A word Hopf algebra based on the selection/quotient principle

Nguyen HOANG-NGHIA

LIPN, Univ. Paris XIII, Sorbonne Paris Cité

based on arXiv:1207.6522v1

Séminaire Lotharingien de Combinatoire 69

Strobl, September 10, 2012

Plan

- Introduction - Combinatorial Hopf algebra classification (for this purpose)

Plan

- Introduction - Combinatorial Hopf algebra classification (for this purpose)
- Algebra structure of WMat

Plan

- Introduction - Combinatorial Hopf algebra classification (for this purpose)
- Algebra structure of WMat
- Bialgebra structure of WMat

Plan

- Introduction - Combinatorial Hopf algebra classification (for this purpose)
- Algebra structure of WMat
- Bialgebra structure of WMat
- Primitive elements of WMat

Plan

- Introduction - Combinatorial Hopf algebra classification (for this purpose)
- Algebra structure of WMat
- Bialgebra structure of WMat
- Primitive elements of WMat
- Hilbert series of WMat

Plan

- Introduction - Combinatorial Hopf algebra classification (for this purpose)
- Algebra structure of WMat
- Bialgebra structure of WMat
- Primitive elements of WMat
- Hilbert series of WMat
- Conclusion and perspectives

Introduction - Combinatorial Hopf algebra classification

1. Combinatorial Hopf algebras of type I - the selection/quotient principle

Introduction - Combinatorial Hopf algebra classification

1. Combinatorial Hopf algebras of type I - the selection/quotient principle
2. Combinatorial Hopf algebras of type II - the selection/complementation principle

Combinatorial Hopf algebras of type I

The selection/quotient principle

$$\Delta(S) = \sum_{A \subseteq S (+Conditions)} S[A] \otimes S/_A \, , \qquad (1)$$

$\hookrightarrow$ mostly used in combinatorial physics

- Connes-Kreimer Hopf algebra of Feynman graphs

A. Connes, D. Kreimer, Commun. Math. Phys. **210** (2000), no. 1, 249-273, [arXiv:hep-th/0003188].

Combinatorial Hopf algebras of type I

The selection/quotient principle

$$\Delta(S) = \sum_{A \subseteq S (+Conditions)} S[A] \otimes S/_A \ , \qquad (1)$$

$\hookrightarrow$ mostly used in combinatorial physics

- Connes-Kreimer Hopf algebra of Feynman graphs

  A. Connes, D. Kreimer, Commun. Math. Phys. **210** (2000), no. 1, 249-273, [arXiv:hep-th/0003188].

- Hopf algebra of Feynman ribbon graphs

  A. Tanasa, Vignes-Tourneret, J. Noncomm. Geom. (2008). arXiv:0707.4143[math-ph]

  A. Tanasa, Kreimer, J. Noncomm. Geom. (in press), arXiv:0907.2182 [hep-th].

Combinatorial Hopf algebras of type I

The selection/quotient principle

$$\Delta(S) = \sum_{A \subseteq S (+Conditions)} S[A] \otimes S/_A \ , \qquad (1)$$

$\hookrightarrow$ mostly used in combinatorial physics

- Connes-Kreimer Hopf algebra of Feynman graphs

  A. Connes, D. Kreimer, Commun. Math. Phys. **210** (2000), no. 1, 249-273, [arXiv:hep-th/0003188].

- Hopf algebra of Feynman ribbon graphs

  A. Tanasa, Vignes-Tourneret, J. Noncomm. Geom. (2008). arXiv:0707.4143[math-ph]

  A. Tanasa, Kreimer, J. Noncomm. Geom. (in press), arXiv:0907.2182 [hep-th].

- Hopf algebra of quantum gravity spin-foam models

  A. Tanasa, Class. Quant. Grav. **27**, 095008 (2010), [arXiv:0909.5631 [gr-qc]].

Combinatorial Hopf algebras of type I

The selection/quotient principle

$$\Delta(S) = \sum_{A \subseteq S (+Conditions)} S[A] \otimes S/_A \,, \qquad (1)$$

