Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 12 (2017), 193 – 201

A NEW CLASS OF INTEGRAL RELATION INVOLVING $\aleph$-FUNCTIONS

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Abstract. The present article is devoted to evaluate new integral relations involving the $\aleph$-functions. The results are expressed in the terms of the Psi functions. Being unified and general in nature, these integrals yield a number of known and new results as special cases. For the sake of illustration, six cases are also recorded here as special case of our main results.

1 Introduction and Preliminaries

Aleph-function is very significant special functions and their closely related ones are widely used in physics and engineering; therefore they are interest physicists and engineers as well as mathematicians. In recent years, many integral formulas involving a diversity of special functions have been expanded by many authors. Here we aim at presenting four generalized integral formulas, which are expressed in terms of the Psi (or digamma) functions $\psi(z)$ with $\aleph$-functions. Some interesting special cases of our main results are also considered.
The $\aleph$-function, which is a general higher transcendental function and was introduced by Südland et al. ([14], [15]), is defined in terms of the Mellin- Barnes type integrals as following manner (see, e.g., [11], [12])

$$\aleph\,(z) = \aleph^{m,n}_{p_i,\,q_i,\,\tau_i;\,r}\left(z \,\middle|\, \begin{matrix} (a_j,\,\alpha_j)_{1,n}\,,\,\ldots,\,[\tau_i\,(a_{ji},\,\alpha_{ji})\,]_{n+1,\,p_i;\,r} \\ (b_j,\,\beta_j)_{1,m}\,,\,\ldots,\,[\,\tau_i\,(b_{ji}\,,\,\beta_{ji})\,]_{\,m+1,q_i;\,r} \end{matrix}\right)$$
$$= \frac{1}{2\pi i}\int_L \Omega^{\,m,n}_{\,p_i,q_i,\,\tau_i;\,r}\,(\zeta)\,z^{-\zeta}d\zeta, \tag{1.1}$$

where $z \in \mathbb{C}$ - 0, $i = \sqrt{-1}$ and

$$\Omega^{m,\,n}_{p_i,q_i,\tau_i;\,r}\,(\zeta) = \frac{\Pi^m_{j=1}\Gamma\,(b_j+\beta_j\zeta)\,\Pi^n_{j=1}\Gamma\,(1-a_j-\alpha_j\zeta)}{\sum^r_{i=1}\;\tau_i\,\Pi^{q_i}_{j=m+1}\Gamma\,(1-b_{ji}-\beta_{ji}\,\zeta)\,\Pi^{p_i}_{j=n+1}\Gamma\,(a_{ji}+\alpha_{ji}\zeta)}, \tag{1.2}$$

2010 Mathematics Subject Classification: 33C45; 33C60.
Keywords: $\aleph$-function; Psi (or digamma) functions $\psi(z)$.

The integration path $L = L_{i\gamma\infty}\,(\gamma \in \mathbb{R})$ ranging from $\gamma - i\infty$ to $\gamma + i\infty$; is a contour of the Mellin-Barnes type, which separates the poles of $\Gamma\left(1 - a_j - \alpha_j\zeta\right),\ (j = 1, \ldots, n)$ from $\Gamma\left(b_j + \beta_j\zeta\right),\ (j = 1, \ldots, m)$. The empty product in (1.2) is interpreted as unity. The parameters $p_i,\, q_i \in \mathbb{N}_0$ with $0 \le n \le p_i,\, 1 \le m \le q_i,\, \tau_i > 0\ (i = 1, \ldots, r)$ and $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji} > 0$ where as $a_j, b_j, a_{ji}, b_{ji} \in \mathbb{C}$. The existence of the $\aleph$-function defined on (1.1) depends on the following conditions.

