Séminaire Lotharingien de Combinatoire **71** (2014), Article B71d

IRREDUCIBLE REPRESENTATIONS OF $SU(n)$ WITH PRIME POWER DEGREE

Sarah Peluse[1]

Department of Mathematics, University of Chicago
5734 S. University Ave., Chicago, Illinois 60637, USA

Abstract. The correspondence between irreducible representations of the symmetric group S_n and partitions of n is well-known. Less well-known is the connection between irreducible representations of the special unitary group $SU(n)$ and partitions with less than n parts. This paper uses this correspondence to classify the irreducible representations of $SU(n)$ with prime power degree.

1. Introduction and Statement of Results

It is well-known that there is a natural one-to-one correspondence between irreducible representations of the symmetric group S_n and partitions of n (see [3].) In fact, these representations are characterized by the action of S_n on the set of tableaux for each partition of n. The famous Frame–Robinson–Thrall hook-length formula computes the degree of an irreducible representation of S_n using only properties of the partition associated to it.

Let $\lambda = (a_k, \ldots, a_1)$ be a partition of n, so $a_k \geq a_{k-1} \geq \cdots \geq a_1 \geq 1$ and $a_k + a_{k-1} + \cdots + a_1 = n$. The *Young diagram* for λ is an array of k left-justified rows of boxes such that the i^{th} row contains a_i boxes.

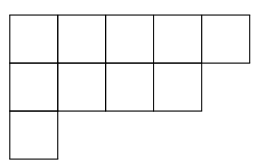

Figure 1. Young diagram of $\lambda = (5, 4, 1)$.

We attach a coordinate system to a Young diagram as follows. The upper left-hand box is labeled by $(1, 1)$, and the x and y coordinates increase moving down and right, respectively (as one labels entries in a matrix.) For a given coordinate (i, j) in a Young diagram, we define the *hook-length*, $h_{i,j}$, to be the number of boxes directly to the right or below the box with coordinates (i, j) plus 1. For more on the beautiful subject of partition hook-lengths, see Han's excellent article [2]. Figure 2 shows a Young diagram where each box is labeled with its hook-length.

[1]The author is grateful for the NSF's support of the REU at Emory University.
Current address: Department of Mathematics, Stanford University, 450 Serra Mall, Building 380, Stanford, California 94305, USA..

$$\begin{array}{|c|c|c|c|c|}\hline 7 & 5 & 4 & 3 & 1 \\ \hline 5 & 3 & 2 & 1 \\ \cline{1-4} 1 \\ \cline{1-1}\end{array}$$

FIGURE 2. Hook lengths for $\lambda = (5, 4, 1)$.

Denote the degree of a group representation T by d_T. If T is an irreducible representation of S_n and λ is its associated partition of n, then the Frame–Robinson–Thrall formula is

$$d_T = \frac{n!}{\prod_{(i,j)} h_{i,j}}. \tag{1.1}$$

It is less well-known that the irreducible representations of $SU(n)$ are naturally indexed by partitions with less than n parts. Further, similar to the case of S_n, $SU(n)$ too possesses a degree formula depending only on n and properties of an irreducible representation's associated partition. For a detailed exposition of this topic, see pages 246–254 and 327–352 of [5].

Let $\lambda = (a_k, \ldots, a_1)$ be a partition and $n > k$. Suppose that T is the irreducible representation of $SU(n)$ corresponding to λ. Draw the Young diagram of λ and fill in the $(1, 1)$ box with n. Then fill in the remaining boxes in the diagram by increasing by 1 moving right and decreasing by 1 moving down. The result of this is that the (i, j) box is filled in with $m_{i,j} = n - i + j$. Then, we have (see page 253 of [5])

$$d_T = \frac{\prod_{(i,j)} m_{i,j}}{\prod_{(i,j)} h_{i,j}}. \tag{1.2}$$

Remark. We shall reformulate (1.2) in Section 3.

We define an *admissible diagram* to be a Young diagram filled in according to the rules in the preceding paragraph. The diagram below is an example of an admissible diagram.

$$\begin{array}{|c|c|c|c|c|}\hline 8 & 9 & 10 & 11 & 12 \\ \hline 7 & 8 \\ \cline{1-2} 6 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$$

FIGURE 3. An admissible diagram for $\lambda = (5, 2, 1, 1)$

The degree of the representation associated to the above diagram is

$$\frac{8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 7 \cdot 8 \cdot 6 \cdot 5}{8 \cdot 5 \cdot 3 \cdot 2 \cdot 1 \cdot 4 \cdot 1 \cdot 2 \cdot 1} = 83160.$$

Balog, Bessenrodt, Olsson, and Ono [1] classified all of the irreducible representations of S_n with prime power degree by combining combinatorial observations related to the Frame–Robinson–Thrall formula (1.1) with results on the distribution of prime numbers. The aim of this paper is to classify the irreducible representations of $SU(n)$ of prime power degree. We have the following theorem.

