BOUNDEDNESS OF LITTLEWOOD-PALEY OPERATORS WITH VARIABLE KERNEL ON THE WEIGHTED HERZ-MORREY SPACES WITH VARIABLE EXPONENT

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Abstract. Let Ω ∈ $L^\infty(\mathbb{R}^n) \times L^2(S^{n-1})$ be a homogeneous function of degree zero. In this article, we obtain some boundedness of the parameterized Littlewood–Paley operators with variable kernels on weighted Herz-Morrey spaces with variable exponent. As a supplement, the boundedness of fractional integral operators with variable kernel is also obtained on these spaces.

1 Introduction

It is well known that the boundedness of Littlewood–Paley operators on function spaces are one of the very important tools not only in harmonic analysis but also in potential theory and in partial differential equations (see [1],[2],[23],[27],[33]) for details). In 2004, Ding, Lin and Shao [3] investigated the $L^2$-boundedness for a class of Marcinkiewicz integral operators with variable kernels $\mu\Omega$ and $\mu_{\Omega,s}$ related to the Littlewood–Paley function $\mu_{\Omega,\lambda}^*$ and the area integral $g_{\lambda}^*$. In 2006, the authors [4] have proved the $L^p$-boundedness of the Littlewood–Paley operators with variable kernels. In 2009, Xue and Ding [5] established the weighted estimate for Littlewood–Paley operators and their commutators.

In 1960, Hörmander [6] introduced the parameterized Littlewood–Paley operators for the first time. Now, let us recall the definitions of the parameterized Lusin area integral and the Littlewood-Paley $g_{\lambda}^*$ function.

Let $S^{n-1}(n \geq 2)$ be the unit sphere in $\mathbb{R}^n$ with normalized Lebesgue measure $d\sigma(x')$. Denote $\Omega(x,z') \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})(r \geq 1)$ to be a homogeneous function of degree zero and 

$$\int_{S^{n-1}} \Omega(x,z')d\sigma(z') = 0, \quad \text{for all} \quad x \in \mathbb{R}^n,$$

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where \( \Omega \) satisfies the following conditions:

(i) For any \( x, z \in \mathbb{R}^n \) and any \( \lambda > 0 \), one has \( \Omega(x, \lambda z) = \Omega(x, z) \);

(ii) \( \| \Omega \|_{L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})} := \sup_{r \geq 0, y \in \mathbb{R}^n} (\int_{S^{n-1}} |\Omega(rz' + y, z')|^r \, d\sigma(z'))^{\frac{1}{r}} < \infty \).

The parameterized Littlewood–Paley operators \( \mu_{\Omega,s}^\rho \) and \( \mu_{\Omega,\lambda}^{*,\rho} \) with variable kernels, which are related to the Lusin area integral and the Littlewood–Paley \( g_\lambda^s \) function are defined by

\[
\mu_{\Omega,s}^\rho(f)(x) = \left( \int \int_{\Gamma(x)} \frac{1}{t^\rho} \int_{|y-z| \leq t} \frac{\Omega(y, y-z)}{|y-z|^{n-\rho}} f(z) \, dz \, \, dy \, dt \right)^\frac{1}{2} \frac{d\sigma dt}{t^{n+1}},
\]

and

\[
\mu_{\Omega,\lambda}^{*,\rho}(f)(x) = \left( \int \int_{\mathbb{R}^{n+1}_+} \frac{1}{t^\rho} \int_{|y-z| \leq t} \frac{\Omega(y, y-z)}{|y-z|^{n-\rho}} f(z) \, dz \, \, dy \, dt \right)^\frac{1}{2} \frac{d\sigma dt}{t^{n+1}},
\]

where \( \Gamma(x) = \{(y, t) \in \mathbb{R}^{n+1} : |x - y| < t \text{ and } \lambda > 1\} \).

In 2013, Wei and Tao [7] investigated the boundedness of parameterized Littlewood–Paley operators on weighted weak Hardy spaces. Lin and Xuan [8] established the boundedness for commutators of parameterized Littlewood–Paley operators and area integrals on weighted Lebesgue spaces \( L^p(w) \).

On the other hand, the theory of the variable exponent function spaces has been rapidly developed after the work [9] where Kovačić and Rákosník have clarified fundamental properties of Lebesgue spaces with variable exponent. After that, many researchers have been interested in the theory of the variable exponent spaces (see [10]–[14]).

The generalization of the Muckenhoupt weights with variable exponent \( A_{p(\cdot)} \) has been considered in ([15]–[18]). We note that the equivalence between the Muckenhoupt condition and the boundedness of the Hardy-Littlewood maximal operator on weighted Lebesgue spaces with variable exponent were discussed in ([15], [16]). After that, Cruz-Uribe and Wang [19] proved the boundedness of some classical operators on weighted Lebesgue spaces with variable exponent \( L^{p(\cdot)}(w) \).

Very recently, Mitsuo and Takahiro [20] introduced the weighted Herz spaces with variable exponent, and also studied the boundedness of fractional integrals on those spaces.

In this paper, by applying the theory of Banach function spaces and the variable exponent Muckenhoupt weight theory, we establish the boundedness of parameterized Littlewood–Paley operators with variable kernels on weighted Herz-Morrey spaces with variable exponent \( M\mathring{K}^{\alpha,q}_{p(\cdot)}(w) \). The similar results for the boundedness of the fractional integral operators with variable kernel are also discussed. Let \( E \) be a Lebesgue measurable set in \( \mathbb{R}^n \) with measure \( |E| > 0 \), \( \chi_E \) means its characteristic function. We shall recall some definitions.

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Definition 1 ([2]). Let $p(\cdot) : E \to [1, \infty)$ be a measurable function, the variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(E) = \{ f \text{ is measurable} : \int_E \left( \frac{|f(x)|}{\eta} \right)^{p(x)} \, dx < \infty \text{ for some constant } \eta > 0 \}.$$ 

The space $L^{p(\cdot)}_{\text{loc}}(E)$ is defined by

$$L^{p(\cdot)}_{\text{loc}}(E) = \{ f \text{ is measurable} : f \in L^{p(\cdot)}(K) \text{ for all compact } K \subset E \}.$$

The Lebesgue spaces $L^{p(\cdot)}(E)$ is a Banach spaces with the norm defined by

$$\|f\|_{L^{p(\cdot)}(E)} = \inf \left\{ \eta > 0 : \int_E \left( \frac{|f(x)|}{\eta} \right)^{p(x)} \, dx \leq 1 \right\}.$$ 

We denote $p_- = \text{ess inf}\{p(x) : x \in E\}$, $p_+ = \text{ess sup}\{p(x) : x \in E\}$, then $\mathcal{P}(E)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$.

