The Poupard statistics on tangent and secant trees

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(Based on some recent papers with Dominique Foata)

Tangent numbers

Taylor expansion of $\tan u$:

$$\tan u = \sum_{n\geq 0} \frac{u^{2n+1}}{(2n+1)!} T_{2n+1}$$

$$= \frac{u}{1!}1 + \frac{u^3}{3!}2 + \frac{u^5}{5!}16 + \frac{u^7}{7!}272 + \frac{u^9}{9!}7936 + \cdots$$

The coefficients T_{2n+1} $(n \geq 0)$ are called the *tangent numbers*

Secant numbers

Taylor expansion of $\sec u$:

$$\sec u = \frac{1}{\cos u} = \sum_{n\geq 0} \frac{u^{2n}}{(2n)!} E_{2n}$$

$$= 1 + \frac{u^2}{2!}1 + \frac{u^4}{4!}5 + \frac{u^6}{6!}61 + \frac{u^8}{8!}1385 + \frac{u^{10}}{10!}50521 + \cdots$$

The coefficients E_{2n} $(n \geq 0)$ are called the *secant numbers*

Alternating permutations

Désiré André's (1879):

A permutation

$$\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$$

of $12\cdots n$ with the property that

$$\sigma(1) > \sigma(2),\ \sigma(2) < \sigma(3),\ \sigma(3) > \sigma(4),\ \text{etc.}$$

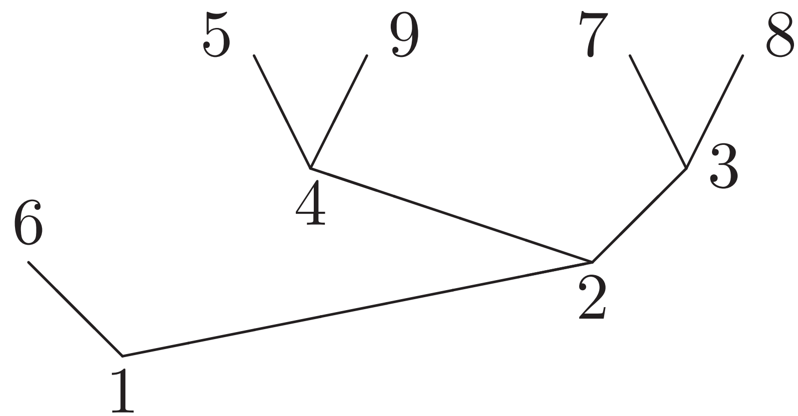
in an alternating way is called *alternating permutation*.

Let $\mathfrak{A}_n$ denote the set of all alternating permutations of $12\cdots n$.

Theorem:

$$\#\mathfrak{A}_{2n-1} = T_{2n-1}, \qquad \#\mathfrak{A}_{2n} = E_{2n}.$$

Tangent tree



$2n+1$ vertices,

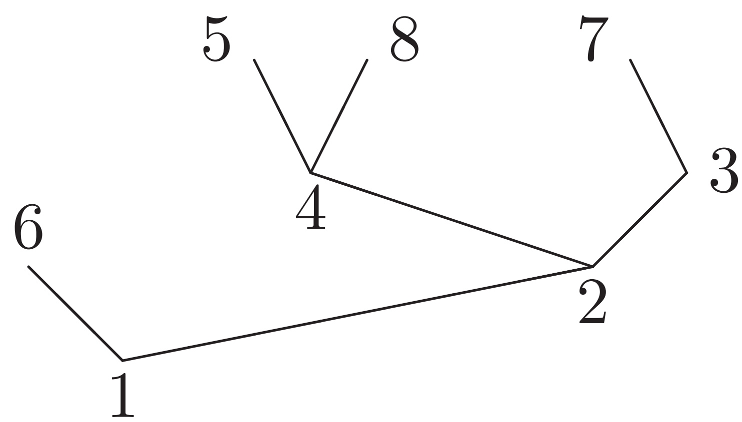
complete,

binary, rooted, planar, labeled, increasing

The set all off tangent trees : $\mathfrak{T}_{2n+1}$.

$$\#\mathfrak{A}_{2n+1} = \#\mathfrak{T}_{2n+1} = T_{2n+1}$$

Secant tree



$2n$ vertices,

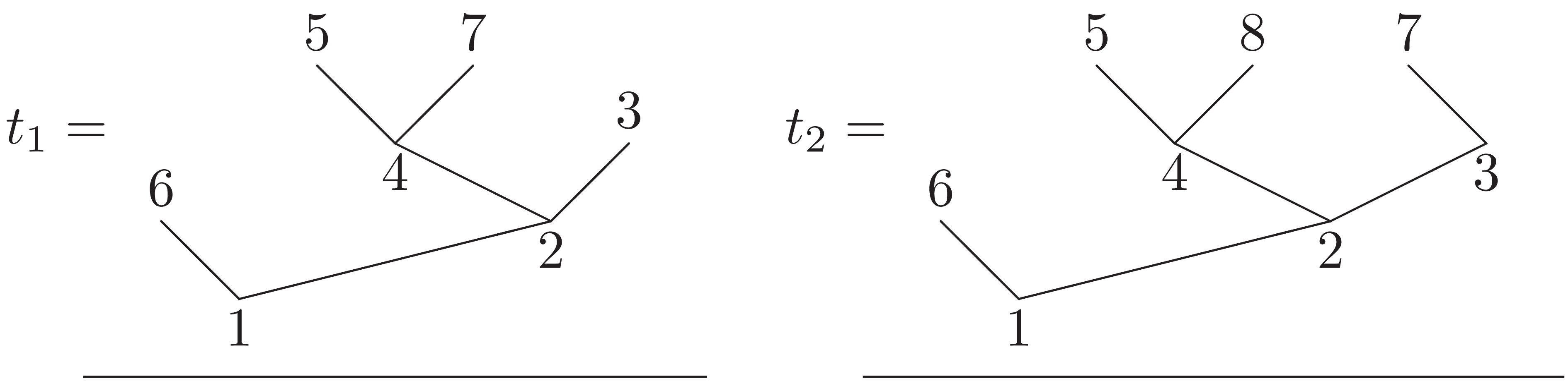
complete (execpt that the rightmost vertice is missing),

binary, rooted, planar, labeled, increasing

The set all off tangent trees : $\mathfrak{T}_{2n}$.

$$\#\mathfrak{A}_{2n} = \#\mathfrak{T}_{2n} = E_{2n}.$$

Bijection



$$\sigma_1 = 6 \quad 1 \quad 5 \quad 4 \quad 7 \quad 2 \quad 3$$

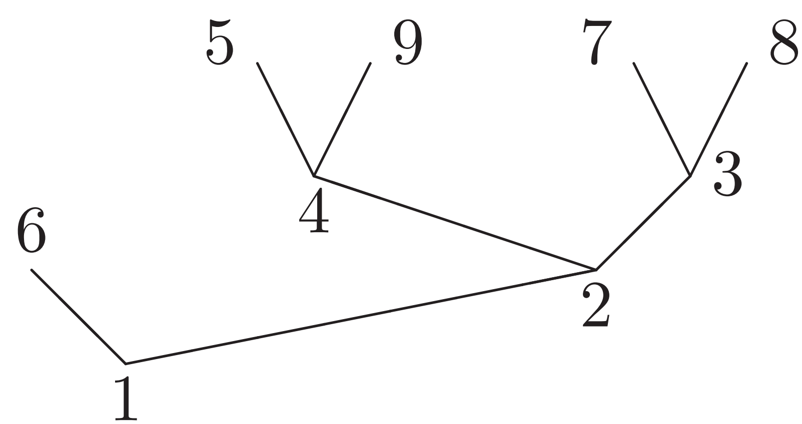
$$\sigma_2 = 6 \quad 1 \quad 5 \quad 4 \quad 8 \quad 2 \quad 7 \quad 3$$

Tangent, secant trees and alternating permutations

Poupard statistics: eoc

Poupard (1989)

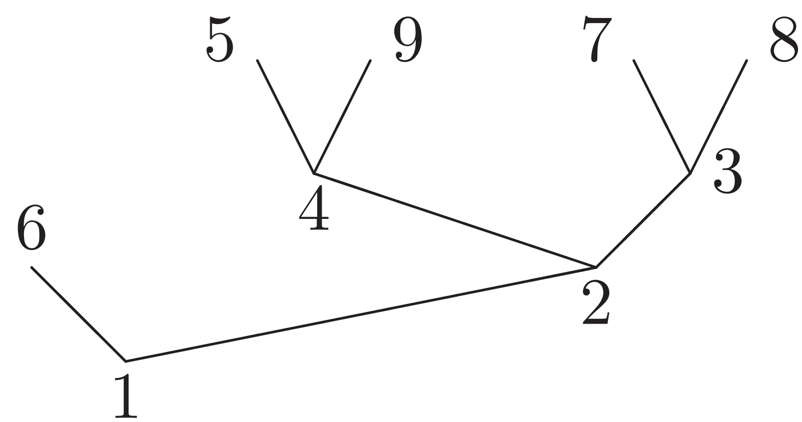
Let $1 = a_1 \to a_2 \to a_3 \to \cdots \to a_{j-1} \to a_j$ be the minimal chain of a tree $t \in \mathfrak{T}_n$, the *"end of the minimal chain"* of t is defined to be $\mathrm{eoc}(t) := a_j$.



For example, the minimal chain of the tree t is $1 \to 2 \to 3 \to 7$, so that $\mathrm{eoc}(t) = 7$

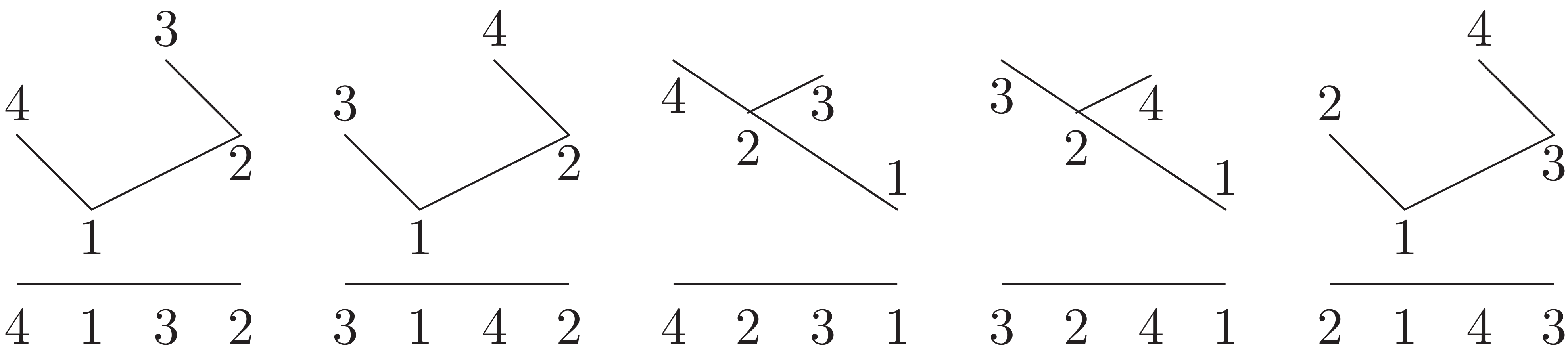
Poupard statistics: pom

If the leaf with the maximum label n is incident to a node labeled k, define its *"parent of the maximum leaf"* to be $\mathrm{pom}(t) := k$.



