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STABILITY FOR PERIODIC EVOLUTION FAMILIES OF BOUNDED LINEAR OPERATORS

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Abstract The long time behavior for solutions of evolution periodic equations are reviewed.

1 Introduction

The study of differential autonomous systems, especially when the "coefficients" are unbounded (the case of infinitesimal generators, for example) is well represented in the contemporary mathematics literature. The study of asymptotic behavior of solutions of systems with time-varying coefficients is more difficult since, in this context, the well known spectral criteria (such as those existing in the autonomous case) are no longer valid. In the following, we describe the history of the issue and also we state new results concerning asymptotic behavior of solutions of non-autonomous periodic systems.

Let A be a bounded linear operator acting on a Banach space X, x an arbitrary vector in X and let μ be a real parameter.

First consider the autonomous system

$$\dot{y}(t) = Ay(t), \quad t \geq 0 \tag{1.1}$$

and the associated inhomogeneous Cauchy Problem

$$\begin{cases} \dot{y}(t) = Ay(t) + e^{i\mu t}x, & t \geq 0 \\ y(0) = 0. \end{cases} \tag{1.2}$$

It is well-known, [30], [6], that the system (1.1) is uniformly exponentially stable (i.e. there exist two positive constants N and ν such that $\|e^{tA}\| \leq Ne^{-\nu t}$) if and

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only if for each $\mu \in \mathbb{R}$ and each $x \in X$, the solution of the Cauchy Problem (1.2) is bounded, i.e. if and only if

$$\sup_{t>0} \left\| \int_0^t e^{-i\mu s} e^{sA} x ds \right\| := M(\mu, x) < \infty. \tag{1.3}$$

We complete the previous result as follows:

Theorem 1. *The following four statements are equivalent:*

(1) The semigroup $\{e^{tA}\}_{t\geq 0}$ is uniformly exponentially stable.

(2) The relation (1.3) occurs.

(3) The relation (1.3) occurs and for a given positive q and each $\omega \in \mathbb{R}$, the operator $V_\omega(q) := \int_0^q e^{-i\omega s} e^{sA} ds$, is invertible.

(4) For each $x \in X$, one has

$$\sup_{\mu \in \mathbb{R}} \sup_{t>0} \left\| \int_0^t e^{-i\mu s} e^{sA} x ds \right\| := M(x) < \infty. \tag{1.4}$$

It is easy to see that the map $s \mapsto V_\omega(s) : [0, \infty) \to \mathcal{L}(X)$ is the solution of the following operatorial Cauchy Problem

$$\begin{cases} \dot{Y}(t) = AY(t) + e^{i\omega t} I, \quad t \geq 0, \quad Y(t) \in \mathcal{L}(X) \\ Y(0) = 0. \end{cases}$$

Here and in the following, $\mathcal{L}(X)$ denotes the Banach algebra of all linear and bounded operators acting on X endowed with the operator norm.

Clearly, in the above Theorem 1, the assertions (**3**) and (**4**) contain additional assumptions in comparison with (**2**). The presentation was made in this manner because we are interested in formulating of a similar result in the time varying periodic case. In this latter case, the first two assertions are no longer equivalent.

Proof of Theorem 1. The equivalence between (**1**) and (**2**) was settled in [**6**], and the implications (**4**) $\Rightarrow$ (**2**), (**1**) $\Rightarrow$ (**4**) and (**3**) $\Rightarrow$ (**2**) are straightforward. It remains to show that (**1**) $\Rightarrow$ (**3**). A simple calculation leads to

$$\begin{aligned} V_\omega(q) &= \int_0^q e^{-i\omega s} e^{sA} ds = \int_0^q e^{-i\omega s I} e^{sA} ds \\ &= \int_0^q e^{s(A - i\omega I)} ds = (A - i\omega I)^{-1} \left(e^{q(A - i\omega I)} - I \right). \end{aligned}$$

Suppose for a contradiction that $1 \in \sigma\left(e^{q(A - i\omega I)}\right)$. From the spectral mapping theorem we get $\sigma\left(e^{q(A-i\omega I)}\right) = e^{q\sigma(A - i\omega I)}$. Thus, $e^{2k\pi i} = e^{q\lambda}$, for $k \in \mathbb{Z}$ and some $\lambda \in \sigma(A - i\omega I)$. Therefore, $\left(\frac{2k\pi}{q} + \omega\right) i \in \sigma(A)$, which is a contradiction because, from hypothesis, the spectrum of A does not intersect the imaginary axis. These complete the proof.

In the following, we analyze if the time varying periodic version of the above Theorem 1, remains valid or not.

2 The issue for bounded linear operators

Let X be a complex Banach space. Assume that the map

$$t \mapsto A(t) : \mathbb{R} \to \mathcal{L}(X)$$

is continuous and q-periodic, i.e. $A(t+q) = A(t)$ for $t \geq 0$ and some positive q. The evolution family associated to the family $\{A(t)\}$ is denoted and defined by

$$\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}, \quad U(t,s) := \Phi(t)\Phi^{-1}(s), \quad \forall t \geq s \geq 0.$$

Here and in as follows, $\Phi(\cdot)$ is the unique solution of the operatorial Cauchy Problem:

$$\begin{cases} \dot{X}(t) = A(t)X(t), & t \in \mathbb{R} \\ X(0) = I. \end{cases}$$

For each given $t \geq 0$, the operator $\Phi(t)$ is invertible. Its inverse is $\Psi(t)$, where $\Psi(\cdot)$ is the solution of the Cauchy Problem

$$\begin{cases} \dot{X}(t) = -X(t)A(t), & t \in \mathbb{R} \\ X(0) = I. \end{cases}$$

The family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ satisfy:

1. $U(t,t) = I$, for all $t \in \mathbb{R}$. Here I is the identity operator on X.
2. $U(t,s)U(s,r) = U(t,r)$, for all $t \geq s \geq r$.
3. The map $(t,s) \mapsto U(t,s) : \{(t,s) \in \mathbb{R}^2 : t \geq s\} \to \mathcal{L}(X)$ is continuous.
4. $U(t+q, s+q) = U(t,s)$, for all $t, s \in \mathbb{R}$.
5. $\frac{\partial}{\partial t}U(t,s) = A(t)U(t,s)$, for all $t, s \in \mathbb{R}$.
6. $\frac{\partial}{\partial s}U(t,s) = -U(t,s)A(s)$, for all $t, s \in \mathbb{R}$.
7. There exist two constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that

$$\|U(t,s)\| \leq Me^{\omega(t-s)}, \forall t \geq s.$$

In particular, the family $\mathcal{U}$ is strongly continuous and exponentially bounded.

Clearly, $V_\mu(\cdot) = \int_0^\cdot e^{i\mu s}U(\cdot, s)ds$, is the unique solution of the next inhomogeneous operatorial problem

$$\begin{cases} \dot{X}(t) = A(t)X(t) + e^{i\mu t}I, & t \geq 0 \\ X(0) = 0. \end{cases}$$

Some details concerning the proof of the next two theorems are presented in as follows and full details can be found in [4]. The next lemma is the key tool in the proof of the Theorem 3 below. It belongs to the general ergodic theory which is well developed in [41]

Lemma 2. *([19, Lemma 1]) A strongly continuous, q-periodic and exponentially bounded evolution family acting on a Banach space X $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ is uniformly exponentially stable if and only if for each $\mu \in \mathbb{R}$ occurs*

$$\sup_{\nu\geq 1} \left\| \sum_{k=1}^{\nu} e^{i\mu k} U(q,0)^k \right\| := L(\mu) < \infty.$$

Theorem 3. *The following two statements hold true:*

1. *If the evolution family $\mathcal{U}$ is uniformly exponentially stable then, for each $\mu \in \mathbb{R}$ and each $x \in X$, have that*

$$\sup_{t\geq 0} \left\| \int_0^t e^{i\mu s} U(t,s)x ds \right\| := M(\mu, x) < \infty. \tag{2.1}$$

2. *Conversely, if (2.1) occurs and for each $\omega \in \mathbb{R}$, the operator*

$$V_\omega(q) := \int_0^q e^{i\omega s}\Phi(q)\Phi^{-1}(s)ds \text{ is invertible,} \tag{2.2}$$

then, the family $\mathcal{U}$ is uniformly exponentially stable.

Proof. The proof of the first statement is obvious. To prove the second one, let $t \geq 0$, $\nu \in \mathbb{Z}_+$ and $r \in [0, q]$ such that $= \nu q + r$. A simple calculation yields:

$$\Phi_\mu(t) = U(r,0)\sum_{k=0}^{\nu-1} e^{i\mu qk}U(q,0)^{\nu-k-1}\Phi_\mu(q) + \int_{q\nu}^{q\nu+r} e^{i\mu s}U(t,s)ds.$$

Now, we use the fact that the operators $U(r,0)$ and $\Phi_\mu(q)$ are invertible and apply Lemma 2.