$\hookrightarrow$ mostly used in combinatorial physics

- Connes-Kreimer Hopf algebra of Feynman graphs

  A. Connes, D. Kreimer, Commun. Math. Phys. **210** (2000), no. 1, 249-273, [arXiv:hep-th/0003188].

- Hopf algebra of Feynman ribbon graphs

  A. Tanasa, Vignes-Tourneret, J. Noncomm. Geom. (2008). arXiv:0707.4143[math-ph]

  A. Tanasa, Kreimer, J. Noncomm. Geom. (in press), arXiv:0907.2182 [hep-th].

- Hopf algebra of quantum gravity spin-foam models

  A. Tanasa, Class. Quant. Grav. **27**, 095008 (2010), [arXiv:0909.5631 [gr-qc]].

$\hookrightarrow$ A Hopf algebraic structure of matroids:

W. Schmitt, J. of Pure and Applied Alg. **96** (1994), 299-330.

Combinatorial Hopf algebra of type II

The selection/complemention principle:

$$\Delta(S) = \sum_{A \subseteq S (+Conditions)} S[A] \otimes [S - A] \,. \qquad (2)$$

- The Loday-Ronco Hopf algebra of planar binary trees

J.- L. Loday, M. O. Ronco, Adv. Math. **139** (1998), 293-309.

Combinatorial Hopf algebra of type II

The selection/complemention principle:

$$\Delta(S) = \sum_{A \subseteq S (+Conditions)} S[A] \otimes [S - A] \,. \tag{2}$$

- The Loday-Ronco Hopf algebra of planar binary trees

  J.- L. Loday, M. O. Ronco, Adv. Math. **139** (1998), 293-309.

- The Hopf algebra on matrix quasi-symmetric functions **MQSym**, the Hopf algebra on the free quasi-symmetric functions **FQSym**, etc

  I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retakh, J.Y. Thibon, *NCSF*, Adv. Math. **112** (1995), 218-348.

  G.H.E. Duchamp, A. Klyachko, D. Krob, J.Y. Thibon, *NCSF III: Deformations of Cauchy and convolution algebras* (1997).

  G.H.E. Duchamp, F. Hivert, J.Y. Thibon, *Some generalizations of quasi-symmetric functions and noncommutative symmetric functions* (2000).

  G.H.E. Duchamp, F. Hivert, J.Y. Thibon, *NCFS VI: Free quasi-symmetric functions and related algebras* (2002).

  G.H.E. Duchamp, F. Hivert, J.C. Novelli, J.Y. Thibon, *NCFS VII: Free quasi-symmetric functions revisited* (2008).

Objective

Define a combinatorial Hopf algebra structure of type I - WMat - on objects familiar to type II (words)

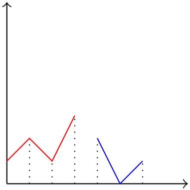
Algebra structure of WMat

$X = \{x_i\}_{i \geq 0}$ (the alphabet)
X^* - the set of words with letters in the alphabet X.
The shifted concatenation $*$ - the product:

$$u * v = uT_{sup(u)}(v) \ , \tag{3}$$

where, for $t \in \mathbb{N}$, $T_t(v)$ - the image of w by S_ϕ for $\phi(n) = n + t$ if $n > 0$ and $\phi(0) = 0$ (in general, all letters can be reindexed except x_0).
Ex.

$$x_1x_2x_1x_3 * x_2x_0x_1 = x_1x_2x_1x_3x_5x_0x_4 \ .$$



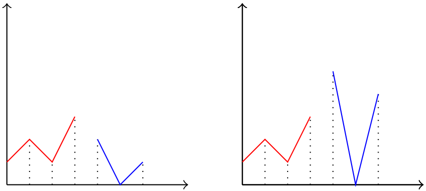
Algebra structure of WMat

$X = \{x_i\}_{i \geq 0}$ (the alphabet)
X^* - the set of words with letters in the alphabet X.
The shifted concatenation $*$ - the product:

$$u * v = u T_{sup(u)}(v) \; , \tag{3}$$

where, for $t \in \mathbb{N}$, $T_t(v)$ - the image of w by S_ϕ for $\phi(n) = n + t$ if $n > 0$ and $\phi(0) = 0$ (in general, all letters can be reindexed except x_0).
Ex.