The parent of its maximum leaf (equal to $n = 9$) is $\mathrm{pom}(t) = 4$

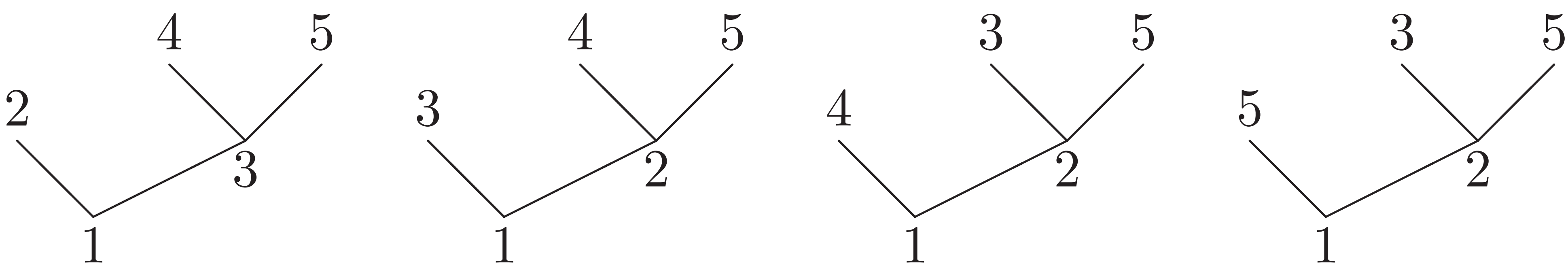
5 secant trees with 4 vertices



eoc =	3	4	3	3	2
pom =	1	2	2	2	3

16 tangent trees with 5 vertices

There 16 tangent trees from $\mathfrak{T}_5$. Only 4 of them (reduced trees) are displayed, but each of them gives rise to three other tangent trees, having the same "eoc" and "pom" statistics, by pivoting each pair of subtrees.



	2 1 4 3 5	3 1 4 2 5	4 1 3 2 5	5 1 3 2 4
eoc =	2	4	3	3
pom =	3	2	2	1

Equidistribution

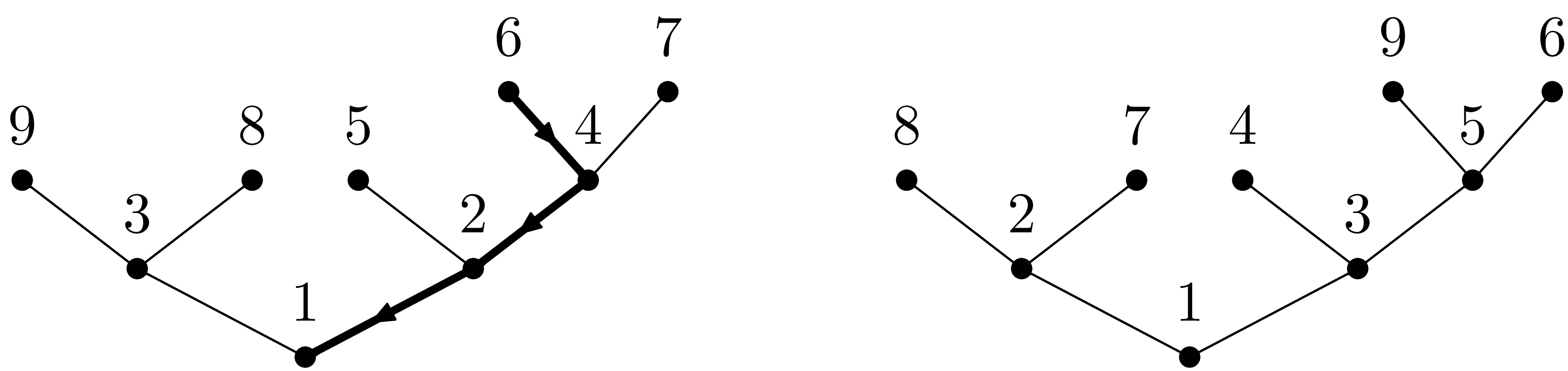
Theorem.
The statistics "$\text{eoc} - 1$" and "pom" are equidistributed on each set $\mathfrak{T}_n$.

The tangent tree case was obtained by Poupard (1989). Her original proof, not of combinatorial nature, makes use of a clever finite difference analysis argument.

Our proof: Bijection inspired from the classical "jeu de taquin" on directed acyclic graphs, (Schützenberger, 1972)

Proof

Let $1 = a_1 \to a_2 \to a_3 \to \cdots \to a_{j-1} \to a_j$ be the minimal chain of t.
(i) for $i = 1, 2, \ldots, j-1$ replace each node label a_i of the minimal chain by $a_{i+1} - 1$;
(ii) replace the node label a_j by n;
(iii) replace each other node label b by $b - 1$.



eoc=6 pom=5

Poupard numbers for tagent trees: $g_n(k)$

$$g_n(k) := \#\{t \in \mathfrak{T}_{2n-1} : \mathrm{pom}(t) = k\}$$
$$= \#\{t \in \mathfrak{T}_{2n-1} : \mathrm{eoc}(t) = k+1\}$$

$k =$	1	2	3	4	5	6	7	Sum	
$n = 1$	1							1	$= T_1$
2	0	2	0					2	$= T_3$
3	0	4	8	4	0			16	$= T_5$
4	0	32	64	80	64	32	0	272	$= T_7$

Theorem (Poupard, 1989).

$$1 + \sum_{n \geq 1} \sum_{1 \leq k \leq 2n+1} g_{n+1}(k) \frac{x^{2n+1-k}}{(2n+1-k)!} \frac{y^{k-1}}{(k-1)!} = \frac{\cos(x-y)}{\cos(x+y)}$$

Poupard numbers for secant trees: $h_n(k)$

$$
\begin{aligned}
h_n(k) &:= \#\{t \in \mathfrak{T}_{2n} : \mathrm{pom}(t) = k\} \\
&= \#\{t \in \mathfrak{T}_{2n} : \mathrm{eoc}(t) = k+1\}
\end{aligned}
$$

$k =$	1	2	3	4	5	6	7	Sum
$n = 1$	1							$1 = E_2$
2	1	3	1					$5 = E_4$
3	5	15	21	15	5			$61 = E_6$
4	61	183	285	327	285	183	61	$1385 = E_8$

Theorem (Foata-H., 2013).

$$
1 + \sum_{n\geq 1} \sum_{1\leq k\leq 2n+1} h_{n+1}(k) \frac{x^{2n+1-k}}{(2n+1-k)!} \frac{y^{k-1}}{(k-1)!} = \frac{\cos(x-y)}{\cos^2(x+y)}
$$

Proof

Lemma.

Let $Z(x,y) := \sum_{i\geq 0,\, j\geq 0} \gamma_{i,j} \frac{x^i}{i!}\frac{y^j}{j!}$ satisfying the partial differential equation

$$\frac{\partial^2 Z(x,y)}{\partial x\,\partial y} = 2\,Z(x,y) + \frac{1}{2}\frac{\partial^2 Z(x,y)}{\partial x^2} + \frac{1}{2}\frac{\partial^2 Z(x,y)}{\partial y^2}.$$

Then,

$$Z(x,y) = f(x+y)\,\sec(x+y)\,\cos(x-y)$$

for some formal power series in one variable $f(x) = 1 + \sum_{n\geq 1} f_{2n}\frac{x^{2n}}{(2n)!}$.

Reduced tangent trees

We will work with the reduced tangent trees.

Recycle the notation:

$$\mathfrak{T}_{2n+1} := \frac{\mathfrak{T}_{2n+1}}{2^n}$$

$$\begin{aligned} f_n(k) &:= \#\{t \in \mathfrak{T}_{2n+1} \mid \mathrm{pom}(t) = k\} \\ &= \#\{t \in \mathfrak{T}_{2n+1} \mid \mathrm{eoc}(t) = k+1\} \end{aligned}$$

$g_n(k)$:

$k =$	1	2	3	4	5	6	7	Sum	
$n = 1$	1							1	$= T_1$
2	0	2	0					2	$= T_3$
3	0	4	8	4	0			16	$= T_5$
4	0	32	64	80	64	32	0	272	$= T_7$

$f_n(k) = g_{n+1}(k)/2^n$:

$k =$	1	2	3	4	5	6	7	Sum	
$n = 0$	1							1	$= T_1/2^0$
1	0	1	0					1	$= T_3/2^1$
2	0	1	2	1	0			4	$= T_5/2^2$
3	0	4	8	10	8	4	0	34	$= T_7/2^3$

2D-Distribution on reduced tangent trees

$$f_n(m,k) := \#\{t \in \mathfrak{T}_{2n+1} : \mathrm{eoc}(t) = m,\ \mathrm{pom}(t) = k\}$$

Matrix

$$M_n := (f_n(m,k))_{1 \le m,k \le 2n}$$

$$M_1 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 4 & 2 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{pmatrix}$$

$$M_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 8 & 10 & 8 & 4 & 0 \\ 4 & 4 & 0 & 16 & 20 & 16 & 8 & 0 \\ 8 & 12 & 16 & 0 & 28 & 20 & 10 & 0 \\ 10 & 18 & 24 & 28 & 0 & 16 & 8 & 0 \\ 8 & 18 & 24 & 24 & 16 & 0 & 4 & 0 \\ 4 & 12 & 18 & 18 & 12 & 4 & 0 & 0 \\ 0 & 4 & 8 & 10 & 8 & 4 & 0 & 0 \end{pmatrix}$$

$$M_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 34 & 68 & 94 & 104 & 94 & 68 & 34 & 0 \\ 34 & 34 & 0 & 136 & 188 & 208 & 188 & 136 & 68 & 0 \\ 68 & 102 & 136 & 0 & 274 & 296 & 262 & 188 & 94 & 0 \\ 94 & 162 & 222 & 274 & 0 & 352 & 296 & 208 & 104 & 0 \\ 104 & 198 & 276 & 330 & 352 & 0 & 274 & 188 & 94 & 0 \\ 94 & 198 & 282 & 330 & 330 & 274 & 0 & 136 & 68 & 0 \\ 68 & 162 & 240 & 282 & 276 & 222 & 136 & 0 & 34 & 0 \\ 34 & 102 & 162 & 198 & 198 & 162 & 102 & 34 & 0 & 0 \\ 0 & 34 & 68 & 94 & 104 & 94 & 68 & 34 & 0 & 0 \end{pmatrix}$$

Guess'n'Prove

Guess'n'Prove

M_3

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 4 & 2 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{pmatrix}$$

M_4

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 8 & 10 & 8 & 4 & 0 \\ 4 & 4 & 0 & 16 & 20 & 16 & 8 & 0 \\ 8 & 12 & 16 & 0 & 28 & 20 & 10 & 0 \\ 10 & 18 & 24 & 28 & 0 & 16 & 8 & 0 \\ 8 & 18 & 24 & 24 & 16 & 0 & 4 & 0 \\ 4 & 12 & 18 & 18 & 12 & 4 & 0 & 0 \\ 0 & 4 & 8 & 10 & 8 & 4 & 0 & 0 \end{pmatrix}$$

Guess'n'Prove

M_3

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & \color{red}{1} & 0 \\ 1 & 1 & 0 & 4 & \color{blue}{2} & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{pmatrix}$$