□

Theorem 4. *The family $\mathcal{U}$ is uniformly exponentially stable if and only if for each $x \in \mathbb{C}^n$, one has*

$$\sup_{\mu\in\mathbb{R}} \sup_{t\geq 0} \left\| \int_0^t e^{i\mu s}\Phi(t)\Phi^{-1}(s)x ds \right\| := K(x) < \infty. \tag{2.3}$$

To prove Theorem 4 we need two lemmas. In order to introduce them, let denote by $\mathcal{F}_{\mathbf{\Phi}}(\mathbb{R}_+, X)$ the set of all continuous and q-periodic functions $f : \mathbb{R}_+ \to X$, whose restriction to the interval $[0, q]$ are given by $f(s) = h(s)\Phi(s)b$, with $b \in X$. The map $h(\cdot)$ belongs to $\{h_1(\cdot), h_2(\cdot)\}$, where h_1 and h_2 are scalar-valued functions, defined on $[0, q]$ by (see Figure 1):

$$h_1(s) = \begin{cases} s, & s \in [0, \frac{q}{2}); \\ q - s, & s \in [\frac{q}{2}, q] \end{cases} \qquad \text{and } h_2(s) = s(q - s).$$

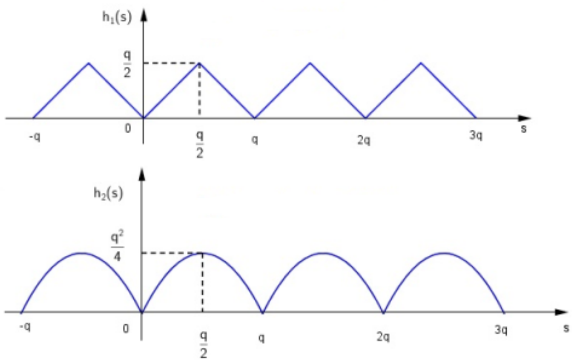

Figure 1: Graphic representation of function h_1 and h_2

Lemma 5. *Every function $f \in \mathcal{F}_\Phi(\mathbb{R}_+, X)$ satisfy a Lipschitz condition on $\mathbb{R}_+$.*

Proof. Let $b \in X$, $j \in \{1, 2\}$ and $f \in \mathcal{F}_\Phi(\mathbb{R}_+, X)$. The function h_j is bounded and

$$\max_{s\in[0,q]} h_j(s) = \rho, \text{ where } \rho := \max\left\{\frac{q}{2}, \frac{q^2}{4}\right\}. \tag{2.4}$$

Integrating the equality $\frac{\partial}{\partial t}\Phi(t) = A(t)\Phi(t)$ between the positive numbers t_1 and t_2, with $t_2 \geq t_1$, $t_1, t_2 \in [0, q]$ and taking into account that $\sup_{s\in[0,q]} \|\Phi(s)\| := M < \infty$, we get

$$\|\Phi(t_2) - \Phi(t_1)\| \leq M\|A(\cdot)\|_\infty |t_2 - t_1|.$$

Clearly, the last inequality remains valid for $t_2 \leq t_1$. Moreover, using the inequality $|h_j(t_2) - h_j(t_1)| \leq \widetilde{\rho}|t_2 - t_1|$, $\widetilde{\rho} := \max\{1, q\}$, we get

$$\|f(t_2) - f(t_1)\| \quad \leq \quad M\left(\widetilde{\rho} + \rho\|A(\cdot)\|_\infty\right)\|b\||t_2 - t_1|.$$

If $t_2 \in [(k-1)q, kq]$ and $t_1 \in [kq, (k+1)q]$, with $t_2 < t_1$ and $k \in \mathbb{Z}$, then from

$$\|f(t_2) - f(kq)\| \leq c \cdot |t_2 - kq| \quad \text{and} \quad \|f(kq) - f(t_1)\| \leq c \cdot |kq - t_1|$$

it follows

$$\|f(t_2) - f(t_1)\| \leq c \cdot [|t_2 - kq| + |kq - t_1|] = c \cdot |t_2 - t_1|.$$

Finally, if $t_1, t_2 \in [kq, (k+1)q]$ or $|t_2 - t_1| \geq q$, considering $t_1 = t_1^* + k_1 q$ and $t_2 = t_2^* + k_2 q$, with $t_1^*, t_2^* \in [0, q]$ and $k_1, k_2 \in \mathbb{Z}$, we obtain

$$\|f(t_2) - f(t_1)\| = \|f(t_2^*) - f(t_1^*)\| \leq c \cdot |t_2 - t_1|.$$

□

Now, denote by $\mathcal{F}_{\mathcal{U}}(\mathbb{R}_+, X)$ the set of all continuous and q-periodic functions $f : \mathbb{R}_+ \to X$, whose restrictions to the interval $[0, q]$ are given by $f(s) = h(s)U(s, 0)b$, with $b \in X$. The second lemma is stated as follows:

Lemma 6. *[15, Thm. 1] Let $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ be a strongly continuous and q-periodic evolution family acting on a Banach space X. If for each $\mu \in \mathbb{R}$ and each $f \in \mathcal{F}_{\mathcal{U}}(\mathbb{R}_+, X)$, occurs*

$$\sup_{t>0} \left\| \int_0^t e^{i\mu s} U(t, s) f(s) ds \right\| := K(\mu, f) < \infty, \tag{2.5}$$

then the family $\mathcal{U}$ is uniformly exponentially stable.

Sketch of the proof of Theorem 4:

We show that (2.5) is a consequence of (2.3). First, according to Lemma 5, every function from $\mathcal{F}_{\mathbf{\Phi}}(\mathbb{R}_+, X)$ satisfy a Lipschitz condition on $\mathbb{R}_+$. Then, using the well known Fourier theorem, [60, Ex. 16, pp. 92-93], such functions belong to the space $AP_1(\mathbb{R}_+, X)$. The desired assertion is obtained by applying Lemma 6.

□

Extensions of the above results to the general case of exponential dichotomy has been obtained in [71], [42], [72].

Next, we analyze the issue when the "coefficient" A is the infinitesimal generator of a strongly continuous semigroup $\mathbf{T} = \{T(t)\}_{t \geq 0}$.

3 The semigroup case

Let X be a Banach space and let $\mathcal{L}(X)$ be the Banach algebra of all bounded linear operators acting on X. The norms in X and in $\mathcal{L}(X)$ will be denoted by the same symbol, namely by $\|\cdot\|$.

A family $\mathbf{T}=\{T(t): t \geq 0\} \subseteq \mathcal{L}(X)$ is called *semigroup of operators* if $T(0)=I$ and $T(t+s)=T(t)\circ T(s)$, for all $t,s \geq 0$. Here I denotes the identity operator in $\mathcal{L}(X)$. Recall that a semigroup $\mathbf{T}$ is called *uniformly continuous* (or uniformly continuous at 0) if $\lim_{t\to 0^+}\|T(t)-T(0)\|_{\mathcal{L}(X)}=0$, it is called *strongly continuous* (or strongly continuous at 0) if $\lim_{t\to 0^+}\|T(t)x-x\|_X=0$, for all $x \in X$ and it is *weakly continuous* if $\lim_{t\to 0^+}|x^*(T(t)x)-x^*(T(0)x)|=0$, for all $x\in X$ and all $x^* \in X^*$. Here X^* is the (strong) dual space of X. It is well-known that a semigroup of operators is strongly continuous if and only if it is weakly continuous, [32]. Obviously, uniformly continuous semigroups are strongly continuous, but the converse statement is not true. The continuity at 0 of a trajectory $t\mapsto T(t)x, (x\in X)$, implies its continuity everywhere in $\mathbb{R}_+$. More exactly, in such circumstances, the map $t\mapsto T(t)x$ is continuous on $\mathbb{R}_+$. If $\mathbf{T}$ is a strongly continuous semigroup then it is *exponentially bounded*, i.e., there exist two positive constants $\omega\in\mathbb{R}$ and $M\geq 1$ such that

$$\|T(t)\| \leq Me^{\omega t}, \text{ for all } t\geq 0. \tag{3.1}$$

If the inequality (3.1) is fulfilled, the semigroup $\mathbf{T}$ has an *uniform growth bound* denoted by

$$\begin{aligned}\omega_0(\mathbf{T}) &= \inf\{\omega\in\mathbb{R}: \text{ there exists } M\geq 1 \text{ such that } \|T(t)\|\leq Me^{\omega t}, \forall t\geq 0\}\\ &= \inf\{\omega\in\mathbb{R}: \quad \text{the map } t\mapsto e^{-\omega t}\|T(t)\| \text{ is bounded on } \mathbb{R}_+\}.\end{aligned} \tag{3.2}$$

Let $\mathbf{T}=\{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup acting on a Banach space X. The linear operator $A: D(A)\subset X\to X$, defined by

$$Ax = \lim_{t\to 0^+}\frac{T(t)x-x}{t}, \quad x\in D(A),$$

is called the *infinitesimal generator* of the semigroup $\mathbf{T}$. Here, $D(A)$ denotes *the maximal domain of* A, and it consists of all $x\in X$ for what the limit: $\lim_{t\to 0^+}\frac{T(t)x-x}{t}$, exists in X. The infinitesimal generator $(A, D(A))$ of the semigroup $\mathbf{T}$ verify:

- A is a linear operator defined on the linear subspace $D(A)$.
- If $x\in D(A)$ then $T(t)x\in D(A)$ and the map $t\mapsto T(t)x$ is continuous differentiable on $\mathbb{R}_+$. In addition,

$$\frac{d}{dt}[T(t)x] = T(t)Ax = AT(t)x, \text{ for all } t\geq 0.$$

In particular, the map $T(\cdot)x$ is the *classical solution* of the abstract Cauchy Problem

$$\dot{u}(t) = Au(t), \quad t \geq 0, \quad u(0) = x.$$

When $y \in X \backslash D(A)$, the map $T(\cdot)y$ is called *mild solution* of the above problem.