$$x_1 x_2 x_1 x_3 * x_2 x_0 x_1 = x_1 x_2 x_1 x_3 x_5 x_0 x_4 \; .$$



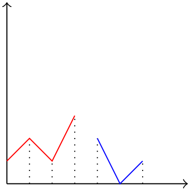
Algebra structure of WMat

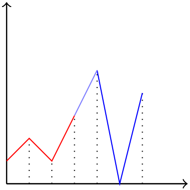
$X = \{x_i\}_{i\geq 0}$ (the alphabet)
X^* - the set of words with letters in the alphabet X.
The shifted concatenation $*$ - the product:

$$u * v = uT_{sup(u)}(v) \ , \qquad (3)$$

where, for $t \in \mathbb{N}$, $T_t(v)$ - the image of w by S_ϕ for $\phi(n) = n + t$ if $n > 0$ and $\phi(0) = 0$ (in general, all letters can be reindexed except x_0).
Ex.

$$x_1x_2x_1x_3 * x_2x_0x_1 = x_1x_2x_1x_3x_5x_0x_4 \ .$$





Algebra structure of WMat

Definition (Packed words)

$w \in X^*$ - a word, $I = IAlph(w)\backslash\{0\}$ - the set of indices of w. Let $I = \{j_1, \cdots j_k\}$ with $j_1 < j_2 < \cdots < j_k$ and define ϕ_0 as $\phi_0(j_m) = m$ and $\phi_0(0) = 0$. The pack of word is $S_{\phi_0}(w)$ - $pack(w)$. A word $w \in X^*$ is called *packed* iff $w = pack(w)$.

Ex. : $w = x_1x_3x_4x_0$, $pack(w) = x_1x_2x_3x_0$.

Algebra structure of WMat

Definition (Packed words)

$w \in X^*$ - a word, $I = IAlph(w) \backslash \{0\}$ - the set of indices of w. Let $I = \{j_1, \cdots j_k\}$ with $j_1 < j_2 < \cdots < j_k$ and define ϕ_0 as $\phi_0(j_m) = m$ and $\phi_0(0) = 0$. The pack of word is $S_{\phi_0}(w)$ - $pack(w)$. A word $w \in X^*$ is called *packed* iff $w = pack(w)$.

Ex. : $w = x_1 x_3 x_4 x_0$, $pack(w) = x_1 x_2 x_3 x_0$.
$\mathcal{H} = span_k(pack(X^*)) \subset k\langle X \rangle$

$$\mu : \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H}, \quad u \otimes v \longmapsto u * v.$$

Algebra structure of WMat

Definition (Packed words)

$w \in X^*$ - a word, $I = IAlph(w) \backslash \{0\}$ - the set of indices of w. Let $I = \{j_1, \cdots j_k\}$ with $j_1 < j_2 < \cdots < j_k$ and define ϕ_0 as $\phi_0(j_m) = m$ and $\phi_0(0) = 0$. The pack of word is $S_{\phi_0}(w)$ - $pack(w)$. A word $w \in X^*$ is called *packed* iff $w = pack(w)$.

Ex. : $w = x_1 x_3 x_4 x_0$, $pack(w) = x_1 x_2 x_3 x_0$.
$\mathcal{H} = span_k(pack(X^*)) \subset k\langle X \rangle$

$$\mu : \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H},$$
$$u \otimes v \longmapsto u * v.$$

The set $pack(X^*)$ with the empty word and the shifted concatenation is a monoid. Therefore

Proposition

$(\mathcal{H}, \mu, 1_{X^*})$ is an associative algebra with unit (AAU).

Definition (Packed words)

$w \in X^*$ - a word, $I = IAlph(w) \setminus \{0\}$ - the set of indices of w. Let $I = \{j_1, \cdots j_k\}$ with $j_1 < j_2 < \cdots < j_k$ and define ϕ_0 as $\phi_0(j_m) = m$ and $\phi_0(0) = 0$. The pack of word is $S_{\phi_0}(w)$ - $pack(w)$. A word $w \in X^*$ is called *packed* iff $w = pack(w)$.