M_4

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 8 & \color{red}{10} & \color{red}{8} & \color{red}{4} & 0 \\ 4 & 4 & 0 & 16 & \color{blue}{20} & \color{blue}{16} & \color{blue}{8} & 0 \\ 8 & 12 & 16 & 0 & 28 & 20 & 10 & 0 \\ 10 & 18 & 24 & 28 & 0 & 16 & 8 & 0 \\ 8 & 18 & 24 & 24 & 16 & 0 & 4 & 0 \\ 4 & 12 & 18 & 18 & 12 & 4 & 0 & 0 \\ 0 & 4 & 8 & 10 & 8 & 4 & 0 & 0 \end{pmatrix}$$

Guess'n'Prove

$$M_3 \qquad\qquad M_4$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 4 & 2 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 8 & 10 & 8 & 4 & 0 \\ 4 & 4 & 0 & 16 & 20 & 16 & 8 & 0 \\ 8 & 12 & 16 & 0 & 28 & 20 & 10 & 0 \\ 10 & 18 & 24 & 28 & 0 & 16 & 8 & 0 \\ 8 & 18 & 24 & 24 & 16 & 0 & 4 & 0 \\ 4 & 12 & 18 & 18 & 12 & 4 & 0 & 0 \\ 0 & 4 & 8 & 10 & 8 & 4 & 0 & 0 \end{pmatrix}$$

$$\underset{k}{\Delta}^2 f_n(m,k) + 2\, f_{n-1}(m,k) = 0$$

Difference operators

The *partial difference operators* $\underset{m}{\Delta}$, $\underset{k}{\Delta}$, act as follows on the entries of the matrices M_n:

$$\underset{m}{\Delta} f_n(m,k) := f_n(m+1,k) - f_n(m,k);$$
$$\underset{k}{\Delta} f_n(m,k) := f_n(m,k+1) - f_n(m,k).$$

Consider the following four triangles of each square $\{(m,k) : 1 \le m,k \le 2n\}$:

$$L_n^{(1)} := \{2 \le k+1 \le m \le 2n-2\};$$
$$L_n^{(2)} := \{4 \le k+3 \le m \le 2n\};$$
$$U_n^{(1)} := \{2 \le m+1 \le k \le 2n-2\};$$
$$U_n^{(2)} := \{4 \le m+3 \le k \le 2n\}.$$

Fundamental relations

Theorem

$$(R1)\quad \Delta_m^2 f_n(m,k) + 2\,f_{n-1}(m,k) = 0 \qquad ((m,k) \in L_n^{(1)});$$

$$(R2)\quad \Delta_k^2 f_n(m,k) + 2\,f_{n-1}(m,k) = 0 \qquad ((m,k) \in U_n^{(1)}).$$

$$(R3)\quad \Delta_m^2 f_n(m,k) + 2\,f_{n-1}(m,k-2) = 0 \qquad ((k,m) \in U_n^{(2)});$$

$$(R4)\quad \Delta_k^2 f_n(m,k) + 2\,f_{n-1}(m-2,k) = 0 \qquad ((k,m) \in L_n^{(2)});$$

Generating function: lower triangle

Theorem.

The triple exponential generating function for the lower triangles of the matrices M_n is given by

$$\sum_{2\le k+1\le m\le 2n} f_n(m,k)\frac{x^{m-k-1}}{(m-k-1)!}\frac{y^{k-1}}{(k-1)!}\frac{z^{2n-m}}{(2n-m)!}$$

$$=\frac{\cos(\sqrt{2}\,x)+\cos(\sqrt{2}\,y)\;\cos(\sqrt{2}\,z)}{2\;\cos^2\left(\dfrac{x+y+z}{\sqrt{2}}\right)}.$$

Generating function: upper triangle

Theorem

The triple exponential generating function for the upper triangles of the matrices M_n is given by

$$\sum_{2\le m+1\le k\le 2n-1} f_n(m,k)\,\frac{x^{2n-k}}{(2n-k)!}\,\frac{y^{k-m-1}}{(k-m-1)!}\,\frac{z^{m-1}}{(m-1)!}$$

$$= \sin(\sqrt{2}\,x)\ \sin(\sqrt{2}\,z)\ \frac{1}{2\ \cos^2\left(\frac{x+y+z}{\sqrt{2}}\right)}.$$

Symmetry property

Theorem

The matrices M_n are symmetric with respect to their counter-diagonals:

$$f_n(m,k) = f_n(2n+1-k, 2n+1-m).$$

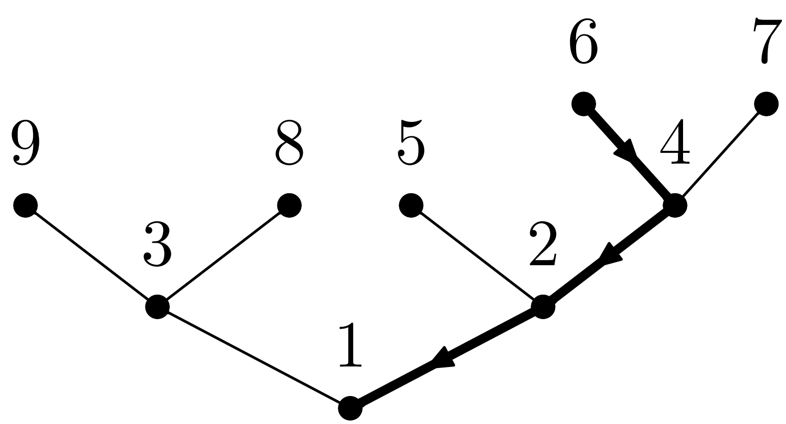
Symmetry property

Theorem

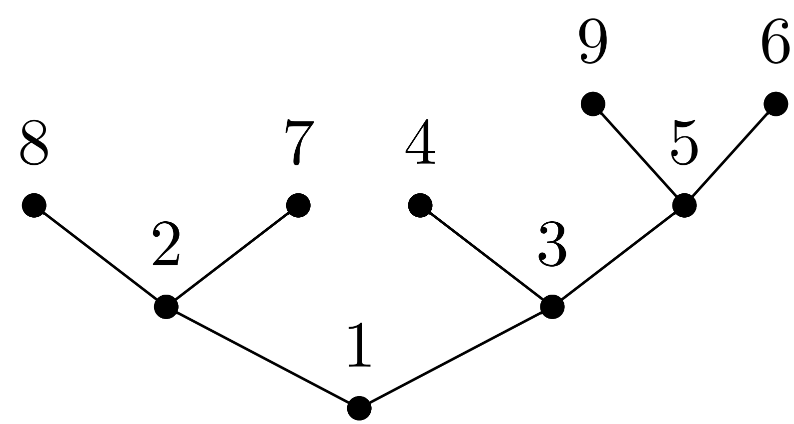
The matrices M_n are symmetric with respect to their counter-diagonals:

$$f_n(m,k) = f_n(2n+1-k, 2n+1-m).$$

Open problem: Find a direct proof:



eoc=6 (pom=3)



(eoc=7) pom=5

I strongly think that the fundamental relations are a miracle of the tree structure.

Our proof is

- primitive,
- tedious,
- error-prone.

It would be interesting to

- find a "nice" short proof that explains the nature of the fundamental relations,
- develop an algebraic structure based on them (I think of Coxeter group, of the "121=212" relation:

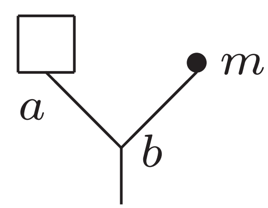
$(k, k+1)(k+1, k+2)(k, k+1) = (k+1, k+2)(k, k+1)(k+1, k+2)$
...),

- find a computer-assisted proof.

Family of trees

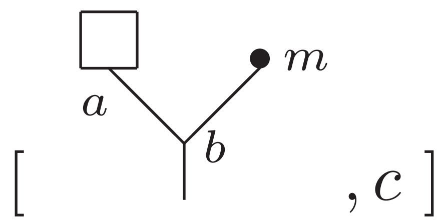
- Subtrees (possibly leaves) are indicated by "$\bigcirc$," "$\bigtriangledown$", "$\square$."
- The end of the minimal chain in each tree is represented by a bullet "$\bullet$."
- Letters occurring below or next to subtrees are labels of their roots.

Example 1.



designate the *family* of all trees t from the underlying set $\mathfrak{T}_{2n+1}$ having a node labeled b [in short, a node b], parent of both a subtree of root a and the leaf m, which is also the end of the minimal chain;

Example 2.



designate the *family* of all trees t from the underlying set $\mathfrak{T}_{2n+1}$ having a node labeled b [in short, a node b], parent of both a subtree of root a and the leaf m, which is also the end of the minimal chain;

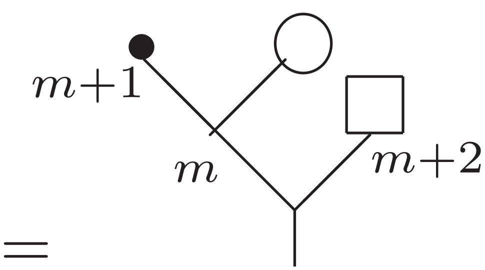
moreover, the symbol on the right has the further property that the node labeled c does not belong, *either* to the subtree of root b, *or* to the path going from root 1 to b.

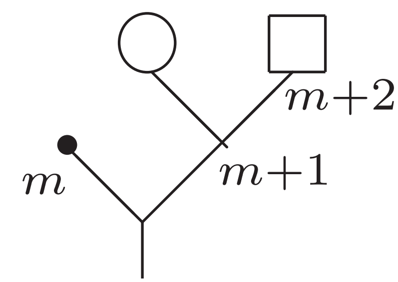
Notation. In the sequel, the letter "m" is always used to designate the end of the minimal chain, unless explicitly indicated by a letter next to •.

Tree Calculus consists of two steps:

- decomposing the sets $\mathfrak{T}_{2n+1,m,k}$ into smaller subsets by considering the mutual positions of the nodes m, $(m+1)$, $(m+2)$ (resp. k, $(k+1)$, $(k+2)$);

- setting up bijections between those subsets by a simple display of certain subtrees. Example:

$A :=$ 

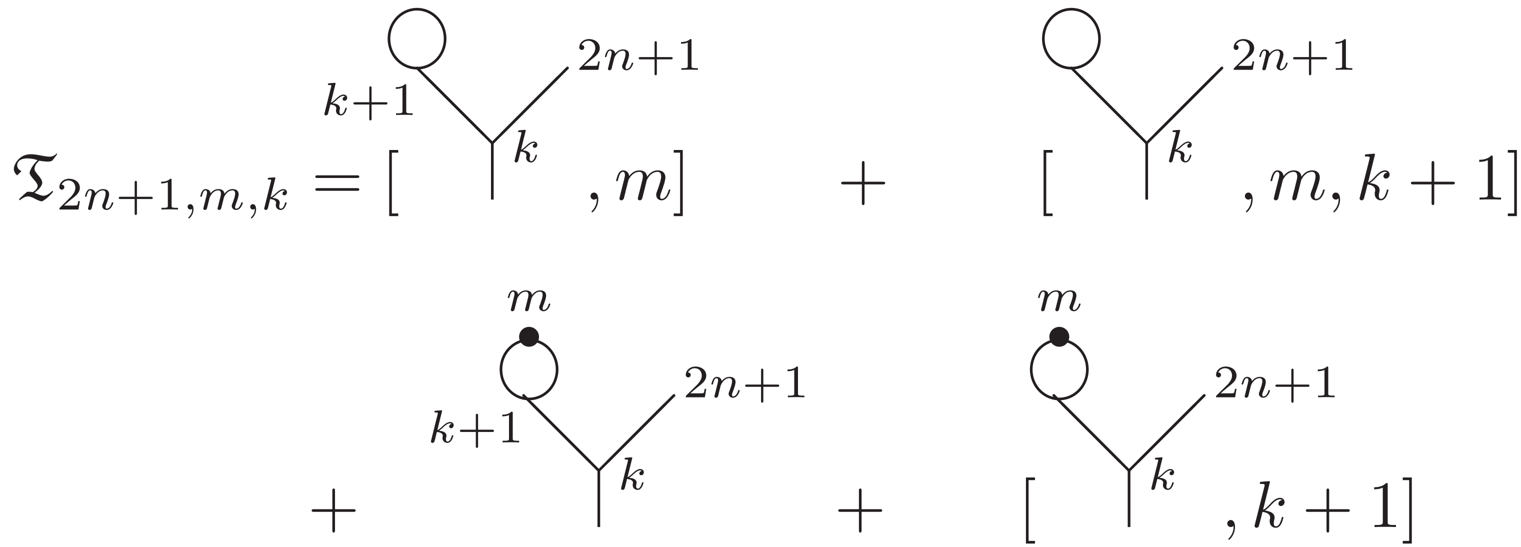
$B :=$ 

To each pair $(\underset{m+2}{\square}, \bigcirc)$ there correspond a unique tree from A and a unique tree from B. This defines a bijection of A onto B.

