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# COMBINATORIAL ASPECTS OF ELLIPTIC CURVES

GREGG MUSIKER

Abstract. Given an elliptic curve $C$, we study here $N_k = \#C(\mathbb{F}_{q^k})$, the number of points of $C$ over the finite field $\mathbb{F}_{q^k}$. This sequence of numbers, as $k$ runs over positive integers, has numerous remarkable properties of a combinatorial flavor in addition to the usual number theoretical interpretations. In particular, we prove that $N_k = -\mathcal{W}_k(q, -N_1)$, where $\mathcal{W}_k(q,t)$ is a $(q,t)$-analogue of the number of spanning trees of the wheel graph. Additionally we develop a determinantal formula for $N_k$, where the eigenvalues can be explicitly written in terms of $q$, $N_1$, and roots of unity. We also discuss here a new sequence of bivariate polynomials related to the factorization of $N_k$, which we refer to as elliptic cyclotomic polynomials because of their various properties.

## Contents



## 1. Introduction

An interesting problem at the cross-roads between combinatorics, number theory, and algebraic geometry, is that of counting the number of points on an algebraic curve over a finite field. Over a finite field, the locus of solutions of an algebraic equation is a

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discrete subset, but since they satisfy a certain type of algebraic equation this imposes a lot of extra structure beneath the surface. One of the ways to detect this additional structure is by looking at field extensions: the infinite sequence of cardinalities is only dependent on a finite set of data. Specifically the number of points over $\mathbb{F}_q$, $\mathbb{F}_{q^2}$, $\dots$, and $\mathbb{F}_{q^g}$ will be sufficient data to determine the number of points on a genus $g$ algebraic curve over any other algebraic field extension. This observation motivates the question of how the points over higher field extensions correspond to points over the first $g$ extensions.

To see this more clearly, we specialize to the case of elliptic curves, where $g = 1$, and examine the expressions for $N_k$, the number of points on $C$ over $\mathbb{F}_{q^k}$, as functions of $q$ and $N_1$. It follows from the well-known rationality of the zeta function that

$$N_k(q, N_1) = 1 + q^k - \alpha_1^k - \alpha_2^k, \tag{1}$$

where $\alpha_1$ and $\alpha_2$ are the two roots of the quadratic $1-(1+q-N_1)T+qT^2$. Additionally, we observe, see Theorem 1, that

(2) $N_k(q, N_1)$ are integral polynomials whose coefficients alternate in sign.

In this paper, we use formulas arising from (1) and (2) to connect elliptic curves to several different areas of combinatorics. Specifically, (1) implies that the family of polynomials $1 + q^k - N_k$ are Chebyshev polynomials of the first kind, a well-studied example of orthogonal polynomials. In Section 4, we describe this perspective in further detail. Alternatively, we can interpret statement (1) as the plethystic expression $N_k = p_k[1+q-\alpha_1-\alpha_2]$, where the $p_k$'s are the power symmetric functions. In summary, we exploit both the fields of orthogonal polynomials and symmetric functions to illustrate numerous identities involving the $N_k$'s.

Moreover, we find that the polynomial expressions for $N_k$ due to (2) are related to a $(q,t)$-deformation of the Lucas numbers (Theorem 2), and also lead to a combinatorial interpretation involving spanning trees of the wheel graph (Theorem 3). Thus the aforementioned identities also indicate properties of the Lucas numbers and spanning trees as well.

Using these new combinatorial interpretations for $N_k$, we develop further properties of this sequence, obtaining determinantal formulas (Theorem 5), as well as formulas involving a certain bivariate version of the Fibonacci polynomials (Theorem 4). Another surprising by-product of our analysis is a factorization of $N_k$ into a new sequence of polynomials, which we refer to as elliptic cyclotomic polynomials. Both of these families of polynomials are interesting in their own right and have numerous properties which justify their names. We give a geometric interpretation of the elliptic cyclotomic polynomials as Theorem 7 and close with some combinatorial identities involving this new family of expressions.

## 2. $N_k$ AS AN ALTERNATING SUM

The zeta function of a curve $C$ is defined to be the exponential generating function

$$Z(C,T) = \exp\left(\sum_{k\geq 1} N_k \frac{T^k}{k}\right). \tag{3}$$

A result due to Weil [22] is that the zeta function of a curve is rational with specific formula given as

$$Z(C,T) = \frac{(1-\alpha_1 T)(1-\alpha_2 T)\cdots(1-\alpha_{2g}T)}{(1-T)(1-qT)}. \tag{4}$$

Here $g$ is the genus of curve $C$, and the numerator is sometimes written as $L(C,T)$, a degree $2g$ polynomial with integer coefficients. Moreover when $E$ is an elliptic curve, $Z(E,T)$ can be expressed as

$$\frac{1-(\alpha_1+\alpha_2)T+\alpha_1\alpha_2 T^2}{(1-T)(1-qT)}.$$

The zeta function of a curve also satisfies a functional equation which in the elliptic case is simply equivalent to

$$\alpha_1\alpha_2 = q.$$

Among other things, (3) and (4) imply that $N_k = 1+q^k-\alpha_1^k-\alpha_2^k-\cdots-\alpha_{2g}^k$, which can be written in plethystic notation as $p_k[1+q-\alpha_1-\alpha_2]$. We describe symmetric functions and plethystic notation in more depth in Section 3. In the case that $E$ is a curve of genus one and $k=1$ we get

$$\alpha_1+\alpha_2 = 1+q-N_1.$$

Hence we can rewrite the zeta function $Z(E,T)$ totally in terms of $q$ and $N_1$ and as a consequence, all the $N_k$'s are actually dependent on these two quantities. The first few formulas are given below:

$$\begin{aligned}
N_2 &= (2+2q)N_1 - N_1^2,\\
N_3 &= (3+3q+3q^2)N_1 - (3+3q)N_1^2 + N_1^3,\\
N_4 &= (4+4q+4q^2+4q^3)N_1 - (6+8q+6q^2)N_1^2 + (4+4q)N_1^3 - N_1^4,\\
N_5 &= (5+5q+5q^2+5q^3+5q^4)N_1 - (10+15q+15q^2+10q^3)N_1^2\\
&\quad + (10+15q+10q^2)N_1^3 - (5+5q)N_1^4 + N_1^5.
\end{aligned}$$

This data gives rise to the following observation of Adriano Garsia.

**Theorem 1.**

$$N_k = \sum_{i=1}^{k} (-1)^{i-1} P_{i,k}(q) N_1^i,$$

*where the $P_{i,k}$'s are polynomials with* positive integer *coefficients.*

This theorem is proved by Garsia using induction and the fact that the sequence of $N_k$'s satisfy a simple recurrence. For the details, see [7, Chap. 7]. This result motivates the combinatorial question: what are the objects that the family of polynomials, $\{P_{i,k}\}$, enumerate? We answer this question in due course in multiple ways, thus providing an alternate, combinatorial, proof of Theorem 1.

### 2.1. **The Lucas numbers and a $(q,t)$-analogue.**

**Definition 1.** Let $S_1^{(n)}$ be the circular shift of set $S \subseteq \{1,2,\dots,n\}$ modulo $n$, i.e., element $x \in S_1^{(n)}$ if and only if $x - 1 \pmod{n} \in S$. We define the $(q,t)$-**Lucas polynomials** to be the sequence of polynomials in variables $q$ and $t$

$$L_n(q,t) = \sum_{S \subseteq \{1,2,\dots,n\} \,:\, S \cap S_1^{(n)} = \emptyset} q^{\#\text{even elements in } S} \; t^{\lfloor \frac{n}{2} \rfloor - \#S}. \tag{5}$$

Note that this sum is over subsets $S$ with no two numbers circularly consecutive.

These polynomials are a generalization of the sequence of Lucas polynomials $L_n$ which have the initial conditions $L_1 = 1$, $L_2 = 3$ (or $L_0 = 2$ and $L_1 = 1$) and satisfy the Fibonacci recurrence $L_n = L_{n-1} + L_{n-2}$. The first few Lucas numbers are

$$1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \dots$$

As described in numerous sources, e.g. [1], $L_n$ is equal to the number of ways to color an $n$-beaded necklace black and white so that no two black beads are consecutive. You can also think of this as choosing a subset of $\{1,2,\dots,n\}$ with no consecutive elements, nor the pair $1, n$. (We call this circularly consecutive.) Thus letting $q$ and $t$ both equal one, we get by definition that $L_n(1,1,) = L_n$.

We prove the following theorem, which relates our newly defined $(q,t)$-Lucas polynomials to the polynomials of interest, namely the $N_k$'s.

**Theorem 2.** *We have*

$$1 + q^k - N_k = L_{2k}(q, -N_1) \tag{6}$$

*for all $k \geq 1$.*

To prove this result it suffices to prove that both sides are equal for $k \in \{1,2\}$, and that both sides satisfy the same three-term recurrence relation. Since

$$L_2(q,t) = 1 + q + t$$

and

$$L_4(q,t) = 1 + q^2 + (2q+2)t + t^2,$$

we have proven that the initial conditions agree. Note that the sets of (5) yielding the terms of these sums are respectively

$$\{1\},\ \{2\},\ \{\ \} \text{ and } \{1,3\},\ \{2,4\},\ \{1\},\ \{2\},\ \{3\},\ \{4\},\ \{\ \}.$$

It remains to prove that both sides of (6) satisfy the recursion

$$G_{k+1} = (1 + q - N_1)G_k - qG_{k-1}$$

for $k \geq 1$.