$$\varphi_l > 0\,,\quad |\arg(z)| < \frac{\pi}{2}\varphi_l\,, l = 1, \ldots, r\,, \tag{1.3}$$

and

$$\varphi_l \ge 0\,,\ |\arg(z)| < \frac{\pi}{2}\varphi_l\,, l = 1, \ldots, r \quad \Re(\zeta) + 1 < 0, \tag{1.4}$$

where

$$\varphi_l = \sum_{j=1}^{n}\alpha_j + \sum_{j=1}^{m}\beta_j - \tau_l\left(\sum_{j=n+1}^{p_l}\alpha_{jl} + \sum_{j=m+1}^{q_l}\beta_{jl}\right), \tag{1.5}$$

and

$$\zeta = \sum_{j=1}^{m} b_j - \sum_{j=1}^{n} a_j + \tau_l\left(\sum_{j=m+1}^{q_l} b_{jl} - \sum_{j=n+1}^{p_l} a_{jl}\right) + \frac{1}{2}(p_l - q_l)\,, l = 1, \ldots, r. \tag{1.6}$$

For the convergence conditions and other details of $\aleph$-function, see Südland et al. [14]. The series representation of $\aleph$-function is introduced by Chaurasia et al. [3] as follow:

$$\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[z \left| \begin{matrix} (a_j,\,\alpha_j)_{1,n}\,, \ldots, [\tau_i\,(a_{ji},\,\alpha_{ji})]_{n+1,p_i;r} \\ (b_j,\,\beta_j)_{1,m}\,, \ldots, [\tau_i\,(b_{ji},\,\beta_{ji})]_{m+1,q_i;r} \end{matrix} \right.\right] = \sum_{h=1}^{m}\sum_{k=0}^{\infty}\frac{(-1)^k\Omega^{m,n}_{p_i,q_i,\tau_i;r}(\zeta)}{\beta_h\,k!}(z)^{-\zeta}, \tag{1.7}$$

with $\zeta = \frac{b_h + k}{\beta_h}$, $p_i < q_i$, $|z| < 1$ and $\Omega^{m,n}_{p_i,q_i,\tau_i;r}(\zeta)$ is given in (1.2).

Remark 1: On setting $\tau_i = 1\,(i = 1, \ldots, r)$ in (1.1), yields the I-function due to Saxena [10], defined in following manner:

$$I^{m,n}_{p_i,q_i;r}[z] = \aleph^{m,n}_{p_i,q_i,1;r}[z] = \aleph^{m,n}_{p_i,q_i,1;r}\left[z \left| \begin{matrix} (a_j,\,\alpha_j)_{1,n}\,, \ldots, [(a_{ji},\,\alpha_{ji})]_{n+1,p_i} \\ (b_j,\,\beta_j)_{1,m}\,, \ldots, [(b_{ji},\,\beta_{ji})]_{m+1,q_i} \end{matrix} \right.\right]$$

$$= \frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i,1;r}(\zeta)\, z^{-\zeta} d\zeta. \tag{1.8}$$

where the kernel $\Omega^{m,n}_{p_i,q_i,1;r}(\zeta)$ is given in (1.2). The existence conditions for the integral in (1.8) are the same as given in (1.4)-(1.6) with $\tau_i = 1\,(i = 1, \ldots, r)$.

Remark 2: If we set $r = 1$, then (1.8) reduces to the familiar H-function [4]:

$$H_{p,q}^{m,n}[z] = \aleph_{p_i,q_i,1;1}^{m,n}[z] = \aleph_{p_i,q_i,1;1}^{m,n}\left[z \left| \begin{matrix} (a_j,\alpha_j) \\ (b_j,\beta_j) \end{matrix} \right.\right] = \frac{1}{2\pi i}\int_L \Omega_{p_i,q_i,1;1}^{m,n}(\zeta)\, z^{-\zeta} d\zeta. \tag{1.9}$$

are the kernel $\Omega_{p_i,q_i,1;1}^{m,n}(\zeta)$ can be obtained from (1.2).