Proof of the fundamental relations

$$\underset{k}{\Delta}^2\,\mathfrak{T}_{2n+1,m,k} + 2\,\mathfrak{T}_{2n-1,m-2,k} = 0, \quad \textit{if}\ \ (m,k) \in L_n^{(2)}$$

Proof.

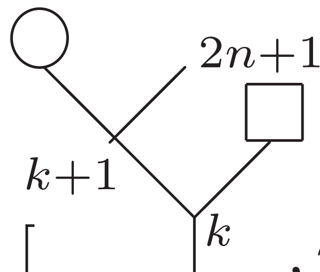


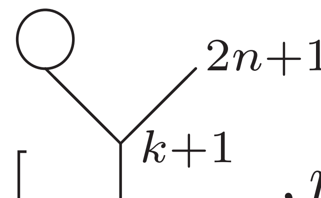
$$:= A_1 + A_2 + A_3 + A_4,$$

meaning that each tree from $\mathfrak{T}_{2n+1,m,k}$ has one of the four forms: either $k+1$ is incident to k, or not, and m is outside or not the subtree of root k; furthermore, the leaf m is the end of the minimal chain.

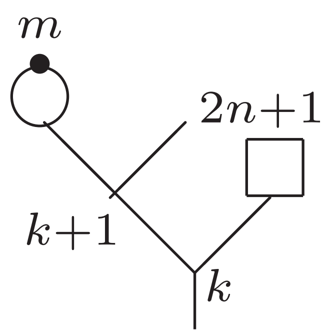
$$\mathfrak{T}_{2n+1,m,k} = A_1 + A_2 + A_3 + A_4$$

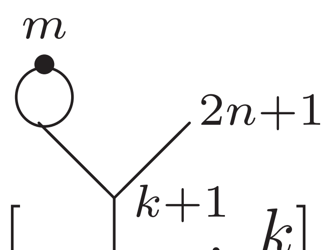
Replace k by $k+1$:

$\mathfrak{T}_{2n+1,m,k+1} = [$ 

$, m]$ $+$ $[$ 

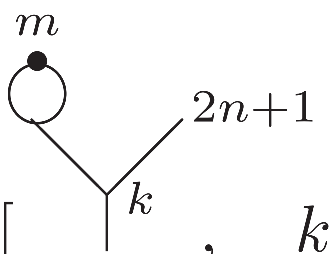
$, m, k]$

$+$ 

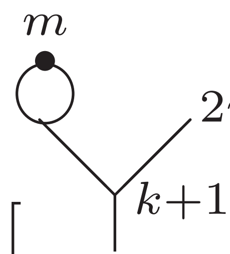
$+$ $[$ 

$, k]$

$$:= B_1 + B_2 + B_3 + B_4.$$

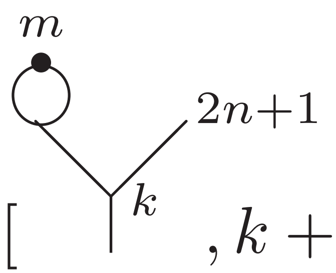
$A_4 = [$ 

$, \quad k+1]$

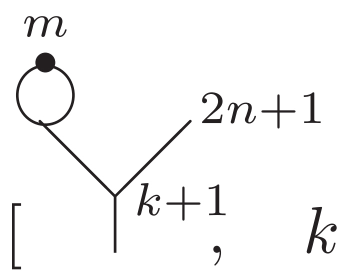
$B_4 = [$ 

$, \quad k]$

Exercise: Which is bigger, A_4 or B_4 ?

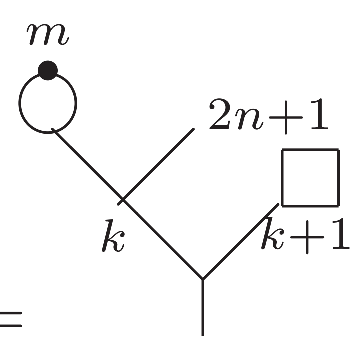
$A_4 = [$ 

$, k + 1]$

$B_4 = [$ 

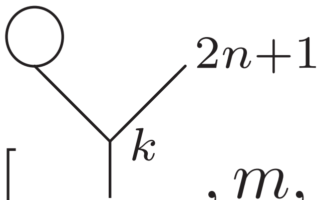
$, k]$

Answer: A_4 is bigger.

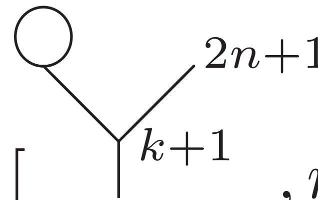
Consider the subsets $A_4' :=$ 

of A_4

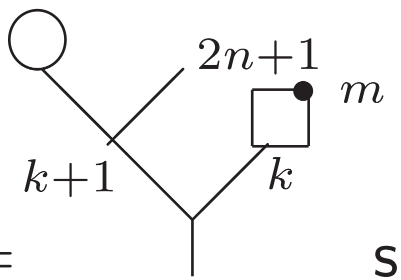
The transposition $(k, k+1)$ maps $A_4 \setminus A_4'$ onto B_4 in a bijective manner.

$A_2 = [$ 

$, m, k+1]$

$B_2 = [$ 

$, m, k]$

$B_2' :=$ 

subset of B_2.

The transposition $(k, k+1)$ maps A_2 onto $B_2 \setminus B_2'$ in a bijective manner.

Difference:

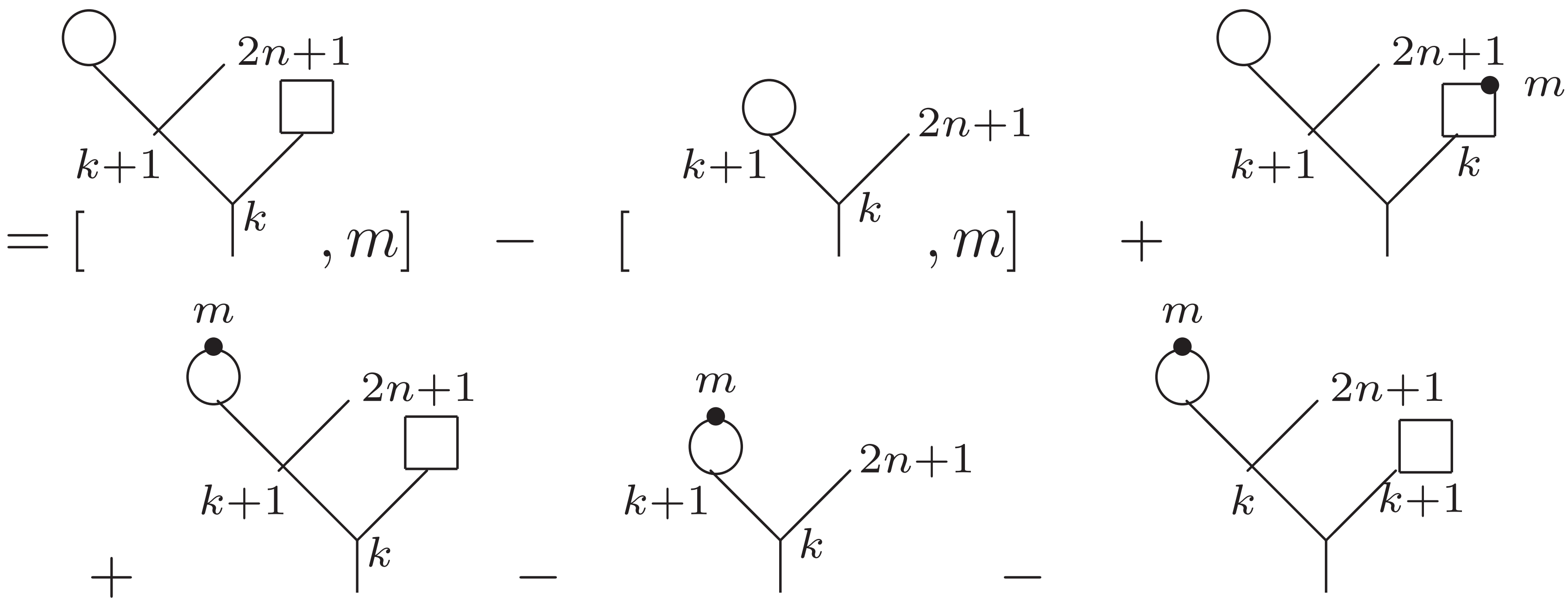
$$\mathfrak{T}_{2n+1,m,k+1} - \mathfrak{T}_{2n+1,m,k}$$

$$= (B_1 - A_1) + (B_2 - A_2) + (B_3 - A_3) + (B_4 - A_4)$$

$$= (B_1 - A_1) + ((B_2 - B_2' - A_2) + B_2')$$

$$+ (B_3 - A_3) + (B_4 - (A_4 - A_4') - A_4')$$

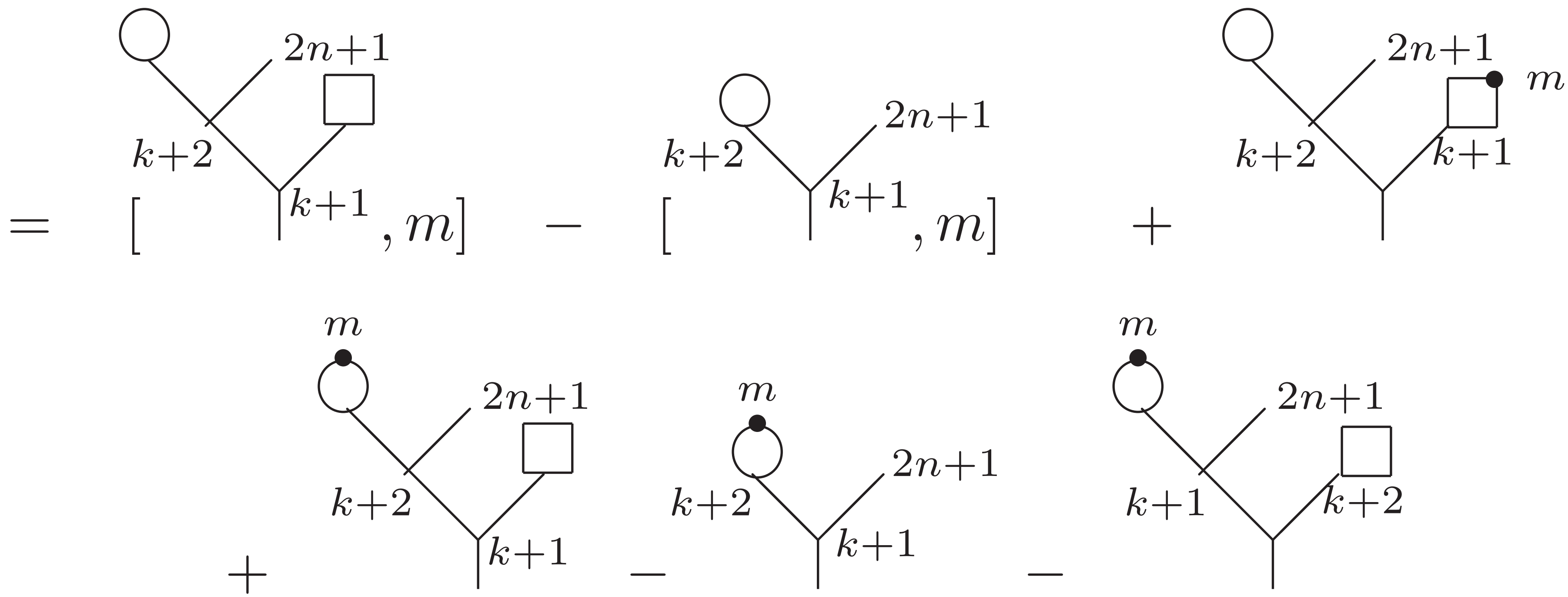
$$= B_1 - A_1 + B_2' + B_3 - A_3 - A_4'$$