**Proposition 1.** *For the $(q,t)$-Lucas polynomials $L_k(q,t)$ defined as above,*

$$L_{2k+2}(q,t) = (1+q+t)L_{2k}(q,t) - qL_{2k-2}(q,t). \tag{7}$$

*Proof.* To prove this we actually define an auxiliary set of polynomials, $\{\tilde{L}_{2k}\}$, such that

$$L_{2k}(q,t) = t^k \tilde{L}_{2k}(q,t^{-1}).$$

Thus recurrence (7) for the $L_{2k}$'s translates into

$$\tilde{L}_{2k+2}(q,t) = (1+t+qt)\tilde{L}_{2k}(q,t) - qt^2\tilde{L}_{2k-2}(q,t) \tag{8}$$

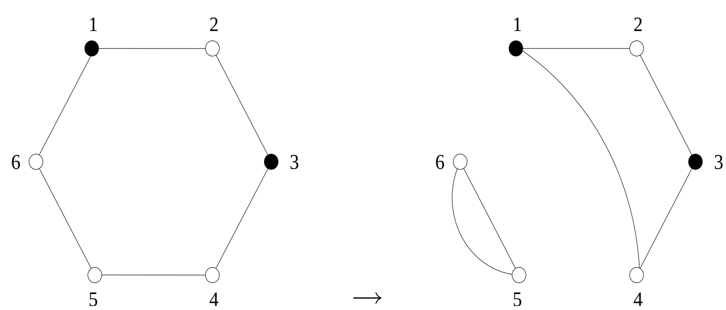
for the $\tilde{L}_{2k}$'s. The $\tilde{L}_{2k}$'s happen to have a nice combinatorial interpretation also, namely

$$\tilde{L}_{2k}(q,t) = \sum_{S\subseteq\{1,2,\dots,2k\}\ :\ S\cap S_1^{(2k)}=\emptyset} q^{\#\text{even elements in } S}\ t^{\#S}.$$

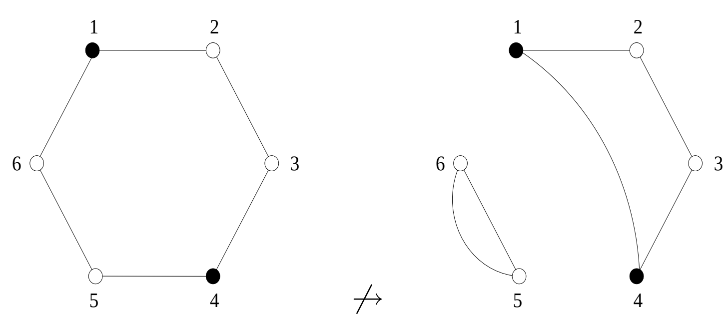
Recall our slightly different description which considers these as the generating function of 2-colored, labeled necklaces. We find this terminology slightly easier to work with. We can think of the beads labeled 1 through $2k+2$ to be constructed from a pair of necklaces; one of length $2k$ with beads labeled 1 through $2k$, and one of length 2 with beads labeled $2k+1$ and $2k+2$.

Almost all possible necklaces of length $2k+2$ can be decomposed in such a way since the coloring requirements of the $2k+2$ necklace are more stringent than those of the pairs. However not all necklaces can be decomposed this way, nor can all pairs be pulled apart and reformed as a $(2k+2)$-necklace. For example, if $k=2$:

Decomposable



Not Decomposable



In these figures, the first necklace is decomposable but the second one is not since black beads 1 and 4 would be adjacent, thus violating the rule. It is clear enough that the number of pairs is $\tilde{L}_2(q,t)\tilde{L}_{2k}(q,t) = (1+t+qt)\tilde{L}_{2k}(q,t)$. To get the third term of the recurrence, i.e., $qt^2\tilde{L}_{2k-2}$, we must define linear analogues, $\tilde{F}_n(q,t)$'s, of the previous generating function. Just as the $\tilde{L}_n(1,1)$'s were Lucas numbers, the $\tilde{F}_n(1,1)$'s are Fibonacci numbers.

**Definition 2.** The (twisted) $(q,t)$-Fibonacci polynomials, denoted as $\tilde{F}_n(q,t)$, are defined as

$$\tilde{F}_k(q,t) = \sum_{S\subseteq\{1,2,\ldots,k-1\}\ :\ S\cap(S_1^{(k-1)}-\{1\})=\emptyset} q^{\#\text{even elements in } S}\ t^{\#S}.$$

The summands here are subsets of $\{1,2,\ldots,k-1\}$ such that no two elements are *linearly* consecutive, i.e., we now allow a subset with both the first and last elements. An alternate description of the objects involved are as (linear) chains of $k-1$ beads which are black or white with no two consecutive black beads. With these new polynomials at our disposal, we can calculate the third term of the recurrence, which is the difference between the number of pairs that cannot be recombined and the number of necklaces that cannot be decomposed.

**Lemma 1.** *The number of pairs that cannot be recombined into a longer necklace is* $2qt^2\tilde{F}_{2k-2}(q,t)$.

*Proof.* We have two cases: either both 1 and $2k+2$ are black, or both $2k$ and $2k+1$ are black. These contribute a factor of $qt^2$, and imply that beads 2, $2k$, and $2k+1$ are white, or that 1, $2k-1$, and $2k+2$ are white, respectively. In either case, we are left counting chains of length $2k-3$, which have no consecutive black beads. In one case we start at an odd-labeled bead and go to an evenly labeled one, and the other case is the reverse, thus summing over all possibilities yields the same generating function in both cases. □

**Lemma 2.** *The number of* $(2k+2)$*-necklaces that cannot be decomposed into a* 2*-necklace and a* $2k$*-necklace is* $qt^2\tilde{F}_{2k-3}(q,t)$.

*Proof.* The only ones that cannot be decomposed are those which have beads 1 and $2k$ both black. Since such a necklace would have no consecutive black beads, this implies that beads 2, $2k-1$, $2k+1$, and $2k+2$ are all white. Thus we are reduced to looking at chains of length $2k-4$, starting at an odd, 3, which have no consecutive black beads. □

**Lemma 3.** *The difference of the quantity referred to in Lemma 2 from the quantity in Lemma 1 is exactly* $qt^2\tilde{L}_{2k-2}(q,t)$.

*Proof.* It suffices to prove the relation

$$qt^2\tilde{L}_{2k-2}(q,t) = 2qt^2\tilde{F}_{2k-2}(q,t) - qt^2\tilde{F}_{2k-3}(q,t),$$

which is equivalent to

$$qt^2\tilde{L}_{2k-2}(q,t) = qt^2\tilde{F}_{2k-2}(q,t) + q^2t^3\tilde{F}_{2k-4}(q,t), \tag{9}$$

since

$$\tilde{F}_{2k-2}(q,t) = qt\tilde{F}_{2k-4}(q,t) + \tilde{F}_{2k-3}(q,t). \tag{10}$$

Note that identity (10) simply comes from the fact that the $(2k-2)$nd bead can be black or white. Finally we prove (9) by dividing by $qt^2$, and then breaking it into the cases where bead 1 is white or black. If bead 1 is white, we remove that bead and cut the necklace accordingly. If bead 1 is black, then beads 2 and $2k+2$ must be white, and we remove all three of the beads. □

With this lemma proven, the recursion for the $\tilde{L}_{2k}$'s, hence the $L_{2k}$'s follows immediately. □

**Proposition 2.** *For an elliptic curve $C$ with $N_k$ points over $\mathbb{F}_{q^k}$ we have that*

$$1 + q^{k+1} - N_{k+1} = (1 + q - N_1)(1 + q^k - N_k) - q(1 + q^{k-1} - N_{k-1}).$$

*Proof.* Recalling that for an elliptic curve $C$ we have the identity

$$N_k = 1 + q^k - \alpha_1^k - \alpha_2^k,$$

we can rewrite the statement of this proposition as

$$\alpha_1^{k+1} + \alpha_2^{k+1} = (\alpha_1 + \alpha_2)(\alpha_1^k + \alpha_2^k) - q(\alpha_1^{k-1} + \alpha_2^{k-1}). \tag{11}$$

Noting that $q = \alpha_1\alpha_2$ we obtain this proposition after expanding out algebraically the right-hand-side of (11). □

With the proof of Propositions 1 and 2, we have proven Theorem 2.

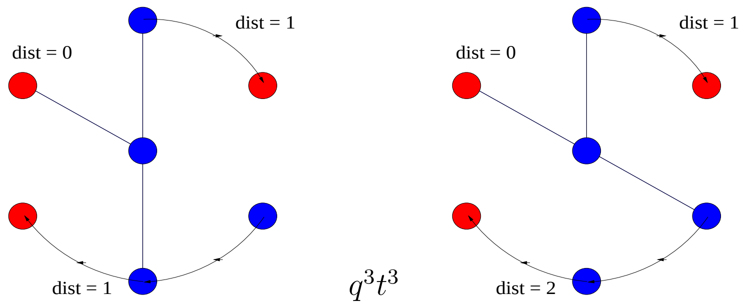
2.2. $(q,t)$**-Wheel polynomials.** Given that the Lucas numbers are related to the polynomial formulas $N_k(q, N_1)$, a natural question concerns how alternative interpretations of the Lucas numbers can help us better understand $N_k$. As noted in [1], [14], and [18, Seq. A004146], the sequence $\{L_{2n} - 2\}$ counts the number of spanning trees in the wheel graph $W_n$; a graph which consists of $n+1$ vertices, $n$ of which lie on a circle and one vertex in the center, a hub, which is connected to all the other vertices.

We note that a spanning tree $T$ of $W_n$ consists of spokes and a collection of disconnected arcs on the rim. Further, since there are no cycles and $T$ is connected, each spoke intersects exactly one arc. (Since it will turn out to be convenient in the subsequent considerations, we make the – somewhat counter-intuitive – convention that an isolated vertex is considered to be an arc of length 1, and more generally, an arc consisting of $k$ *vertices* is considered as an arc of *length* $k$.) We imagine the circle being oriented clockwise, and imagine the tail of each arc being the vertex which is the sink for that arc. In the case of an isolated vertex, the lone vertex is the tail of that arc. Since the spoke intersects each arc exactly once, if an arc has length $k$, meaning that it contains $k$ vertices, there are $k$ choices of where the spoke and the arc meet. We define the $q$-weight of an arc to be $q^{\text{number of edges between the spoke and the tail}}$, abbreviating this exponent as $spoke - tail$ distance. We define the $q$-weight of the tree to be the product of the $q$-weights for all arcs on the rim of the tree. This combinatorial interpretation motivates the following definition.

**Definition 3.**

$$\mathcal{W}_n(q,t) = \sum_{T \text{ a spanning tree of } W_n} q^{\text{sum of spoke-tail distance in } T}\, t^{\#\text{ spokes of } T}.$$

Here the exponent of $t$ counts the number of edges emanating from the central vertex, and the exponent of $q$ is as above.