The notation of $\aleph$-function (1.1) does not follow wholly the usual of the Fox's H-function notation. Namely, in the kernel $\Omega_{p_i,q_i,\tau_i;r}^{m,n}(\zeta)$ of $\aleph$-function (1.2), parameter couples $(a_j, \alpha_j)_{1,n}$, $(b_j, \beta_j)_{1,m}$ construct the Gamma function terms exclusively in the numerator, and $(\tau_i(a_{ji}, \alpha_{ji}))_{n+1,p_i;r}$, $(\tau_i(b_{ji}, \beta_{ji}))_{m+1,q_i;r}$ construct the linear combination exclusively in the denominator, while for the $H_{p,q}^{m,n}[z]$ both upper $(a_j, \alpha_j)_{1,p}$ and lower couples of parameters $(b_j, \beta_j)_{1,q}$ play roles in forming both numerator and denominator terms according to m and n. However for $r = 1$, being $\tau_r > 0$, the definitions (1.1) and (1.8) clearly coincide.

A thorough and wide-ranging account of the H-function is obtainable from the monographs written by Kilbas and Saigo [7], Mathai et al. [8], Prudnikov et al. [9] and Srivastava et al. [13].

2 Main Integral formula

In this section, we establish four general integrals, which are expressed in terms of $\psi(z)$, by inserting the $\aleph$-functions (1.1) with suitable arguments into the integrand.

Lemma 1. *Let $s, t, \mu, \upsilon > 0$, $\varphi_l \geq 0$ and $|\arg z| < \varphi_l \pi/2$, Further let the constraint satisfy the conditions $\Re(s) + \mu \min(b_j/\beta_j) > 0$ and $\Re(t) + \upsilon \min(b_j/\beta_j) > 0$, then the following formula holds:*

$$\int_{-1}^{1} \frac{(1-x)^{s-1}(1+x)^{t-1}}{\{2+\lambda_1(1-x)+\lambda_2(1-x)\}^{s+t}} \aleph_{p_i,q_i,\tau_i;r}^{m,n}\left[y\frac{(1-x)^{\mu}(1+x)^{\upsilon}}{\{2+\lambda_1(1-x)+\lambda_2(1-x)\}^{\mu+\upsilon}}\right] dx$$

$$\equiv \frac{1}{2}(1+\lambda_1)^{-s}(1+\lambda_2)^{-t}\,\aleph_{p_i+2,q_i+1,\tau_i;r}^{m,n+2}\left[y(1+\lambda_1)^{\mu}(1+\lambda_2)^{\upsilon}\left| \begin{matrix} \mathrm{H}_1 \\ \mathrm{H}_2 \end{matrix} \right.\right]. \tag{2.1}$$

where

$$\mathrm{H}_1 \equiv (1-s,\mu);(1-t,\upsilon);(a_j,\alpha_j)_{1,n};[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i;r},$$

and

$$\mathrm{H}_2 \equiv (b_j,\beta_j)_{1,m};[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i;r};(1-s-t,\ \mu+\upsilon),$$

The result (2.1) *is also represented in the subsequent form with the help of equation* (1.7)*, as follows:*

$$\int_{-1}^{1} \frac{(1-x)^{s-1}(1+x)^{t-1}}{\{2+\lambda_1(1-x)+\lambda_2(1-x)\}^{s+t}} \aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[y\frac{(1-x)^{\mu}(1+x)^{\upsilon}}{\{2+\lambda_1(1-x)+\lambda_2(1-x)\}^{\mu+\upsilon}}\right]dx$$

$$\equiv \frac{1}{2}\sum_{h=1}^{m}\sum_{k=0}^{\infty}\frac{(-1)^k}{\beta_h\,k\,!}\frac{\Pi_{j=1}^{m}\Gamma\left(b_j+\beta_j\zeta\right)\,\Pi_{j=1}^{n}\Gamma\left(1-a_j-\alpha_j\zeta\right)}{\sum_{i=1}^{r}\left\{\tau_i\,\Pi_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}-\beta_{ji}\,\zeta\right)\,\Pi_{j=n+1}^{p_i}\Gamma\left(a_{ji}+\alpha_{ji}\zeta\right)\right\}}y^{-\zeta}$$

$$\times\,(1+\lambda_1)^{-s-\mu\,\zeta}(1+\lambda_2)^{-t-\upsilon\,\zeta}\frac{\Gamma\left(s+\mu\,\zeta\right)\,\Gamma\left(t+\upsilon\,\zeta\right)}{\Gamma\left(s+t+(\mu+\upsilon)\,\zeta\right)}. \tag{2.2}$$