- For each $t \geq 0$ and each $x \in X$ have that $\int_0^t T(s)xds \in D(A)$ and

$$T(t)x - x = A\int_0^t T(s)xds.$$

  Moreover, when $x \in D(A)$ one has

$$T(t)x - x = \int_0^t T(s)Axds. \tag{3.3}$$

- The generator A is densely defined, i.e. $\overline{D(A)} = X$. This property help us to define the adjoint of A. We define $D(A^2)$ as the set of all $x \in D(A)$ with $Ax \in D(A)$. The domain of the other powers of A are defined in a similar manner. We mention that even the smaller set $D := \bigcap_{n=1}^{\infty} D(A^n)$ is dense in X.

- A is a closed operator, i.e. its graph is a closed set in $X \times X$. With other words, if $(x_n), x$ and y are such that $x_n \in D(A), x, y \in X$ and $x_n \to x, Ax_n \to y$ as $n \to \infty$, in the norm of X, then $x \in D(A)$ and $Ax = y$.

3.0.1 Elements of spectral theory

Let $T : D(T) \subset X \to X$ be a linear operator. The set of all $\lambda \in \mathbb{C}$ for what $(\lambda I - T) : D(T) \to X$ is invertible in $\mathcal{L}(X)$, (i.e. there exists $L \in \mathcal{L}(X)$ such that $(\lambda I - T)Lx = x$ for all $x \in X$ and $L(\lambda I - T)y = y$ for all $y \in D(T)$) is denoted by $\rho(T)$, and is called the *resolvent set* of T. The set $\sigma(T) := \mathbb{C} \setminus \rho(T)$ is called *the spectrum* of T. Generally speaking, the sets $\rho(T), \sigma(T)$ can be empty. When T is closed and $\lambda \in \rho(T)$ then $R(\lambda, T) := (\lambda I - T)^{-1}$ is a bounded linear operator on X called *the resolvent operator* (of T in λ). If T is a closed operator then $\lambda \in \rho(T)$ if and only if the map $\lambda I - T : D(T) \to X$ is bijective. If A generates a strongly continuous semigroup then $\rho(A)$ is a non-empty open set. In fact, in this case $\rho(A)$ contains an half-plane of the complex plane. However, the spectrum of a generator can be the empty set. The *spectral bound* of a infinitesimal generator denoted by $s(A)$ is the supremum of the set $\{\Re(z) : z \in \sigma(A)\}$. Clearly, we have $-\infty \leq s(A) < \infty$. For a bounded linear operator A, defined on the Banach space X, the spectrum $\sigma(A)$ is always a compact and nonempty set; therefore, its *spectral radius*, denoted by $r(A)$ is given by

$$r(A) := \sup\{|\lambda| : \quad \lambda \in \sigma(A)\} = \lim_{n\to\infty} \|A^n\|^{1/n}.$$

Moreover it is finite and satisfy $r(A) \leq \|A\|$ ([63, Chapter V, Theorem 3.5] or [69, XIII.2, Theorem 3]).

In the next two theorems, several connections between a strongly continuous semigroup and the resolvent of its infinitesimal generator, are established.

Theorem 7. *Let* $\boldsymbol{T}= \{T(t)\}_{t\geq 0}$ *be a strongly continuous semigroup defined on a Banach space* X *and let* $\omega \in \mathbb{R}$ *and* $M \geq 1$ *such that* $\|T(t)\| \leq Me^{\omega t}$, *for* $t \geq 0$. *Let* $(A, D(A))$ *be the infinitesimal generator of* $\mathbf{T}$. *The following statements hold true.*

- *If for* $\lambda \in \mathbb{C}$ *there exists (as improper integral)* $R(\lambda)x := \int_0^\infty e^{-\lambda s}T(s)xds$ *for all* $x \in X$, *then* $\lambda \in \rho(A)$ *and* $R(\lambda, A) = R(\lambda)$.
- *If* $\Re(\lambda) > \omega$, *then* $\lambda \in \rho(A)$ *and* $R(\lambda, A) = R(\lambda)$.
- $\|R(\lambda, A)^n\| \leq \frac{M}{(\Re(\lambda)-\omega)^n}$, *for each* $\Re(\lambda) > \omega$ *and each* $n \in \mathbb{Z}_+$.

The proof of the above Theorem 7 can be found in [32, Cap. II]. We mention that a closed and densely defined operator A, verifying the third statement from the previous theorem, generates a strongly continuous semigroup.

Theorem 8. *Let* A *be a closed operator. The following statements hold true.*

- *For every* $\lambda \in \rho(A)$, *the following spectral mapping theorem for operator resolvent one has:*

$$\sigma(R(\lambda, A)) = \frac{1}{\sigma(\lambda I - A)} := \left\{\frac{1}{\lambda - \mu} : \mu \in \sigma(A)\right\}. \tag{3.4}$$

- *An obvious consequence of (3.4) is the next useful inequality:*

$$dist\ (\lambda, \sigma(A))\|R(\lambda, A)\| \geq 1. \tag{3.5}$$

  In particular, this shows that the operator resolvent "exploding" in the near of the spectrum. An important property of the operator resolvent is presented in the following.

- *Assume that the resolvent set* $\rho(A)$ *contains the half-plane* $\{\Re(z) > 0\}$ *and that*

$$\sup_{\Re(z)>0} \|R(z, A)\| := M < \infty.$$

  Thus, $\pi := \{\Re(z) > -\frac{1}{M}\} \subset \rho(A)$ *and* $\|R(z, A)\| \leq M$ *for all* $z \in \pi$.

The proof of the above Theorem 8 can be found, for example, in the van Neerven monograph [65].

3.0.2 Characterizations of the uniform exponential stability

The semigroup of operators $\mathbf{T}=\{T(t)\}_{t\geq 0}$, defined on a Banach space X is called *uniformly exponential stable* if there exist the nonnegative constants N and ν such that

$$\|T(t)\| \leq Ne^{-\nu t}, \text{ for each } t \geq 0.$$

From the definition of the uniform growth bound of a semigroup $\mathbf{T}=\{T(t)\}_{t\geq 0}$ (3.2), it follows that the semigroup $\mathbf{T}$ is uniformly exponentially stable if and only if $\omega_0(\mathbf{T})$ is negative.

Next, we analyze the concept of stability from two perspectives: the analytical continuation of the operator resolvent and the admissibility of Perron type.

Obviously, if $\mathbf{T}$ is an uniformly exponentially stable semigroup then, for each $x \in X$ and each $1 \leq p < \infty$, the map $T(\cdot)x$ belongs to $L^p(\mathbb{R}_+, X)$.

The following theorem shows that the converse statement is also true. See [25], [51] and [52, Chapter.4, Theorem 4.1].

Theorem 9. *([Datko-Pazy Theorem]) A strongly continuous semigroup* $\mathbf{T}=\{T(t)\}_{t\geq 0}$*, acting on a Banach space* X *is uniformly exponentially stable if and only if for a given* $p \in [1, \infty)$ *(so for all) and for any* $x \in X$*, occurs*

$$\int_0^\infty \|T(t)x\|^p dt < \infty.$$

With other words, the trajectories of a strongly continuous semigroup $\mathbf{T}$ belong to the space $L^p(\mathbb{R}_+, X)$ if and only if the semigroup $\mathbf{T}$ is uniformly exponentially stable.

Based on this theorem it follows that the uniform growth bound $\omega_0(\mathbf{T})$ is the abscissa of absolute convergence of the improper Laplace transform of $\mathbf{T}$, i.e.

$$\omega_0(\mathrm{T}) = \inf\left\{\omega \in \mathbb{R} : \lim_{t\to\infty} \int_0^t e^{-\omega s}\|T(s)x\|ds \text{ exists, for each } x \in X\right\}.$$

An important role in the study of asymptotic behavior of the trajectories of a semigroup $\mathbf{T}$, is also played by the index $\omega_1(\mathbf{T})$ which monitors the growth bound of the classical solutions, i.e. of solutions that have started from $D(A)$. Frank Neubrander has shown in [48] that $\omega_1(\mathbf{T})$ is the abscissa of convergence of the improper Laplace transform of the semigroup $\mathbf{T}$, i.e. it is the infimum of all $\omega \in \mathbb{R}$ for what the limit $\lim_{t\to\infty} \int_0^t e^{-\lambda s}T(s)xds$ exists in X for all $x \in X$ and all $\lambda \in \mathbb{C}$ with $\Re(\lambda) > \omega$. As is well known, such complex numbers λ belong to the resolvent set $\rho(A)$ and the above limit is equal to $R(\lambda, A)x$. In [3, A.IV, Corolarul 1.5], it was shown that always occurs

$$s(A) \leq \omega_1(\mathbf{T}) \leq \omega_0(\mathbf{T}).$$

When the semigroup $\mathbf{T}$ is uniformly continuous the above indices are equal, but in the general case they can be mutually different.