$q^2t^3$ $q^3t^3$

This definition actually provides exactly the generating function that we desired.

**Theorem 3.**

$$N_k = -\mathcal{W}_k(q, -N_1)$$

*for all* $k \geq 1$.

Notice that this yields an exact interpretation of the $P_{i,k}$ polynomials as follows:

$$P_{i,k}(q) = \sum_{T \text{ a spanning tree of } W_n \text{ with exactly } i \text{ spokes}} q^{\text{sum of spoke}-\text{tail distance in } T}.$$

We prove this theorem in two different ways. The first method utilizes Theorem 2 and an analogue of the bijection given in [1] which relates perfect and imperfect matchings of the circle of length $2k$ and spanning trees of $W_k$. Our second proof uses the observation that we can categorize the spanning trees based on the sizes of the various connected arcs on the rims. Since this categorization corresponds to partitions, this method exploits formulas for decomposing power symmetric function $p_k$ into a linear combination of $h_\lambda$'s, as described in Section 2.4.

2.3. **First proof of Theorem 3: Bijective.** There is a simple bijection between subsets of $\{1, 2, \ldots, 2n\}$ with size at most $n-1$ as well as no two elements circularly consecutive and spanning trees of the wheel graph $W_n$. We use this bijection to give our first proof of Theorem 3. The bijection is as follows:

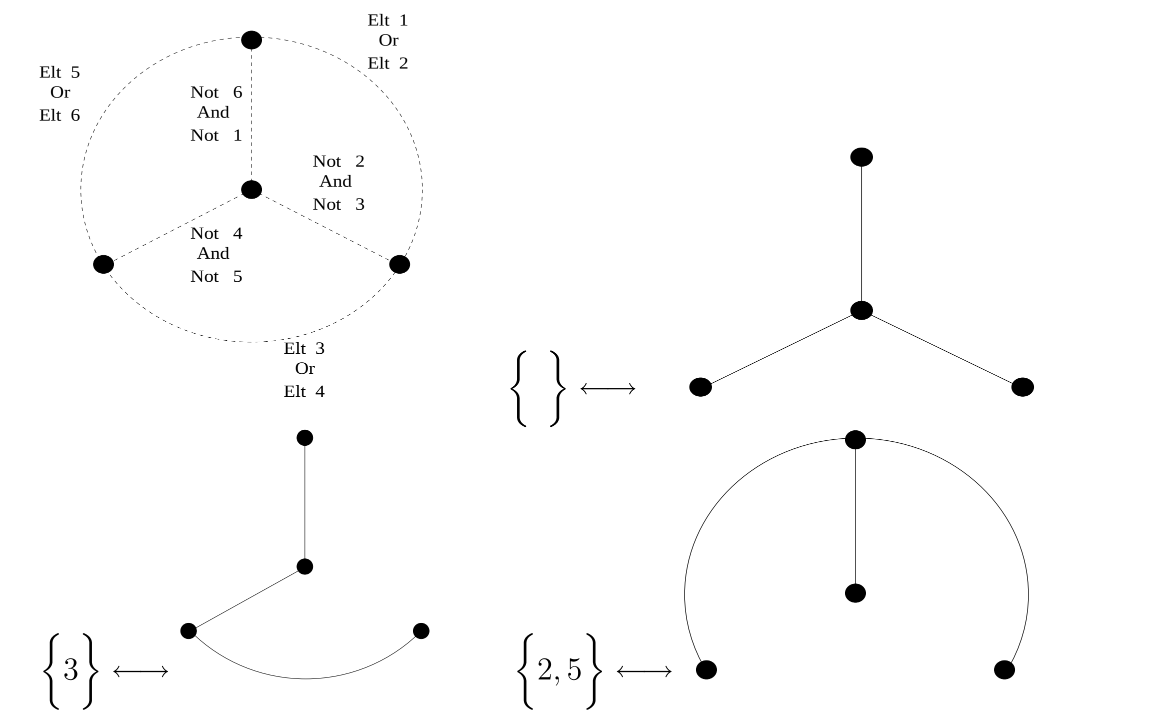
Given a subset $S$ of the set $\{1, 2, \ldots, 2n-1, 2n\}$ with no circularly consecutive elements, we define the corresponding spanning tree $T_S$ of $W_n$ (with the correct $q$ and $t$ weight) in the following way:

1) We use the convention that the vertices of the graph $W_n$ are labeled so that the vertices on the rim are $w_1$ through $w_n$, and the central vertex is $w_0$.

2) We exclude the two subsets which consist of all the odds or all the evens from this bijection. Thus we only look at subsets which contain $n-1$ or fewer elements.

3) For $1 \leq i \leq n$, an edge exists from $w_0$ to $w_i$ if and only if neither $2i-2$ nor $2i-1$ (element 0 is identified with element $2n$) is contained in $S$.

4) For $1 \leq i \leq n$, an edge exists from $w_i$ to $w_{i+1}$ ($w_{n+1}$ is identified with $w_1$) if and only if element $2i-1$ or element $2i$ is contained in $S$.




**Proposition 3.** *Given this construction, $T_S$ is in fact a spanning tree of $W_n$ and further, tree $T_S$ has the same $q$-weights and $t$-weights as set $S$.*

*Proof.* Suppose that set $S$ contains $k$ elements. From our above restriction, we have that $0 \leq k \leq n-1$. Since $S$ is a $k$-subset of a $2n$ element set with no circularly consecutive elements, there are $(n-k)$ pairs $\{2i-2, 2i-1\}$ with neither element in set $S$, and $k$ pairs $\{2i-1, 2i\}$ with one element in set $S$. Consequently, subgraph $T_S$ consists of exactly $(n-k)+k = n$ edges. Since $n = (\#\text{ vertices of } W_n) - 1$, to prove $T_S$ is a spanning tree, it suffices to show that each vertex of $W_n$ is included. For every oddly-labeled element of $\{1, 2, \ldots, 2n\}$, i.e., $2i-1$ for $1 \leq i \leq n$, we have the following rubric:

1) If $(2i-1) \in S$ then the subgraph $T_S$ contains the edge from $w_i$ to $w_{i+1}$.

2) If $(2i-1) \notin S$ and additionally $(2i-2) \notin S$, then $T_S$ contains the spoke from $w_0$ to $w_i$.

3) If $(2i-1) \notin S$ and additionally $(2i-2) \in S$, then $T_S$ contains the edge from $w_{i-1}$ to $w_i$.

Since one of these three cases happens for all $1 \leq i \leq n$, vertex $w_i$ is incident to an edge in $T_S$. Also, the central vertex, $w_0$, has to be included since by our restriction, $0 \leq k \leq n-1$, there are $(n-k) \geq 1$ pairs $\{2i-2, 2i-1\}$ which contain no elements of $S$.

The number of spokes in $T_S$ is $(n-k)$ which agrees with the $t$-weight of a set $S$ with $k$ elements. Finally, we prove that the $q$-weight is preserved, by induction on the number of elements in the set $S$. If set $S$ has no elements, the $q$-weight should be $q^0$, and spanning tree $T_S$ will consist of $n$ spokes which also has $q$-weight $q^0$.

Now given a $k$ element subset $S$ ($0 \leq k \leq n-2$), it is only possible to adjoin an odd number if there is a sequence of three consecutive numbers starting with an even, i.e., $\{2i-2, 2i-1, 2i\}$, which is disjoint from $S$. Such a sequence of $S$ corresponds to a segment of $T_S$ where a spoke and tail of an arc intersect. (Note this includes the case of vertex $w_i$ being an isolated vertex.)

In this case, subset $S' = S \cup \{2i-1\}$ corresponds to $T_{S'}$, which is equivalent to spanning tree $T_S$ except that one of the spokes $w_0$ to $w_i$ has been deleted and replaced with an edge from $w_i$ to $w_{i+1}$. The arc corresponding to the spoke from $w_i$ will now be connected to the next arc, clockwise. Thus the distance between the spoke and the tail of this arc will not have changed, hence the $q$-weight of $T_{S'}$ will be the same as the $q$-weight of $T_S$.

Alternatively, it is only possible to adjoin an even number to $S$ if there is a sequence $\{2i-1, 2i, 2i+1\}$ which is disjoint from $S$. Such a sequence of $S$ corresponds to a segment of $T_S$ where a spoke meets the *end* of an arc. (Note this includes the case of vertex $w_i$ being an isolated vertex.)

Here, subset $S'' = S \cup \{2i\}$ corresponds to $T_{S''}$, which is equivalent to spanning tree $T_S$ except that one of the spokes $w_0$ to $w_{i+1}$ has been deleted and replaced with an edge from $w_i$ to $w_{i+1}$. The arc corresponding to the spoke from $w_{i+1}$ will now be connected to the *previous* arc, clockwise. Thus the cumulative change to the total distance between spokes and the tails of arcs will be an increase of one, hence the $q$-weight of $T_{S''}$ will be $q^1$ times the $q$-weight of $T_S$.

Since any subset $S$ can be built up this way from the empty set, our proof is complete via this induction. □

Since the two sets we excluded, of size $k$ had $(q,t)$-weights $q^0t^0$ and $q^kt^0$ respectively, we have proven Theorem 3.

2.4. **Second proof of Theorem 3: Via generating function identities.** For our second proof of Theorem 3, we consider writing the zeta function as an ordinary generating function instead, i.e.,

$$Z(C,T) = 1 + \sum_{k\geq 1} H_k T^k. \tag{12}$$

In such a form, the $H_k$'s are positive integers which enumerate the number of effective $C(\mathbb{F}_q)$-divisors of degree $k$, as noted in several places, such as [13].