Proof. By making use of (1.1) in the integrate of (2.1) and then interchanging the order of integral sign and summation with the conditions $\alpha\neq\beta$, $\Re\,(p)>0$, $\Re\,(q)>0$ and the x-integral with the help of the known result [5]:

$$\int_{\alpha}^{\beta}\frac{(t-\alpha)^{p}\,(\beta-t)^{q-1}}{\{\,\beta-\alpha+\lambda\,(t-\alpha)+\mu\,(\beta-t)\}^{p+q}}dt\;=\;\frac{(1+\lambda)^{-p}\,(1+\mu)^{-q}}{\beta-\alpha}\frac{\Gamma\,(p)\;\Gamma\,(q)}{\Gamma\,(p+q)},$$

In accordance with the contour integral in terms of Aleph function, we obtain the result (2.1), this result also can be expressed in series of by the help of equation (1.7), we easily arrive at (2.2). This completes the proof of the lemma. □

Now, we establish four general integrals as follows:

First Integral:

$$I_1\equiv\int_{-1}^{1}\Theta(x)\log\left(\frac{1-x}{\{2+\lambda_1(1-x)+\lambda_2(1+x)\}}\right)dx$$

$$=\frac{1}{2}\left[\sum_{h=1}^{m}\sum_{k=0}^{\infty}\frac{(-1)^k}{\beta_h\,k\,!}\frac{\Pi_{j=1}^{m}\Gamma\left(b_j+\beta_j\zeta\right)\,\Pi_{j=1}^{n}\Gamma\left(1-a_j-\alpha_j\zeta\right)}{\sum_{i=1}^{r}\;\tau_i\;\Pi_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}-\beta_{ji}\iota\zeta\right)\,\Pi_{j=n+1}^{p_i}\Gamma\left(a_{ji}+\alpha_{ji}\zeta\right)}\;(z)^{-\zeta}\right.$$

$$\times\left\{\,(1+\lambda_1)^{-s-\mu\zeta}\,(1+\lambda_2)^{-t-\upsilon\zeta}\,B(s+\mu\zeta,t+\upsilon\zeta)\right.$$

$$\times\left[-\log(1+\lambda_1)+\psi(s+\mu\zeta)-\psi(s+t+(\mu+\upsilon)\zeta\,]\}\right], \tag{2.3}$$

where

$$\Theta(x)=\frac{(1-x)^{s-1}(1+x)^{t-1}}{\{2+\lambda_1(1-x)+\lambda_2(1-x)\}^{s+t}}\,\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[y\frac{(1-x)^{\mu}(1+x)^{\upsilon}}{\{2+\lambda_1(1-x)+\lambda_2(1+x)\}^{\mu+\upsilon}}\right].$$

Proof. By taking the partial derivative of both side of (2.2) with respect to s, and also using Psi function defined in the form of digamma function as $\psi(z) = \Gamma'(z)/\Gamma(z)$, further, we get the desired result (2.3). ☐

