now suppose $u_i \in \{t_{i1}, ..., t_{im}\}$ for each $i \in \{1, ..., n\}$. Let $\mathcal{C}$ be a category from the class

$$\mathrm{Loc}\left(\bigcup_{x \in \mathbf{X}} \mathbf{Cat}_{(M,x)}\right).$$

We should show that $\mathcal{C}$ has M-closed relations. Consider an arbitrary internal relation $R \longrightarrow A^n$ in $\mathcal{C}$. Let S denote the reflexive relation $S \longrightarrow R^n$ constructed in Proposition 2.2. Take $x \in \mathbf{X}$ for which $\mathrm{Pt}(\mathcal{C} \downarrow R)$ has (M, x)-closed relations. Let M_x be the extended matrix of variables, constructed in the proof of Theorem 3.3, from the pair (M, x). We then have:

- for any variable y, the pair (M_x, y) is admissible,
- $\mathbf{Cat}_{(M,x)} = \mathbf{Cat}_{(M_x,x)}$,
- strict M-closedness of any relation is the same as strict M_x-closedness.

Now, $\mathrm{Pt}(\mathcal{C} \downarrow R)$ has (M_x, x)-closed relations, which implies, by Proposition 2.5, that the relation S is M_x-closed. By Proposition 2.2, this gives that the relation R is strictly M_x-closed. Hence, R is strictly M-closed. This shows that $\mathcal{C}$ has M-closed relations, concluding the proof. ■

From this Bourn localization theorem we obtain:

3.7. Corollary. *The class of all Mal'tsev categories is the Bourn localization of the union of the class of all unital categories and the class of all subtractive categories.*

Proof. Let M be the Mal'tsev matrix

$$M = \left(\begin{array}{ccc|c} x & y & y & x \\ y & y & z & z \end{array}\right).$$

Then, $\mathbf{Cat}_M$ is the class of Mal'tsev categories, $\mathbf{Cat}^*_{(M,x)}$ is the class of subtractive categories (see [11]), and $\mathbf{Cat}^*_{(M,y)}$ is the class of unital categories. At the same time, both pairs (M, x) and (M, y) are admissible. Note that $\mathbf{Cat}^*_{(M,z)} = \mathbf{Cat}^*_{(M,x)}$. So for this M the equality in Theorem 3.6 states precisely that the class of Mal'tsev categories is the Bourn localization of the union of the classes of unital and subtractive categories. ■

Theorems 3.3 and 3.6 can be derived from the following more general Bourn localization theorem:

3.8. THEOREM. *For any extended matrix M of variables with at least one left column, one has*

$$\mathbf{Cat}_M = \mathrm{Loc}(\mathbf{Cat}_M) = \mathrm{Loc}\left(\bigcup_{P\in\mathbf{P}} \mathbf{Cat}_P\right),$$

where $\mathbf{P}$ denotes the set of all pointed matrices P such that $P = (M', x)$ for some extended matrix M' of variables, which is row-wise similar to M, and for some variable x such that the pair (M', x) is admissible.

PROOF. We already know $\mathbf{Cat}_M = \mathrm{Loc}(\mathbf{Cat}_M)$. Further, for each admissible pair (M', x), where M' is row-wise similar to M, $\mathbf{Cat}_M = \mathbf{Cat}_{M'} = \mathrm{Loc}(\mathbf{Cat}_{(M',x)})$. This gives

$$\mathbf{Cat}_M = \bigcup_{P\in\mathbf{P}} \mathrm{Loc}(\mathbf{Cat}_P) \subseteq \mathrm{Loc}\left(\bigcup_{P\in\mathbf{P}} \mathbf{Cat}_P\right).$$

It remains to show

$$\mathbf{Cat}_M \supseteq \mathrm{Loc}\left(\bigcup_{P\in\mathbf{P}} \mathbf{Cat}_P\right).$$

Suppose first there exists a row

$$\left(\begin{array}{ccc|c} t_{i1} & \cdots & t_{im} & u_i \end{array}\right)$$

in M such that $u_i \notin \{t_{i1}, ..., t_{im}\}$. Then the same statement is true also for any M' which is row-wise similar to M. If we then take an admissible pair (M', x), we would have $x \neq u_i'$, where u_i' denotes the right column entry of the i-th row of M'. This gives, by Proposition 2.4, that $\mathbf{Cat}^*_{(M',x)}$ is the class of all codiscrete categories. Hence,

$$\bigcup_{P\in\mathbf{P}} \mathbf{Cat}^*_P$$

is also the class of all codiscrete categories. Its localization is, as we already know, the class of all preorders, which, by Proposition 2.4 again, is precisely the class $\mathbf{Cat}_M$.