Theorem 1.1. *The only prime power degree irreducible representations of $SU(n)$ are those corresponding to the following families of admissible diagrams:*

(1) *The first family is the collection of admissible diagrams where $\lambda = (1)$ and $n = p^k$ for some prime p and $k \in \mathbb{N}$.*
(2) *The second family is the collection of admissible diagrams where*
$$\lambda = ((n-1)(p^\ell - 1), (n-2)(p^\ell - 1), \dots, p^\ell - 1),$$
p is a prime, and $\ell \in \mathbb{N}$.
(3) *The third family is the collection of admissible diagrams where*
$$\lambda = (n(p^\ell - 1) + 1, (n-2)(p^\ell - 1), \dots, (p^\ell - 1))$$
or
$$\lambda = (n(p^\ell - 1) + 1, (n-1)(p^\ell - 1) + 1, \dots, 2(p^\ell - 1) + 1)$$
where p is a prime, $\ell \in \mathbb{N}$, and n is a power of p.

It is not difficult to check that the three infinite families correspond to prime power degree irreducible representations. The bulk of the proof of Theorem 1.1 consists of a reformulation of (1.2) and an inductive argument. The proof then boils down to a characterization of what we call the "hands" of a partition and our ability to detect inadmissible prime factors.

This paper is organized as follows. In Section 2, we write down the degree formula for $SU(n)$ in a tangible form and deal with the families listed in Theorem 1.1, proving that they do in fact correspond to prime power degree representations. In Section 3, we rewrite the degree formula in terms of certain entries in an admissible diagram. This reformulation helps to illuminate how the divisibility properties of a representation's degree depend on the shape of its associated partition. In Section 4 we prove Theorem 1.1 using this formula.

Acknowledgments

The author would like to thank Ken Ono for his encouragement throughout this project, the referee for many helpful comments, and Christian Krattenthaler for pointing out references for the derivation of formula (2.1).

2. The Three Families

In this section we prove that the families described in Theorem 1.1 do indeed correspond to prime power degree representations of $SU(n)$. First, we will translate the degree formula for $SU(n)$ into a closed-form formula depending on n and the parts of a partition. Let $\lambda = (a_k, a_{k-1}, \dots, a_1)$ be a partition and $n > k$. Then

$$d_T = \prod_{i=1}^{k} \frac{(n - (k-i+1) + a_i)!\,(a_i - a_{i-1} + 1) \cdots (a_i - a_1 + i - 1)}{(n - (k-i+1))!\,(a_i + i - 1)!}. \tag{2.1}$$

This follows from well-known manipulations involving specializations of Schur functions. See pages 44 and 45 in [4].

We will now use this formula for the degree to show that all of the families described in Theorem 1.1 correspond to prime power degree representations.

Proposition 2.1. *Let p be a prime and $k, \ell, m \in \mathbb{N}$, and let $\lambda = (a_k, a_{k-1}, \dots, a_1)$.*

(1) *The partition*
$$\lambda = (1)$$
corresponds to a prime power degree irreducible representation of $SU(p^m)$.

(2) *The partition*

$$\lambda = (k(p^\ell - 1), (k-1)(p^\ell - 1), \ldots, p^\ell - 1)$$

corresponds to a prime power degree irreducible representation of $SU(k+1)$.

(3) *The partition*

$$\lambda = (p^m(p^\ell - 1) + 1, (p^m - 2)(p^\ell - 1), \ldots, p^\ell - 1)$$

corresponds to a prime power degree irreducible representation of $SU(p^m)$.

(4) *The partition*

$$\lambda = (p^m(p^\ell - 1) + 1, (p^m - 1)(p^\ell - 1) + 1, \ldots, 2(p^\ell - 1) + 1)$$

corresponds to a prime power degree irreducible representation of $SU(p^m)$.

Proof. For each family, we will call the representation specified by the family T.

(1) Clearly, in this case

$$d_T = \frac{p^m}{1} = p^m.$$

(2) Note that, in this case, $a_i = i(p^\ell - 1)$. So, using Formula (2.1), we have

$$\begin{aligned} d_T &= \prod_{i=1}^{k} \frac{(ip^\ell)!\, p^\ell \cdots (i-1)p^\ell}{i!\,(ip^\ell - 1)!} \\ &= \prod_{i=1}^{k} \frac{ip^\ell p^{\ell(i-1)}(i-1)!}{i!} \\ &= \prod_{i=1}^{k} p^\ell p^{\ell(i-1)}. \end{aligned}$$