**Proposition 4.**

$$N_k = \sum_{\lambda \vdash k} (-1)^{l(\lambda)-1} \frac{k}{l(\lambda)} \binom{l(\lambda)}{d_1,\ d_2,\ \ldots\ d_m} \prod_{i=1}^{l(\lambda)} H_{\lambda_i}. \tag{13}$$

*Proof.* Comparing formulas (3) and (12) for $Z(C,T)$ and taking logarithms, we obtain

$$\frac{N_k}{k} = \log Z(C,T)\Big|_{T^k} = \log\left(1 + \sum_{n\geq 1} H_n T^n\right)\Big|_{T^k}$$

$$= \sum_{m\geq 1} \frac{(-1)^{m-1}\Big(\sum_{n=1}^{k} H_n T^n\Big)^m}{m}\Bigg|_{T^k}.$$

To obtain the coefficient of $T^k$ in

$$\Big(H_1 T + H_2 T^2 + \cdots + H_k T^k\Big)^m, \tag{14}$$

we first select a partition of $k$ with length $\ell(\lambda) = m$. In other words, $\lambda$ is a vector of positive integers satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$. Each occurrence of $\lambda_i = j$ in this partition corresponds to choosing summand $H_j T^j$ in the $i$th term in product (14). Secondly, since the order of these terms does not matter, we include multinomial coefficients. Finally, multiplying through by $k$ yields formula (13) for $N_k$. □

*Remark* 1. The same manipulations done above for the generating functions are analogous to identities which relate the power symmetric functions and homogeneous symmetric functions. See for example [5], [12], or [20, pg. 21]. This is no coincidence, and in particular the terminology of plethysm provides a rigorous connection between symmetric functions and the enumeration of points on curves. See Section 3 below, [7], or [15] for more details on plethysm and this connection.

*Remark* 2. The above algebraic reasoning can also be translated into a combinatorial description of how points on $C$ over $\mathbb{F}_{q^k}$ can be enumerated using inclusion-exclusion, and points over smaller extension fields. See [15, Chap. 4] for more details.

We now specialize to the case of $g = 1$. Here we can write $H_k$ in terms of $N_1$ and $q$. We expand the series

$$Z(E,T) = \frac{1-(1+q-N_1)T + qT^2}{(1-T)(1-qT)} = 1 + \frac{N_1 T}{(1-T)(1-qT)} \tag{15}$$

with respect to $T$, and obtain $H_0 = 1$ and $H_k = N_1(1+q+q^2+\cdots+q^{k-1})$ for $k \geq 1$. Plugging these into formula (13), we get polynomial formulas for $N_k$ in terms of $q$ and $N_1$

$$N_k = \sum_{\lambda\vdash k} (-1)^{l(\lambda)-1} \frac{k}{l(\lambda)} \binom{l(\lambda)}{d_1,\ d_2,\ \ldots\ d_k} \Bigg(\prod_{i=1}^{l(\lambda)} (1+q+q^2+\cdots+q^{\lambda_i-1})\Bigg) N_1^{l(\lambda)}.$$

Consequently, Theorem 3 is true if and only if we can replace $N_1$ with $-t$ and then multiply by $(-1)$ and get a true expression for $\mathcal{W}_k$, the $(q,t)$-weighted number of spanning trees on the wheel graph $W_k$. We thus provide the following combinatorial argument for the required formula.

**Proposition 5.**

$$\mathcal{W}_k = \sum_{\lambda\vdash k} \frac{k}{l(\lambda)} \binom{l(\lambda)}{d_1,\ d_2,\ \ldots\ d_k} \Bigg(\prod_{i=1}^{l(\lambda)} (1+q+q^2+\cdots+q^{\lambda_i-1})\Bigg) t^{l(\lambda)}. \tag{16}$$

*Proof.* We construct a spanning tree of $W_k$ from the following choices: First we choose a partition $\lambda = 1^{d_1}2^{d_2}\cdots k^{d_m}$ of $k$. We let this dictate how many arcs of each length occur, i.e., we have $d_1$ isolated vertices, $d_2$ arcs of length 2, etc. Note that this choice also dictates the number of spokes, which is equal to the number of arcs, i.e., the length of the partition.

Second, we pick an arrangement of the $l(\lambda)$ arcs on the circle. After picking one arc to start with, without loss of generality since we are on a circle, we have

$$\frac{1}{l(\lambda)}\binom{l(\lambda)}{d_1,\ d_2,\ \dots\ d_m}$$

choices for such an arrangement. Third, we pick which vertex $w_i$ of the rim to start with. There are $k$ such choices. Fourth, we pick where the $l(\lambda)$ spokes actually intersect the arcs. There are $|\mathrm{arc}|$ choices for each arc, and the $q$-weight of this sum is $(1+q+q^2+\cdots+q^{|\mathrm{arc}|})$ for each arc. Summing up all the possibilities yields (16) as desired. □

Thus we have given a second proof of Theorem 3.

## 3. More on bivariate Fibonacci polynomials via duality

In this section we explore further properties of various sequences of coefficients arising from the zeta function of a curve, and also more properties regarding bivariate Fibonacci polynomials. Our tools for such investigations consists of two different manifestations of duality.

### 3.1. Duality between the symmetric functions $h_k$ and $e_k$.

Given the usefulness of symmetric functions in discovering the identities described by Propositions 4 and 5, we now illustrate further applications of the plethystic view of the zeta function.

The symmetric functions that we utilize in this paper are the power symmetric functions $p_k$, the complete homogeneous symmetric functions $h_k$, and the elementary symmetric functions $e_k$. Given the alphabet $\{x_1, x_2, \dots, x_n\}$, each of these can be written as

$$\begin{aligned} p_k &= x_1^k + x_2^k + \cdots + x_n^k, \\ h_k &= \sum_{\substack{0\le i_1,i_2,\dots,i_n\le k \\ i_1+i_2+\cdots+i_n=k}} x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}, \text{ and} \\ e_k &= \sum_{1\le i_1<i_2<\cdots<i_k\le n} x_{i_1}x_{i_2}\cdots x_{i_k}. \end{aligned}$$

In general, a plethystic substitution of a formal power series $F(t_1, t_2, \dots)$ into a symmetric polynomial $A(x)$, denoted as $A[E]$, is obtained by setting

$$A[E] = Q_A(p_1, p_2, \dots)|_{p_k \rightarrow E(t_1^k, t_2^k, \dots)},$$

where $Q_A(p_1, p_2, \dots)$ gives the expansion of $A$ in terms of the power sums basis $\{p_\alpha\}_\alpha$. The main example of this technique that we use is $N_k = p_k[1+q-\alpha_1-\alpha_2-\cdots-\alpha_{2g}]$ for a genus $g$ curve.

To begin, we use the following well-known symmetric function identity

$$\prod_{k\in\mathcal{I}} \frac{1}{1-t_kT} = \exp\left(\sum_{n\geq 1} p_n \frac{T^n}{n}\right)$$
$$= \sum_{n\geq 0} h_n T^n$$
$$= \frac{1}{\sum_{n\geq 0}(-1)^n e_n T^n},$$

where $h_n$, $p_n$, and $e_n$ are symmetric functions in the variables $\{t_k\}_{k\in\mathcal{I}}$. [20, pgs. 21, 296] The zeta function $Z(C,T)$ is equal to all of these for a certain choice of $\{t_k\}_{k\in\mathcal{I}}$ and consequently, we get that

$$Z(C,T) = \frac{1}{\sum_{k\geq 0}(-1)^k E_k \cdot T^k}, \tag{17}$$

where $E_k = e_k[1+q-\alpha_1-\alpha_2-\cdots-\alpha_{2g}]$.

*Remark* 3. Like the $N_k$'s and $H_k$'s, the $E_k$'s also have an algebraic geometric interpretation, namely $E_k$ equals the *signed* number of positive divisors $D$ of degree $k$ on curve $C$ *such that no prime divisor appears more than once in* $D$. This follows from the reciprocity between $h_k$ and $e_k$ which is analogous to the reciprocity between *choose* and *multi-choose*, i.e., choice with replacement.

Recall that in Section 2.1, we defined $\tilde{F}_k(q,t)$, i.e., the twisted $(q,t)$-Fibonacci polynomials. Here we define $F_k(q,t)$, an alternative bivariate analogue of the Fibonacci numbers. The definition of $F_k(q,t)$ is identical to that of $\tilde{F}_k(q,t)$ except for the weighting of parameter $t$.

**Definition 4.** We define the $(q,t)$-**Fibonacci polynomials** to be the sequence of polynomials in variables $q$ and $t$ given by

$$F_k(q,t) = \sum_{S\subseteq\{1,2,\ldots,k-1\}\ :\ S\cap(S_1^{(k-1)}-\{1\})=\emptyset} q^{\#\text{even elements in } S}\ t^{\lceil\frac{k}{2}\rceil-\#S}.$$

From this definition we obtain the following formulas for the $E_k$'s in the elliptic case.

**Theorem 4.** *If $C$ is a genus one curve, and the $E_k$'s are as above, then for $n\geq 1$, $E_{-n}=0$, $E_0=1$, and*

$$E_n = (-1)^n F_{2n-1}(q,-N_1),$$

*where $E_k$ and $F_k(q,t)$ are as defined above.*

The expansions for the first several $E_k$'s, i.e., $F_{2k-1}(q,t)$'s, are given below.

$E_1 = N_1,$

$E_2 = -(1+q)N_1 + N_1^2,$

$E_3 = (1+q+q^2)N_1 - (2+2q)N_1^2 + N_1^3,$

$E_4 = -(1+q+q^2+q^3)N_1 + (3+4q+3q^2)N_1^2 - (3+3q)N_1^3 + N_1^4,$

$$E_5 = (1+q+q^2+q^3+q^4)N_1 - (4+6q+6q^2+4q^3)N_1^2 + (6+9q+6q^2)N_1^3 - (4+4q)N_1^4 + N_1^5.$$

Before proving Theorem 4 we develop two key propositions.

**Proposition 6.** *$F_{2n+1}(q,t) = (1+q+t)F_{2n-1}(q,t) - qF_{2n-3}(q,t)$ for $n \geq 2$.*

*Proof.* This follows the similar logic as the proof of Proposition 1 except we can use a more direct method. (One can use the $t$-weighting of the twisted $(q,t)$-Fibonacci polynomials instead to see this recursion more clearly, but we omit this detour.) The polynomial $F_{2n+1}$ is a $(q,t)$-enumeration of the number of chains of $2n$ beads, with each bead either black or white, and no two consecutive beads both black. Similarly $(1+q+t)F_{2n-1}$ enumerates the concatenation of such a chain of length $2n-2$ with a chain of length 2. One can recover a legal chain of length $2n$ this way except in the case where the $(2n-2)$nd and $(2n-1)$st beads are both black. Such cases are enumerated by $qF_{2n-3}$ and this completes the proof. □

**Proposition 7.** *$(-1)^{n+1}E_{n+1} = (1+q-N_1)(-1)^nE_n - q(-1)^{n-1}E_{n-1}$ for $n \geq 2$.*

*Proof.* One can prove this via plethysm, but it also follows directly from the generating function for the $E_n$'s which is given by

$$\sum_{n\geq 0}(-1)^nE_nT^n = \frac{(1-T)(1-qT)}{1-(1+q-N_1)T+qT^2}.$$

The denominator of this series, also known as the series' characteristic polynomial, yields the desired linear recurrence for the coefficients of $T^{n+1}$, whenever $n+1$ exceeds the degree of the numerator. □

With these two propositions verified, we can also now prove Theorem 4.