$$\mathfrak{T}_{2n+1,m,k+1} - \mathfrak{T}_{2n+1,m,k} = B_1 - A_1 + B_2' + B_3 - A_3 - A_4'$$

Replace k by $k+1$:

$$\mathfrak{T}_{2n+1,m,k+2} - \mathfrak{T}_{2n+1,m,k+1}$$



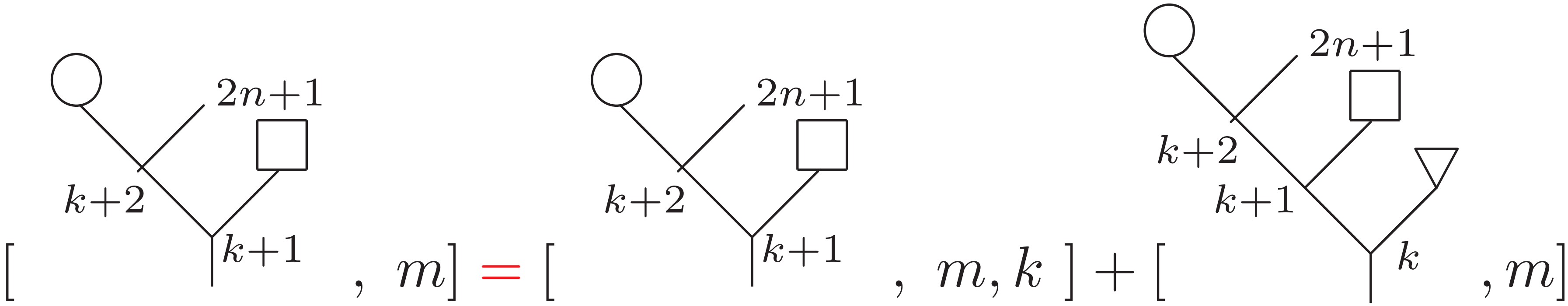
$$:= D_1 - C_1 + D_2' + D_3 - C_3 - C_4'.$$

Difference of the difference :

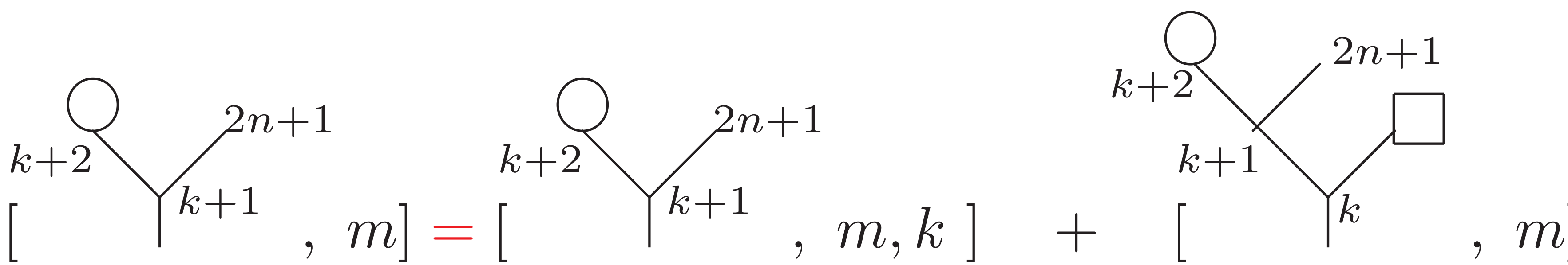
$$\begin{aligned}
&\Delta_k^2 \mathfrak{T}_{2n+1,m,k} \\
&= \left(\mathfrak{T}_{2n+1,m,k+2} - \mathfrak{T}_{2n+1,m,k+1}\right) - \left(\mathfrak{T}_{2n+1,m,k+1} - \mathfrak{T}_{2n+1,m,k}\right) \\
&= D_1 - C_1 + D_2' + D_3 - C_3 - C_4' \\
&\qquad - B_1 + A_1 - B_2' - B_3 + A_3 + A_4'.
\end{aligned}$$

The further decompositions of the components of the previous sum depend on the mutual positions of the nodes k, $(k+1)$, $(k+2)$.

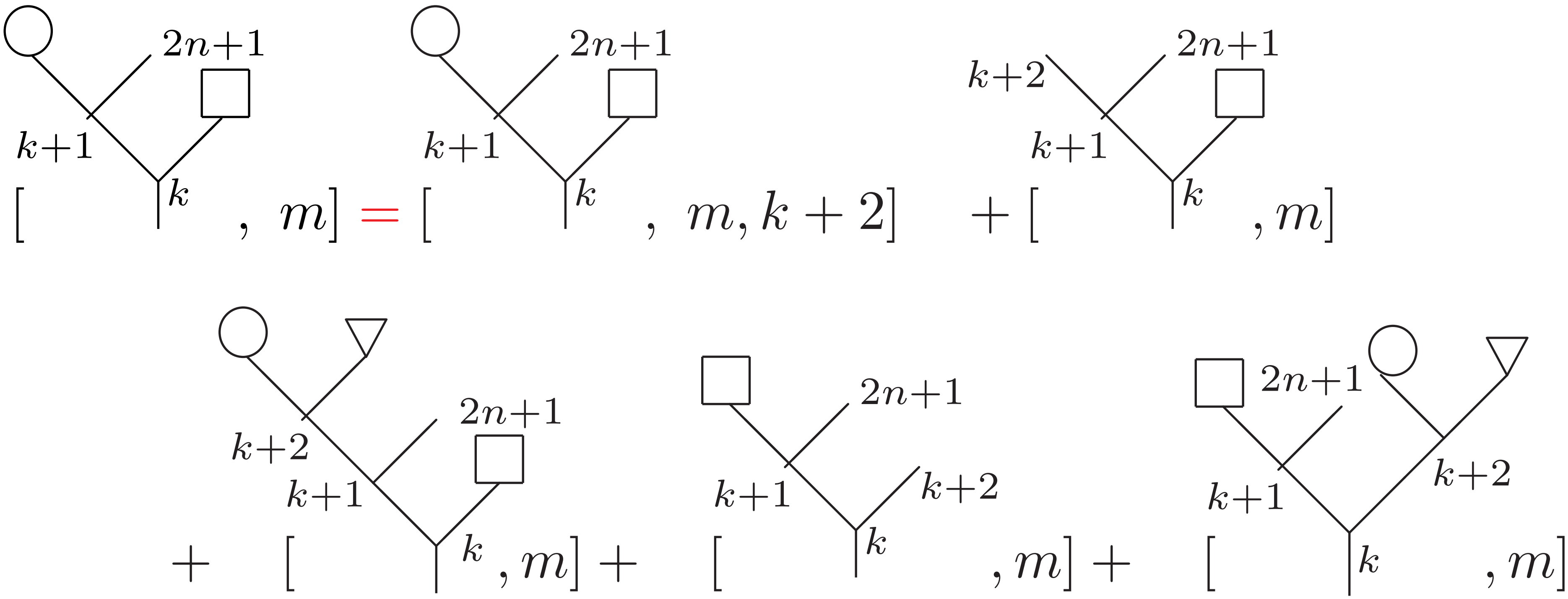
First, evaluate the subsum: $S_1 := D_1 - C_1 - B_1 + A_1$:



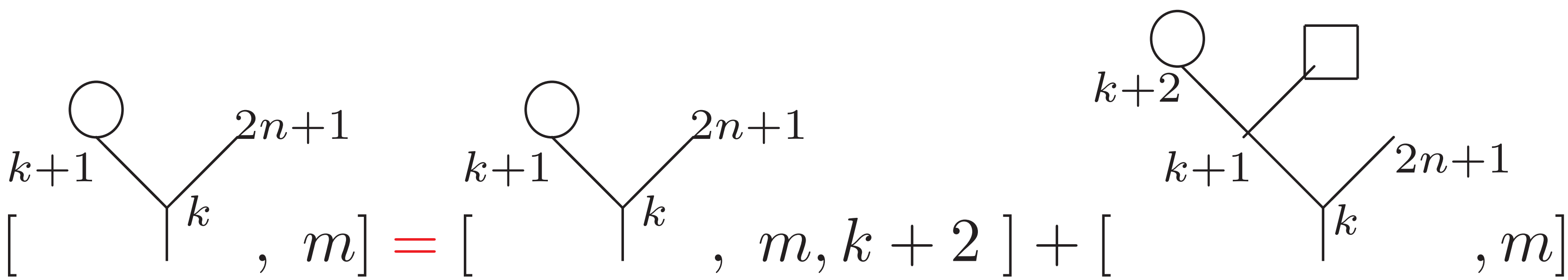
$$D_1 = D_{1,1} + D_{1,2};$$



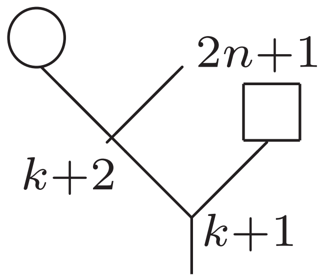
$$C_1 = C_{1,1} + C_{1,2};$$

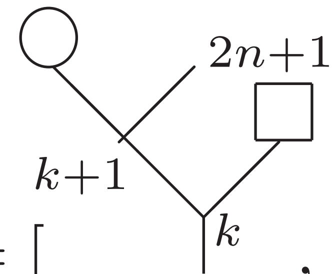


$$B_1 = B_{1,1} + B_{1,2}$$
$$+ B_{1,3} + B_{1,4} + B_{1,5};$$

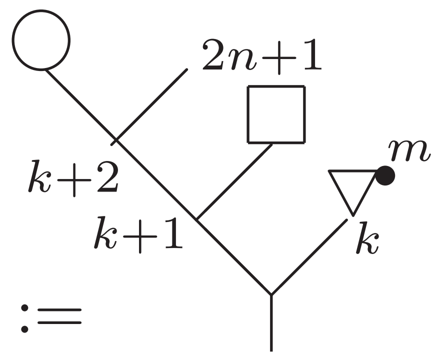


$$A_1 = A_{1,1} + A_{1,2}.$$

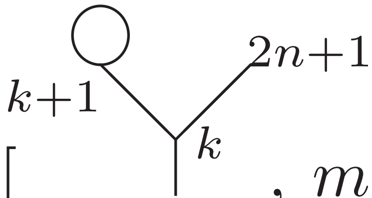
$D_{1,1} = [$  $, m, k]$

$B_{1,1} = [$  $, m, k+2]$

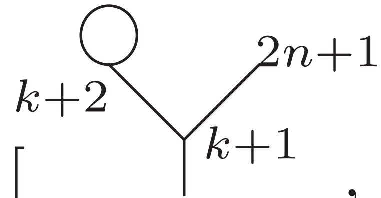
Also, let

$D'_{1,1} :=$  ;

The permutation $\binom{k \;\; k+1 \;\; k+2}{k+2 \;\; k \;\; k+1}$ maps $D_{1,1} \setminus D'_{1,1}$ onto $B_{1,1}$

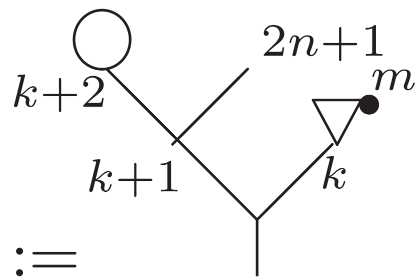
$A_{1,1} = [$ 

$, \; m, k{+}2 \,]$

$C_{1,1} = [$ 

$, \; m, k \,]$

Also, let

$C'_{1,1} :=$ 

.