(3) Note that, in this case, $a_{p^m-1} = p^m(p^\ell - 1) + 1$ and $a_i = i(p^\ell - 1)$ for $i \neq p^m - 1$. So, using (2.1), we have

$$\begin{aligned} d_T &= \frac{p^{m+\ell}!\,(2p^\ell)\cdots(p^m-1)(p^\ell)}{(p^m-1)!\,(p^{m+\ell}-1)!} \prod_{i=1}^{p^m-2} \frac{(ip^\ell)!\, p^\ell \cdots (i-1)p^\ell}{i!\,(ip^\ell - 1)!} \\ &= \frac{p^{m+\ell}p^{\ell(p^m-2)}(p^m-1)!}{(p^m-1)!} \prod_{i=1}^{p^m-2} \frac{(ip^\ell)p^{\ell(i-1)}(i-1)!}{i!} \\ &= p^{m+\ell}p^{\ell(p^m-2)} \prod_{i=1}^{p^m-2} p^\ell p^{\ell(i-1)}. \end{aligned}$$

(4) Note that, in this case, $a_{p^m-1} = p^m(p^\ell - 1) + 1$ and $a_i = (i+1)(p^\ell - 1) + 1$ for $i \neq p^m - 1$. So, using (2.1), we have

$$\begin{aligned} d_T &= \frac{p^{m+\ell}!\,(2p^\ell)\cdots(p^m-1)(p^\ell)}{(p^m-1)!\,(p^{m+\ell}-1)!}\prod_{i=1}^{p^m-2}\frac{(i(p^\ell))!\,p^\ell\cdots(i-1)p^\ell}{i!\,(i(p^\ell)-1)!} \\ &= \frac{p^{m+\ell}p^{\ell(p^m-2)}(p^m-1)!}{(p^m-1)!}\prod_{i=1}^{p^m-2}\frac{i(p^\ell)p^{\ell(i-1)}(i-1)!}{i!} \\ &= p^{m+\ell}p^{\ell(p^m-2)}\prod_{i=1}^{p^m-2}p^\ell p^{\ell(i-1)}. \end{aligned}$$

□

3. A Reformulation of the Degree Formula

In this section, we derive a version of the degree formula depending on the "hands" of a partition which is much more compact than (2.1). Let $\lambda = (a_k, a_{k-1}, \ldots, a_i)$ be a partition and define the i^{th} *hand* to be $q_i = a_i + i$. In an admissible diagram for λ when $n = k+1$, i.e., when n is minimal, q_i is also the number in the last box in the i^{th} row.

5	6	7	8	9
4	5			
3				
2				

Figure 4. The hands of the partition $(5,2,1,1)$ are $2, 3, 5$, and 9

Substitution of the definition of the hands into (2.1) yields the formula in the following lemma.

Lemma 3.1. *Let $\lambda = (a_k, a_{k-1}, \ldots, a_1)$ be a partition, $q_i = a_i + i$, and $j \geq 0$. If T is the irreducible representation of $SU(k+1+j)$ corresponding to λ, then we have*

$$d_T = \prod_{i=1}^{k}\frac{(q_i+j)!\,(q_i-q_{i-1})\cdots(q_i-q_1)}{(i+j)!\,(q_i-1)!}. \tag{3.1}$$

Proof. By (2.1), we have

$$\begin{aligned} d_T &= \prod_{i=1}^{k}\frac{(j+a_i+i)!\,(a_i-a_{i-1}+i-(i-1))\cdots(a_i-a_1+i-1)}{(i+j)!\,(a_i+i-1)!} \\ &= \prod_{i=1}^{k}\frac{(q_i+j)!\,(q_i-q_{i-1})\cdots(q_i-q_1)}{(i+j)!\,(q_i-1)!}. \end{aligned}$$

□

In the case that n is as small as possible, i.e., when $j = 0$, expression (3.1) becomes

$$d_T = \prod_{i=1}^{k} \frac{q_i(q_i - q_{i-1}) \cdots (q_i - q_1)}{i!}. \tag{3.2}$$

This suggests the key idea used in the proof of Theorem 1.1: the prime factors of d_T depend highly on the distribution of the hands modulo $i = 1, 2, \ldots, k$.

4. Proof of Theorem 1.1

In this section we first use (3.2) to prove that the only λ that correspond to prime power degree representations for minimal n are those listed in Theorem 1.1. We then use a similar technique and (3.1) to prove that, if $\lambda \neq (1)$, then λ has k parts, and $n > k + 1$, then λ does not correspond to a prime power degree representation of $SU(n)$. These two results will complete the proof of Theorem 1.1.

We begin with a simple combinatorial lemma.