Generally speaking, it is difficult to apply Theorem 9 in concrete applications, since the knowledge of the semigroup $\mathbf{T}$ is needed. Most often, we only have information about its infinitesimal generator A. Therefore, characterizations of uniform exponential stability in terms regarding the generator A or the operator resolvent $R(\lambda, A)$ are useful and effective.

It is known that the location of the spectrum of a generator of a strongly continuous semigroup in the half-plane $\mathbb{C}_- := \{\lambda : \Re(\lambda) < 0\}$, does not ensure convergence towards 0 of the semigroup trajectories. Moreover, the spectral mapping theorem (i.e. the equality $\sigma(T(t)) = e^{t\sigma(A)} \setminus \{0\}$) does not work for strongly continuous semigroups, but it works for eventually norm continuous semigroups, i.e. the semigroups $\mathbf{T}$ for which there exists $t_0 \geq 0$ such that the map $t \mapsto T(t)$ is continuous on (t_0, ∞) in the norm topology of $\mathcal{L}(X)$, [26].

However, Gearhart, [34], has obtained a characterization of $\sigma(T(t))$ for strongly continuous semigroups of contractions acting on a complex Hilbert space. He has shown that a nonzero complex number z belongs to the resolvent set $\rho(T(t))$ if and only if the set $\mathcal{M}$ of all complex scalars, verifying the equality $e^{\lambda t} = z$, is contained in the resolvent set of A and $R(\cdot, A)$ is uniformly bounded on $\mathcal{M}$.

Theorem 10. *([34], [38], [55], Greiner 1985) Let H be a complex Hilbert space and $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup on H having A as a generator. Assume that $\sigma(A)$ lies in $\mathbb{C}_- := \{z \in \mathbb{C} : \Re(z) < 0\}$. The following statements are equivalent:*

- *The semigroup $\mathbf{T}$ is uniformly exponentially stable.*
- $$\sup_{\mu\in\mathbb{R}} \|R(i\mu, A)\| < \infty, \tag{3.6}$$
- *The resolvent $R(\cdot, A)$ is the Laplace transform of $T(\cdot)$ on the imaginary axis, and*
$$\sup_{\mu\in\mathbb{R}} \left\| \int_0^\infty e^{-i\mu t} T(t)x dt \right\| := M(x) < \infty, \quad \forall x \in X. \tag{3.7}$$

Denote by

$$s_0(\mathbf{T}) := \inf \left\{ \omega \in \mathbb{R} : \sup_{\Re(\lambda)\geq\omega} \left\| \int_0^\infty e^{-\lambda t} T(t)x dt \right\| < \infty, \text{ for all } x \in X \right\},$$

the uniform growth bound of the resolvent. Obviously we have:

$$s(A) \leq \omega_1(\mathbf{T}) \leq \mathbf{s_0}(\mathbf{T}) \leq \omega_\mathbf{0}(\mathbf{T}). \tag{3.8}$$

First example of semigroup whose generator verifies the inequality $s(A) < \omega_0(A)$ was provided by Foiaş, [33]. An important example, on Hilbert spaces, verifying

$$s(A) < \omega_1(\mathbf{T}) = \omega_0(\mathbf{T})$$

was offered by Zabczyk, [70].

From the Gearhart theorem follows that for any strongly continuous semigroup acting on a Hilbert space the equality $s_0(\mathbf{T}) = \omega_{\mathbf{0}}(\mathbf{T})$ holds.

Huang, [39], has conjectured that under the same Gearhart theorem assumptions, but on Banach spaces, the semigroup $\mathbf{T}$ is exponentially stable, i.e. $\omega_1(\mathbf{T})$ is negative. The result was settled by Weis and Wrobel, [66], [68].

From the boundedness condition (3.7) to the assumptions of the next van Neerven theorem, is just one step.

Theorem 11. *([64, Corolarul 5]) Let $\boldsymbol{T}= \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup acting on a complex Banach space X, having A as infinitesimal generator. If*

$$\sup_{\mu\in\mathbb{R}}\sup_{t\geq 0}\left\|\int_0^t e^{-i\mu s}T(s)x ds\right\| := M(x) < \infty, \quad \forall x \in X. \tag{3.9}$$

then $\boldsymbol{T}$ is exponentially stable, *i.e. there exist two positive constants N and ν such that*

$$\|T(t)x\| \leq Ne^{-\nu t}\|x\|_{D(A)}, (\|x\|_{D(A)} := \|x\| + \|Ax\|), \quad \forall x \in D(A).$$

The ideas of the proof of the above van Neerven theorem, [64], comes from the Complex Analysis for vector-valued functions. Using the Phragmen-Lindelöf and Vitali theorems, [36], he has shown that for a given $x_0 \in X$, the map $\lambda \mapsto R(\lambda, A)x_0$ has a bounded analytical continuation on $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \Re(\lambda) > 0\}$ provided that

$$\sup_{\mu\in\mathbb{R}}\sup_{t\geq 0}\left\|\int_0^t e^{-i\mu s}T(s)x_0 ds\right\| := M < \infty.$$

Further, if the map $\lambda \mapsto R(\lambda, A)x_0$ has an bounded analytical continuation on $\mathbb{C}_+$, then for each $\lambda \in \mathbb{C}$, with $\Re(\lambda) > \omega_0(\mathbf{T})$, one has

$$\|T(t)R(\lambda, A)x_0\| \leq M(1 + t), \quad \forall t \geq 0. \tag{3.10}$$

To prove (3.10) he used the inversion formula, [2],

$$T(t)R(\lambda, A)x_0 = \int_{\alpha-i\infty}^{\alpha+i\infty} e^{zt}R(z, A)R(\lambda, A)x_0 dz, \quad \Re(\lambda) > \omega_0(\mathbf{T}).$$

From the Cauchy theorem, the integral on the closed curve $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ (oriented as in Figure 2) is equal to 0, so the integral on Γ_1 is equal to the sum of

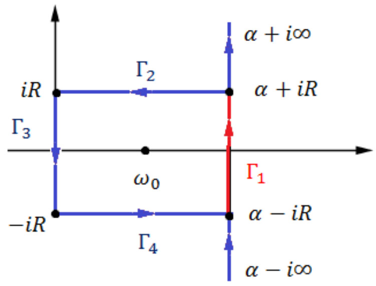

Figure 2: The deformation of the integration contour

integrals on Γ_2, Γ_3 and Γ_4. Evaluating the integrals separately the desired assertion is obtained.

On the other hand, it is shown that if

$$\sup_{\mu\in\mathbb{R}}\sup_{t\geq 0}\left\|\int_0^t e^{-i\mu s}T(s)xds\right\| := M(x) < \infty, \quad \forall x \in X,$$

then the resolvent $R(\cdot, A)$, as operator-valued function, admits a bounded analytical continuation on $\mathbb{C}_+$. A well known result assures that if

$$\sup_{\Re(\lambda)\geq 0}\|R(\lambda, A)\| \leq L < \infty,$$

then the resolvent is bounded by the same constant L, on the half-plane $\{\Re(z) > -\frac{1}{2L}\}$. Choosing $\delta = \frac{1}{4L} > 0$ and considering the semigroup $S(t) = e^{\delta t}T(t)$, generated by $B := \delta I + A$, whose resolvent, given by $R(\lambda, B) = R(\lambda - \delta, A)$, is bounded, is obtained

$$\|T(t)R(\lambda - \delta, A)x\| \leq M_x e^{-\delta t}(1 + t), \quad \forall t \geq 0, \forall x \in X.$$

Finally have

$$\omega_1(\mathbf{T}) = \limsup_{t\to\infty}\frac{\ln\|T(t)R(\lambda - \delta, A)\|}{t} \leq -\delta < 0.$$

Originally, the condition (3.9), without the uniformity in respect to the parameter μ, was introduced by Krein, [30]. Next example, shows that the condition (3.9) does not imply the uniform exponential stability of the semigroup.