Ex. : $w = x_1x_3x_4x_0$, $pack(w) = x_1x_2x_3x_0$.
$\mathcal{H} = span_k(pack(X^*)) \subset k\langle X \rangle$

$$\mu : \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H}, \quad u \otimes v \longmapsto u * v.$$

The set $pack(X^*)$ with the empty word and the shifted concatenation is a monoid. Therefore

Proposition

$(\mathcal{H}, \mu, 1_{X^*})$ is an associative algebra with unit (AAU).

The product is non-commutative, Ex. : $x_1 * x_1x_1 \neq x_1x_1 * x_1$.

Algebra structure of WMat

Definition (Packed words)

$w \in X^*$ - a word, $I = IAlph(w) \backslash \{0\}$ - the set of indices of w. Let $I = \{j_1, \cdots j_k\}$ with $j_1 < j_2 < \cdots < j_k$ and define ϕ_0 as $\phi_0(j_m) = m$ and $\phi_0(0) = 0$. The pack of word is $S_{\phi_0}(w)$ - $pack(w)$. A word $w \in X^*$ is called *packed* iff $w = pack(w)$.

Ex. : $w = x_1x_3x_4x_0$, $pack(w) = x_1x_2x_3x_0$.
$\mathcal{H} = span_k(pack(X^*)) \subset k\langle X \rangle$

$$\mu : \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H},$$
$$u \otimes v \longmapsto u * v.$$

The set $pack(X^*)$ with the empty word and the shifted concatenation is a monoid. Therefore

Proposition

$(\mathcal{H}, \mu, 1_{X^*})$ is an associative algebra with unit (AAU).

The product is non-commutative, Ex. : $x_1 * x_1x_1 \neq x_1x_1 * x_1$.

The mapping pack $k < X > \xrightarrow{pack} \mathcal{H}$ is a morphism AAU.

WMat is a subalgebra of an extension of **MQSym**

The "free basis" of **MQSym**: $FB_M = \mathbf{MS}_{M_1}\mathbf{MS}_{M_2}\ldots\mathbf{MS}_{M_k}$, where

$$M = \begin{pmatrix} M_1 & 0 & \cdots & \cdots \\ 0 & M_2 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_k \end{pmatrix}.$$

is a maximal decomposition (k maximal, i.e. the M_i are irreducible).

$$\pi_1 : pack(X) \longrightarrow k\langle Y\rangle$$
$$x \longmapsto \begin{cases} y \text{ if } x \neq x_0 \\ y_0 \text{ otherwise} \end{cases} .$$

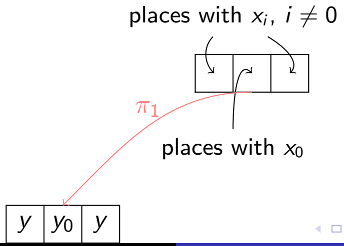
WMat is a subalgebra of an extension of **MQSym**

The "free basis" of **MQSym**: $FB_M = \mathbf{MS}_{M_1}\mathbf{MS}_{M_2}\dots\mathbf{MS}_{M_k}$, where

$$M = \begin{pmatrix} M_1 & 0 & \cdots & \cdots \\ 0 & M_2 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_k \end{pmatrix}.$$

is a maximal decomposition (k maximal, i.e. the M_i are irreducible).

$$\begin{aligned} \pi_1 : pack(X) &\longrightarrow k\langle Y\rangle \\ x &\longmapsto \begin{cases} y \text{ if } x \neq x_0 \\ y_0 \text{ otherwise} \end{cases} \end{aligned}.$$



WMat is a subalgebra of an extension of **MQSym**

$$\begin{aligned}\pi_2 : pack(X) &\longrightarrow \mathbf{MQSym}\\ w &\longmapsto FB_{M_w}\ ,\end{aligned}$$

where j^{th} column of the finite matrix $M_w$is e_k if $w[j] = x_k$, $k > 0$.

$$e_k = \begin{pmatrix} 0 \\ \dots \\ 1 \\ \dots \\ 0 \end{pmatrix}$$

place k

Ex. :

$$\pi_2(x_2x_1x_0x_3) = FB_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} FB_{(1)} = FB_{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}\ .$$

WMat is a subalgebra of an extension of **MQSym**

Let $\mathcal{A} = k\langle y, y_0 \rangle \otimes \mathbf{MQSym}$.