Definition 2 ([32]). Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$. A measurable function $p(\cdot)$ is said to be globally log-Hölder continuous if it satisfies

1. $|p(x) - p(y)| \leq \frac{1}{-\log(|x-y|)}$, $x, y \in \mathbb{R}^n, |x-y| \leq 1/2$;
2. $|p(x) - p_\infty| \leq \frac{1}{\log(e + |x|)}$, $x \in \mathbb{R}^n$,

for some $p_\infty \geq 1$. The set of $p(\cdot)$ satisfying (1) and (2) is denoted by $LH(\mathbb{R}^n)$. We know that, if $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, the Hardy–Littlewood maximal operator $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (see [34]).

Definition 3 ([24]). Suppose that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $w$ is a weight function. The weighted Lebesgue spaces with variable exponent $L^{p(\cdot)}(w)$ is the set of all complex-valued measurable function $f$ such that $f w^{1/p(\cdot)} \in L^{p(\cdot)}(\mathbb{R}^n)$. The space $L^{p(\cdot)}(w)$ is a Banach space equipped with the norm

$$\|f\|_{L^{p(\cdot)}(w)} = \|f w^{1/p(\cdot)}\|_{L^{p(\cdot)}}.$$ 

$p'(\cdot)$ is the conjugate of $p(\cdot)$ such that $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$. Next, we introduce the classical Muckenhoupt $A_p$ weight.

Definition 4 ([21]). Let $1 < p < \infty$, then $w \in A_p$ for every cube $Q$,

$$\left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{1-p'} \, dx \right)^{p-1} \leq C < \infty.$$ 

We say that $w \in A_1$ if it satisfies $Mw(x) \leq w(x)$ for all $x \in \mathbb{R}^n$. The set of $A_1$ consists of all Muckenhoupt $A_1$ weights.
Definition 5 ([15],[19]). Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and a weight $w$, then $w \in A_{p(\cdot)}$ if
\[
\sup_{B:\text{ball}} |B|^{-1} \|w^{1/p(\cdot)} \chi_B\|_{L^p(\cdot)(w)} \|w^{-1/p(\cdot)} \chi_B\|_{L^{p'}(\cdot)(w)} < \infty.
\]

Definition 6 ([19]). Given $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $1/p_1(x) - 1/p_2(x) = \mu/n$ such that $0 < \mu < n$. Then $w \in A_{p_1(\cdot), p_2(\cdot)}$ if
\[
\|w \chi_B\|_{L^{p_2}(\cdot)} \|w^{-1/N} \chi_B\|_{L^{p_1}(\cdot)} \leq |B|^{\frac{N-n}{n}}
\]
holds for all balls $B \in \mathbb{R}^n$.

Definition 7 ([19]). Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $w$ be a weight. We say that $(p(\cdot), w)$ is an $M$-pair if the maximal operator $M$ is bounded on $L^{p(\cdot)}(w)$ and $L^{p'(\cdot)}(w^{-1})$.

Now, we need give the definition of weighted Herz space with variable exponent. For all $k \in \mathbb{Z}$, we denote $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $C_k = B_k \setminus B_{k-1}$, $\chi_k = \chi_{C_k}$.

Definition 8 ([20]). Suppose that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < q < \infty$, $\alpha \in \mathbb{R}$. The homogeneous weighted Herz space with variable exponent $K_{p(\cdot)}^{\alpha,q}(w)$ is the collection of $f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}, w)$ such that
\[
\|f\|_{K_{p(\cdot)}^{\alpha,q}(w)} := \left( \sum_{k=\infty}^{\infty} 2^{\alpha q k} \|f \chi_k\|_{L^{p(\cdot)}(w)}^q \right)^{1/q} < \infty.
\]

Definition 9. Suppose that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $\alpha \in \mathbb{R}$, $0 \leq \lambda < \infty$, $0 < q < \infty$. The homogeneous weighted Herz-Morrey space with variable exponent $M^{\alpha,\lambda}_{q,p(\cdot)}(w)$ is the collection of $f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}, w)$ such that
\[
\|f\|_{M^{\alpha,\lambda}_{q,p(\cdot)}(w)} := \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k \alpha q} \|f \chi_k\|_{L^{p(\cdot)}(w)}^q \right)^{1/q} < \infty.
\]

Obviously, when $\lambda = 0$, then $M^{\alpha,0}_{q,p(\cdot)}(w) = K_{p(\cdot)}^{\alpha,q}(w)$. Obviously, when $\lambda = 0$, then $M^{\alpha,0}_{q,p(\cdot)}(w) = K_{p(\cdot)}^{\alpha,q}(w)$.

Definition 10. We say a kernel function $\Omega(x,z)$ satisfies the $L^r$-Dini condition ($r \geq 1$), if
\[
\int_0^1 \frac{\omega_r(\delta)}{\delta} (1 + |\log \delta|^\sigma) d\delta < \infty,
\]
where $\omega_r(\delta)$ denotes the integral modulus of continuity of order $r$ of $\Omega$ defined by
\[
\omega_r(\delta) = \sup_{x \in \mathbb{R}^n, \|\rho\| < \delta} \left( \int_{\mathbb{R}^{n-1}} |\Omega(x, \rho z') - \Omega(x, z')|^r d\sigma(z') \right)^{1/r},
\]
where $\rho$ is the rotation in $\mathbb{R}^n$, $\|\rho\| = \sup_{z' \in S^{n-1}} \|\rho z' - z'\|$.
2 Preliminaries and notations

In order to prove our main theorems, we need the following Lemmas.