Now suppose $u_i \in \{t_{i1}, ..., t_{im}\}$ for each $i \in \{1, ..., n\}$, i.e. every right column entry of each row is also a left column entry of the same row. Then any M' which is row-wise similar to M also has the same property. Take any category $\mathcal{C}$ from the class

$$\mathrm{Loc}\left(\bigcup_{P\in\mathbf{P}} \mathbf{Cat}_P\right).$$

We should show that $\mathcal{C}$ has M-closed relations. Consider an arbitrary internal relation $R \longrightarrow A^n$ in $\mathcal{C}$. Let S denote the reflexive relation $S \longrightarrow R^n$ constructed in Proposition

2.2. Take $P \in \mathbf{P}$ for which $\mathrm{Pt}(\mathcal{C} \downarrow R)$ has P-closed relations. Then $P = (M', x)$ where M' is row-wise similar to M and the pair (M', x) is admissible. Let M'_x be the extended matrix of variables, constructed from M' and x, as in the proof of Theorem 3.3 (where we had M instead of M'). We then have:

- for any variable y, the pair (M'_x, y) is admissible,
- $\mathbf{Cat}_{(M',x)} = \mathbf{Cat}_{(M'_x,x)}$,
- strict M'-closedness of any relation is the same as strict M'_x-closedness.

Now, $\mathrm{Pt}(\mathcal{C} \downarrow R)$ has (M'_x, x)-closed relations, which implies, by Proposition 2.5, that the relation S is M'_x-closed. By Proposition 2.2, this gives that the relation R is strictly M'_x-closed. Hence, R is strictly M'-closed. But M' is row-wise similar to M, so we obtain that R is strictly M-closed. This shows that $\mathcal{C}$ has M-closed relations, concluding the proof. ∎

Note that, as this follows from Remark 3.4, Theorem 3.8 does not give us any new information on the class of Mal'tsev categories (if we let M in Theorem 3.8 to be any Mal'tsev matrix). However, for each particular Mal'tsev matrix M, the equality in Theorem 3.8 may be more general than the equality in Theorem 3.6. For instance, let

$$M = \left(\begin{array}{ccc|c} x & y & y & x \\ u & u & v & v \end{array} \right).$$

Then the equality in 3.6 does not imply that the Bourn localization of the class of all unital categories is the class of all Mal'tsev categories (it only implies that the Bourn localization of the class of all subtractive categories is the class of all Mal'tsev categories). However, for each Mal'tsev matrix M, the equality in 3.8 implies the assertion 3.7.

Let M and $\mathbf{P}$ be as in Theorem 3.8. Take any matrix M' which is row-wise similar to M, but has the property that no variable belongs to two different rows of M' at the same time, i.e. the sets of entries of any two different rows of M' are disjoint. Call a pointed matrix P *a pointed matrix obtained from M' by a coherent substitution of* 0*'s in M'*, if P can be obtained from M' by substituting 0's in certain variables, all of which can be found in one fixed left column of M'. It is easy to show that any pointed matrix P from $\mathbf{P}$ is row-wise similar to some pointed matrix P', which is obtained from M' by a coherent substitution of 0's. Thus, Theorem 3.8 can be restated in the following way:

3.9. Theorem. *Let M be an extended matrix of variables, with at least one left column, and such that the sets of entries of any two different rows of M are disjoint. Then,*

$$\mathbf{Cat}_M = \mathrm{Loc}(\mathbf{Cat}_M) = \mathrm{Loc}\left(\bigcup_{P \in \mathbf{P}} \mathbf{Cat}_P \right),$$

where $\mathbf{P}$ denotes the set of all pointed matrices P which can be obtained from M by a coherent substitution of 0*'s in M, i.e. P can be obtained from M by substituting* 0*'s in* *(every instance of) certain variables* $x_1, ..., x_j$ *such that there exists a left column of* M *among whose entries are* $x_1, ..., x_j$*. In particular, this gives that for each* $P \in \mathbf{P}$*,*

$$\mathbf{Cat}_M = \mathrm{Loc}(\mathbf{Cat}_P).$$

Let us also add in conclusion that for *any* pointed matrix P, the Bourn localization of $\mathbf{Cat}_P$ is a class of categories determined by an extended matrix of variables, i.e. $\mathrm{Loc}(\mathbf{Cat}_P) = \mathbf{Cat}_M$ for some matrix M of variables. Indeed, we can always present P as $P = (M, x)$, where M is an extended matrix of variables. If the pair (M, x) is admissible, then, from Theorem 3.3 we get $\mathrm{Loc}(\mathbf{Cat}_P) = \mathrm{Loc}(\mathbf{Cat}_{(M,x)}) = \mathbf{Cat}_M$. If the pair (M, x) is not admissible, then we should add to M a column consisting of just x's. Then, for the resulting matrix M' of variables, we would have: (i) the pair (M', x) is admissible, (ii) $\mathbf{Cat}^*_P = \mathbf{Cat}^*_{(M',x)}$ (since P is different from (M', x) only by a left column consisting of just 0's). Hence, from Theorem 3.3 again, we get $\mathrm{Loc}(\mathbf{Cat}_P) = \mathrm{Loc}(\mathbf{Cat}_{(M',x)}) = \mathbf{Cat}_{M'}$.

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