Example 12. *[11, Ex. 2] Let $X_1 := C_0(\mathbb{R}_+, \mathbb{C})$ be the set of all continuous functions $x : \mathbb{R}_+ \to \mathbb{C}$ which tend to zero to infinity and let $X_2 := L_1(\mathbb{R}_+, \mathbb{C}, e^s ds)$ be the set of all measurable functions $x : \mathbb{R}_+ \to \mathbb{C}$ for which $\int_0^\infty e^t|x(t)|dt < \infty$. Let $X := X_1 \cap X_2$. The norm in X is given by $\|x\|_X := |x|_\infty + \int_0^\infty e^t|x(t)|dt$, $x \in X$. For* *each $t \geq 0$, define the operator $S(t) : X \to X$, $(S(t)x)(\tau) = x(t+\tau)$, for $\tau \geq 0$. Let $T(t) = e^{\frac{t}{2}} S(t)$, $t \geq 0$. For the semigroup $\boldsymbol{T}$ the condition (3.9) is fulfilled although $\omega_0(\mathbf{T}) = \frac{1}{2}$.*

Actually, the roots of Example 12 are in [35].

For semigroups defined on Hilbert spaces verifying (3.9), Quoc Phong Vu, [54, Theorem 2], settled the following:

Theorem 13. *A strongly continuous semigroup $\mathbf{T} = \{T(t)\}$ acting on a complex Hilbert space H is uniformly exponentially stable if and only if (3.9) is fulfilled.*

Now we review the main ideas of the proof of Vu Phong theorem. In [56] (see also [54]) it was shown that if

$$\sup_{t \geq 0} \left\| \int_0^t T(s)x ds \right\| = M(x) < \infty, \quad \forall x \in X,$$

then $0 \in \rho(A)$, i.e. there exists $A^{-1} = R(0, A)$. Previous result remains valid when the semigroup $S(t) = e^{i\mu t}T(t)$, with $\mu \in \mathbb{R}$, replaces $T(t)$, in which case, the conclusion says that $i\mu \in \rho(A)$. Vu Phong has completed the proof of this result showing that for every real number μ,

$$\|R(i\mu, A)\| \leq M_\mu := \sup_{t \geq 0} \left\| \int_0^t e^{-i\mu s} T(s) ds \right\|_{\mathcal{L}(X)}.$$

By the assumption there exists an absolute constant M such that

$$\sup_{\mu \in \mathbb{R}} \|R(i\mu, A)\| \leq M := \sup_{\mu \in \mathbb{R}} \sup_{t \geq 0} \left\| \int_0^t e^{-i\mu s} T(s) ds \right\|_{\mathcal{L}(X)}$$

and the desired assertion is obtained via Theorem 10.

We mention that (1.3) is not enough to imply the uniform exponential stability of a semigroup $\mathbf{T}$ which acts on a Hilbert space, [17, Example 4.2.2].

3.1 Strong stability. A possible new approach.

Recall that the trajectory started from $x \in X$ of a strongly continuous semigroup $\mathbf{T} = \{T(t)\}_{t \geq 0}$ is strongly stable if $\lim_{t \to \infty} T(t)x = 0$, and that the semigroup is strongly stable if all its trajectories are strongly stable.

A famous result, known as ABLV theorem, see [1], [44], provides a sufficient condition for strong stability of semigroups (acting on Banach spaces) in terms of countability of the boundary spectrum $\sigma(A) \cap i\mathbb{R}$. In particular, lack of the spectrum of the infinitesimal generator of an uniformly bounded semigroup which acts on a Banach space on the imaginary axis implies the strong stability of semigroup.

Theorem 14. *([10]) Let* $\mathbf{T} = \{T(t)\}_{t\geq 0}$ *be a strongly continuous semigroup of bounded linear operators acting on a complex Hilbert space* H*, and let* $(A, D(A))$ *be its infinitesimal generator. If there exists a positive absolute constant* R *such that*

$$\sup_{\mu\in\mathbb{R}}\sup_{t\geq 0}\left\|\int_0^t e^{i\mu s}T(t-s)yds\right\| \leq R\|y\|_{D(A)}, \quad \forall y\in D(A),$$

then, for every $x \in D := D(A^2)$*, one has:*

$$\sup_{z\in\mathbb{C},|z|=1}\sup_{n\in\mathbb{Z}_+}\left\|\sum_{k=0}^{n} z^k T(q)^k x\right\| := N(x) < \infty.$$

The key tool in the proof of the Theorem 14 is the next lemma whose proof can be found in [10, Lemma 2.10].

Lemma 15. *Let* $x \in H$ *such that the map* $s \mapsto U(s,0)x$ *satisfies a Lipschitz condition on* $(0,q)$*. For each* $j \in \{1,2\}$ *let us consider the q-periodic function* $f_j : \mathbb{R} \to H$*, given on* $[0,q]$ *by*

$$f_j(t) := h_j(t)U(t,0)x.$$

The following two statements are true:

(1) *Each function* f_j *satisfies a Lipschitz condition on* $\mathbb{R}$*.*

(2) *The Fourier series associated to* f_j *is absolutely and uniformly convergent on* $\mathbb{R}$*.*

In order to do a consequence of Theorem 14, we state the following useful Lemma, [10].

Lemma 16. *Let* T *be a bounded linear operator acting on the complex Hilbert space* H*. If for some* $x \in H$*, one has*

$$\sup_{z\in\mathbb{C},|z|=1}\sup_{n\in\mathbb{Z}_+}\left\|\sum_{k=0}^{n}\frac{T^k x}{z^{k+1}}\right\| := N(x) < \infty, \tag{3.11}$$

then $\lim_{k\to\infty} T^k x = 0$.

Corollary 17. *Let* $\mathbf{T} = \{T(t)\}_{t\geq 0}$ *be a uniformly bounded and strongly continuous semigroup acting on a Hilbert space* H*, and let* $D(A)$ *the maximal domain of its infinitesimal generator. If there exists a positive constant* K *such that*

$$\sup_{\mu\in\mathbb{R}}\sup_{t\geq 0}\left\|\int_0^t e^{i\mu s}T(t-s)yds\right\| \leq K\|y\|_{D(A)}, \quad \forall y\in D(A), \tag{3.12}$$

then, the semigroup $\mathbf{T}$ *is strongly stable.*

Proof. As is well-known, the map $T(\cdot)x$ is differentiable for every, $x \in D(A)$. Therefore, for a given positive q, it satisfies a Lipschitz condition on $(0,q)$. Indeed, for any pair (t,s) with $t \geq s$, have that:

$$\|T(t)x - T(s)x\| = \left\| \int_s^t \frac{d}{dr}[T(r)x]dr \right\| = \left\| \int_s^t T(r)Axdr \right\| \leq \|Ax\| \sup_{r \in [0,q]} \|T(r)\||t-s|. \tag{3.13}$$

Clearly, (3.13) remains valid for all pairs (t,s) with $t < s$. Then, by applying the above Theorem it follows that $T(\cdot)x$ is strongly stable for every $x \in D(A)$. Let now, $y \in H$ and $x_n \in D(A)$ such that $x_n \to y$, as $n \to \infty$, in the norm of H. Then,

$$\|T(t)y\| \leq \|T(t)(y - x_n)\| + \|T(t)x_n\|$$

$$\leq (\sup\{\|T(s)\| : s \geq 0\})\|y - x_n\| + \|T(t)x_n\| \to 0, \text{ as } t, n \to \infty.$$

□

4 The issue for periodic evolution families

The content of this section is based on the articles [10], [21] and [58]. A family $\mathcal{U} = \{U(t,s) : t \geq s\} \subset \mathcal{L}(X)$ is called *evolution family* on X if $U(t,s)U(s,r) = U(t,r)$ for all $t \geq s \geq r$ and $U(t,t) = I$ for every $t \in \mathbb{R}$. The evolution family $\mathcal{U}$ is *strongly continuous* if for every $x \in X$ the map

$$(t,s) \mapsto U(t,s)x : \{(t,s) \in \mathbb{R}^2 : t \geq s\} \to X,$$

is continuous. If the family $\mathcal{U}$ is strongly continuous and satisfy the convolution condition (i.e. $U(t,s) = U(t-s,0)$ for every pair (t,s), with $t \geq s$, then the family $\{T(t)\}_{t \geq 0}$, given by $T(t) := U(t,0)$, is a strongly continuous semigroup on X. We say that the family $\mathcal{U}$ has *exponential growth* if there exist the constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|U(t,s)\| \leq Me^{\omega(t-s)}, \quad \forall t \geq s.$$

When $U(t+q, s+q) = U(t,s)$, for all $t \geq s$ and a given $q > 0$, the family $\mathcal{U}$ is called q-*periodic*. Obviously a q-periodic evolution family verify: $U(pq+q, pq+u) = U(q,u)$, for all $p \in \mathbb{N}$, $u \in [0,q]$ and $U(pq, rq) = U((p-r)q, 0) = U(q,0)^{p-r}$, for all $p \in \mathbb{N}, r \in \mathbb{N}, p \geq r$. Every q-periodic evolution family $\mathcal{U} = \{U(t,s)\}$, defined for all pairs (t,s), with $t \geq s \geq 0$, can be extended to a q-periodic evolution family for all pairs (t,s), with $t \geq s \in \mathbb{R}$, by choosing $U(t,s) = U(t+kq, s+kq)$, where k is the smallest positive integer for which $s + kq \geq 0$. Every strongly continuous q-periodic evolution family $\mathcal{U}$ has exponential growth, [24, Lemma 4.1]. We say that the evolution family $\mathcal{U}$ is *uniformly exponentially stable* if there exist two nonnegative constants N and ν such that

$$\|U(t,s)\| \leq Ne^{-\nu(t-s)}, \quad \forall t \geq s \geq 0. \tag{4.1}$$

A q-periodic strongly continuous evolution family $\mathcal{U}$ is uniformly exponentially stable if and only if there exist two nonnegative constants N_1 and ν_1 such that

$$\|U(t,0)\| \leq N_1 e^{-\nu_1 t}, \quad \forall t \geq 0.$$

In [12, Lemma 2], it was shown that a strongly continuous q-periodic evolution family $\mathcal{U}$ is uniformly exponentially stable if and only if $r(U(q,0)) < 1$. Further details concerning periodic evolution families can be found in [31]

Let $J \in \{\mathbb{R}, \mathbb{R}_+\}$. The spaces listed below will be intensively used in the sequel.