**Lemma 11** ([23]). Suppose that \(X \subset M\) is a Banach function space

1. (The generalized Hölder’s inequality) For all \(f \in X\) and \(g \in X'\), we have

\[
\int_{\mathbb{R}^n} |f(x)g(x)| \, dx \leq \|f\|_X \|g\|_{X'}.
\]

2. For all \(f \in X\), we have

\[
\sup \left\{ \left| \int_{\mathbb{R}^n} f(x)g(x) \, dx \right| : \|g\|_{X'} \leq 1 \right\} = \|f\|_X.
\]

In particular, space \((X')' = X\).

As an application of the generalized Hölder’s inequality above, we have the following Lemma.

**Lemma 12.** Let \(X\) be a Banach function space, we have

\[
1 \leq \frac{1}{|B|} \|\chi_B\|_X \|\chi_B\|_{X'},
\]

holds for all balls \(B\).

**Lemma 13** ([18]). Let \(X\) be a Banach function space. If the Hardy-Littlewood maximal operator \(M\) is weakly bounded on \(X\), that is

\[
\|\chi_{\{Mf > \lambda\}}\|_X \leq \lambda^{-1} \|f\|_X,
\]

it holds for all \(f \in X\) and \(\lambda > 0\). Then we get

\[
\sup_{B: \text{Ball}} \frac{1}{|B|} \|\chi_B\|_X \|\chi_B\|_{X'} < \infty.
\]

**Remark 14.** ([24]) The weighted Banach function space \(X(\mathbb{R}^n, W)\) is a Banach function space equipped the norm \(\|f\|_{X(\mathbb{R}^n, W)} := \|fW\|_X\). The associated space of \(X(\mathbb{R}^n, W)\) is a Banach function space and equals \(X'(\mathbb{R}^n, W^{-1})\).

**Remark 15.** If \(p_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)\), by comparing the definition of the weighted Banach function space with weighted variable Lebesgue space, we have

1. If \(X = L^{p_1(\cdot)}(\mathbb{R}^n)\) and \(W = w\), then we obtain \(L^{p_1(\cdot)}(\mathbb{R}^n, w) = L^{p_1(\cdot)}(w^{p_1(\cdot)})\).
2. If \(X = L^{p_1(\cdot)}(\mathbb{R}^n)\) and \(W = w^{-1}\), by Remark 14, then we obtain

\[
L^{p_1(\cdot)}(\mathbb{R}^n, w^{-1}) = L^{p_1(\cdot)}(w^{-p_1(\cdot)}) = (L^{p_1(\cdot)}(w^{p_1(\cdot)}))'.
\]
Lemma 16 ([25]). Suppose that \( X \) is a Banach space. Let \( M \) be bounded on the associated space \( X' \). Then there exists a constant \( 0 < \delta < 1 \) such that

\[
\frac{\| \chi_E \|_X}{\| \chi_B \|_X} \leq \left( \frac{|E|}{|B|} \right)^\delta
\]

holds for all balls \( B \) and all measurable sets \( E \subset B \).

Lemma 17 ([20]). Suppose that \( p_1(\cdot) \in LH(\mathbb{R}^n) \cap P(\mathbb{R}^n) \) and \( w^{p_1(\cdot)} \in A_1 \). Let \( M \) be a bounded on \( L^{p_1(\cdot)}(w^{p_1(\cdot)}) \) and \( L^{p_1(\cdot)}(w^{-p_1(\cdot)}) \), then there exist constants \( \delta_1, \delta_2 \in (0, 1) \) such that

\[
\frac{\| \chi_S \|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}}{\| \chi_B \|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}} \leq C \left( \frac{|S|}{|B|} \right)^{\delta_1}, \quad \frac{\| \chi_S \|_{L^{p_1(\cdot)}(w^{-p_1(\cdot)})}}{\| \chi_B \|_{L^{p_1(\cdot)}(w^{-p_1(\cdot)})}} \leq C \left( \frac{|S|}{|B|} \right)^{\delta_2}
\]

holds for all balls \( B \) and all measurable sets \( S \subset B \).

Lemma 18 ([26]). Suppose that \( \Omega \in L^\infty(\mathbb{R}^n) \times L^2(S^{n-1}) \) satisfies (1) and (2), \( \lambda > 2, 2\rho - n > 0, 1 < p < \infty \). Then for all \( f \in L^p(w) \) there exists \( C > 0 \) independent of \( f \) such that

\[
\| \mu_{\Omega, s}^{\rho} f \|_{L^p(w)} \leq C \| f \|_{L^p(w)}
\]

and

\[
\| \mu_{\Omega, \lambda}^{\rho} f \|_{L^p(w)} \leq C \| f \|_{L^p(w)}.
\]

Lemma 19 ([19]). Assume that for \( p_0, 1 < p_0 < \infty \) and every \( w_0 \in A_{p_0} \),

\[
\int_{\mathbb{R}^n} f(x)^{p_0} w_0(x) dx \leq \int_{\mathbb{R}^n} g(x)^{p_0} w_0(x) dx, \quad (f, g) \in \mathcal{F}.
\]

Then for any \( M \)-pair \( (p(\cdot), w) \),

\[
\| f \|_{L^{p(\cdot)}(w)} \leq C \| g \|_{L^{p(\cdot)}(w)}, \quad (f, g) \in \mathcal{F},
\]

Lemma 19 holds for \( p_0 = 1 \) and the maximal operator is bounded on \( L^{p(\cdot)}(w^{-1}) \). We know the Hardy–Littlewood maximal operator is bounded on \( L^{p(\cdot)}(w) \) and \( L^{p(\cdot)}(w^{-p(\cdot)}) \) (see [25]).

Combining Lemma 18 with Lemma 19, we obtain the following conclusion.

Corollary 20. Let \( p(\cdot) \in LH(\mathbb{R}^n) \cap P(\mathbb{R}^n) \), \( \Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})(r \geq 1) \) and \( w \in A_{p(\cdot)} \). Then the parameterized Littlewood–Paley type operators \( \mu_{\Omega, s}^{\rho} \) and \( \mu_{\Omega, \lambda}^{\rho} \) with variable kernel are bounded on \( L^{p(\cdot)}(w) \).

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3 Main Theorems and their proofs

In this section, we will prove the boundedness of the parameterized Littlewood–Paley operators with variable kernels on variable weighted Herz-Morrey spaces.