Second Integral:

$$I_2 \equiv \int_{-1}^{1} \Theta(x) \log\left(\frac{1+x}{\{2+\lambda_1(1-x)+\lambda_2(1+x)\}}\right) dx$$

$$= \frac{1}{2}\left[\sum_{h=1}^{m}\sum_{k=0}^{\infty} \frac{(-1)^k}{\beta_h\, k\,!} \frac{\Pi_{j=1}^{m}\Gamma\left(b_j+\beta_j\zeta\right)\, \Pi_{j=1}^{n}\Gamma\left(1-a_j-\alpha_j\zeta\right)}{\sum_{i=1}^{r}\ \tau_i\ \Pi_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}-\beta_{ji}\iota\zeta\right)\, \Pi_{j=n+1}^{p_i}\Gamma\left(a_{ji}+\alpha_{ji}\zeta\right)} (z)^{-\zeta}\right.$$
$$\times \left\{ (1+\lambda_1)^{-s+\mu\zeta}\,(1+\lambda_2)^{-t+\upsilon\zeta}\, B(s+\mu\zeta,\ t+\upsilon\zeta)\right.$$
$$\times\ [-\log(1+\lambda_2)+\psi(t+\upsilon\,\zeta)-\psi(s+t+(\mu+\upsilon)\zeta\,]\,\}\,]\,. \tag{2.4}$$

Proof. Similarly taking the partial derivative of both side of (2.2) with respect to t, and also using Psi function defined in first integral with simplification, we get the desired result (2.3). ☐

Third Integral:

$$I_3 \equiv \int_{-1}^{1} \Theta(x) \log\left(\frac{1-x^2}{\{2+\lambda_1(1-x)+\lambda_2(1+x)\}^2}\right) dx$$

$$= \frac{1}{2}\left[\sum_{h=1}^{m}\sum_{k=0}^{\infty} \frac{(-1)^k}{\beta_h\, k\,!} \frac{\Pi_{j=1}^{m}\Gamma\left(b_j+\beta_j\zeta\right)\, \Pi_{j=1}^{n}\Gamma\left(1-a_j-\alpha_j\zeta\right)}{\sum_{i=1}^{r}\ \tau_i\ \Pi_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}-\beta_{ji}\iota\zeta\right)\, \Pi_{j=n+1}^{p_i}\Gamma\left(a_{ji}+\alpha_{ji}\zeta\right)} (z)^{-\zeta}\right.$$
$$\left.\begin{array}{l}\times\{\ (1+\lambda_1)^{-s+\mu\zeta}\,(1+\lambda_2)^{-t+\upsilon\zeta}\, B(s+\mu\zeta,\, t+\upsilon\zeta)\\ \times\,[-\log\,(1+\lambda_1)(1+\lambda_2)+\psi(s+\mu\,\zeta)+\psi(t+\upsilon\,\zeta)-2\,\psi(s+t+(\mu+\upsilon)\zeta\,]\,\}\end{array}\right]. \tag{2.5}$$

Proof. For obtain third integral, adding first and second integral formula, we obtain the required result (2.5). ☐

Fourth Integral:

$$I_4 \equiv \int_{-1}^{1} \Theta(x) \log\left(\frac{1-x}{1+x}\right) dx$$

$$= \frac{1}{2}\left[\sum_{h=1}^{m}\sum_{k=0}^{\infty} \frac{(-1)^k}{\beta_h\, k\,!} \frac{\Pi_{j=1}^{m}\Gamma\left(b_j+\beta_j\zeta\right)\, \Pi_{j=1}^{n}\Gamma\left(1-a_j-\alpha_j\zeta\right)}{\sum_{i=1}^{r}\ \tau_i\ \Pi_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}-\beta_{ji}\iota\zeta\right)\, \Pi_{j=n+1}^{p_i}\Gamma\left(a_{ji}+\alpha_{ji}\zeta\right)} (z)^{-\zeta}\right.$$
$$\times \left\{ (1+\lambda_1)^{-s+\mu\zeta}\,(1+\lambda_2)^{-t-\upsilon\zeta}\, B(s+\mu\zeta, t+\upsilon\zeta)\right.$$
$$\times\ [-\log\{(1+\lambda_1)/(1+\lambda_2)\}+\psi(s+\mu\zeta)-\psi(t+\upsilon\,\zeta\,]\,\}\,]\,. \tag{2.6}$$

Proof. For obtain third integral, subtracting second from first integral formula, we obtain the desired result (2.6). □

3 Special cases

In this section, we consider other variations of integral formulas for one to four. In detail on account of the general nature of the ℵ-function occurring in our main integrals (2.3) to (2.6), a huge number of integrals involving straightforward functions of one can simply be obtained as their special cases. We present here numerous special cases.