- $BUC(J,X)$ is the linear space consisting of all X-valued functions defined on J, which are uniformly continuous and bounded on J. Endowed with the norm $\|f\|_\infty := \sup_{t\in J} \|f(t)\|$, it becomes a Banach space.

- $AP(J,X)$ is the smallest linear closed subspace of $BUC(J,X)$ which contains all functions $f_\mu(t) = e^{i\mu t}x, t \in J$, with $\mu \in \mathbb{R}$ and $x \in X$.

- $AP_1(J,X)$ the space consisting of all functions $f \in AP(J,X)$ having the property that the Fourier coefficients series associated to the function f is absolutely convergent. For further information concerning the spaces $AP(J,X)$ and $AP_1(J,X)$ we refer the reader to the monographs [29], [43], [73].

- $P_q(J,X)$ is the space of all X-valued, continuous q-periodic functions defined on J.

- $P_q^0(J,X)]$ is the subspace of $f \in P_q(J,X)$ consisting by all functions f verifying the condition $f(0) = 0$.

Next, for a given $t \geq 0$, denote by $\mathcal{A}_t$ the set of all X-valued functions defined on $\mathbb{R}$ having the property that there exists a function F from $P_q(\mathbb{R},X) \cap AP_1(\mathbb{R},X)$ such that $F(t) = 0$, $f = F_{|_{[t,\infty)}}$ and $f(s) = 0$, for all $s < t$. Denote by $E(\mathbb{R},X)$ the closure in respect to the "sup" norm of the linear span of the set $A_0 := \{\cup_{t\geq 0}\mathcal{A}_t\}$, i.e. $E(\mathbb{R},X) = \overline{\text{span}}A_0$. With other words, $E(\mathbb{R},X)$ is the smallest closed subspace of $BUC(\mathbb{R},X)$ containing A_0. The evolution semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ associated to a strongly continuous and q-periodic evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ on $E(\mathbb{R},X)$ is formally defined by

$$(\mathcal{T}(t)f)(s) := \begin{cases} U(s,s-t)f(s-t), & s \geq t \\ 0, & s < t. \end{cases} \tag{4.2}$$

Obviously, it acts on $E(\mathbb{R},X)$ and it is strongly continuous.

First result of this section is stated as follows.

Theorem 18. *Let $\mathcal{U}$ be a strongly continuous and q-periodic evolution family acting on a Banach space X and let $\mathcal{T}$ be the evolution semigroup associated to $\mathcal{U}$ on $E(\mathbb{R}, X)$. Denote by G its infinitesimal generator. Consider the statements:*

(1) *$\mathcal{U}$ is uniformly exponentially stable.*

(2) *$\mathcal{T}$ is uniformly exponentially stable.*

(3) *G is invertible.*

(4) *For each $f \in E(\mathbb{R}, X)$, the map $t \mapsto (\mathcal{U} * f)(t) := \int_0^t U(t,s)f(s)ds$ belongs to $E(\mathbb{R}, X)$.*

(5) *For each $f \in E(\mathbb{R}, X)$, the function $\mathcal{U} * f$ belongs to $BUC(\mathbb{R}, X)$.*

Then

$$\mathbf{(1)} \Leftrightarrow \mathbf{(2)} \Rightarrow \mathbf{(3)} \Rightarrow \mathbf{(4)} \Rightarrow \mathbf{(5)}.$$

If, in addition, the family $\mathcal{U}$ is uniformly bounded then $\mathbf{(5)} \Rightarrow \mathbf{(3)}$.

From structural point of view, this theorem is known. See, for example, [14], [16] [27], [28],[46] [59] and their references. Everywhere in these references, the evolution semigroup is defined on a function space having the property that it contains the set $C_0^\infty((0,\infty), X)$; the set of all indefinite differentiable X-valued functions having compact support in the open interval $(0,\infty)$. In addition, in many of those spaces the set $C_0^\infty((0,\infty), X)$ is dense in respect with the corresponding norm of the base space. This particularity of the spaces allow us to deduce the linear stability principle for evolution semigroups, i.e. $s(G) = \omega_0(\mathcal{T})$. First details concerning the evolution semigroups and their applications was provided in [37]. Further developments in this area was made for example in [22], [47], [8].

By contrast, the space $E(\mathbb{R}, X)$, does not contain the set $C_0^\infty((0,\infty), X)$. In this context, we can not prove (3)⇒(2). The implication (3) ⇒ (2) is proved in [58, Theorem 4.1] using a wider space instead of $E(\mathbb{R}, X)$. Under additional assumptions, namely that the evolution family is uniformly bounded and that the map $s \mapsto U(s,0)x$ satisfies a Lipschitz condition on $\mathbb{R}_+$ for each x in a dense subset D of X, we can state that the set $\left\{e^{\frac{2ji\pi}{q}} : j \in Z\right\}$ is included in the resolvent set of the operator $U(q,0)$. Here $i \in \mathbb{C}$ and $i^2 = -1$.

Proposition 19. *Suppose that the family $\mathcal{U}$ is uniformly bounded and that there exists a dense subset D of X such that for each $x \in D$ the map $s \mapsto U(s,0)x : \mathbb{R}_+ \to X$ satisfies a Lipschitz condition on $(0,q)$. If the condition **(5)** from the previous theorem is fulfilled, then the set $\left\{e^{\frac{2ji\pi}{q}} : j \in \mathbb{Z}\right\}$ is contained in $\rho(U(q,0))$.*

For the proof of Proposition 19 we refer the reader to [21, Proposition 3.8] with the mention that "i" is missing there.

The following lemmas are used in the proof of Theorems 18.

Lemma 20. *Let* $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ *be a strongly continuous and q-periodic evolution family acting on* X. *Suppose that the family* $\mathcal{U}$ *is uniformly bounded, i.e.*

$$\sup_{t\geq s\geq 0} ||U(t,s)|| := K < \infty.$$

If for a real number μ *and a function* $f \in E(\mathbb{R}, X)$ *occurs*

$$\sup_{t\geq 0} \left\| \int_0^t e^{i\mu s} U(t,s) f(s) ds \right\| := M(\mu, f) < \infty,$$

then

$$\sup_{t\geq 0} \left\| \int_0^t e^{-i\mu s} \mathcal{T}(s) f ds \right\| \leq K(\mu, f) < \infty.$$

Proof. Let $\tau \in \mathbb{R}$ and $J_{f,t,\mu}(\tau) = \left(\int_0^t e^{-i\mu s} \mathcal{T}(s) f ds \right)(\tau)$. Evaluate the "sup" norm of $J_{f,t,\mu}(\cdot)$ and obtain the conclusion for $K(\mu, f) := (1 + K)M(\mu, f)$. □

The next lemma has a similar proof as [46, Lemma 1.1].

Lemma 21. *Let* $f, u \in E(\mathbb{R}, X)$. *The following two statements are equivalent:*

1. $u \in D(G)$ *and* $Gu = -f$.
2. $u(t) = \int_0^t U(t,s) f(s) ds$, *for all* $t \geq 0$.

The main result of this section can be stated as follows.

Theorem 22. *Let* $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ *be a strongly continuous and q-periodic evolution family acting on a Banach space* X. *If*

$$\sup_{\mu\in\mathbb{R}} \sup_{t\geq\tau\geq 0} \left\| \int_\tau^t e^{i\mu s} U(t,s) x ds \right\| \leq M\|x\|, \quad \forall x \in X, \tag{4.3}$$

for some positive constant M, *then*

$$\sup_{\mu\in\mathbb{R}} \sup_{t\geq 0} \left\| \int_0^t e^{-i\mu s} \mathcal{T}(s) f ds \right\| \leq L(f) < \infty, \quad \forall f \in span\{\cup_{t\geq 0} \mathcal{A}_t\}.$$

More, the family $\mathcal{U}$ *is uniformly exponentially bounded if for each* $x \in X$ *the map* $s \mapsto U(s,0)x$ *satisfies a Lipschitz condition on* $(0,q)$.