Lemma

A mapping

$$\begin{aligned} \nu : pack(X) &\longrightarrow \mathcal{A} \\ w &\longmapsto \pi_1(w) \otimes \pi_2(w) \ , \end{aligned} \tag{4}$$

is an injective morphism.

WMat is a subalgebra of an extension of **MQSym**

Let $\mathcal{A} = k\langle y, y_0 \rangle \otimes \mathbf{MQSym}$.

Lemma

A mapping

$$\begin{aligned} \nu : pack(X) &\longrightarrow \mathcal{A} \\ w &\longmapsto \pi_1(w) \otimes \pi_2(w) \ , \end{aligned} \tag{4}$$

is an injective morphism.

$\hookrightarrow$ The algebra WMat can thus be shown to be a subalgebra of $\mathcal{A}$.

WMat is a free algebra

Objective: Check that $pack(X)$ is a free monoid on its irreducibles (see just below).

Definition

A packed word w in $pack(X)$ is an irreducible word iff it can not be written under the form $w = u * v$ where u and v are two non trivial packed words.

Ex. : $x_1 x_1$ is an irreducible word. $x_1 x_2 = x_1 * x_1$ - a reducible word.

WMat is a free algebra

Objective: Check that $pack(X)$ is a free monoid on its irreducibles (see just below).

Definition

A packed word w in $pack(X)$ is an irreducible word iff it can not be written under the form $w = u * v$ where u and v are two non trivial packed words.

Ex. : x_1x_1 is an irreducible word. $x_1x_2 = x_1 * x_1$ - a reducible word.

Lemma

w - a packed word, then w can be written uniquely as $w = v_1 * v_2 * \cdots * v_n$ where v_i, $1 \leq i \leq n$, are non-trivial irreducible words.

WMat is a free algebra

Objective: Check that $pack(X)$ is a free monoid on its irreducibles (see just below).

Definition

A packed word w in $pack(X)$ is an irreducible word iff it can not be written under the form $w = u * v$ where u and v are two non trivial packed words.

Ex. : x_1x_1 is an irreducible word. $x_1x_2 = x_1 * x_1$ - a reducible word.

Lemma

w - a packed word, then w can be written uniquely as $w = v_1 * v_2 * \cdots * v_n$ where v_i, $1 \leq i \leq n$, are non-trivial irreducible words.

$\hookrightarrow$ WMat is a free monoids on its letters.

Bialgebra structure of WMat

Definition

Let $A \subset X$, one defines $w/A = S_{\phi_A}(w)$ with $\phi_A(i) = \begin{cases} i \text{ if } x_i \notin A, \\ 0 \text{ if } x_i \in A \end{cases}$.

Let $w/u = w/Alph(u)$.

Definition

Let $A \subset X$, one defines $w/A = S_{\phi_A}(w)$ with $\phi_A(i) = \begin{cases} i \text{ if } x_i \notin A, \\ 0 \text{ if } x_i \in A \end{cases}$.

Let $w/u = w/Alph(u)$.

Definition

The coproduct:

$$\Delta(w) = \sum_{I+J=[1\ldots|w|]} pack(w[I]) \otimes pack(w[J]/w[I]) \ , \forall w \in \mathcal{H} \ . \qquad (5)$$

Ex. : $\Delta(x_1x_2x_3) = x_1x_2x_3 \otimes 1_{\mathcal{H}} + 3x_1 \otimes x_1x_2 + 3x_1x_2 \otimes x_1 + 1_{\mathcal{H}} \otimes x_1x_2x_3$.

Bialgebra structure of WMat

Definition

Let $A \subset X$, one defines $w/A = S_{\phi_A}(w)$ with $\phi_A(i) = \begin{cases} i \text{ if } x_i \notin A, \\ 0 \text{ if } x_i \in A \end{cases}$.

Let $w/u = w/Alph(u)$.