**Theorem 21.** Let \( p(\cdot) \in LH(\mathbb{R}^n) \cap P(\mathbb{R}^n) \), \( 0 < q_1 \leq q_2 < \infty \), \( \lambda > 2, 2p - n > 0 \) and \( \Omega \in L^\infty(\mathbb{R}^n) \times L^2(S^{n-1}) \) satisfies (1) and (2). If \( w^{p(\cdot)}_1 \in A_1 \) and \( -n\delta_1 < \alpha < n\delta_2 \), where \( \delta_1, \delta_2 \) are the constants in Lemma \( 17 \), then the operator \( \mu_{\Omega,s} \) is bounded from \( M^{\alpha,\lambda}\Omega_{q_2,p_1(\cdot)}(w^{p(\cdot)}) \) to \( M^{\alpha,\lambda}_p\Omega_{q_1,p_1(\cdot)}(w^{p(\cdot)}) \).

**Theorem 22.** Let \( p(\cdot) \in LH(\mathbb{R}^n) \cap P(\mathbb{R}^n) \), \( 0 < q_1 \leq q_2 < \infty \), \( \lambda > 2, 2p - n > 0 \) and \( \Omega \in L^\infty(\mathbb{R}^n) \times L^2(S^{n-1}) \) satisfies (1) and (2). If \( w^{p(\cdot)}_1 \in A_1 \) and \( -n\delta_1 < \alpha < n\delta_2 \), where \( \delta_1, \delta_2 \) are the constants in Lemma \( 17 \), then the operator \( \mu_{\Omega,s}^p \) is bounded from \( M^{\alpha,q_2}_p\Omega_{p_1(\cdot)}(w^{p(\cdot)}) \) to \( M^{\alpha,q_1}_p\Omega_{p_1(\cdot)}(w^{p(\cdot)}) \).

**Remark 23.** As is well known that, \( \mu_{\Omega,s}^p f(x) \leq 2^{n\lambda} \mu_{\Omega,s} f(x) \) (see [27], p.89). Therefore, we give only the proof of Theorem 22.

**Proof.** Let \( f \in M^{\alpha,\lambda}_p\Omega_{q_2,p_1(\cdot)}(w^{p(\cdot)}) \). By the Jensen’s inequality, we have

\[
\|\mu_{\Omega,s}^p f\|_{M^{\alpha,\lambda}_p\Omega_{q_2,p_1(\cdot)}(w^{p(\cdot)})} \leq \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda q_1} \left( \sum_{k=\infty}^{k_0} 2^{\alpha q_2 k} \| (\mu_{\Omega,s}^p f) \chi_k \|_{L^{p_1(\cdot)}(w^{p(\cdot)})}^q \right)^{q_1/q_2} 
\]

Denote \( f_j := f \chi_j \) for each \( j \in \mathbb{Z} \), then \( f := \sum_{j=-\infty}^{\infty} f_j \), so we have

\[
\|\mu_{\Omega,s}^p f\|_{M^{\alpha,\lambda}_p\Omega_{q_2,p_1(\cdot)}(w^{p(\cdot)})} \leq \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda q_1} \left( \sum_{k=\infty}^{k_0} 2^{\alpha q_1 k} \| (\mu_{\Omega,s}^p f) \chi_k \|_{L^{p_1(\cdot)}(w^{p(\cdot)})}^q \right)^{q_1} 
\]

\[
= \sum_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda q_1} \left( \sum_{k=\infty}^{k_0} 2^{\alpha q_1 k} \| (\mu_{\Omega,s}^p f) \chi_k \|_{L^{p_1(\cdot)}(w^{p(\cdot)})}^q \right)^{q_1} 
\]

\[
= F_1 + F_2 + F_3. 
\]
First, we consider $F_2$. Using Corollary 20 and $-2 \leq k - j \leq 2$, it is easy to get

\[
F_2 = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k = -\infty}^{k_0} 2^{\alpha k} \left( \sum_{j = k-2}^{k+2} \|\nu_{\Omega, \lambda}^{s, \rho} f_j\|_{L^p(U_1)} \right)^{q_1} \\
\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k = -\infty}^{k_0} 2^{\alpha (k-j)} \|f_j\|_{L^p(U_1)}^{q_1} \\
\leq C\|f\|_{MK^{n, q_1}_{U_1}}^{q_1},
\]

Now we need to consider $\nu_{\Omega, \lambda}^{s, \rho} f_j$. Applying the Minkowski inequality, we conclude that

\[
|\nu_{\Omega, \lambda}^{s, \rho} (f_j)(x)| = \left( \int_0^\infty \int_{\mathbb{R}^n} \frac{t}{t + |x - y|} |\Omega(y, y - z)| \left| \frac{1}{|y - z|^{n - \rho}} \Omega(y, y - z) \right|^2 dy dt \frac{1}{t^{n+1}} \right)^{\frac{1}{2}} \\
\leq \int_{\mathbb{R}^n} f_j(z) \left( \int_0^\infty \int_{|y - z| \leq t} \frac{t}{t + |x - y|} |\Omega(y, y - z)| \left| \frac{1}{|y - z|^{n - \rho}} \Omega(y, y - z) \right|^2 dy dt \frac{1}{t^{n+1}} \right)^{\frac{1}{2}} dz \\
\leq \int_{\mathbb{R}^n} f_j(z) \left( \int_0^{|x - z|} \int_{|y - z| \leq t} \frac{t}{t + |x - y|} |\Omega(y, y - z)| \left| \frac{1}{|y - z|^{n - \rho}} \Omega(y, y - z) \right|^2 dy dt \frac{1}{t^{n+1}} \right)^{\frac{1}{2}} dz \\
+ \int_{\mathbb{R}^n} f_j(z) \left( \int_{|x - z|}^\infty \int_{|y - z| \leq t} \frac{t}{t + |x - y|} |\Omega(y, y - z)| \left| \frac{1}{|y - z|^{n - \rho}} \Omega(y, y - z) \right|^2 dy dt \frac{1}{t^{n+1}} \right)^{\frac{1}{2}} dz.
\]