Lemma 4.1. *If $a, k \in \mathbb{N}$ and μ is a strictly decreasing sequence $q_k > q_{k-1} > \cdots > q_1 > 0$, for each $b = 0, \ldots, a-1$ let k_b^μ denote the number of q_i in μ that are congruent to b modulo a. Define*

$$m_{a,\mu} := \#\{i : q_i \equiv 0 \pmod{a}\} + \#\{(i_1, i_2) : q_{i_1} \equiv q_{i_2} \pmod{a}, i_1 > i_2\}.$$

Let $m_{a,k}$ be the minimum over all such sequences μ of the value of $m_{a,\mu}$. Then $m_{a,k} = m_{a,\mu}$ if and only if $k_b^\mu - 2 \leq k_0^\mu \leq k_b^\mu$ and $|k_b^\mu - k_{b'}^\mu| \leq 1$ for all $b, b' \in \{1, \ldots, a-1\}$. Further,

$$m_{a,k} = \sum_{i=1}^{k} \#\{j : 1 \leq j \leq i, j \equiv 0 \pmod{a}\}.$$

Proof. Let μ be a strictly decreasing sequence $q_k > q_{k-1} > \cdots > q_1 > 0$. Then

$$m_{a,\mu} = k_0^\mu + \sum_{b=0}^{a-1} \binom{k_b^\mu}{2}.$$

Note that, if $k_b^\mu \geq k_{b'}^\mu + 1$, then since $\binom{n+1}{2} - \binom{n}{2} = n$ for all $n \in \mathbb{N}$, we have

$$\binom{k_b^\mu}{2} + \binom{k_{b'}^\mu}{2} \geq \binom{k_b^\mu - 1}{2} + \binom{k_{b'}^\mu + 1}{2},$$

where equality holds exactly when $k_b^\mu = k_{b+1}^\mu + 1$. Hence, $m_{a,\mu} = m_{a,k}$ exactly when $k_b^\mu - 2 \leq k_0^\mu \leq k_b^\mu$ for all $b = 0, 1, \ldots, a-1$ and the distance between k_b^μ and $k_{b'}^\mu$ is at most 1 for all b and b' in $\{1, \ldots, a-1\}$. Clearly, $q_i = i$ is one sequence with this property. Set $m = \sum_{i=1}^{k} \#\{j : 1 \leq j \leq i, j \equiv 0 \pmod{a}\}$. So, letting $U = \{1, \ldots, k\}$ and $U_i = \{u \in U : u \equiv 0 \pmod{a}, i \geq u\}$, we have

$$\begin{aligned} m_{a,k} &= \#U_k + \sum_{i=2}^{k} \#\{j : q_i - q_j \equiv 0 \pmod{a}, i > j\} \\ &= \#U_k + \sum_{i=2}^{k} \#\{j : i - j \equiv 0 \pmod{a}, i > j\} \\ &= \#U_k + \sum_{i=2}^{k} \#U_{i-1} = \sum_{i=1}^{k} \#U_i = m. \end{aligned}$$

□

This lemma says that $m_{a,k}$ is equal to the number of multiples of a in $\prod_{i=1}^{k} i!$, which is the denominator in (3.2). There are two important consequences of the lemma. The first is that it shows that d_T is an integer without any representation theory. The second is that it shows what is required for a representation to have prime power degree: $q_k > \cdots > q_1 > 1$ must achieve the minimum in the definition of $m_{a,k}$ for all a not equal to a multiple of some fixed prime. This motivates the following definition.

Definition 4.2. *We say a strictly decreasing sequence of k integers μ is* **good with respect to** p *if $m_{a,k} = m_{a,\mu}$ for all $a \not\equiv 0 \pmod{p}$.*

The following lemma makes the observation from the previous paragraph precise.

Lemma 4.3. *Suppose that T is an irreducible representation of $SU(k+1)$ corresponding to the partition λ with hands $\mu = q_k > \cdots > q_1$. The degree d_T is a power of some prime p if and only if μ is good with respect to p.*

Proof. To see this, simply note that μ is good with respect to p if and only if any power of a prime $q \neq p$ appearing in the numerator of d_T is canceled by something in the denominator. This is because, if q is a prime distinct from p, then Lemma 4.1 says that μ is good with respect to p if and only if, for all $\ell \in \mathbb{N}$, there are the same number of multiples of q^ℓ in the numerator as in the denominator, and hence $q \nmid d_T$. □

Suppose that $\mu = q_k > \cdots > q_1 > 0$ is good with respect to p. The following three facts follow easily from Lemma 4.1, and will be used repeatedly throughout the proofs of the next several lemmas:

(1) $m_{k,k} = 1$.
(2) If $p \nmid q_i$, then $q_i \leq k$.
(3) If $p \nmid q_i - q_j$ for $i > j$, then $q_i - q_j \leq k$.