The permutation $\left(\begin{smallmatrix} k & k+1 & k+2 \\ k+2 & k & k+1 \end{smallmatrix}\right)$ maps $C_{1,1} \setminus C'_{1,1}$ onto $A_{1,1}$.

Evaluate S_1

Hence, $D_{1,1} = B_{1,1} + D'_{1,1}$, $C_{1,1} = A_{1,1} + C'_{1,1}$.

Evaluate S_1

Hence, $D_{1,1} = B_{1,1} + D'_{1,1}$, $C_{1,1} = A_{1,1} + C'_{1,1}$.

Moreover,

$D_{1,2} = 2\,B_{1,3}$, $C_{1,2} = A_{1,2}$, $B_{1,2} = B_{1,4}$, $B_{1,3} = B_{1,5}$.

Evaluate S_1

Hence, $D_{1,1} = B_{1,1} + D'_{1,1}$, $C_{1,1} = A_{1,1} + C'_{1,1}$.

Moreover,

$D_{1,2} = 2\, B_{1,3}$, $C_{1,2} = A_{1,2}$, $B_{1,2} = B_{1,4}$, $B_{1,3} = B_{1,5}$.

Altogether, $S_1 = D_1 - C_1 - B_1 + A_1 = (B_{1,1} + D'_{1,1} + 2\, B_{1,3}) - (A_{1,1} + C'_{1,1} + A_{1,2}) - (B_{1,1} + B_{1,2} + B_{1,3} + B_{1,2} + B_{1,3}) + (A_{1,1} + A_{1,2})$.

Evaluate S_1

Hence, $D_{1,1} = B_{1,1} + D'_{1,1}$, $C_{1,1} = A_{1,1} + C'_{1,1}$.

Moreover,

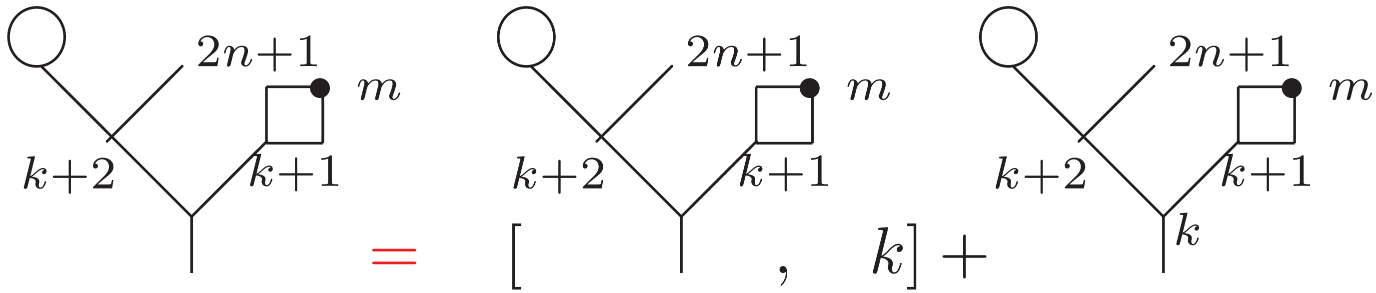
$D_{1,2} = 2\,B_{1,3}$, $C_{1,2} = A_{1,2}$, $B_{1,2} = B_{1,4}$, $B_{1,3} = B_{1,5}$.

Altogether, $S_1 = D_1 - C_1 - B_1 + A_1 = (B_{1,1} + D'_{1,1} + 2\,B_{1,3}) - (A_{1,1} + C'_{1,1} + A_{1,2}) - (B_{1,1} + B_{1,2} + B_{1,3} + B_{1,2} + B_{1,3}) + (A_{1,1} + A_{1,2})$.

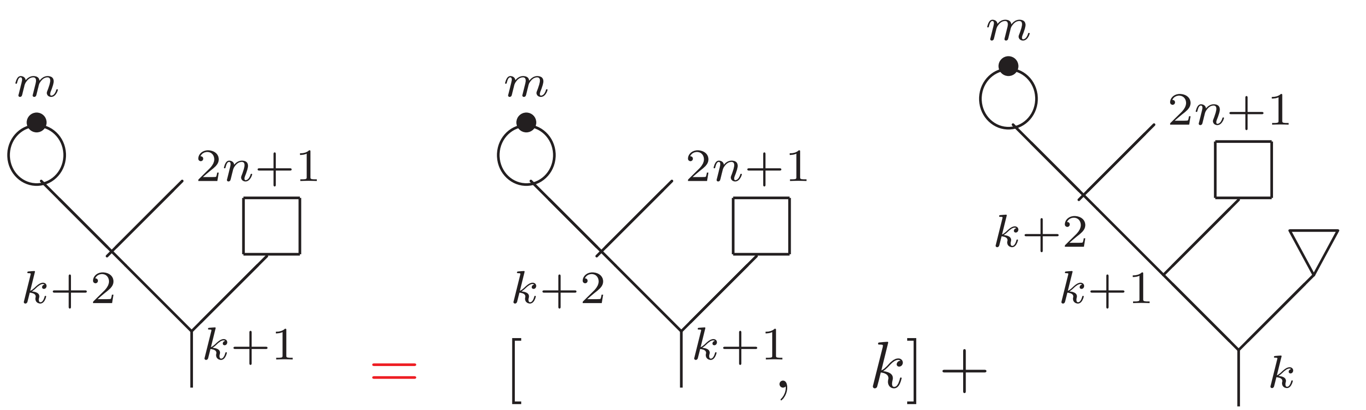
Thus,

$$S_1 = -2\,B_{1,2} + D'_{1,1} - C'_{1,1}.$$

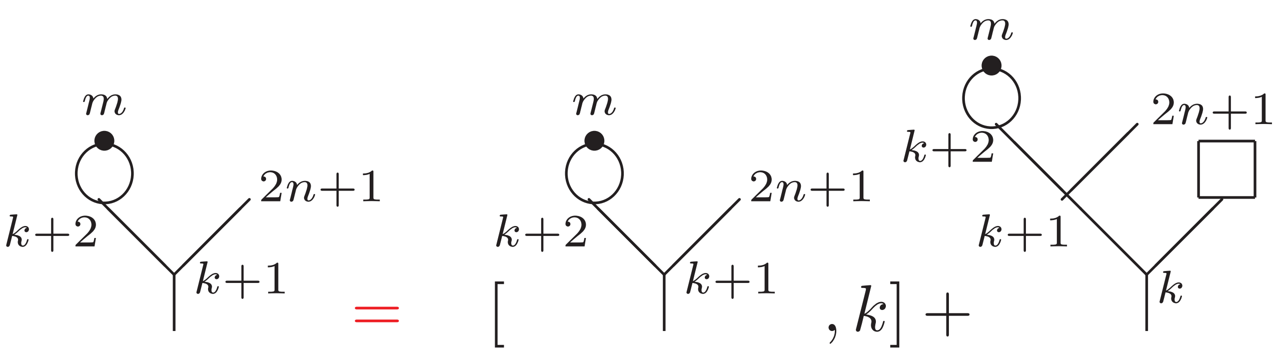
Next, evaluate the sum $S_2 := D_2' + D_3 - C_3 - C_4' - B_2' - B_3 + A_3 + A_4'$