(1) On setting $\tau_i = 1\,(i = 1, ..., r)$, in integral (I_1)-(I_4), we obtain the following new interesting results.

$$C_1 \equiv \int_{-1}^{1} \frac{(1-x)^{s-1}(1+x)^{t-1}}{\{\lambda(x)\}^{s+t}} \log\left(\frac{1-x}{\lambda(x)}\right) I^{m,n}_{p_i,q_i;r}\left[y\frac{(1-x)^{\mu}(1+x)^{\upsilon}}{\{\lambda(x)\}^{\mu+\upsilon}}\right] dx$$

$$= \frac{1}{2}\left[\sum_{h=1}^{m}\sum_{k=0}^{\infty} \frac{(-1)^k\,\Omega^{m,n}_{p_i,q_i;r}(\zeta)}{\beta_h\,k!}\,(z)^{-\zeta}\,(1+\lambda_1)^{-s+\mu\zeta}\,(1+\lambda_2)^{-t-\upsilon\zeta}\,B(s+\mu\zeta, t+\upsilon\zeta)\right.$$
$$\times\left[-\log(1+\lambda_1) + \psi(s+\mu\zeta) - \psi(s+t+(\mu+\upsilon)\zeta\right]\Bigg], \tag{3.1}$$

where $\lambda(x) = 2 + \lambda_1(1-x) + \lambda_2(1-x)$.

$$C_2 \equiv \int_{-1}^{1} \frac{(1-x)^{s-1}(1+x)^{t-1}}{\{\lambda(x)\}^{s+t}} \log\left(\frac{1+x}{\lambda(x)}\right) I^{m,n}_{p_i,q_i;r}\left[y\frac{(1-x)^{\mu}(1+x)^{\upsilon}}{\{\lambda(x)\}^{\mu+\upsilon}}\right] dx$$

$$= \frac{1}{2}\left[\sum_{h=1}^{m}\sum_{k=0}^{\infty} \frac{(-1)^k\,\Omega^{m,n}_{p_i,q_i;r}(\zeta)}{\beta_h\,k!}\,(z)^{-\zeta}\,(1+\lambda_1)^{-s+\mu\zeta}\,(1+\lambda_2)^{-t-\upsilon\zeta}\,B(s+\mu\zeta, t+\upsilon\zeta)\right.$$
$$\times\left[-\log(1+\lambda_2) + \psi(t+\upsilon\,\zeta) - \psi(s+t+(\mu+\upsilon)\zeta\right]\Bigg]. \tag{3.2}$$

$$C_3 \equiv \int_{-1}^{1} \frac{(1-x)^{s-1}(1+x)^{t-1}}{\{\lambda(x)\}^{s+t}} \log\left(\frac{1-x^2}{\{\lambda(x)\}^2}\right) I^{m,n}_{p_i,q_i;r}\left[y\frac{(1-x)^{\mu}(1+x)^{\upsilon}}{\{\lambda(x)\}^{\mu+\upsilon}}\right] dx$$

$$= \frac{1}{2}\left[\sum_{h=1}^{m}\sum_{k=0}^{\infty} \frac{(-1)^k\,\Omega^{m,n}_{p_i,q_i;r}(\zeta)}{\beta_h\,k!}\,(z)^{-\zeta}\,(1+\lambda_1)^{-s+\mu\zeta}\,(1+\lambda_2)^{-t-\upsilon\zeta}\,B(s+\mu\zeta, t+\upsilon\zeta)\right.$$

$$\times \left[-\log(1+\lambda_1)(1+\lambda_2)+\psi(s+\mu\,\zeta)+\psi(t+\upsilon\,\zeta)-2\,\psi(s+t+(\mu+\upsilon)\zeta\,\right]. \quad (3.3)$$