*Proof of Theorem 4.* It is clear that $E_1 = -F_1(q,-N_1)$, $E_2 = F_3(q,-N_1)$, and $E_3 = -F_5(q,-N_1)$. Propositions 6 and 7 show that both satisfy the same recurrence relations. Thus we have verified that

$$E_n = (-1)^nF_{2n-1}(q,-N_1).$$

□

*Remark* 4. We can utilize plethysm and obtain results of a similar flavor to Proposition 7, for example see Lemma 4 below. With this result in mind, we obtain the following table of symmetric function $e_k$ and $h_k$ in terms of various alphabets.

| poly. \ alphabet | $1+q-\alpha_1-\alpha_2$ | $1+q$ | $\alpha_1+\alpha_2$ |
|---|---|---|---|
| $e_k$ | $E_k$ | $e_1 = 1+q,\ e_2 = q$ | $e_1 = 1+q-N_1,\ e_2 = q$ |
| $h_k$ | $H_k$ | $1+q+\cdots+q^k$ | $(-1)^kE_{k+1}/N_1$ |

Notice that the formulas for $e_k[1+q]$ and $h_k[1+q]$ are precisely the $N_1 = 0$ cases of $e_k[\alpha_1+\alpha_2]$ and $h_k[\alpha_1+\alpha_2]$. This should come at no surprise since 1 and $q$ are the two roots of $T^2-(1+q)T+q$.

**Lemma 4.** *Letting $E_k$ be defined as $e_k[1+q-\alpha_1-\alpha_2]$, where $\alpha_1$ and $\alpha_2$ are roots of $T^2-(1+q-N_1)T+q$, we obtain*

$$h_k[\alpha_1+\alpha_2] = (-1)^k E_{k+1}/N_1.$$

*Proof.* We have for $n \geq 2$ that

$$N_1 E_n = E_{n+1} + (1+q)E_n + qE_{n-1}$$

since $(-1)^{n+1}E_{n+1} = (1+q-N_1)(-1)^n E_n - q(-1)^{n-1}E_{n-1}$ by Proposition 7. However, by

$$e_k[A-B] = \sum_{i=0}^{k} e_i[A](-1)^{k-i}h_{k-i}[B],$$

we get

$$E_{n+1} = (-1)^{n+1}\Big(h_{n+1}[\alpha_1+\alpha_2] - (1+q)h_n[\alpha_1+\alpha_2] + qh_{n-1}[\alpha_1+\alpha_2]\Big)$$

using $A = 1+q$ and $B = \alpha_1+\alpha_2$. After verifying initial conditions and comparing with

$$(-1)^{n+1}E_{n+1} = (-1)^{n+1}E_{n+2}/N_1 - (-1)^n(1+q)E_{n+1}/N_1 + (-1)^{n-1}qE_n/N_1,$$

we get

$$h_{n+1}[\alpha_1+\alpha_2] = (-1)^{n+1}E_{n+2}/N_1$$

by induction. □

We apply the above $H_k$–$E_k$ (i.e., $h_k$–$e_k$) duality to obtain an exponential generating function for the weighted number of spanning trees of the wheel graph,

$$W(q,N_1,T) = \exp\left(\sum_{k\geq 1}\mathcal{W}_k(q,N_1)\frac{T^k}{k}\right).$$

Using $\mathcal{W}_k = -N_k|_{N_1\to -N_1}$, and the fact this is an exponential, we use (15) to obtain

$$W(q,N_1,T) = \frac{1}{1-\frac{N_1T}{(1-qT)(1-T)}} = \frac{(1-qT)(1-T)}{1-(1+q+N_1)T+qT^2}.$$

Also, rewriting $W(q,t,T)$as an ordinary generating function, we get

$$W(q,t,T) = \sum_{k\geq 0} E_k\Big|_{N_1\to -N_1}(-T)^k = 1 + \sum_{k\geq 1} F_{2k-1}(q,t)T^k.$$

We summarize our results as the following dictionary between elliptic curves and spanning trees accordingly.

| | Elliptic Curves | Spanning Trees |
|---|---|---|
| Generating Function | $\frac{1-(1+q-N_1)T+qT^2}{(1-qT)(1-T)}$ | $\frac{(1-qT)(1-T)}{1-(1+q+N_1)T+qT^2}$ |
| Factors of $1-(1+q\mp N_1)T+qT^2$ | $(1-\alpha_1 T)(1-\alpha_2 T)$ | $(1-\beta_1 T)(1-\beta_2 T)$ |
| $N_k$ (*resp.* $\mathcal{W}_k$ ) | $p_k[1+q-\alpha_1-\alpha_2]$ | $p_k[-1-q+\beta_1+\beta_2]$ |
| $H_k = N_1(1+q+\cdots+q^{k-1})$ | $h_k[1+q-\alpha_1-\alpha_2]$ | $(-1)^{k-1}e_k[-1-q+\beta_1+\beta_2]$ |
| $(-1)^k E_k = F_{2k-1}(q,\mp N_1)$ | $(-1)^k e_k[1+q-\alpha_1-\alpha_2]$ | $h_k[-1-q+\beta_1+\beta_2]$ |

3.2. **Duality between Lucas and Fibonacci numbers.** In addition to the above discussion of how $H_k$ and $E_k$ are dual, this dictionary also highlights a comparison between *elliptic curve–spanning tree* duality and duality between Lucas numbers and Fibonacci numbers. As an application, we obtain a formula for $E_k$, i.e., $F_{2k-1}(q,t)$, in terms of the polynomial expansion for the $L_{2k}(q,t)$'s. If we recall our definition of $P_{i,k}$'s such that $N_k = \sum_{i=1}^k (-1)^{i+1} P_{i,k}(q) N_1^i$, or equivalently $L_{2k}(q,t) = 1 + q^k + \sum_{i=1}^k P_{i,k}(q)t^i$, then we have the following identity.

**Proposition 8.** *We have*

$$E_k = \sum_{i=1}^{k} \frac{(-1)^{k+i} \cdot i}{k} P_{i,k}(q) N_1^i.$$

*Proof.* We use the identities as above, and the fact that $\frac{1}{Z(E,T)} = \sum_{n\geq 0} (-1)^n E_n T^n$. Thus we have

$$\begin{aligned}
\sum_{n\geq 1} (-1)^n E_n T^n &= \frac{1}{Z(E,T)} - 1 = \frac{1}{1 + \frac{N_1 T}{(1-qT)(1-T)}} - 1 \\
&= \sum_{n\geq 1} (-1)^n \left( \frac{N_1 T}{(1-qT)(1-T)} \right)^n \\
&= -N_1 \frac{\partial}{\partial N_1} \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \left( \frac{N_1 T}{(1-qT)(1-T)} \right)^n \\
&= -N_1 \frac{\partial}{\partial N_1} \left( \log \left( 1 + \frac{N_1 T}{(1-qT)(1-T)} \right) \right) \\
&= -N_1 \frac{\partial}{\partial N_1} \log \Big( Z(E,T) \Big),
\end{aligned}$$

which equals $-N_1 \frac{\partial}{\partial N_1}\left( \sum_{k\geq 1} \frac{N_k}{k} T^k \right)$. Rewriting the $N_k$'s using the polynomial formulas of Theorem 1, we have

$$\begin{aligned}\sum_{n\geq 1}(-1)^n E_n T^n &= -N_1 \frac{\partial}{\partial N_1}\left( \sum_{k\geq 1} \frac{1}{k} \sum_{i=1}^{k} (-1)^{i-1} P_{i,k}(q) N_1^i T^k \right) \\ &= \sum_{k\geq 1}\sum_{i=1}^{k} \frac{i}{k} (-1)^i P_{i,k}(q) N_1^i T^k.\end{aligned}$$

Comparing the coefficients of $T^k$ on both sides completes the proof. □

Proposition 8 can also be given a combinatorial proof by the following lemma which contrasts the circular nature of our combinatorial interpretation for the Lucas numbers with the linear nature of the Fibonacci numbers.

**Lemma 5.** *For $1 \leq i \leq k$ and $0 \leq j \leq i$, we have the number, which we denote as $c_{i,j}$, of subsets $S_1$ of $\{1, 2, \ldots, 2k\}$ with $k-i-j$ odd elements, $j$ even elements, and no two elements circularly consecutive equals*

$$\frac{k}{i} \cdot \#\Big(\text{subsets } S_2 \text{ of } \{1, 2, \ldots, 2k-2\} \text{ with } k-i-j \text{ odd elements, } j \text{ even elements, and no two elements consecutive}\Big).$$

This notation might seem non-intuitive, but we use these indices so that the total number of elements is $k-i$ and the number of even elements is $j$. Thus the number of subsets $S_1$ (respectively $S_2$) directly describes the coefficient of $q^j t^i$ in $L_{2k}(q,t)$ (respectively $F_{2k-1}(q,t)$).

*Proof.* To prove this result we note that there is a bijection between the number of subsets of the first kind that do not contain $2k-1$ or $2k$ and those of the second kind. Thus it suffices to show that the number of sets $S_1$ which *do* contain element $2k-1$ or $2k$ is precisely fraction $\frac{k-i}{k}$ of all sets $S_1$ satisfying the above hypotheses.