Definition

The coproduct:

$$\Delta(w) = \sum_{I+J=[1\ldots|w|]} pack(w[I]) \otimes pack(w[J]/w[I]) \ , \forall w \in \mathcal{H} \ . \qquad (5)$$

Ex. : $\Delta(x_1x_2x_3) = x_1x_2x_3 \otimes 1_{\mathcal{H}} + 3x_1 \otimes x_1x_2 + 3x_1x_2 \otimes x_1 + 1_{\mathcal{H}} \otimes x_1x_2x_3$.

$\Delta(x_1x_2x_1) = x_1x_2x_1 \otimes 1_{\mathcal{H}} + x_1 \otimes (x_1x_0 + x_1x_1 + x_0x_1) + x_1x_2 \otimes x_0 + x_1x_1 \otimes x_1$
$+ x_2x_1 \otimes x_0 + 1_{\mathcal{H}} \otimes x_1x_2x_1$.

Bialgebra structure of WMat

Proposition

The coproduct (5) *is coassociative.*
The counit is given by $\epsilon(w) = \delta_{1_{X^*},w}$.
Therefore $(\mathcal{H}, \Delta, \epsilon)$ *is a coassociative coalgebra with counit (co-AAU).*

Bialgebra structure of WMat

Proposition

The coproduct (5) *is coassociative.*
The counit is given by $\epsilon(w) = \delta_{1_{X^*},w}$.
Therefore $(\mathcal{H}, \Delta, \epsilon)$ *is a coassociative coalgebra with counit (co-AAU).*

WMat is not cocommutative. Ex. : $\tau \circ \Delta'(x_1x_1) \neq \Delta'(x_1x_1)$, where $\tau(u \otimes v) = v \otimes u$.

Bialgebra structure of WMat

Proposition

The coproduct (5) *is coassociative.*
The counit is given by $\epsilon(w) = \delta_{1_{X^*},w}$.
Therefore $(\mathcal{H}, \Delta, \epsilon)$ *is a coassociative coalgebra with counit (co-AAU).*

WMat is not cocommutative. Ex. : $\tau \circ \Delta'(x_1 x_1) \neq \Delta'(x_1 x_1)$, where $\tau(u \otimes v) = v \otimes u$.

Theorem (Main result)

$(\mathcal{H}, *, 1_{\mathcal{H}}, \Delta, \epsilon)$ *is a* ($\mathbb{N}$*-graded) bialgebra. And, hence a Hopf algebra.*

Bialgebra structure of WMat

Proposition

The coproduct (5) *is coassociative.*
The counit is given by $\epsilon(w) = \delta_{1_{X^*},w}$.
Therefore $(\mathcal{H}, \Delta, \epsilon)$ *is a coassociative coalgebra with counit (co-AAU).*

WMat is not cocommutative. Ex. : $\tau \circ \Delta'(x_1x_1) \neq \Delta'(x_1x_1)$, where $\tau(u \otimes v) = v \otimes u$.

Theorem (Main result)

$(\mathcal{H}, *, 1_{\mathcal{H}}, \Delta, \epsilon)$ *is a* ($\mathbb{N}$*-graded) bialgebra. And, hence a Hopf algebra.*

The antipode:

$$S(w) = -w - \sum_{I+J=[1\ldots|w|],I,J\neq\emptyset} S(pack(w[I]))pack({}^{w[J]}/_{w[I]}) \ . \qquad (6)$$

Primitive elements of WMat

Lemma

$\mathrm{Prim}(\mathrm{WMat})$ *is a Lie subalgebra of* WMat, *graded by the word's length:*

$$\mathrm{Prim}(\mathrm{WMat})_n = \mathrm{Prim}(\mathrm{WMat}) \cap \mathrm{WMat}_n \ . \tag{7}$$

Primitive elements of WMat

Lemma

$\mathrm{Prim}(\mathrm{WMat})$ *is a Lie subalgebra of* WMat, *graded by the word's length:*

$$\mathrm{Prim}(\mathrm{WMat})_n = \mathrm{Prim}(\mathrm{WMat}) \cap \mathrm{WMat}_n \ . \tag{7}$$

The words x_0 and x_1 are primitive elements. Thus, the element $p_1 = [x_0, x_1] = x_0 x_1 - x_1 x_0$ (ordinary or shifted concatenation in this case) is also a primitive element.