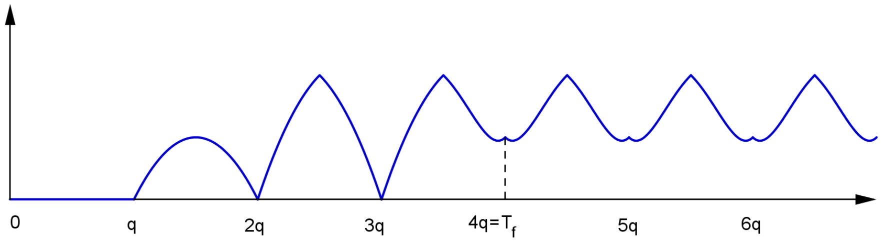

Figure 3: The graphical representation of a particular function from $\text{span}\{\cup_{t\geq 0}\mathcal{A}_t\}$

Sketch of the proof.

We prove the theorem with the supplementary assumption that the evolution family $\mathcal{U}$ is uniformly bounded by the constant K. This complementary assumption does not affect the generality.

Let f in $\text{span}\{\cup_{t\geq 0}\mathcal{A}_t\}$. Then there exists a positive number T_f and a function $F \in P_q(\mathbb{R}, X) \cap AP_1(\mathbb{R}, X)$ such that for any $s \geq T_f$ have that $f(s) = F(s)$ (see Figure 3).

The representation $F(s) = \sum_{k=-\infty}^{k=\infty} e^{i\mu_k s} c_k(F)$, with $c_k(F)$ in X was used. Let $\tau \in \mathbb{R}$ and $J_{f,t,\mu}(\tau)$ as in the proof of Lemma 20. We evaluate the "sup" norm of $J_{f,t,\mu}(\cdot)$ analysing three cases which leads us to the first statement by choosing $L(f) := KT_f\|f\|_\infty + M\|F\|_1$.

To prove the second statement, consider $h \in P_q(\mathbb{R}, \mathbb{R})$ having the following properties: h satisfies a Lipschitz condition on $\mathbb{R}$, $h(0) = h(q) = 0$ and for each real number μ, the map $\mu \mapsto \gamma(\mu) := \int_0^q e^{i\mu s} h(s) ds$ is different from 0. For each $x \in X$, consider the map $t \mapsto h_x(t)$ from $P_q(\mathbb{R}, X)$, given on $[0, q]$ by $h_x(s) := h(s)U(s, 0)x$. We define the function f_x by

$$s \mapsto f_x(s) := \begin{cases} h_x(s), & s \geq 0 \\ 0, & s < 0. \end{cases}$$

According to the Lemma 5, f_x belongs to $AP_1(\mathbb{R}, X)$. Therefore f_x belongs to $\text{span}\{\cup_{t\geq 0}\mathcal{A}_t\}$. Let $t = nq$, for $n = 0, 1, 2, \cdots$ we obtain

$$\int_0^{nq} e^{i\mu\rho} U(nq, \rho) f_x(\rho) d\rho \quad = \quad \gamma(\mu) \sum_{k=0}^{n-1} e^{i\mu k q} U(q, 0)^{n-k} x.$$

The desired assertion is obtained by passing to the norms and applying the Uniform Boundedness Principle and Lemma 2.

□

In Theorem 18 are established several connections between the exponential stability of a q-periodic evolution family of linear operators acting on a Banach space and the spectral properties of its infinitesimal generator. In that theorem we could not close the chain equivalents because the function spaces, $E(\mathbb{R}, X)$, that act the evolution semigroup, was not rich enough. In his section, we extend this function space such that under certain conditions we can close the chain equivalents.

In the following, denote by $\mathcal{A}$ the set of all functions $e^{i\mu\cdot} \otimes f$, with $\mu \in \mathbb{R}$ and $f \in \cup_{t\geq 0}\mathcal{A}_t$. The set $\mathcal{A}_t$ was defined in the previous section. Denote by $\widetilde{E}(\mathbb{R}, X)$ the closure in relation to the "sup" norm of the linear coverage of the set $\mathcal{A}$, i.e. $\widetilde{E}(\mathbb{R}, X) := \overline{\text{span}}(\mathcal{A})$. The evolution semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ associated to the strongly continuous and q-periodic evolution family $\mathcal{U} = \{U(t, s)\}_{t\geq s}$ on $\widetilde{E}(\mathbb{R}, X)$, is formal defined by

$$(\mathcal{T}(t)\widetilde{f})(s) := \begin{cases} U(s, s-t)\widetilde{f}(s-t), & s \geq t \\ 0, & s < t \end{cases}, \quad \text{pentru } \widetilde{f} \in \widetilde{E}(\mathbb{R}, X). \tag{4.4}$$

The evolution semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ acts on $\widetilde{E}(\mathbb{R}, X)$ and it is strongly continuous.

Theorem 23. *Let $\mathcal{U}$ be a strongly continuous and q-periodic evolution family acting on a Banach space X and let $\mathcal{T}$ be the evolution semigroup associated to $\mathcal{U}$ on $\widetilde{E}(\mathbb{R}, X)$. Denote by $\widetilde{G}$ its infinitesimal generator. Consider the following statements:*

(1) *$\mathcal{U}$ is uniformly exponentially stable.*

(2) *$\mathcal{T}$ is uniformly exponentially stable.*

(3) *$\widetilde{G}$ is invertible.*

(4) *For each $\widetilde{f} \in \widetilde{E}(\mathbb{R}, X)$ the map $t \mapsto g_{\widetilde{f}}(t) := \int_0^t U(t, s)\widetilde{f}(s)ds$ belongs to $\widetilde{E}(\mathbb{R}, X)$.*

(5) *For each $\widetilde{f} \in \widetilde{E}(\mathbb{R}, X)$ the map $g_{\widetilde{f}}$ belongs to $BUC(\mathbb{R}_+, X)$.*

Then,

$$\mathbf{(1)} \Leftrightarrow \mathbf{(2)} \Rightarrow \mathbf{(3)} \Rightarrow \mathbf{(4)} \Rightarrow \mathbf{(5)}.$$

In addition, if there exists a dense subset D of X such that for each $x \in D$ the map $s \mapsto U(s, 0)x : \mathbb{R}_+ \to X$ satisfies a Lipschitz condition on $(0, q)$, then **(5)** $\Rightarrow$ **(1)**.

Corollary 24. *Let $\boldsymbol{T}= \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup acting on X and let $\mathcal{T}$ be the evolution semigroup associated to $\boldsymbol{T}$ on $\widetilde{E}(\mathbb{R}, X)$. Denote by $\widetilde{G}$ the infinitesimal generator of $\mathcal{T}$. The following four statements are equivalent.*

(1) *$\boldsymbol{T}$ is uniformly exponentially stable.*

(2) *$\mathcal{T}$ is uniformly exponentially stable.*

(3) *$\widetilde{G}$ is invertible.*

(4) *For each $\widetilde{f} \in \widetilde{E}(\mathbb{R}, X)$ the map $t \mapsto g_{\widetilde{f}}(t) := \int_0^t T(t-s)\widetilde{f}(s)ds$ belongs to $\widetilde{E}(\mathbb{R}, X)$.*

(5) *For each $\widetilde{f} \in \widetilde{E}(\mathbb{R}, X)$ the map $g_{\widetilde{f}}$ is bounded on $\mathbb{R}_+$.*

In the proof of (5) $\Rightarrow$ (1) we no longer need additional conditions because for $x \in D(A)$ the map $t \mapsto T(t)x$ satisfies a Lipschitz condition on $\mathbb{R}_+$.

An immediate consequence of the Theorem 23 is the spectral mapping theorem for the evolution semigroup $\mathcal{T}$ on $\widetilde{E}(\mathbb{R}, X)$.

Theorem 25. *[58] Let $\mathcal{U}$ be a q-periodic strongly continuous evolution family acting on X and let $\mathcal{T}$ be the evolution semigroup associated to $\mathcal{U}$ on $\widetilde{E}(\mathbb{R}, X)$. Denote by $\widetilde{G}$ the infinitesimal generator of the semigroup $\mathcal{T}$. Suppose that there exists a dense subset D of X such that for each x in D, the map $s \mapsto U(s,0)x : \mathbb{R}_+ \to X$ satisfies a Lipschitz condition on $(0,q)$. Then,*

$$e^{t\sigma(\widetilde{G})} = \sigma(\mathcal{T}(t)) \setminus \{0\}, \quad t \geq 0.$$

Moreover,

$$\sigma(\widetilde{G}) = \{z \in \mathbb{C} : \ \Re(z) \leq s(\widetilde{G})\}$$

and

$$\sigma(\mathcal{T}(t)) = \{\lambda \in \mathbf{C} : \ |\lambda| \leq \mathbf{r}(\mathcal{T}(\mathbf{t}))\}, \quad \forall \mathbf{t} \geq \mathbf{0}.$$

Another result of this paper is the following extension of Vu Phong's theorem to the nonautonomous periodic case. It can read as follows.