For $\Omega \in L^\infty(\mathbb{R}^n) \times L^2(S^{n-1})$ and $2\rho - n > 0$, the following inequality holds

\[
\int_{|y - z| \leq t} \frac{|\Omega(y, y - z)|^2}{|y - z|^{2n - 2\rho}} dy \\
\leq \int_{s^{n-1}} \int_0^t \frac{|\Omega(sy' + z, y')|^2}{s^{2n - 2\rho}} s^{n-1} ds d\sigma(y') \\
\leq \|\Omega\|^2_{L^\infty(\mathbb{R}^n) \times L^2(S^{n-1})} t^{2\rho - n}.
\]

Since $|x - z| \leq |x - y| + |y - z| \leq |x - y| + t$. For $\lambda > 2$, taking $0 < \delta < (\lambda - 2)n$, so
we have

\[
\int_0^{[x-z]}\int_{|y-z|\leq t} \left(\frac{t}{t + |x-y|}\right)^{\lambda n} \frac{|\Omega(y, y - z)|^2}{|y - z|^{2n-2\rho}} \frac{dydt}{t^{2\rho + n+1}} \\
\leq \int_0^{[x-z]}\int_{|y-z|\leq t} \left(\frac{t}{t + |x-y|}\right)^{\lambda n-2n-\delta} \frac{1}{|x-z|^{2n+\delta}} \frac{|\Omega(y, y - z)|^2}{|y - z|^{2n-2\rho}} \frac{dydt}{t^{2\rho-n-\delta+1}} \\
\leq \frac{1}{|x-z|^{2n+\delta}} \int_0^{[x-z]} \int_{|y-z|\leq t} \frac{|\Omega(y, y - z)|^2}{|y - z|^{2n-2\rho}} \frac{dydt}{t^{2\rho-n-\delta+1}} \\
\leq \frac{\|\Omega\|_{L^\infty(\mathbb{R}^n) \times L^2(S^{n-1})}^2}{|x-z|^{2n+\delta}} \int_0^{[x-z]} t^{\delta-1} dt \\
\leq C|x-z|^{-2n}.
\]

If we take \(1 < \lambda_1 < 2\), then \(\lambda_1 n - n > 0\) and \(\lambda_1 n - 2n < 0\), we have

\[
\int_{[x-z]} \int_{|y-z|\leq t} \left(\frac{t}{t + |x-y|}\right)^{\lambda n} \frac{|\Omega(y, y - z)|^2}{|y - z|^{2n-2\rho}} \frac{dydt}{t^{2\rho + n+1}} \\
\leq \int_{[x-z]} \int_{|y-z|\leq t} |x-z|^{-\lambda_1 n} \frac{|\Omega(y, y - z)|^2}{|y - z|^{2n-2\rho}} \frac{dydt}{t^{2\rho-\lambda_1 n+n+1}} \\
\leq \int_{[x-z]} |x-z|^{-\lambda_1 n} \int_{|y-z|\leq t} \frac{|\Omega(y, y - z)|^2}{|y - z|^{2n-2\rho}} \frac{dydt}{t^{\rho+1}} \\
\leq \int_{[x-z]} |x-z|^{-\lambda_1 n} \int_{s^{n-1}} \int_0^t \frac{|\Omega(y, y - z)|^2}{s^{2n-\lambda_1 n}} \frac{ds}{s^{n-1}} dsd\sigma(y') \frac{dt}{t^{\rho+1}} \\
\leq C\|\Omega\|_{L^\infty(\mathbb{R}^n) \times L^2(S^{n-1})}^2 |x-z|^{-\lambda_1 n} \int_{[x-z]} t^{\lambda_1 n - 2n - 1} dt \\
\leq C|x-z|^{-2n}.
\]

Combining the above two estimates, we obtain

\[
\mu_{\Omega,\lambda}^n(f)(x) \leq \int_{\mathbb{R}^n} \frac{|f(z)|}{|x-z|^n} dz. \tag{3}
\]

Next, we consider \(F_1\). Noting that for \(x \in A_k, z \in A_j\) and \(j \leq k - 2\), then \(|x - z| \sim |x|\). By the virtue of the generalized Hölder’s inequality, we have

\[
\mu_{\Omega,\lambda}^n(f_j)(x) \leq C2^{-kn} \|f_j\|_{L^{p_1}(\omega^{p_1})} \|\chi_j\|_{L^{p_1}(\omega^{p_1})'}.
\]
Applying Lemma 13 and Lemma 17, we take \( \| \cdot \|_{L^{p_1}(w^{p_1}(\cdot))} \) for each side, we have

\[
\| \mu^{\alpha}_{\Omega,\lambda}(f_j)(x) \chi_k \|_{L^{p_1}(w^{p_1}(\cdot))} \\
\leq C 2^{-kn} \| f_j \|_{L^{p_1}(w^{p_1}(\cdot))} \| \chi_k \|_{L^{p_1}(w^{p_1}(\cdot))} \\
\leq C 2^{-kn} \| f_j \|_{L^{p_1}(w^{p_1}(\cdot))} \| \chi B_k \|_{L^{p_1}(w^{p_1}(\cdot))} \| \chi B_k \|_{L^{p_1}(w^{p_1}(\cdot))} \\
\leq C \| f_j \|_{L^{p_1}(w^{p_1}(\cdot))} \| \chi B_k \|_{L^{p_1}(w^{p_1}(\cdot))} \| \chi B_k \|_{L^{p_1}(w^{p_1}(\cdot))}^{-1} \\
\leq C \| f_j \|_{L^{p_1}(w^{p_1}(\cdot))} \| \chi B_k \|_{L^{p_1}(w^{p_1}(\cdot))}^{-1} \\
\leq C 2^{j-k}n \delta_2 \| f_j \|_{L^{p_1}(w^{p_1}(\cdot))}.
\]

On the other side, we have the following fact:

\[
\| f_j \|_{L^{p_1}(w^{p_1}(\cdot))} = 2^{-j \alpha} \left( \sum_{n=\infty}^{j} 2^{nq_1} \| f_n \|_{L^{p_1}(w^{p_1}(\cdot))} \right)^{1/q_1} \\
\leq C 2^{-j \alpha} \left( \sum_{n=\infty}^{j} 2^{nq_1} \| f_n \|_{L^{p_1}(w^{p_1}(\cdot))} \right)^{1/q_1} \\
= C 2^{j(\lambda-\alpha)} \left( \sum_{n=\infty}^{j} 2^{-\lambda j} \left( 2^{nq_1} \| f_n \|_{L^{p_1}(w^{p_1}(\cdot))} \right)^{1/q_1} \right) \\
\leq C 2^{j(\lambda-\alpha)} \| f \|_{M^{\alpha,q_1}_{p_1}(w^{p_1}(\cdot))}.
\]