Circularly shifting every element of set $S_1$ by an even amount $r$, i.e., $\ell \mapsto \ell + r - 1 \pmod{2k} + 1$, does not affect the number of odd elements and even elements. Furthermore, out of the $k$ possible even shifts, $(k-i)$ of the sets, i.e., the cardinality of set $S_1$, will contain $2k-1$ or $2k$. This follows since for a given element $\ell$ there is exactly one shift which makes it $2k-1$ (or $2k$) if $\ell$ is odd (or even), respectively. Since elements cannot be consecutive, there is no shift that sends two different elements to both $2k-1$ and $2k$ simultaneously and thus we get the full $(k-i)$ possible shifts. □

Using this relationship, we can derive formulas involving binomial coefficients for $P_{i,k}(q)$ using our combinatorial interpretation for the $(q,t)$-Lucas polynomials and $(q,t)$-Fibonacci polynomials.

**Proposition 9.** *For $k \geq 1$ and $1 \leq i \leq k$, we have*

$$P_{i,k}(q) = \sum_{j=0}^{i} \frac{k}{i} \binom{k-1-j}{i-1} \binom{i+j-1}{j} \ q^j.$$

*Proof.* See [23, Theorem 2.2] or [16, Theorem 3] which show by algebraic and combinatorial arguments, respectively, that the number of ways to choose a subset $S \subset \{1, 2, \ldots, 2n\}$ such that $S$ contains $q$ odd elements, $r$ even elements, and no consecutive elements is

$$\binom{n-r}{q}\binom{n-q}{r}.$$

Letting $n = k - 1$, $q = k - i - j$ and $r = j$, we obtain

$$\frac{i}{k} P_{i,k}(q) = F_{2k-1}(q, N_1)\Big|_{N_1^i} = \sum_{j=0}^{i} \binom{k-1-j}{i-1} \binom{i+j-1}{j} q^j.$$

□

**Corollary 1.** *We have*

$$N_k(q, N_1) = \sum_{i=1}^{k} \sum_{j=0}^{i} \frac{(-1)^{i+1} \cdot k}{i} \binom{k-1-j}{i-1} \binom{i+j-1}{j} N_1^i \ q^j.$$

*and*

$$E_k = \sum_{i=1}^{k} \sum_{j=0}^{i} (-1)^{k+i} \ \binom{k-1-j}{i-1} \binom{i+j-1}{j} N_1^i \ q^j.$$

*Remark* 5. From the proof in Section 2.4, we have that

$$\mathcal{W}_k(q, N_1) = \sum_{\lambda \vdash k} \frac{k}{l(\lambda)} \binom{l(\lambda)}{d_1, \ d_2, \ \ldots \ d_r} \left( \prod_{i=1}^{l(\lambda)} (1 + q + q^2 + \cdots + q^{\lambda_i - 1}) \right) N_1^{l(\lambda)}$$
$$= \sum_{i=1}^{k} \frac{k}{i} \left( \sum_{\substack{\lambda \vdash k \\ l(\lambda) = i}} \binom{i}{d_1, \ d_2, \ \ldots \ d_r} \prod_{j=1}^{i} (1 + q + q^2 + \cdots + q^{\lambda_j - 1}) \right) N_1^i$$

which implies also that

$$P_{i,k}(q) = \frac{k}{i} \sum_{\substack{\lambda \vdash k \\ l(\lambda) = i}} \binom{i}{d_1, \ d_2, \ \ldots \ d_r} \prod_{j=1}^{i} (1 + q + q^2 + \cdots + q^{\lambda_j - 1}).$$

Comparing the coefficients of this identity with the coefficients in Proposition 9 seems to give a combinatorial identity that seems interesting in its own right.

## 4. Factorizations of $N_k$

We now introduce a family of $k$-by-$k$ matrices $M_k$ which, for elliptic curves, yield a determinantal formula for $N_k$ in terms of $q$ and $N_1$.

**Theorem 5.** *Let* $M_1 = [-N_1]$, $M_2 = \begin{bmatrix} 1+q-N_1 & -1-q \\ -1-q & 1+q-N_1 \end{bmatrix}$, *and for* $k \geq 3$, *let* $M_k$ *be the* $k$*-by-*$k$ *"three-line" circulant matrix*

$$\begin{bmatrix} 1+q-N_1 & -1 & 0 & \dots & 0 & -q \\ -q & 1+q-N_1 & -1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & -q & 1+q-N_1 & -1 & 0 \\ 0 & \dots & 0 & -q & 1+q-N_1 & -1 \\ -1 & 0 & \dots & 0 & -q & 1+q-N_1 \end{bmatrix}.$$

*The sequence of integers* $N_k = \#C(\mathbb{F}_{q^k})$ *satisfies the relation*

$$N_k = -\det M_k \text{ for all } k \geq 1.$$

We provide two proofs of this theorem, one which utilizes the three term recurrence from Section 2.1, and one which introduces a new sequence of polynomials which are interesting in their own right.

### 4.1. Connection to orthogonal polynomials.

Recall from the zeta function of an elliptic curve, $Z(E, T)$, we derived a three term recurrence relation for the sequence $\{G_k = 1 + q^k - N_k\}$:

$$G_{k+1} = (1 + q - N_1)G_k - qG_{k-1}. \tag{18}$$

Such a relation is indicative of an interpretation of the $(1 + q^k - N_k)$'s as a sequence of orthogonal polynomials. In particular, any sequence of orthogonal polynomials, $\{P_k(x)\}$, satisfies

$$P_{k+1}(x) = (a_k x + b_k)P_k(x) + c_k P_{k-1}(x), \tag{19}$$

where $a_k$, $b_k$ and $c_k$ are constants that depend on $k \in \mathbb{N}$. Additionally, it is customary to initialize $P_{-k}(x) = 0$, $P_0(x) = 1$, and $P_1(x) = a_0 x + b_0$.

Since we can think of the bivariate $N_k(q, N_1)$ as univariate polynomials in variable $N_1$ with constants from field $\mathbb{Q}(q)$, it follows that recurrence (18) is a special case of recurrence (19), therefore $\{P_k(x)\}_{k=1}^{\infty} = \{(1 + q^k - N_k)(N_1)\}_{k=1}^{\infty}$ are a family of orthogonal polynomials. In particular, we plug in the following values for the $a_k$, $b_k$, and $c_k$'s:

$$\begin{aligned} a_k &= -1 && \text{for } k \geq 0 \\ b_k &= 1 + q && \text{for } k \geq 0, \\ c_1 &= -2q && \text{and} \\ c_k &= -q && \text{for } k \geq 2. \end{aligned}$$

(Note that we take $c_1$ to be $-2q$ since $G_0 = 1 + q^0 - N_0 = 2$, but we wish to normalize so that $P_0(x) = 1$.)

In fact, the family $\{1+q^k-N_k\}_{k=1}^{\infty}$ can be described in terms of a classical sequence of orthogonal polynomials. Namely $T_k(x)$ denotes the $k$th Chebyshev (Tchebyshev) polynomials of the first kind, which are defined as $\cos(k\theta)$ written out in terms of $x$ such that $\theta = \arccos x$. Equivalently, we can define $T_k(x)$ as the expansion of $\alpha^k + \beta^k$ in terms of powers of $\cos\theta$, where

$$\alpha = \cos\theta + i\sin\theta$$
$$\beta = \cos\theta - i\sin\theta.$$

**Theorem 6.** *Considering the $(1+q^k-N_k)$'s as univariate polynomials in $N_1$ over the field $\mathbb{Q}(q)$, we obtain*

$$1 + q^k - N_k = 2q^{k/2} T_k\bigg((1+q-N_1)/2q^{1/2}\bigg).$$

*Proof.* We note that Chebyshev polynomials satisfy initial conditions $T_0(x) = 1$, and $T_1(x) = x$ and the three-term recurrence

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)$$

for $k \geq 1$ since

$$\begin{aligned} T_{k+1}(x) &= \alpha^{k+1} + \beta^{k+1} \\ &= (\alpha+\beta)(\alpha^k+\beta^k) - \alpha\beta(\alpha^{k-1}+\beta^{k-1}) \\ &= 2\cos\theta\ T_k(x) - T_{k-1}(x) \\ &= 2xT_k(x) - T_{k-1}(x). \end{aligned}$$

Let $x = \frac{1+q-N_1}{2}\sqrt{q}$. Clearly Theorem 6 holds for $k = 1$, and additionally, by Proposition 1, the $\frac{1+q^k-N_k}{2q^{k/2}}$'s satisfy the same recurrence as the $T_k(x)$'s. Namely

$$\begin{aligned} \frac{1+q^{k+1}-N_{k+1}}{2q^{(k+1)/2}} &= \frac{(1+q-N_1)(1+q^k-N_k) - q(1+q^{k-1}-N_{k-1})}{2q^{(k+1)/2}} \\ &= 2\bigg(\frac{1+q-N_1}{2q^{1/2}}\bigg)\bigg(\frac{1+q^k-N_k}{2q^{k/2}}\bigg) - \bigg(\frac{1+q^{k-1}-N_{k-1}}{2q^{(k-1)/2}}\bigg). \end{aligned}$$

□

Another way to foresee the appearance of Chebyshev polynomials is by noting that in the case that we plug in $q = 0$ or $q = 1$, we obtain a family of univariate polynomials $\tilde{N}_k$ with the property $\tilde{N}_{mk} = \tilde{N}_m(\tilde{N}_k) = \tilde{N}_k(\tilde{N}_m)$. It is a fundamental theorem of Chebyshev polynomials that families of univariate polynomials with such a property are very restrictive. In particular, from [2] as described on page 33 of [4]: If $\{\tilde{N}_k\}$ is a sequence of integral univariate polynomials of degree $k$ with the property

$$\tilde{N}_{mn} = \tilde{N}_m(\tilde{N}_n) = \tilde{N}_n(\tilde{N}_m)$$

for all positive integers $m$ and $n$, then $\tilde{N}_k$ must either be a linear transformation of

(1) $x^k$ or
(2) $T_k(x)$, the Chebyshev polynomial of the first kind,

where a linear transformation of a polynomial $f(x)$ is of the form

$$A \cdot f\Big((x-B)/A\Big) + B \text{ or equivalently } \Big(f(\overline{A}x+\overline{B})-\overline{B}\Big)\Big/\overline{A}.$$

In particular, we get formulas for $\mathcal{W}_k(0,N_1)$ and $\mathcal{W}_k(1,N_1)$ (respectively $N_k(0,N_1)$ and $N_k(1,N_1)$) which are indeed linear transformations of $x^k$ and $T_k(x)$ respectively.