Fact 1 is immediate from the Lemma 4.1. Note also that $m_{a,k} > 0$ only when $k \geq a$. If $p \nmid q_i$, then $m_{q_i,k} = m_{q_i,\mu} \geq 1$ since μ is good with respect to p and q_i contributes to $m_{q_i,\mu}$. This implies that $q_i \leq k$. The proof of Fact 3 is similar.

Lemma 4.4. *Suppose that $k \geq 2$ and $\mu = q_k > \cdots > q_1 > 0$ is good with respect to some prime p. Then either q_k and q_{k-1} are both divisible by p, $q_i = i$ for all $i = 1, \ldots, k$, $q_i = i+1$ for all $i = 1, \ldots, k$, or $q_k = k+1$ and $q_i = i$ for all $i = 1, \ldots, k-1$.*

Proof. Suppose that $p \nmid q_k$. Then $q_k \leq k$ by Fact 2. This forces $q_i = i$ for all $i = 1, \ldots, k$ since $k \geq q_k > q_{k-1} > \cdots > q_1 \geq 1$.

Now suppose that $p \mid q_k$ and $p \nmid q_{k-1}$. By Fact 2, $q_{k-1} \leq k$. Since the q_i are strictly increasing, $q_{k-1} \geq q_1 + k - 2$, which implies that $q_1 = 1$ or $q_1 = 2$. If $q_k = k+1$ and $q_{k-1} = k$, then, if $q_i \neq i+1$ for some i (i.e., we are not in the third case in the statement of the lemma), we must have $q_1 = 1$ because the q_i are strictly decreasing. In this case, since $p \nmid q_{k-1}$, $p \nmid k$ as well. However, $k = q_{k-1} = q_k - q_1$, so $m_{k,\mu} \geq 2$. Since μ is good with respect to p, $m_{k,k} = m_{k,\mu} \geq 2$, which contradicts Fact 1. If $q_k = k+1$ and $q_{k-1} = k-1$, this forces $q_i = i$ for all $i = 1, 2, \ldots, k-1$, the final case in the statement of the lemma. If $q_k \geq k+2$ and $q_1 = 1$, then $p \nmid q_k - 1$ and $q_k - 1 > k$, contradicting Fact 3. Finally, if $q_k \geq k+2$ and $q_1 = 2$, then $q_{k-1} = k$ and $p \nmid k$. Either $q_k > k+2$ or $q_k = k+2$. In the first case, $q_k - q_1 > k$, contradicting Fact 3. In the second case, $m_{k,k} = m_{k,\mu} \geq 2$ since $k = q_{k-1} = q_k - q_1$, contradicting Fact 1. □

The goal of the next collection of lemmas is to show that, if $k \geq 3$, then $\mu = q_k > q_{k-1} > \cdots > q_1 > 0$ is good with respect to a prime p, and p divides both q_k and q_{k-1}, then p must in fact divide q_i for every $i = 1, \ldots, k$. The next lemma gives an upper bound for the smallest q_i in μ which is not divisible by p based on how many consecutive large terms in μ are known to be multiples of p.

Lemma 4.5. *Suppose that $\mu = q_k > \cdots > q_1 > 0$ is good with respect to some prime p and that $p \mid q_k, q_{k-1}, \ldots, q_{k-m+1}$ with $m \geq 2$, but not every term in μ is divisible by p. Then there exists an $i = 1, \ldots, k$ such that $p \nmid q_i$ and $q_i \leq \frac{p}{p-1}(q_{k-m} + m + 2 - k - p) + p - 1$.*

Proof. To construct the sequence $\mu' = q'_k > q'_{k-1} > \cdots > q'_1 > 0$ with $q'_i = q_i$ for $i = k-m, \ldots, k$ with maximum possible $\min\{q'_i : p \nmid q'_i\}$, we must set $q'_i = pi$ for as many of the first $k-m$ i's as possible, and then increase q'_i by $p-1$ once and then only by 1 as i increases by 1 up to $k-m-1$ in order to have a sequence of the required length. So let $q'_i = pi$ for $i = 1, \ldots, \ell_p$, $q'_{\ell_p+1} = p\ell_p + p - 1$, and $q'_{\ell_p+1+j} = p\ell_p + p - 1 + j$ for $j = 1, \ldots, k-m-\ell_p-1$. Then, we have $k - m - \ell_p - 1 + p - 1 + p\ell_p = q_{k-m}$, i.e., $\ell_p = \frac{q_{k-m}+m+2-k-p}{p-1}$. Thus, for some $i \leq k-m$, $p \nmid q_i$ with $q_i \leq p\ell_p + p - 1 = \frac{p}{p-1}(q_{k-m} + m + 2 - k - p) + p - 1$.

□

Using the previous lemma in conjunction with Fact 3, we are able to show that, when $p > 2$, if μ is good with respect to p and p divides q_k and q_{k-1}, then p must divide every term in μ. When $p = 2$, the following lemma also gives the same result in all but two cases.