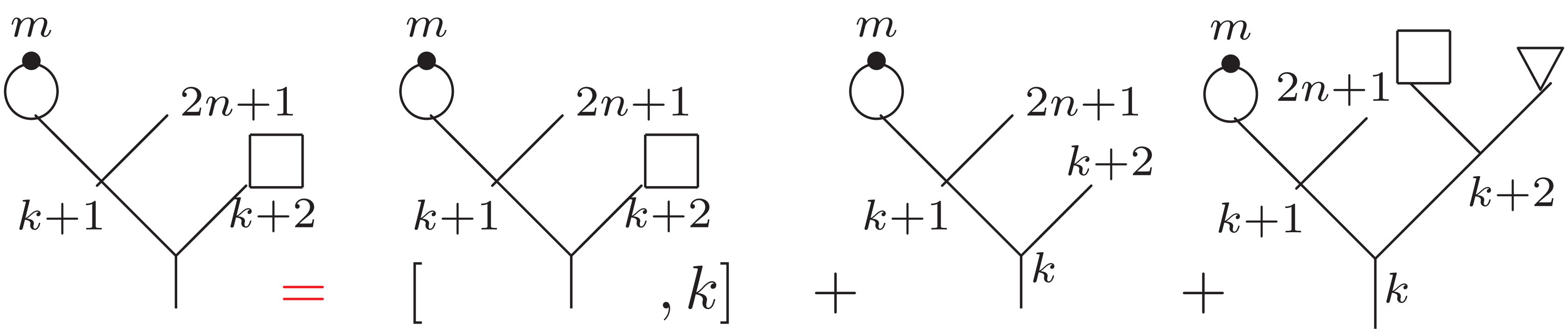
$$D_2' = D_{2,1}' + D_{2,2}'$$



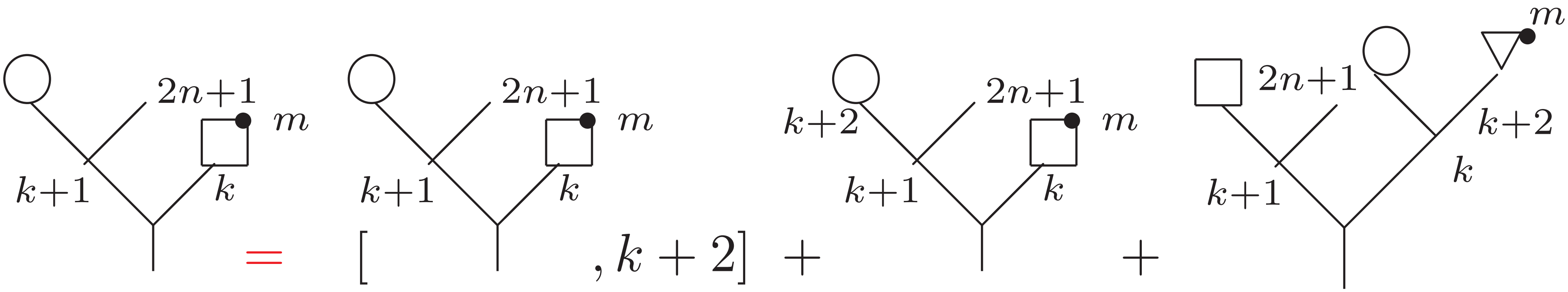
$$D_3 = D_{3,1} + D_{3,2}$$



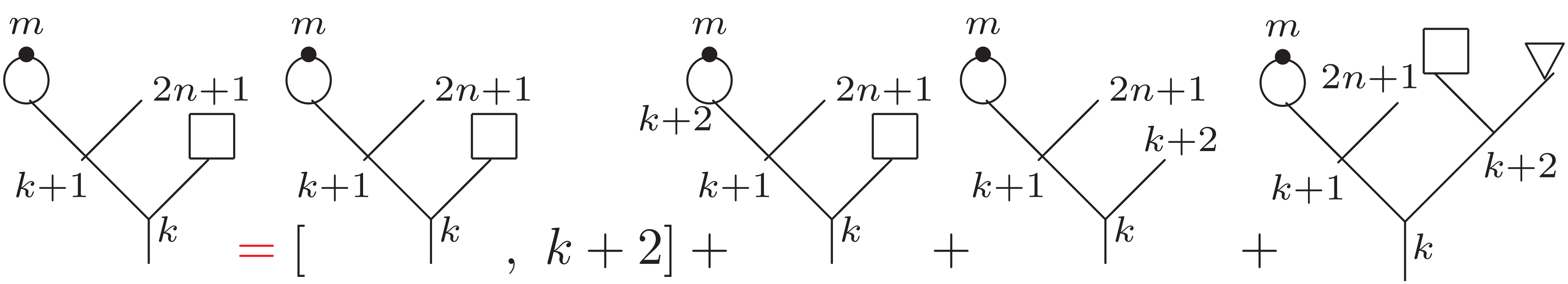
$$C_3 = C_{3,1} + C_{3,2};$$



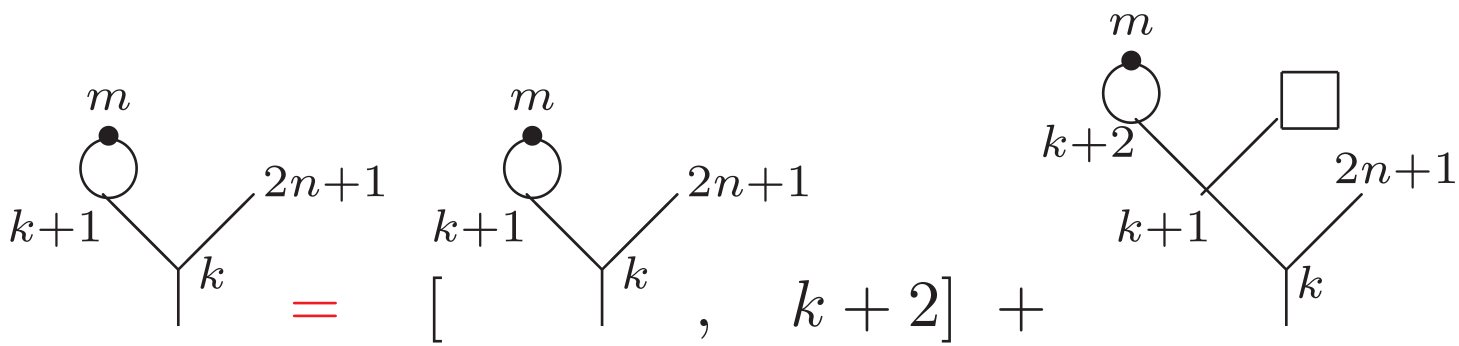
$$C'_4 = C'_{4,1} + C'_{4,2} + C'_{4,3};$$



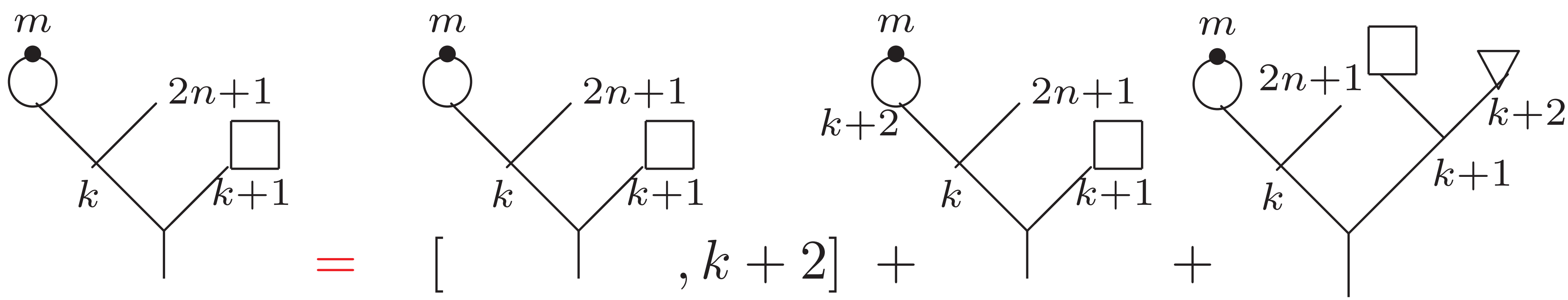
$$B'_2 = B'_{2,1} + B'_{2,2} + B'_{2,3};$$



$$B_3 = B_{3,1} + B_{3,2} + B_{3,3} + B_{3,4};$$



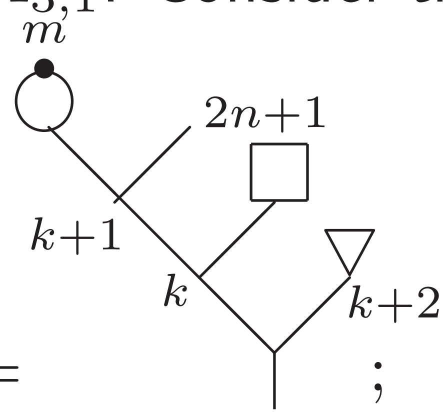
$$A_3 = A_{3,1} + A_{3,2};$$

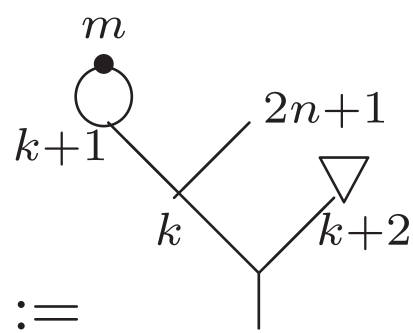


$$A'_4 = A'_{4,1} + A'_{4,2} + A'_{4,3}.$$

Within the sum S_2 there are numerous cancellations we now describe.

(a) *Components of the form* $[t, k]$ *or* $[t, k+2]$*, where* t *is a subtree, whose root is labeled.* There are four of them: $D_{3,1}$, $-C_{3,1}$, $-B_{3,1}$, $A_{3,1}$. Consider the subsets:

$B_{3,1,1} :=$ 

; $A_{3,1,1} :=$ 

;

of $B_{3,1}$ and $A_{3,1}$, respectively. The permutation $\begin{pmatrix} k & k+1 & k+2 \\ k+2 & k & k+1 \end{pmatrix}$ maps $D_{3,1}$ onto $B_{3,1} \setminus B_{3,1,1}$ and $C_{3,1}$ onto $A_{3,1} \setminus A_{3,1,1}$. Hence, $D_{3,1} - C_{3,1} - B_{3,1} + A_{3,1} = (B_{3,1} - B_{3,1,1}) - (A_{3,1} - A_{3,1,1}) - B_{3,1} + A_{3,1} = -B_{3,1,1} + A_{3,1,1}$.

(b) *Components of the form* $[t, k]$ *or* $[t, k{+}2]$*, where* t *is a subtree, whose root is not labeled.* There are four of them: $D'_{2,1}$, $-C'_{4,1}$, $-B'_{2,1}$, $A'_{4,1}$. Again, the permutation $\binom{k \;\; k{+}1 \;\; k{+}2}{k{+}2 \;\; k \;\; k{+}1}$ maps $D'_{2,1}$ onto $B'_{2,1}$, and $C'_{4,1}$ onto $A'_{4,1}$. Hence, $D'_{2,1} - B'_{2,1} = -C'_{4,1} + A'_{4,1} = 0$. Their sum vanish.

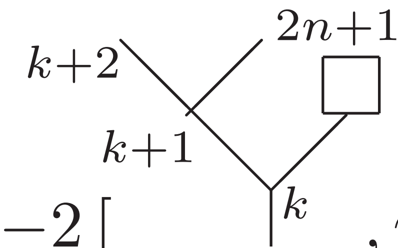
(c) *Components represented by a tree* t*, whose root is unlabeled.* There are four of them: $-B'_{2,2}$, $-B'_{2,3}$, $-A'_{4,2}$, $A'_{4,3}$. As $B'_{2,2} = A'_{4,2}$, the contribution of those components to S_2 is then $-B'_{2,3} + A'_{4,3}$.

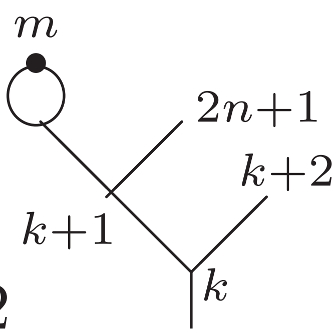
(d) *Components represented by a tree t, whose root is labeled.* There are nine of them: $D'_{2,2}$, $D_{3,2}$, $-C_{3,2}$, $-C'_{4,2}$, $-C'_{4,3}$ $-B_{3,2}$, $-B_{3,3}$, $-B_{3,4}$, $A_{3,2}$. By simply comparing the subtree contents we have: $D'_{2,2} - C_{3,2} = -B_{3,2} + A_{3,2} = 0$, $D_{3,2} - (C'_{4,3} + B_{3,4}) = 0$ and $C'_{4,2} = B_{3,3}$. The contribution of those terms is then $-2\,C'_{4,2}$.

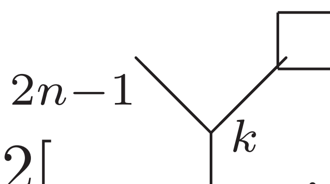
Hence, $S_1+S_2 = (-2\,B_{1,2}+D'_{1,1}-C'_{1,1})+\big((-B_{3,1,1}+A_{3,1,1})+(-B'_{2,3}+A'_{4,3})+(-2\,C'_{4,2})\big)$. As $D'_{1,1} = B'_{2,3}$, $C'_{1,1} = A_{3,1,1}$ and $B_{3,1,1} = A'_{4,3}$, we get

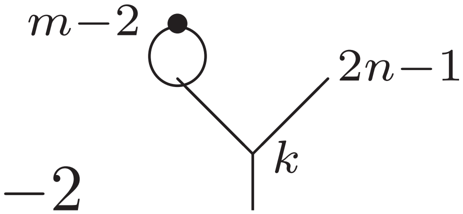
$$S_1 + S_2 = -2\,B_{1,2} - 2\,C'_{4,2}$$

$S_1 + S_2 = -2\,B_{1,2} - 2\,C'_{4,2}$

$\underset{k}{\Delta}^2\, \mathfrak{T}_{2n+1,m,k} = -2\,[$ 

$, m] \quad - 2$ 

$= -2[$ 

$, m-2] \quad -2$ 

$= -2\,\mathfrak{T}_{2n-1,m-2,k}. \quad \square$

Seidel Triangle Sequences

Infinite matrix $A = (a(m,k))_{m,k\geq 0}$

Exponential generating functions

$$A(x,y) := \sum_{m,k\geq 0} a(m,k)\frac{x^m}{m!}\frac{y^k}{k!};$$

$$A_{m,\bullet}(y) := \sum_{k\geq 0} a(m,k)\frac{y^k}{k!};$$

$$A_{\bullet,k}(x) := \sum_{m\geq 0} a(m,k)\frac{x^m}{m!};$$

for A itself, its m-th row, its k-th column.