Thus, by using condition \( \alpha < \lambda + n \delta_2 \), it follows that

\[
F_1 = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k} \left( \sum_{j=-\infty}^{k-2} \| \mu^{\alpha}_{\Omega,\lambda}(f_j)(x) \chi_k \|_{L^{p_1}(w^{p_1}(\cdot))} \right)^{q_1} \\
\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k} \left( \sum_{j=-\infty}^{k-2} 2^{(j-k) n \delta_2} \| f_j \|_{L^{p_1}(w^{p_1}(\cdot))} \right)^{q_1} \\
\leq C \| f \|_{M^{\alpha,q_1}_{p_1}(w^{p_1}(\cdot))} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k} \left( \sum_{j=-\infty}^{k-2} 2^{(j-k) n \delta_2} 2^{j(\lambda-\alpha)} \right)^{q_1} \\
\leq C \| f \|_{M^{\alpha,q_1}_{p_1}(w^{p_1}(\cdot))} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\lambda q_1 k} \left( \sum_{j=-\infty}^{k-2} 2^{(j-k)(\alpha-\lambda n \delta_2)} \right)^{q_1} \\
\leq C \| f \|_{M^{\alpha,q_1}_{p_1}(w^{p_1}(\cdot))} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \left( \sum_{k=-\infty}^{k_0} 2^{\lambda q_1 k} \right). 
\]
Finally, we estimate $F_3$. Noting that for $x \in A_k$, $y \in A_j$ and $j \geq k + 2$, then $|y - x| \sim |y|$. By (3) and the virtue of the generalized Hölder’s inequality, we show that

\[ \mu^{\alpha,\rho}_{\Omega,\lambda}(f_j)(x) \leq C 2^{-jn} \| f_j \|_{L^p(\Omega)} \| \chi_j \|_{L^p(\Omega)} \| \chi_k \|_{L^p(\Omega)} \].

Applying Lemma 13 and Lemma 17, we can take $\| \cdot \|_{L^p(\Omega)}$ for each side, we have

\[
\begin{align*}
&\| \mu^{\alpha,\rho}_{\Omega,\lambda}(f_j)(x) \chi_k \|_{L^p(\Omega)} \\
&\leq C 2^{-jn} \| f_j \|_{L^p(\Omega)} \| \chi_j \|_{L^p(\Omega)} \| \chi_k \|_{L^p(\Omega)} \\
&\leq C 2^{-jn} \| f_j \|_{L^p(\Omega)} \| \chi_B_k \|_{L^p(\Omega)} \| \chi_B_j \|_{L^p(\Omega)}^{-1} \\
&\leq C 2^{(k-j)n\delta_1} \| f_j \|_{L^p(\Omega)}.
\end{align*}
\]

Then, by using condition $\alpha > \lambda - n\delta_1$, it follows that

\[
F_3 = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k = -\infty}^{k_0} 2^{k \lambda q_1} \left( \sum_{j = k+2}^{\infty} \| \mu^{\alpha,\rho}_{\Omega,\lambda}(f_j) \chi_k \|_{L^p(\Omega)} \right)^q \\
\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k = -\infty}^{k_0} 2^{k \lambda q_1} \left( \sum_{j = k+2}^{\infty} 2^{(k-j)n\delta_1} \| f_j \|_{L^p(\Omega)} \right)^q \\
\leq C \| f \|_{M^{\alpha,q_1}_{p,1}(\Omega)} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k = -\infty}^{k_0} 2^{k \lambda q_1} \left( \sum_{j = k+2}^{\infty} 2^{(k-j)(\lambda - n\delta_1)} \right)^q \\
\leq C \| f \|_{M^{\alpha,q_1}_{p,1}(\Omega)} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k = -\infty}^{k_0} 2^{\lambda q_1} \left( \sum_{j = k+2}^{\infty} 2^{(k-j)(\alpha - n\delta_1)} \right)^q \\
\leq C \| f \|_{M^{\alpha,q_1}_{p,1}(\Omega)} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \left( \sum_{k = -\infty}^{k_0} 2^{\lambda q_1} \right)^q
\]

This completes the proof of Theorem 22. \hfill \Box
4 Fractional integral operator with variable kernel on $M\hat{K}^{\alpha,q}_{p(\cdot)}(w^{p(\cdot)})$ spaces

Let $0 < \mu < n$ and $\Omega \in L^{\infty}(\mathbb{R}^n) \times L^{r}(S^{n-1})$ satisfies (i)-(ii), the fractional integral operator with variable kernel $T_{\Omega,\mu}$ is defined by

$$T_{\Omega,\mu}f(y) = \int_{\mathbb{R}^n} \frac{\Omega(y, y - z)f(z)}{|y - z|^{n-\mu}} dz.$$ 

In this section, we will prove some main results of this paper. We begin two preliminary Lemmas.

Lemma 24 ([29],[30]). Let $\Omega \in L^{\infty}(\mathbb{R}^n) \times L^{r}(S^{n-1})(r \geq 1)$ and $w^{r}(x) \in A(p/r', q/r')$. If $0 < \mu < n$, $1 \leq r' < p < \frac{n}{\mu}$, $\frac{1}{p} - \frac{1}{q} = \frac{n}{\mu}$, then we have

$$\left( \int_{\mathbb{R}^n} |T_{\Omega,\mu}f(x)w(x)|^q dx \right)^{1/q} \leq \left( \int_{\mathbb{R}^n} |f(x)w(x)|^p dx \right)^{1/p}.$$ 

Lemma 25 ([19]). Assume that $1 < p_0 \leq q_0 < \infty$ and every $w_0 \in A_{p_0,q_0}$,

$$\left( \int_{\mathbb{R}^n} f(x)^{q_0}w_0(x)^{q_0} dx \right)^{1/q_0} \leq C \left( \int_{\mathbb{R}^n} g(x)^{p_0}w_0(x)^{p_0} dx \right)^{1/p_0},$$

Given $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}.$$ 

Define $\sigma \geq 1$ by $\frac{1}{\sigma} = \frac{1}{p_0} - \frac{1}{q_0}$. If $w \in A_{p(\cdot),q(\cdot)}$ and $(q(\cdot)/\sigma, w^{\sigma})$ is an $M$-pair then

$$\|f\|_{L^{q(\cdot)}(w)} \leq C\|g\|_{L^{p(\cdot)}(w)},$$

$(f, g) \in \mathcal{F}$.