Lemma 4.6. *Suppose that $\mu = q_k > \cdots > q_1 > 0$ is good with respect to some prime p, $p \mid q_k, q_{k-1}, \ldots, q_{k-m+1}$, $p \nmid q_{k-m}$, and $k > m \geq 2$. Then $p = 2$, k is odd, and either $q_{k-m} = k$ or $q_{k-m} = k - 2$.*

Proof. Let q_i be the smallest term in μ which is not divisible by p. We have, by Lemma 4.5,

$$\begin{aligned} q_k - q_i &\geq q_{k-m} + p(m-1) + 1 - \frac{p}{p-1}(q_{k-m} + m + 2 - k - p) - p + 1 \\ &= k + \frac{1}{p-1}(k - q_{k-m}) + m\left(\frac{p^2 - 2p}{p-1}\right) - \frac{p^2}{p-1} + 2 \\ &\geq k + \frac{1}{p-1}(k - q_{k-m}) + \frac{p^2 - 4p}{p-1} + 2, \end{aligned}$$

since $m \geq 2$. When $p \geq 3$, since $q_{k-m} \leq k$, we get that $q_k - q_i \geq k - \frac{3}{2} + 2 = k + \frac{1}{2}$, which contradicts Fact 3 since $p \nmid q_k - q_i$.

When $p = 2$, the inequality above yields $q_k - q_i \geq k - 2 + (k - q_{k-m})$. Because $p \nmid q_k - q_i$, by Fact 3 we have $k \geq q_k - q_i \geq k - 2 + (k - q_{k-m})$, and hence $q_{k-m} \geq k - 2$. Since $q_{k-m} \leq k$ by Fact 2, we see that either $q_{k-m} = k$ or $q_{k-m} = k - 2$, and, in addition, k must be odd.

□

In the next lemma, we show that, if $\mu = q_k > \ldots q_1 > 0$ is good with respect to 2 but not every term in μ is even, then μ must be of a specific form depending on whether $q_k = k$ or $q_k = k - 2$.

Lemma 4.7. *Suppose that $\mu = q_k > \cdots > q_1 > 0$ is good with respect to 2 and that $2 \mid q_k, q_{k-1}, \ldots, q_{k-m+1}$, $2 \nmid q_{k-m}$, and $m \geq 2$. If $q_{k-m} = k$, then*

$$q_j = \begin{cases} 2j, & j = 1, \ldots, m, \\ m + j, & j = m+1, \ldots, m + 2\ell + 1, \\ 2(j - \ell - 1), & j = m + 2\ell + 2, \ldots, 2m + 2\ell + 1, \end{cases}$$

for some $\ell \geq 0$*, and, if* $q_{k-m} = k-2$*, then*

$$q_j = \begin{cases} 2j, & j = 1, \dots, m-2, \\ m+j-2, & j = m-1, \dots, m+2\ell-1, \\ 2(j-\ell-1), & j = m+2\ell, \dots, 2m+2\ell-1, \end{cases}$$

for some $\ell \geq 0$*.*

Proof. Let i be the smallest integer for which $2 \nmid q_i$. Assume that $q_{k-m} = k$. Since $m_{k,\mu} = m_{k,k}$ because μ is good with respect to 2 and k is odd, by Fact 1 $m_{k,\mu} = 1$. Hence, we must have $q_k - q_i = k-2$. It is also clear that $q_k \geq q_{k-m} + 2(m-1)+1 = k+2m-1$. Subtracting q_i from both sides, we see that $k-2 \geq k+2m-1-q_i$, which implies that $q_i \geq 2m+1$. However, by the definition of q_i and Lemma 4.5, we also have that $q_i \leq 2(k+m+2-k-2)+2-1 = 2m+1$, and hence $q_i = 2m+1$.

Since $k = q_{k-m} \geq q_i + (k-m-i)$ because the q_i are strictly increasing, we have $k \geq 2m+1+k-m-i$, i.e., $i \geq m+1$. By the definition of q_i, for all $j < i$, we have $2 \mid q_j$. Hence, $q_i \geq 2(i-1)+1$, i.e., $2m+2 \geq 2i$, and $m+1 \geq i$. This implies that $i = m+1$ and forces $q_j = 2j$ for all $j = 1, \dots, m$.

By Fact 2, since $2 \nmid q_i$, $2m+1 \leq k$. Since k is odd, $k = 2m+2\ell+1$ for some $\ell \geq 0$. Note that, if $k \neq 2m+1$ and if we do not increase q_j by 1 as j increases by 1 from q_i up to q_{k-m}, then we must have $k > 2m+1+(k-m-(m+1)) = k$, which is impossible. Hence, $q_{m+1+j} = 2m+1+j$ for $j = 1, \dots, (k-m-(m+1))$, and a choice of ℓ uniquely determines μ: $q_j = 2j$, $j = 1, \dots, m$, $q_j = m+j$ for $j = m+1, \dots k-m$, and $q_j = 2j+2m-k-1$ for $j = k-m+1, \dots, k$. Replacing k with $2m+2\ell+1$, we get the result.