• A *Seidel matrix* $A = (a(m,k))$ $(m, k \geq 0)$ is defined to be an infinite matrix, whose entries belong to some ring, and obey the following relation holds:

$$a(m,k) = a(m-1,k) + a(m-1,k+1).$$

• the sequence of the entries from the top row $a(0,0)$, $a(0,1)$, $a(0,2), \ldots$ is given; it is called the *initial sequence*;

• The leftmost column $a(0,0)$, $a(1,0)$, $a(2,0)$, $\ldots$ is called the *final sequence*.

Theorem. Let $A = (a_{i,j})$ $(i, j \geq 0)$ be a Seidel matrix. Then,

$$A_{\bullet,0}(x) = e^x A_{0,\bullet}(x) \quad \text{and} \quad A(x,y) = e^x A_{0,\bullet}(x+y).$$

A sequence of square matrices (A_n) $(n \geq 1)$ is called a *Seidel triangle sequence* if the following three conditions are fulfilled:

- each matrix A_n is of dimension n;
- each matrix A_n has null entries along and below its diagonal; let $(a_n(m,k)$ $(0 \leq m < k \leq n-1)$ denote its entries strictly above its diagonal, so that

$$A_1 = (\,\cdot\,)\,; \quad A_2 = \begin{pmatrix} \cdot & a_2(0,1) \\ \cdot & \cdot \end{pmatrix}; \quad A_3 = \begin{pmatrix} \cdot & a_3(0,1) & a_3(0,2) \\ \cdot & \cdot & a_3(1,2) \\ \cdot & \cdot & \cdot \end{pmatrix};$$

$$A_n =$$

$$\begin{pmatrix} \cdot & a_n(0,1) & a_n(0,2) & \cdots & a_n(0,n-2) & a_n(0,n-1) \\ \cdot & \cdot & a_n(1,2) & \cdots & a_n(1,n-2) & a_n(1,n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdots & a_n(n-3,n-2) & a_n(n-3,n-1) \\ \cdot & \cdot & \cdot & \cdots & \cdot & a_n(n-2,n-1) \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \end{pmatrix};$$

the dots "·" along and below the diagonal referring to null entries.

- for each $n \geq 2$, the following relation holds:

$$a_n(m,k) - a_n(m,k+1) = a_{n-1}(m,k) \quad (m < k).$$

Record the last columns of the triangles A_2, A_3, A_4, A_5, $\dots$, read from top to bottom, as counter-diagonals of an infinite matrix $H = (h_{i,j})_{i,j \geq 0}$, as shown next:

$$H := \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} \quad 0 \quad & \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad \end{array} \\ \begin{pmatrix} a_2(0,1) & a_3(1,2) & a_4(2,3) & a_5(3,4) & a_6(4,5) & \cdots \\ a_3(0,2) & a_4(1,3) & a_5(2,4) & a_6(3,5) & & \\ a_4(0,3) & a_5(1,4) & a_6(2,5) & & & \\ a_5(0,4) & a_6(1,5) & & & & \\ a_6(0,5) & & & & & \\ \vdots & & & & & \end{pmatrix} \end{array}$$

In an equivalent manner, the entries of H are defined by:

$$h_{i,j} = a_{i+j+2}(j, i+j+1).$$

Theorem.

The three-variable generating function for the Seidel triangle sequence $(A_n = (a_n(m,k)))_{n\geq 1}$ is equal to

$$\sum_{1\leq m+1\leq k\leq n-1} a_n(m,k)\frac{x^{n-k-1}}{(n-k-1)!}\frac{y^{k-m-1}}{(k-m-1)!}\frac{z^m}{m!}$$

$$= e^x H(x+y,z).$$

Generating functions for Poupard statistics

Take the matrices M_n with the following modifications:
(W1) delete the lower triangle;
(W2) divide by $(-1)^n 2^{n-1}$ for each coefficient in M_n
(W3) replace M_n by W_{2n}

$$W_2 = -\frac{1}{2^0} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$W_4 = \frac{1}{2^1} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$W_6 = -\frac{1}{2^2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & . & 0 & 4 & 2 & 0 \\ 0 & . & . & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & . & . & . & 0 & 0 \end{pmatrix}$$

$$W_8 = \frac{1}{2^3}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 8 & 10 & 8 & 4 & 0 \\ 0 & . & 0 & 16 & 20 & 16 & 8 & 0 \\ 0 & . & . & 0 & 28 & 20 & 10 & 0 \\ 0 & . & . & . & 0 & 16 & 8 & 0 \\ 0 & . & . & . & . & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & . & . & . & . & . & 0 & 0 \end{pmatrix};$$

We have only the even-part of the sequence. However, we can define odd W_{2n-1} such that (W_n) is a Seidel triangle sequence:

$$M_3 = \frac{1}{2^1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$W_5 = -\frac{1}{2^2}\begin{pmatrix} 0 & . & . & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & . & 0 & 2 & 2 \\ 0 & . & . & 0 & 1 \\ 0 & . & . & 0 & 0 \end{pmatrix} \quad ;$$

$$W_7 = \frac{1}{2^3}\begin{pmatrix} 0 & . & . & . & . & 0 & 0 \\ 0 & 0 & -4 & -2 & 2 & 4 & 4 \\ 0 & . & 0 & -4 & 4 & 8 & 8 \\ 0 & . & . & 0 & 8 & 10 & 10 \\ 0 & . & . & . & 0 & 8 & 8 \\ 0 & . & . & . & . & 0 & 4 \\ 0 & . & . & . & . & 0 & 0 \end{pmatrix};$$

$$W_9 = -\frac{1}{2^4}\begin{pmatrix} 0 & . & . & . & . & . & . & 0 & 0 \\ 0 & 0 & -34 & -26 & -10 & 10 & 26 & 34 & 34 \\ 0 & . & 0 & -52 & -20 & 20 & 52 & 68 & 68 \\ 0 & . & . & 0 & -22 & 34 & 74 & 94 & 94 \\ 0 & . & . & . & 0 & 56 & 88 & 104 & 104 \\ 0 & . & . & . & . & 0 & 86 & 94 & 94 \\ 0 & . & . & . & . & . & 0 & 68 & 68 \\ 0 & . & . & . & . & . & . & 0 & 34 \\ 0 & . & . & . & . & . & . & 0 & 0 \end{pmatrix}$$

The infinite matrix H is equal to

$$H = \begin{pmatrix} 0 & \frac{1}{2} & 0 & -\frac{1}{2^2} & 0 & \frac{4}{2^3} \\ 0 & 0 & -\frac{2}{2^2} & 0 & \frac{8}{2^3} & \\ 0 & -\frac{1}{2^2} & 0 & \frac{10}{2^3} & & \\ 0 & 0 & \frac{8}{2^3} & & & \\ 0 & \frac{4}{2^3} & & & & \\ 0 & & & & & \end{pmatrix}$$

By the generating function for the 1D Poupard statistics:

$$H(x,y) = \frac{\partial}{\partial x} \frac{\cos(xI/2 - yI/2)}{\cos(xI/2 + yI/2)} = \frac{-I \sin(yI)}{1 + \cos(xI + yI)}.$$

So that the generating function for the 2D Poupard statistics W_n is

$$\Omega(x,y,z) = e^x H(x+y,z) = \frac{-I e^x \sin(zI)}{1 + \cos(xI + yI + zI)}.$$

The real part of $\Omega(xI, yI, zI)$ is equal to

$$\frac{\sin(x)\sin(z)}{1 + \cos(x+y+z)} = \frac{\sin(x)\sin(z)}{2\cos^2((x+y+z)/2)}. \quad \square$$

Secant trees

$$h_n(m,k) := \#\{t \in \mathfrak{T}_{2n} : \mathrm{eoc}(t) = m \text{ and } \mathrm{pom}(t) = k\}.$$

$M_2 =$

$k =$	1	$h_1(m,.)$
$m = 2$	1	1
$h_1(.,k)$	1	$E_2 = 1$

$M_4 =$

$k =$	1	2	3	$h_2(m,.)$
$m = 2$	.	.	1	1
3	1	2	.	3
4	.	1	.	1
$h_2(.,k)$	1	3	1	$E_4 = 5$

$M_6 =$

$k =$	1	2	3	4	5	$h_3(m, .)$
$m = 2$	.	.	1	3	1	5
3	1	2	.	9	3	15
4	3	7	10	.	1	21
5	1	4	8	2	.	15
6	.	2	2	1	.	5
$h_3(., k)$	5	15	21	15	5	$E_6 = 61$

$M_8 =$

$k =$	1	2	3	4	5	6	7	$h_4(m,.)$
$m = 2$	.	.	5	15	21	15	5	61
3	5	10	.	45	63	45	15	183
4	15	35	50	.	101	63	21	285
5	21	54	86	106	.	45	15	327
6	15	46	82	87	50	.	5	285
7	5	22	46	60	40	10	.	183
8	.	16	16	14	10	5	.	61
$h_4(.,k)$	61	183	285	327	285	183	61	$E_8 = 1385$

Fundamental recurrences for secant trees

Theorem.

The finite difference equation systems hold:

$$\Delta_m^2 h_n(m,k) + 4\,h_{n-1}(m,k-2) = 0 \qquad (2 \le m \le k-3 < k \le 2n-1);$$

$$\Delta_k^2 h_n(m,k) + 4\,h_{n-1}(m,k) = 0 \qquad (2 \le m \le k-1 < k \le 2n-3).$$

Proof.
Secant Tree calculus (more complicate than tangent tree because the missing vertice).

Generating function for secant trees

Theorem.

The triple exponential generating function for the upper triangles of the matrices $(h_n(m,k))$ is given by

$$\sum_{2\le m<k\le 2n-1} h_n(m,k)\frac{x^{2n-k-1}}{(2n-k-1)!}\,\frac{y^{k-m-1}}{(k-m-1)!}\,\frac{z^{m-2}}{(m-2)!}$$

$$=\frac{\cos(2y)+2\,\cos(2(x-z))-\cos(2(z+x))}{2\,\cos^3(x+y+z)}.$$

Remark. No formula for the *lower triangles* $\{h_n(m,k) : 1 \le k < m \le 2n\}$

Other subsets of trees

André Permutation
André Tree

Papers by Foata-H.

Finite Difference Calculus for Alternating Permutations, Journal of Difference Equations and Applications, 2013

Tree Calculus for Bivariate Difference Equations, Journal of Difference Equations and Applications, 2014

Secant Tree Calculus, Central European Journal of Mathematics, 2014

Seidel Triangle Sequences and Bi-Entringer Numbers, European Journal of Combinatorics, 2014

André Permutation Calculus; a Twin Seidel Matrix Sequence, *in preparation*, 2015

Thank you!