Lemma 25 holds for $p_0 = 1$ and the maximal operator is bounded on $L^{(q(\cdot)/q_0)'(w^{-q_0})}$.

Izuki and Noi [20] proved the Hardy–Littlewood maximal operator $M$ is bounded on $L^{q(\cdot)/\sigma}(w^{\sigma}, q(\cdot)/\sigma)$ and $L^{(q(\cdot)/\sigma)'(w^{-\sigma(q(\cdot)/\sigma)'})}$, then $(q(\cdot)/\sigma, w^{\sigma})$ is an $M$-pair when $w \in A_{p(\cdot),q(\cdot)}$ and $p(\cdot), q(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$.

From Lemma 24 and Lemma 25, we have the following conclusion.

Corollary 26. If $p_1(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, $\Omega \in L^{\infty}(\mathbb{R}^n) \times L^{r}(S^{n-1})(r \geq 1)$, $0 < \mu < n/p_1^*$ and $1/p_1(\cdot) - 1/p_2(\cdot) = \mu/n$. Then for $w \in A_{p_1(\cdot),p_2(\cdot)}$, the fractional integral operator with variable kernel $T_{\Omega,\mu}$ is bounded from $L^{p_1(\cdot)}(w^{p_1(\cdot)})$ to $L^{p_2(\cdot)}(w^{p_2(\cdot)})$. 

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Theorem 27. Let $0 \leq \lambda < \infty$, $0 < q_1 \leq q_2 < \infty$, $p_2(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and 
$\Omega \in L^\infty(\mathbb{R}^n) \times L^r(\mathbb{S}^{n-1})(r \geq 1)$. If $w^{p_2(\cdot)} \in A_1$ and $\lambda - n\delta_1 < \alpha < n\delta_2 - \frac{n}{r} + \lambda$, where $\delta_1, \delta_2$ are the constants in Lemma 17. Define $p_2(\cdot)$ by $1/p_1(\cdot) - 1/p_2(\cdot) = \mu/n$, then the operator $T_{\Omega, \mu}$ is bounded from $MK^{\alpha, \lambda}_{q_2, p_2(\cdot)}(w^{p_2(\cdot)})$ to $MK^{\alpha, \lambda}_{q_1, p_1(\cdot)}(w^{p_1(\cdot)})$.

Proof. Let $f \in MK^{\alpha, \lambda}_{q_1, p_1(\cdot)}(w^{p_1(\cdot)})$. By the Jensen’s inequality, we have

$$
\|T_{\Omega, \mu}f\|_{MK^{\alpha, \lambda}_{q_2, p_2(\cdot)}(w^{p_2(\cdot)})} \leq \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda q_1} \left( \sum_{k = -\infty}^{k_0} 2^a q_2 k \left( \sum_{j = -\infty}^{k-2} \| (T_{\Omega, \mu}f) \chi_k \|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \right) \right)^{q_1/q_2}
$$

Denote $f_j := f \chi_j$ for each $j \in \mathbb{Z}$, then $f := \sum_{j = -\infty}^{\infty} f_j$, so we have

$$
\|T_{\Omega, \mu}f\|_{MK^{\alpha, \lambda}_{q_2, p_2(\cdot)}(w^{p_2(\cdot)})} \leq \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k = -\infty}^{k_0} 2^a q_1 k \left( \sum_{j = -\infty}^{k-2} \| (T_{\Omega, \mu}f_j) \chi_k \|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \right)^{q_1/q_2}
$$

First, we consider $H_2$. Using Corollary 26 and $-2 \leq k - j \leq 2$, it is easy to get

$$
H_2 = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k = -\infty}^{k_0} 2^a q_1 k \left( \sum_{j = -\infty}^{k-2} \| (T_{\Omega, \mu}f_j) \chi_k \|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \right)^{q_1/q_2}
$$

The document also includes references and a web address.
Now we turn to estimate of $H_1$. Noting that $j \leq k - 2$ for every $j, k \in \mathbb{Z}$, then we have the following inequality by applying the generalized Hölder’s inequality

$$|T_{\Omega,\mu}f_j| \leq C2^{-k(n-\mu)}\|\Omega(y,y-z)\|_{L^r(\mathbb{R}^n)}\|f_j(z)\|_{L^p(\mathbb{R}^n)}.$$ 

By virtue of the generalized Hölder’s inequality, we have

$$\|f_j(z)\|_{L^{r'}} \leq |B_j|^{-\frac{1}{p'}}\|f_jw\|_{L^{p1'}(\mathbb{R}^n)}\|w^{-1}\chi_j\|_{L^{p1'}(\mathbb{R}^n)},$$

(4)

for any $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$, we get by (4)

$$|T_{\Omega,\mu}f_j| \leq C2^{-k(n-\mu)}2^{(k-j)\frac{n}{p}}\|f_jw\|_{L^{p1'}(\mathbb{R}^n)}\|w^{-1}\chi_j\|_{L^{p1'}(\mathbb{R}^n)}.$$ 

(5)

By 5 and taking $\|(\cdot)\|_{L^{p2'}(w^{p2'})}$ for each side, we have

$$\|((T_{\Omega,\mu}f_j)\chi_k)\|_{L^{p2'}(w^{p2'})} \leq C2^{-k(n-\mu)}2^{(k-j)\frac{n}{p}}\|f_jw\|_{L^{p1'}(\mathbb{R}^n)}\|w^{-1}\chi_j\|_{L^{p1'}(\mathbb{R}^n)}\|\chi_k\|_{L^{p2'}(w^{p2'})} \leq C2^{-k(n-\mu)}2^{(k-j)\frac{n}{p}}\|f_jw\|_{L^{p1'}(\mathbb{R}^n)}\|\chi_k\|_{L^{p2'}(w^{p2'})}.$$ 