Theorem 26. *A strongly continuous q-periodic evolution family $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0}$ of bounded linear operators acting on a complex Hilbert space H is uniformly exponentially stable if for each $x \in X$, the map $U(\cdot,0)x$ satisfy a Lipschitz condition on $(0,q)$ and there exists a positive constant K such that*

$$\sup_{\mu \in \mathbb{R}} \sup_{t \geq 0} \left\| \int_0^t e^{i\mu s} U(t,s)x ds \right\| \leq K\|x\|. \tag{4.5}$$

The additional condition, (in comparison with Vu Phong's theorem), i.e. the Lipschitz condition, is not very restrictive. In the semigroup case, it is automatically checked for every x in the domain of the infinitesimal generator, which we know that is a dense set in the state space. Remains as open problem whether we can replace the assumption concerning the previous Lipschitz condition with a similar one, but for x in a certain dense subset D of H.

All the above results are formulated in the general context given by the functional analysis. However, the condition (1.4) can be translated in terms of differential equations (or partial derivative), taking into account that the solution of the following abstract Cauchy Problem

$$\begin{cases} \dot{y}(t) = Ay(t) + e^{i\mu t}x, & t > 0 \\ y(0) = 0 \end{cases} \tag{4.6}$$

is given by

$$y_{\mu,x}(t) = \int_0^t e^{i\mu s} T(t-s)x ds = e^{i\mu t} \int_0^t e^{-i\mu s} T(s)x ds.$$

In terms of differential equations, the van Neerven-Vu Phong's theorem says that the semigroup $\mathbf{T}$ is exponentially stable (uniformly exponentially stable) if and only if for each $x \in X$, the solution of the Cauchy Problem (4.6) is bounded on $\mathbb{R}_+$, uniformly in respect to the real parameter μ.

From this perspective, the problems exposed above, may be classified in the admissibility theory initiated by Oscar Perron in 1930, [53]. In the general form, it can be formulated as follows:

Let $A(\cdot)$ be a periodic linear operator valued function acting on a Banach space X. What is the connection between the exponential stability of solutions of a homogeneous system

$$\dot{x}(t) = A(t)x(t), \tag{4.7}$$

and the boundedness on $\mathbb{R}_+$ (uniformly in respect to the real parameter μ) of the solutions of the well-posed abstract Cauchy problems

$$\begin{cases} \dot{y}(t) = A(t)y(t) + e^{i\mu t}b, & t > 0, \quad b \in X, \quad \mu \in \mathbb{R} \\ y(0) = 0? \end{cases} \tag{4.8}$$

Let $\{A(t)\}$ be a family of closed linear operators acting on X and let $x_0 \in X$. Further, we are referring to the homogeneous Cauchy problem

$$\begin{cases} \dot{x}(t) = A(t)x(t), & t > 0 \\ x(0) = x_0. \end{cases} \tag{4.9}$$

Using very general terms, we will say that this problem is well-posed if there are an evolution family $\{U(t,s)\}_{t \geq s \geq 0}$, such that, for every solution $x(\cdot)$ of the Cauchy Problem (4.9), we have that

$$x(t) = U(t,s)x(s), \quad \text{for all } t \geq s \geq 0.$$

Details concerning the conditions that must be fulfilled by the family $\{A(t)\}$ so that the problem (4.9) to be well-posed, can be found, for example, in [62], [49], [50].

5 The Barbashin type theorems

The above results have through their consequences new real integral characterizations for the exponential stability of the system (4.7) or even of periodic evolution families. These real integral characterizations are often used in control theory and in stability theory; in the latter case when we need to define Lyapunov functions associated with the system (4.7).

Theorem 27. *([7, Theorem 5.1]): Let $\Phi(\cdot)$ be the unique solution of the following Cauchy problem*

$$\begin{cases} \dot{X}(t) = A(t)X(t), t \geq 0 \\ X(0) = I. \end{cases} \tag{5.1}$$

where $A(t)$ and $X(t)$ are square matrix, I is the unit matrix of the same order with $A(t)$ and the map $t \mapsto A(t)$ is continuous. If $\sup_{t\geq 0} \int_0^t \|\Phi(t)\Phi^{-1}(s)\|ds < \infty$, then, there exist two positive constants N and ν such that $\|\Phi(t)\Phi^{-1}(s)\| \leq Ne^{-\nu(t-s)}$, for all $t \geq s \geq 0$.

In [45], this theorem was formulated for evolution families as follows.

Theorem 28. *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be an evolution family of bounded linear operators acting on a Banach space X and having exponential growth such that for each $t > 0$, the map $s \mapsto \|U(t,s)\| : [0,t] \to \mathbb{R}$ is measurable. The family $\mathcal{U}$ is uniformly exponentially stable if and only if for some $p \in [1,\infty)$, one has*

$$\sup_{t\geq 0} \left(\int_0^t \|U(t,s)\|^p ds \right)^{\frac{1}{p}} < \infty. \tag{5.2}$$

We mention that in Theorem 28, the operator $U(t,s)$ can be non-invertible.

The strong version of Barbashin's theorem states that the evolution family $\{U(t,s)\}_{t\geq s}$ of bounded linear operators acting on a Banach space X is uniformly exponentially stable if for every x in X and some $p \geq 1$, the boundedness condition of strong Barbashin type occurs, i.e.:

$$\sup_{t\geq 0} \int_0^t \|U(t,s)x\|^p ds := M_p(x) < \infty. \tag{5.3}$$

Surprisingly, the proof of the strong Barbashin theorem seems to be more difficult than the uniform, and, as we know, it is still an open question for strongly continuous evolution families acting on an arbitrary Banach space. In this direction, some progress has been made in [23] and [18], where the dual family $\mathcal{U}^*$ of $\mathcal{U}$ was used, and estimations like (5.3) were considered in respect to the strong topology of $\mathcal{L}(X^*)$. For instance , in Theorem 2 from [23] it is shown that an exponentially bounded evolution family, like considered above is uniformly exponentially stable if and only if there exists $1 \leq q < \infty$ such that

$$\sup_{t\geq 0} \int_0^t \|U(t,s)^* x^* ds\|^q < \infty.$$

In the discrete periodic case, the strong variant of the Barbashin theorem is completely solved in [5] and [9].

Corollary 29. *Let us consider the evolution family $\mathcal{U} = \{\Phi(t)\Phi^{-1}(s)\}_{t\geq s\geq 0}$ associated to the system (4.7), where the map $t \mapsto A(t)$ is q-periodic and continuous. The following four statements are equivalent:*

(i) *The family $\mathcal{U}$ is uniformly exponentially stable.*

(ii) $\sup_{t\geq 0} \int_0^t |\langle \Phi(t)\Phi^{-1}(s)x, y\rangle| ds < \infty, \quad \forall x, y \in X.$

(iii) $\sup_{t\geq 0} \int_0^t |\langle \Phi(t)\Phi^{-1}(s)x, x\rangle| ds < \infty, \quad \forall x \in X.$

(iv) $\sup_{t\geq 0} \int_0^t \|\Phi(t)\Phi^{-1}(s)x\| ds < \infty, \quad \forall x \in X.$

This corollary is a consequence of Theorem 4; its proof can be found in [4].

Corollary 30. *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be a q-periodic strongly continuous evolution family acting on a Hilbert space H, having the property that for each $x \in H$, the map $U(\cdot, 0)x$ satisfies a Lipschitz condition on $(0, q)$. The following four statements are equivalent:*

(i) *The family $\mathcal{U}$ is uniformly exponentially stable.*

(ii) $\sup_{t\geq 0} \int_0^t |\langle U(t,s)x, y\rangle|\, ds < \infty, \quad \forall x, y \in H.$

(iii) $\sup_{t\geq 0} \int_0^t |\langle U(t,s)x, x\rangle|\, ds < \infty, \quad \forall x \in H.$

(iv) $\sup_{t\geq 0} \int_0^t \|U(t,s)x\| ds < \infty, \quad \forall x \in H.$

We mention that the third condition in both above corollaries can be deduced from the second one by using the polarization identity as in [13]. Finally, recall a very interesting result in this area, concerning strongly continuous semigroups offered recently by Storozhuk, [61].

Theorem 31. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup acting on a Banach space X such that $s_0(\mathbf{T}) \geq 0$. Then for each nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ which is positive on $(0, \infty)$, there are $x \in X$ and $x^* \in X^*$ such that $\int_0^\infty \phi(| < T(t)x, x^* > |)dt = \infty$.*

A natural consequence of the above Storozhuk theorem, is the next Rolewicz type theorem "rescued" in the weak topology of a Hilbert space H, see [57] for further details.

Corollary 32. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup acting on a Hilbert space H and ϕ as in the previous theorem. If $\int_0^\infty \phi(| < T(t)x, y > |)dt < \infty$ for every $x, y \in H$ then $\omega_0(\mathbf{T})$ is negative.*

Particular cases of Storozhuk theorem has been obtained earlier in [40] and [67]. For discrete versions of Rolewicz theorem see [20].

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