Primitive elements of WMat

Lemma

$\mathrm{Prim}(\mathrm{WMat})$ *is a Lie subalgebra of* WMat, *graded by the word's length:*

$$\mathrm{Prim}(\mathrm{WMat})_n = \mathrm{Prim}(\mathrm{WMat}) \cap \mathrm{WMat}_n \ . \tag{7}$$

The words x_0 and x_1 are primitive elements. Thus, the element $p_1 = [x_0, x_1] = x_0x_1 - x_1x_0$ (ordinary or shifted concatenation in this case) is also a primitive element.
Define $\mathrm{Prim}(\mathrm{WMat})_{n+1}$ by the following recursion: $p_n = [x_0, p_n]_*$, for all $n \geq 1$.

Primitive elements of WMat

Lemma

$\mathrm{Prim}(\mathrm{WMat})$ *is a Lie subalgebra of* WMat*, graded by the word's length:*

$$\mathrm{Prim}(\mathrm{WMat})_n = \mathrm{Prim}(\mathrm{WMat}) \cap \mathrm{WMat}_n \ . \tag{7}$$

The words x_0 and x_1 are primitive elements. Thus, the element $p_1 = [x_0, x_1] = x_0x_1 - x_1x_0$ (ordinary or shifted concatenation in this case) is also a primitive element.
Define $\mathrm{Prim}(\mathrm{WMat})_{n+1}$ by the following recursion: $p_n = [x_0, p_n]_*$, for all $n \geq 1$.

Lemma

$p_n \neq 0$, $\forall n \geq 1$.

Hilbert series of WMat

A word $w = x_{i_1} x_{i_2} \dots x_{i_n}$: length n and supremum k is in one-to-one correspondence with the list $[S_0, S_1, S_2, \dots, S_k]$, where the S_i is the set of positions of x_i in the word w, with $0 \leq i \leq k$.

Hilbert series of WMat

A word $w = x_{i_1}x_{i_2}\dots x_{i_n}$: length n and supremum k is in one-to-one correspondence with the list $[S_0, S_1, S_2, \dots, S_k]$, where the S_i is the set of positions of x_i in the word w, with $0 \leq i \leq k$.
The cardinal of set of packed words with length n, supremum k is given by:

$$d(n,k) = S(n,k)k! + S(n,k+1)(k+1)!\ , \tag{8}$$

where $S(n,k)$ - The Stirling numbers of the second kind.

Hilbert series of WMat

A word $w = x_{i_1}x_{i_2}\ldots x_{i_n}$: length n and supremum k is in one-to-one correspondence with the list $[S_0, S_1, S_2, \ldots, S_k]$, where the S_i is the set of positions of x_i in the word w, with $0 \leq i \leq k$.

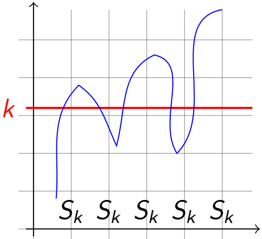
The cardinal of set of packed words with length n, supremum k is given by:

$$d(n,k) = S(n,k)k! + S(n,k+1)(k+1)! \;, \tag{8}$$

where $S(n,k)$ - The Stirling numbers of the second kind.

The cardinal of set of packed words with length n is given by

$$d_n = \sum_{k=0}^{n} d(n,k) = \begin{cases} 1 \text{ if } n = 0, \\ 2\sum_{k=1}^{n} S(n,k)k! \text{ if } n \geq 1 \end{cases} \quad . \tag{9}$$



Conclusion and perspectives

- We have defined here a new non-commutative and non-cocommutative Hopf algebra of type I.

Conclusion and perspectives

- We have defined here a new non-commutative and non-cocommutative Hopf algebra of type I.
- The next step would be to find a polynomial realization of WMat.

Conclusion and perspectives

- We have defined here a new non-commutative and non-cocommutative Hopf algebra of type I.
- The next step would be to find a polynomial realization of WMat.
- This model in easily computed and a help to study the properties of universality of the Tutte polynomial of matroids, using the characteristics of the Schmitt Hopf algebra of matroids.

T. Krajewski, P. Martinetti, *Wilsonian renormalization, differential equations and Hopf algebras* (2007).

Thank you for your attention!