**Proposition 10.** *We have*

$$N_k(0,N_1) = -(1-N_1)^k + 1, \tag{20}$$

$$N_k(1,N_1) = -2T_k(-N_1/2+1) + 2. \tag{21}$$

*Proof.* The coefficient of $N_1^m$ in $\mathcal{W}_k(0,N_1)$ is the number of directed rooted spanning trees of $W_k$ with $m$ spokes and arcs always directed counter-clockwise. In particular, it is only the placement of the spokes that matter at this point since the placement of the arcs is now forced. Thus the coefficient of $N_1^m$ in $\mathcal{W}_k(0,N_1)$ is $\binom{k}{m}$ for all $1 \leq m \leq k$. Thus the generating function $\mathcal{W}_k(0,N_1)$ satisfies

$$\mathcal{W}_k(0,N_1) = (1+N_1)^k - 1$$

since the constant term of $\mathcal{W}_k(0,N_1)$ is zero. Use of the relation $N_k(q,N_1) = -\mathcal{W}_k(q,-N_1)$ completes the proof in the $q=0$ case. We also note that $-(1-x)^k+1$ is a linear transformation of $x^k$ via $A=-1$ and $B=1$. The case for $q=1$ is a corollary of Theorem 6. ☐

4.2. **First proof of Theorem 5: Using orthogonal polynomials.** As an application of Theorem 6, we use the theory of orthogonal polynomials to learn properties of the $(1+q^k-N_k)$'s. For example, one of the properties of a sequence of orthogonal polynomials is an interpretation as the determinants of a family of tridiagonal $k$-by$k$ matrices.

**Proposition 11.** *We have*

$$1+q^k-N_k = \det \begin{bmatrix} 1+q-N_1 & -2q & 0 & 0 & 0 & 0 \\ -1 & 1+q-N_1 & -q & 0 & 0 & 0 \\ 0 & -1 & 1+q-N_1 & -q & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1+q-N_1 & -q \\ 0 & 0 & 0 & \cdots & -1 & 1+q-N_1 \end{bmatrix}.$$

*We denote this matrix as $M_k'$.*

*Proof.* Given a sequence of orthogonal polynomials satisfying $P_0(x) = 1$, $P_1(x) = a_0x + b_0$ and recurrence (19), we have the formula [10]

$$P_k(x) = \det \begin{bmatrix} a_0x+b_0 & c_1 & 0 & 0 & 0 & 0 \\ -1 & a_1x+b_1 & c_2 & 0 & 0 & 0 \\ 0 & -1 & a_2x+b_2 & c_3 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & a_{k-2}x+b_{k-2} & c_k \\ 0 & 0 & 0 & \cdots & -1 & a_{k-1}x+b_{k-2} \end{bmatrix}.$$

Plugging in the $a_i$, $b_i$, and $c_i$'s as in Section 4.1 yields the formula. □

*Remark* 6. Alternatively, we can use symmetric functions and the Newton Identities [20] to obtain these determinant identities, as described in [7, Chap. 7] or [15, Chap. 5].

We can prove Theorem 5 via Proposition 11 followed by an algebraic manipulation of matrix $M_k$. Namely, by using the multilinearity of the determinant, and expansions about the first row followed by the first column, we obtain

$$\det(M_k) = \det(A_k) + \det(B_k) + \det(C_k) + \det(D_k),$$

where $A_k$, $B_k$, $C_k$, and $D_k$ are the following $k$-by-$k$ matrices:

$$A_k = \begin{bmatrix} 1+q-N_1 & -1 & 0 & 0 & 0 & 0 \\ -q & 1+q-N_1 & -1 & 0 & 0 & 0 \\ 0 & -q & 1+q-N_1 & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1+q-N_1 & -1 \\ 0 & 0 & 0 & \cdots & -q & 1+q-N_1 \end{bmatrix},$$

$$B_k = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -q \\ -q & 1+q-N_1 & -1 & 0 & 0 & 0 \\ 0 & -q & 1+q-N_1 & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1+q-N_1 & -1 \\ 0 & 0 & 0 & \cdots & -q & 1+q-N_1 \end{bmatrix},$$

$$C_k = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1+q-N_1 & -1 & 0 & 0 & 0 \\ 0 & -q & 1+q-N_1 & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1+q-N_1 & -1 \\ -1 & 0 & 0 & \cdots & -q & 1+q-N_1 \end{bmatrix},$$

$$D_k = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -q \\ 0 & 1+q-N_1 & -1 & 0 & 0 & 0 \\ 0 & -q & 1+q-N_1 & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1+q-N_1 & -1 \\ -1 & 0 & 0 & \cdots & -q & 1+q-N_1 \end{bmatrix}.$$

Cyclic permutation of the rows of $B_k$ and the columns of $C_k$ yield upper-triangular matrices with $-1$'s (respectively $-q$)'s on the diagonal. Given that the sign of such a cyclic permutation is $(-1)^{k-1}$, we obtain $\det(B_k)+\det(C_k) = -q^k - 1$. Additionally, by expanding $\det(D_k)$ about the first row followed by the first column, we obtain $\det(D_k) = -q\det(A_{k-2})$. In conclusion

$$1 + q^k + \det(M_k) = \det(A_k) - q\det(A_{k-2}).$$

After transposing $M'_k$, by analogous methods we obtain

$$\det M'_k = \det(A_k) - q\det(A_{k-2})$$

and thus the desired formula $\det M_k = -N_k$.

4.3. **Second proof of Theorem 5: Using the zeta function.** Alternatively, we note that we can factor

$$N_k = 1 + q^k - \alpha_1^k - \alpha_2^k$$

using the fact that $q = \alpha_1\alpha_2$. Consequently,

$$N_k = (1-\alpha_1^k)(1-\alpha_2^k)$$

and we can factor each of these two terms using cyclotomic polynomials. We recall that $(1-x^k)$ factors as

$$1 - x^k = \prod_{d|k} Cyc_d(x),$$

where $Cyc_d(x)$ is a monic irreducible polynomial with integer coefficients. We can similarly factor $N_k$ as

$$N_k = \prod_{d|k} Cyc_d(\alpha_1)Cyc_d(\alpha_2).$$

These factors are therefore bivariate analogues of the cyclotomic polynomials, and we refer to them henceforth as **elliptic cyclotomic polynomials**, denoted as $ECyc_d$.

**Definition 5.** We define the elliptic cyclotomic polynomials to be a sequence of polynomials in variables $q$ and $N_1$ such that for $d \geq 1$,

$$ECyc_d = Cyc_d(\alpha_1)Cyc_d(\alpha_2),$$

where $\alpha_1$ and $\alpha_2$ are the two roots of

$$T^2 - (1+q-N_1)T + q.$$

We verify that they can be expressed in terms of $q$ and $N_1$ by the following proposition.

**Proposition 12.** *Writing down $ECyc_d$ in terms of $q$ and $N_1$ yields irreducible bivariate polynomials with integer coefficients.*

*Proof.* Firstly we have

$$\alpha_1^j + \alpha_2^j = (1 + q^j - N_j) \in \mathbb{Z}$$

for all $j \geq 1$ and expanding a polynomial in $\alpha_1$ multiplied by the same polynomial in $\alpha_2$ yields terms of the form $\alpha_1^i\alpha_2^i(\alpha_1^j + \alpha_2^j)$. Secondly the quantity $N_j$ is an integral polynomial in terms of $q$ and $N_1$ by Theorem 1 and $\alpha_1^i\alpha_2^i = q^i$. Putting these relations together, and the fact that $Cyc_d$ is an integral polynomial itself, we obtain the desired expressions for $ECyc_d$.

Now let us assume that $ECyc_d$ is factored as $F(q,N_1)G(q,N_1)$. The polynomial $Cyc_d(x)$ factors over the complex numbers as

$$Cyc_d(x) = \prod_{\substack{j=1\\ \gcd(j,d)=1}}^{d} (1-\omega^j x),$$

where $\omega$ is a $d$th root of unity. Thus $F(q,N_1) = \prod_{i\in S}(1-\omega^i\alpha_1)\prod_{j\in T}(1-\omega^j\alpha_2)$ for some nonempty subsets $S$, $T$ of elements relatively prime to $d$. The only way $F$ can be integral is if $F$ equals its complex conjugate $\overline{F}$. However, $\alpha_1$ and $\alpha_2$ are complex conjugates by the Riemann hypothesis for elliptic curves [9, 17] (Hasse's Theorem), and thus $F = \overline{F}$ implies that the sets $S$ and $T$ are equal. Since $Cyc_d(x)$ is known to be irreducible, the only possibility is $S = T = \{j : \gcd(j,d) = 1\}$, and thus $F(q,N_1) = ECyc_d$, $G(q,N_1) = 1$. □

*Remark* 7. Alternatively, the integrality of the $ECyc_d$'s also follows from the Fundamental Theorem of Symmetric Functions that states that a symmetric polynomial with integer coefficients can be rewritten as an integral polynomial in $e_1, e_2, \dots$. In this case, $Cyc_d(\alpha_1)Cyc_d(\alpha_2)$ is a symmetric polynomial in two variables so $e_1 = \alpha_1+\alpha_2 = 1+q-N_1$, $e_2 = \alpha_1\alpha_2 = q$, and $e_k = 0$ for all $k \geq 3$. Thus we obtain an expression for $ECyc_d$ as a polynomial in $q$ and $N_1$ with integer coefficients.