The $q_{k-m} = k-2$ case is proved analogously. □

The following lemma and proposition comprise one half of Theorem 1.1.

Lemma 4.8. *Suppose that* $\mu = q_k > \cdots > q_1 > 0$ *is good with respect to some prime* p*. Then either the* $q_1, \dots, q_k$ *are all divisible by* p*,* $q_i = i$ *for all* $i = 1, \dots, k$*,* $q_i = i+1$ *for all* $i = 1, \dots, k$*, or* $q_k = k+1$ *and* $q_i = i$ *for all* $i = 1, \dots, k-1$*.*

Proof. By Lemmas 4.4 and 4.6, it suffices to show that both of the two cases described in Lemma 4.7 lead to a contradiction. Our main tool will be Lemma 4.1.

Assume that $q_{k-m} = k$. When $q_{k-m} = k-2$ and $m \geq 3$ (so that we are in not in the last case described in the statement of the lemma), a contradiction is reached in an almost identical manner. Suppose that $\ell = 0$. Then $k = 2m+1$. Since μ is good with respect to 2, $3 = m_{k-2,k} = m_{k-2,\mu}$. However, $q_k - q_{k-m} = k-2$, $q_{k-m} - q_1 = k-2$, $q_{k-2} = 4m-2 = 2(k-2)$, and $q_k - q_1 = 4m-2 = 2(k-2)$. So, $m_{k-2,\mu} \geq 4$, a contradiction. Suppose that $\ell = 1$, so then $k = 2m+2\ell+1 \geq 7$, and since μ is good with respect to 2, $5 = m_{k-4,k} = m_{k-2,\mu}$. However, $q_k - q_2 = 2(k-4)$, $q_k - q_{k-m} = k-4$, $q_{k-1} - q_1 = 2(k-4)$, $q_{k-1} - q_{k-m-2} = k-4$, $q_{k-m-2} - q_1 = k-4$, and $q_{k-m} - q_2 = k-4$. So, $m_{\mu,k-2} \geq 6$, a contradiction.

Now suppose that $\ell \geq 2$. There are two cases: either $m+\ell$ is odd or $m+\ell+1$ is odd. In the first case, it suffices to show that there are three q_j's which are congruent to each of 2 and 4 modulo $m+\ell$, and that, if there is more than one q_j congruent to 1 modulo $m+\ell$, then there are two q_j's divisible by $m+\ell$. Then, in either case, there must be a nonzero congruence class modulo $m+\ell$ which contains at most one element, while the congruence class of 2 contains three elements, contradicting Lemma 4.1. When $m+\ell$ is odd, $q_j \equiv q_{k-m+j} \pmod{m+\ell}$ for all $j = 1, \dots, m$, so we have at least two q_j's equivalent to each of 2 and 4, and two q_j's from this range congruent to 1 if and only if $2m \geq m+\ell$. For $j = m+1, \dots, k-m$, q_j ranges from $2m+1$ to $2m+2\ell+1$, and $2m+2\ell+1 \geq m+\ell+4$ since $m, \ell \geq 2$, and hence there are

q_j's in this range that are equivalent to each of $1, 2$, and 4 modulo $m+\ell$, giving us a total of at least three in the classes of 2 and 4. If $2m \geq m+\ell$, then the class of 1 contains at least three elements, and so there must be a nonzero equivalence class with at most one element. If $2m < m+\ell$, then $m+\ell < 2\ell$, which implies that, since $q_{k-m} \equiv 1 \pmod{m+\ell}$, that at least two of the q_j in the range of $j = m+1, \ldots, k-m$ are divisible by $m+\ell$, and so, again, there must be a nonzero equivalence class with at most one element. The argument when $m+\ell+1$ is odd is similar. □

Proposition 4.9. *Let T be an irreducible representation of $SU(k+1)$ corresponding to the partition with hands $q_k > \cdots > q_1 > 1$. If $d_T = p^\ell$ for $\ell \in \mathbb{N}$, then T is one of the representations in Theorem 1.1.*

Proof. Let m be the highest power of p that divides all of the q_i, so setting $q_i' = q_i/p^m$, define d_T' to be the result of substituting $q_k', \ldots, q_1'$ into (3.2). Then not all of the q_i' are divisible by p, so either $q_i' = i$ or $q_i' = i+1$ for all $i = 1, \ldots, k-1$ by Lemma 4.8, which implies that T is one of the representations in Theorem 1.1. □

We now finish the proof of Theorem 1.1 by proving that, if $n > k+1$ where k is the number of parts of the partition λ and T is the irreducible representation corresponding to λ of $SU(n)$, then d_T is not a prime power unless $\lambda = (1)$. In order to do this, we modify the strategy used to prove the previous proposition. First, we have a result analogous to Lemma 4.1, which has essentially the same proof.