$$C_4 \equiv \int_{-1}^{1} \frac{(1-x)^{s-1}(1+x)^{t-1}}{\{\lambda(x)\}^{s+t}} \log\left(\frac{1-x}{1+x}\right) I^{m,n}_{p_i,q_i;r}\left[y\frac{(1-x)^{\mu}(1+x)^{\upsilon}}{\{\lambda(x)\}^{\mu+\upsilon}}\right] dx$$

$$= \frac{1}{2}\left[\sum_{h=1}^{m}\sum_{k=0}^{\infty} \frac{(-1)^k\,\Omega^{m,n}_{p_i,q_i;r}(\zeta)}{\beta_h\,k\,!}\,(z)^{-\zeta}\,(1+\lambda_1)^{-s+\mu\zeta}\,(1+\lambda_2)^{-t-\upsilon\zeta}\,B(s+\mu\zeta,\ t+\upsilon\zeta)\right.$$

$$\times\left[-\log\{(1+\lambda_1)/(1+\lambda_2)\}+\psi(s+\mu\zeta)-\psi(t+\upsilon\,\zeta\,\right]\,\right]. \quad (3.4)$$

(2) On using $\tau_i = 1\,(i = 1, ..., r)$, and $r = 1$ in integral (I_1)-(I_4), we obtain the similar type of new interesting results as above. For example:

$$C_5 \equiv \int_{-1}^{1} \frac{(1-x)^{s-1}(1+x)^{t-1}}{\{\lambda(x)\}^{s+t}} \log\left(\frac{1-x}{\lambda(x)}\right) H^{m,n}_{p,q}\left[y\frac{(1-x)^{\mu}(1+x)^{\upsilon}}{\{\lambda(x)\}^{\mu+\upsilon}}\right] dx$$

$$= \frac{1}{2}\left[\sum_{h=1}^{m}\sum_{k=0}^{\infty} \frac{(-1)^k\,\Omega^{m,n}_{p_i,q_i}(\zeta)}{\beta_h\,k\,!}\,(z)^{-\zeta}\,(1+\lambda_1)^{-s+\mu\zeta}\,(1+\lambda_2)^{-t-\upsilon\zeta}\,B(s+\mu\zeta, t+\upsilon\zeta)\right.$$

$$\times\left[-\log(1+\lambda_1)+\psi(s+\mu\zeta)-\psi(s+t+(\mu+\upsilon)\zeta\,\right]\,\right]. \quad (3.5)$$

(3) If we take $\lambda_1 = \lambda_2 = 0$ and the integral (2.3) takes the following form after a little simplification;

$$C_6 \equiv \int_{-1}^{1} \frac{(1-x)^{s-1}(1+x)^{t-1}}{(2)^{s+t}} \log\left(\frac{1-x}{2}\right) \aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[z\frac{(1-x)^{\mu}(1+x)^{\upsilon}}{(2)^{\mu+\upsilon}}\right] dx$$

$$= \frac{1}{2}\left[\sum_{h=1}^{m}\sum_{k=0}^{\infty} \frac{(-1)^k\,\Omega^{m,n}_{p_i,q_i;\tau_i;r}(\zeta)}{\beta_h\,k\,!}\,(z)^{-\zeta}\,B(s+\mu\,\zeta,\ t+\upsilon\zeta)\right.$$

$$\times\left[\psi(s+\mu\zeta)-\psi(s+t+(\mu+\upsilon)\zeta\,\right]\,\right]. \quad (3.6)$$

4 Conclusion

We have established four new integral relations involving the $\aleph$-functions. We have also derived analogous result in the form of I-function and H-function, which have been depicted in corollaries. Therefore, the results presented in this article are easily converted in terms of a similar type ([1], [2], [6], [16]) of new interesting integrals with different arguments after some suitable parametric replacements.

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