Noting that $\chi_{B_k}(x) \leq C2^{-k\mu}T_{\Omega,\mu}(\chi_{B_k})(x)$, by the Corollary 26, we obtain

$$\|\chi_{B_k}\|_{L^{p2'}(w^{p2'})} \leq C2^{-k\mu}\|T_{\Omega,\mu}(\chi_{B_k})\|_{L^{p2'}(w^{p2'})} \leq C2^{-k\mu}\|\chi_{B_k}\|_{L^{p1'}(w^{p1'})}.$$ 

From this and using Lemma 13 and Lemma 17, we have

$$\|((T_{\Omega,\mu}f_j)\chi_k)\|_{L^{p2'}(w^{p2'})} \leq C2^{-k(n-\mu)}2^{(k-j)\frac{n}{p}}\|f_jw\|_{L^{p1'}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p1'}(w^{p1'})}\|\chi_{B_k}\|_{L^{p1'}(w^{p1'})}^{2-k\mu}\|\chi_{B_k}\|_{L^{p1'}(w^{p1'})} \leq C2^{(k-j)\frac{n}{p}}\|f_jw\|_{L^{p1'}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p1'}(w^{p1'})}\|\chi_{B_k}\|_{L^{p1'}(w^{p1'})}^{2-k\mu}\|\chi_{B_k}\|_{L^{p1'}(w^{p1'})} \leq C2^{(k-j)\frac{n}{p}}\|f_jw\|_{L^{p1'}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p1'}(w^{p1'})}\|\chi_{B_k}\|_{L^{p1'}(w^{p1'})}\|\chi_{B_k}\|_{L^{p1'}(w^{p1'})}^{2-k\mu}\|\chi_{B_k}\|_{L^{p1'}(w^{p1'})}.$$
On the other side, we have the following fact:

\[
\|f_j\|_{L^{p_1}(\mathbb{w}^{p_1}(\cdot))} = 2^{-j\alpha} \left(2^{j\alpha q_1} \|f_j\|_{L^{p_1}(\mathbb{w}^{p_1}(\cdot))}^{q_1}\right)^{1/q_1}
\]

\[
\leq C2^{-j\alpha} \left(\sum_{n=-\infty}^{n} 2^{n\alpha q_1} \|f_n\|_{L^{p_1}(\mathbb{w}^{p_1}(\cdot))}^{q_1}\right)^{1/q_1}
\]

\[
= C2^{j(\lambda-\alpha)} \left(2^{-\lambda j} \left(\sum_{n=-\infty}^{n} 2^{n\alpha q_1} \|f_n\|_{L^{p_1}(\mathbb{w}^{p_1}(\cdot))}^{q_1}\right)^{1/q_1}\right)\]

\[
\leq C2^{j(\lambda-\alpha)} \|f\|_{M^{\alpha,q_1}_{p_1}(\mathbb{w}^{p_1}(\cdot))}.
\]

Thus, by using condition \(\alpha < \lambda + n\delta_2 - \frac{n}{r}\), it follows that

\[
H_1 = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k} \left(\sum_{j=-\infty}^{j} \|(T_{\Omega,\mu} f_j)\|_{L^{p_1}(\mathbb{w}^{p_1}(\cdot))}^{q_1}\right)
\]

\[
\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k} \left(\sum_{j=-\infty}^{j} 2^{-j(\frac{\alpha}{\lambda} - n\delta_2)} \|f_j\|_{L^{p_1}(\mathbb{w}^{p_1}(\cdot))}^{q_1}\right)
\]

\[
\leq C \|f\|_{M^{\alpha,q_1}_{p_1}(\mathbb{w}^{p_1}(\cdot))} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k} \left(\sum_{j=-\infty}^{j} 2^{-j(1 - n\delta_2)} 2^{j(\lambda-\alpha)}\right)
\]

\[
\leq C \|f\|_{M^{\alpha,q_1}_{p_1}(\mathbb{w}^{p_1}(\cdot))} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k} \left(\sum_{j=-\infty}^{j} 2^{j(\alpha - n\delta_2 + \frac{n}{r})}\right)
\]

\[
\leq C \|f\|_{M^{\alpha,q_1}_{p_1}(\mathbb{w}^{p_1}(\cdot))} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \left(\sum_{k=-\infty}^{k_0} 2^{\alpha q_1 k}\right).
\]

Finally, we estimate \(U_3\). For every \(j, k \in \mathbb{Z}\) with \(j \geq k + 2\) and applying the generalized Hölder’s inequality assures that

\[
|T_{\Omega,\mu} f_j| \leq C2^{-j(n-\mu)} \|\Omega(y, y - z)\|_{L^r(\mathbb{R}^n)} \|f_j(z)\|_{L^{r'}(\mathbb{R}^n)}.
\]

For any \(\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})\), we get by (4)

\[
|T_{\Omega,\mu} f_j| \leq C2^{-j(n-\mu)} \|f_j w\|_{L^{p_1}(\cdot)} \|w^{-1} \chi_j\|_{L^{p_2'}(\cdot)}.
\]

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from this and applying Lemma 17 and taking $\| \cdot \|_{L^p(\omega^p)}$ for each side, we have

$$\| T_{\Omega, \mu} f_j(x) \chi_k \|_{L^p(\omega^p)} \leq C 2^{-j(n-\mu)} \| f_j \|_{L^p(\omega^p)} \| \chi_j \|_{L^p(\omega^p)}$$

Then, we obtain that

$$\| T_{\Omega, \mu} f_j(x) \chi_k \|_{L^p(\omega^p)} \leq C 2^{j-k} \| f_j \|_{L^p(\omega^p)}.$$

The rest of the proof is the same as the proof of $F_3$ in Theorem 22, it is not difficult to obtain

$$H_3 \leq C \| f \|^{\eta_1}_{MK^\alpha_{p_1}(\omega^p)}.$$

Thus, we omit the details there. Then the proof of Theorem 27 is finished. \qed

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