Lemma 4.10. *If $a, k, j \in \mathbb{N}$ and μ is a strictly decreasing sequence $q_k > q_{k-1} > \cdots > q_1 > 0$, define*

$$m_{a,\mu,j} := \#\{(i,s) : q_i + s \equiv 0 \pmod{a}, s = 0, \ldots, j\} + \#\{(i_1, i_2) : q_{i_1} \equiv q_{i_2} \pmod{a}, i_1 > i_2\}.$$

Let $m_{a,k,j}$ be the minimum over all such sequences μ of the value $m_{a,\mu,j}$. Then

$$m_{a,k,j} = \sum_{i=1}^{k} \#\{m : 1 \leq m \leq i+j, m \equiv 0 \pmod{a}\}.$$

In analogy with the $n = k+1$ case, we make the following definition.

Definition 4.11. *We say a decreasing sequence of k integers is* j**-good with respect to** p *if $m_{a,k,j} = m_{a,\mu,j}$ for all $a \not\equiv 0 \pmod{p}$.*

Note that the definition of 0-good is the same as the original definition of good. Corresponding to Lemma 4.3, we have the obvious result.

Lemma 4.12. *Suppose that T is an irreducible representation of $SU(k+1+j)$ corresponding to the partition λ with hands $q_k > \cdots > q_1$. The degree d_T is a power of some prime p if and only if $q_k, \ldots, q_1$ is j-good with respect to p.*

Finally, we complete the proof of Theorem 1.1 with the following proposition.

Proposition 4.13. *Let T be an irreducible representation of $SU(k+1+j)$ with $j > 0$ corresponding to the partition with hands $q_k > \cdots > q_1 > 1$. The degree d_T is a prime power if and only if $k = 1$ and $q_1 = 2$.*

Proof. Suppose that $q_k > \cdots > q_1 > 1$ is j-good with respect to p. Note that either $q_k + j$ is not a multiple of p or $q_k + j - 1$ is not a multiple of p. In the first case, we have $m_{q_k+j,k,j} = 1$, and so by Lemma 4.10,

$$1 = \sum_{i=1}^{k} \#\{m : 1 \leq m \leq i+j, m \equiv 0 \pmod{q_k + j}\},$$

which means that there exists some $1 \leq m \leq j+k$ such that $q_k + j$ divides m, which means that $q_k + j \leq m \leq j + k$. In the second case, we have $m_{q_k+j-1,k,j} = 1$ or 2, and so by a similar argument, $q_k + j - 1 \leq j + k$. So, we certainly have $q_k + j - 1 \leq k + j$, which means that $q_k \leq k+1$. This implies that $q_i = i+1$ for all $i = 1, \ldots, k$ since $q_1 \geq 2$. Now, suppose that $k > 1$. We will show that in this case d_T cannot be a prime power.

We have

$$\begin{aligned} d_T &= (k+1)\prod_{i=1}^{k}\frac{(q_i+1)\cdots(q_i+j)}{(i+1)\cdots(i+j)} \\ &= (k+1)\prod_{i=1}^{k}\frac{(i+2)\cdots(i+j+1)}{(i+1)\cdots(i+j)} \\ &= (k+1)\prod_{i=1}^{k}\frac{(i+j+1)}{(i+1)} \\ &= \frac{1}{k!}\prod_{i=1}^{k}(i+j+1) \\ &= \frac{(j+2)\cdots(j+k+1)}{k!} \\ &= \binom{j+k+1}{j+1}. \end{aligned}$$

But $\binom{j+k+1}{j+1}$ is a prime power only when $j+1 = 1$ or $j+1 = j+k+1-1$. However, $j > 0$ and $k > 1$, so d_T is not a prime power. $\square$

References

[1] A. Balog, C. Bessenrodt, J. Olsson, K. Ono *Prime power degree representations of the symmetric and alternating groups*, J. London Math. Soc. (2) 64 (2001), no. 2, 344–356.

[2] G. Han, *Some conjectures and open problems on partition hook lengths*, Experimental Math. 18 (2009), 97–106.

[3] G. James, A. Kerber, 'The representation theory of the symmetric group', *Encyclopedia of Mathematics and its Applications* 16, Addison Wesley, 1981.

[4] I. G. Macdonald, *Symmetric functions and Hall polynomials*, The Clarendon Press, Oxford University Press, New York, 1995.

[5] S. Sternberg, *Group Theory and Physics*, Cambridge University Press, Cambridge, 1994.