We can factor $N_k$, i.e., the $ECyc_d$'s even further, if we no longer require our expressions to be integral.

$$\begin{aligned} N_k &= \prod_{j=1}^{k}(1-\alpha_1\omega_k^j)(1-\alpha_2\omega_k^j) \\ &= \prod_{j=1}^{k}(1-(\alpha_1+\alpha_2)\omega_k^j + (\alpha_1\alpha_2)\omega_k^{2j}) \\ &= (-1)\prod_{j=1}^{k}(-\omega_k^{k-j})(1-(1+q-N_1)\omega_k^j + (q)\omega_k^{2j}) \\ &= -\prod_{j=1}^{k}\Big((1+q-N_1) - q\omega_k^j - \omega_k^{k-j}\Big). \end{aligned}$$

Furthermore, the eigenvalues of a circulant matrix are well-known, and involve roots of unity analogous to the expression precisely given by the second equation above. (For example Loehr, Warrington, and Wilf [11] provide an analysis of a more general family of three-line-circulant matrices from a combinatorial perspective. Using their notation, our result can be stated as

$$N_k = \Phi_{k,2}(1+q-N_1, -q),$$

where $\Phi_{p,q}(x,y) = \prod_{j=1}^{p}(1 - x\omega^j - y\omega^{qj})$ and $\omega$ is a primitive $p$th root of unity. It is unclear how our combinatorial interpretation of $N_k$, in terms of spanning trees, relates to theirs, which involves permutation enumeration.) In particular, we prove Theorem 5 since $\det M_k$ equals the product of $M_k$'s eigenvalues, which are precisely given as the $k$ factors of $-N_k$ in second equation above.

4.4. **Combinatorics of elliptic cyclotomic polynomials.** In this subsection we further explore properties of elliptic cyclotomic polynomials, noting that they are more than auxiliary expressions that appear in the derivation of a proof. To start with, by Möbius inversion, we can use the identity

$$N_k = \prod_{d|k} ECyc_d(q, N_1) \tag{22}$$

to define elliptic cyclotomic polynomials directly as

$$ECyc_k(q, N_1) = \prod_{d|k} {N_d}^{\mu(k/d)} \tag{23}$$

in addition to the alternative definition

$$ECyc_k(q, N_1) = \prod_{\substack{j=1 \\ \gcd(j,d)=1}}^{k} \left((1 + q - N_1) - q\omega_k^j - \omega_k^{k-j}\right). \tag{24}$$

In particular, $ECyc_1 = N_1$ and $ECyc_p = N_p/N_1$ if $p$ is prime. To get a handle on $ECyc_k$ for $k$ composite, we provide the following table for small values of $k$:

$$\begin{aligned}
ECyc_4 &= N_1^2 - (2 + 2q)N_1 + 2(1 + q^2)\\
ECyc_6 &= N_1^2 - (1 + q)N_1 + (1 - q + q^2)\\
ECyc_8 &= N_1^4 - (4 + 4q)N_1^3 + (6 + 8q + 6q^2)N_1^2 - (4 + 4q + 4q^2 + 4q^3)N_1 + 2(1 + q^4)\\
ECyc_9 &= N_1^6 - (6 + 6q)N_1^5 + (15 + 24q + 15q^2)N_1^4 - (21 + 36q + 36q^2 + 21q^3)N_1^3\\
&\quad + (18 + 27q + 27q^2 + 27q^3 + 18q^4)N_1^2\\
&\quad - (9 + 9q + 9q^2 + 9q^3 + 9q^4 + 9q^5)N_1 + 3(1 + q^3 + q^6)\\
ECyc_{10} &= N_1^4 - (3 + 3q)N_1^3 + (4 + 3q + 4q^2)N_1^2\\
&\quad - (2 + q + q^2 + 2q^3)N_1 + (1 - q + q^2 - q^3 + q^4)\\
ECyc_{12} &= N_1^4 - (4 + 4q)N_1^3 + (5 + 8q + 5q^2)N_1^2\\
&\quad - (2 + 2q + 2q^2 + 2q^3)N_1 + (1 - q^2 + q^4)
\end{aligned}$$

We note several commonalities among these polynomials, as described in the following propositions. These properties are further rationale for our choice of name for this family of polynomials.

**Proposition 13.** *We have*

$$ECyc_d|_{N_1=0} = C(d)Cyc_d(q) \tag{25}$$

$$ECyc_d|_{N_1=2q+2} = C'(d)Cyc_d(-q), \tag{26}$$

*where $C(d)$ and $C'(d)$ are the functions from $\mathbb{Z}_{>0}$ to $\mathbb{Z}_{\geq 0}$ such that*

$$C(d) = \begin{cases} 0 \text{ if } d = 1 \\ p \text{ if } d = p^k \text{ for } p \text{ prime} \\ 1 \text{ otherwise} \end{cases}$$

*and*

$$C'(d) = \begin{cases} -2 \text{ if } d = 1 \\ 0 \text{ if } d = 2 \\ p \text{ if } d = 2p^k \text{ for } p \text{ prime (including 2)} \\ 1 \text{ otherwise} \end{cases}.$$

*Proof.* In the case that $N_1 = 0$, the characteristic quadratic equation factors as

$$1 - (1 + q - N_1)T + qT^2 = (1 - T)(1 - qT).$$

Consequently, $\alpha_1 = 1$ and $\alpha_2 = q$ in this special case. (Note this is strictly formal since $N_1 = 0$ is impossible, and thus it is not contradictory that the Riemann Hypothesis fails.) Nonetheless, we still have $ECyc_d = Cyc_d(\alpha_1)Cyc_d(\alpha_2)$, and consequently,

$$ECyc_d|_{N_1=0} = Cyc_d(1)Cyc_d(q).$$

Finally the value of $Cyc_d(1)$ equals the function defined as $C(d)$ above [18, Seq. A020500].

For the reader's convenience we also provide a simple proof of this equality. It is clear that $Cyc_1(q) = 1 - q$ and $Cyc_p(q) = 1 + q + q^2 + \cdots + q^{p-1}$ so by induction on $k \geq 1$, assume that $Cyc_{p^k}(1) = p$.

$$\frac{1 - q^{p^k}}{1 - q} = 1 + q + q^2 + \cdots + q^{p^k - 1} = \prod_{j=1}^{k} Cyc_{p^j}(q).$$

Plugging in $q = 1$, and by induction we get $p^k = p^{k-1} \cdot Cyc_{p^k}(1)$, thus we have $Cyc_{p^k}(1) = p$. We now proceed to show $Cyc_d(1) = 1$ if $d = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ for any $r \geq 2$. For this we use $k$ such that $d|k$. We assume $k = p_1^{k_1'} p_2^{k_2'} \cdots p_r^{k_r'}$.

$$\begin{aligned} \frac{1 - q^k}{1 - q} &= 1 + q + q^2 + \cdots + q^{k-1} \\ &= \left(\prod_{j_1=1}^{k_1'} Cyc_{p_1^{j_1}}(q)\right)\left(\prod_{j_2=1}^{k_2'} Cyc_{p_2^{j_2}}(q)\right) \cdots \left(\prod_{j_r=1}^{k_r'} Cyc_{p_r^{j_r}}(q)\right) \\ &\qquad \times \left(\prod_{d \text{ is another divisor of } k} Cyc_d(q)\right). \end{aligned}$$

The expression $\left.\frac{1-q^k}{1-q}\right|_{q=1}$ equals $k$, and the first $r$ products on the right-hand-side equal $p_1^{k_1'}$, $p_2^{k_2'}$, $\ldots$, $p_r^{k_r'}$ respectively. Thus the last set of factors, i.e., the cyclotomic polynomials of $d$ with two or more prime factors, must all equal the value 1.

We prove (26) analogously. When $N_1 = 2q + 2$ (again this is strictly formal), the characteristic equation factors as

$$1 - (1 + q - N_1)T + qT^2 = (1 + T)(1 + qT)$$

implying $\alpha_1 = -1$ and $\alpha_2 = -q$. Additionally, $C'(d) = Cyc_d(-1)$ was observed by Ola Veshta on Jun 01 2001, as cited on [18, Seq. A020513]. □

**Proposition 14.** *For $d \geq 2$,*

$$\deg_{N_1} ECyc_d = \deg_q ECyc_d = \phi(d),$$

*where the Euler $\phi$ function which counts the number of integers between 1 and $d - 1$ which are relatively prime to $d$.*

*Proof.* As noted in Remark 7, we can write $ECyc_d$ as an integral polynomial in $e_1 = \alpha_1 + \alpha_2 = 1 + q - N_1$ and $e_2 = \alpha_1\alpha_2 = q$. The highest degree of $N_1$ in $ECyc_d$ is therefore equal to the highest degree of $e_1 = \alpha_1 + \alpha_2$, which is the same as the largest $m$ such that $\alpha_1^m\alpha_2^0$ (respectively $\alpha_1^0\alpha_2^m$) is a term in $Cyc_d(\alpha_1)Cyc_d(\alpha_2)$. Thus $\deg_{N_1} ECyc_d(q, N_1) = \deg_{\alpha_1} Cyc_d(\alpha_1) = \phi(d)$. Analogously, the degree of $q$ comes from the highest power of $(\alpha_1\alpha_2)^m$ in $Cyc_d(\alpha_1)Cyc_d(\alpha_2)$. Thus we have shown

$$\deg_q ECyc_d \leq \phi(d).$$

Equality follows from the first half of Proposition 13 when $d \geq 2$ since the constant term with respect to $N_1$, which equals $C(d)Cyc_d(q)$, has degree $\phi(d)$. □

Finally, if one examines the expressions for $ECyc_d(q, N_1)$, one notes that they appear alternating in sign just as the polynomials for $N_k$, except for the constant term which equals $C(d)Cyc_d(q)$ by Proposition 13. More precisely, the author finds the following empirical evidence for such a claim.

**Proposition 15.** *For $d$ between 2 and 104, we obtain*

$$ECyc_d(q, N_1) = Cyc_d(1) \cdot Cyc_d(q) + \sum_{i=1}^{\phi(d)} (-1)^i Q_{i,d}(q) N_1^i,$$

*where $Q_{i,d}$ is a univariate polynomial with* positive *integer coefficients.*

However, the conjecture fails for $d = 105$. In particular, if we write

$$ECyc_{105}(q, N_1) = Cyc_{105}(1) \cdot Cyc_{105}(q) + \sum_{i=1}^{48} (-1)^i Q_{i,105}(q) N_1^i,$$

where the $Q_{i,105}(q)$'s are univariate polynomials with integer coefficients, then $Q_{2,105}(q)$ through $Q_{48,105}(q)$ indeed have *positive* integer coefficients as expected. However the first univariate polynomial, i.e., the coefficient of $-N_1$ is

$$\begin{aligned} Q_{1,105}(q) = {} & 24q^{47} + 47q^{46} + 69q^{45} + 69q^{44} + 69q^{43} + 50q^{42} + 32q^{41} \\ & - 2q^{40} - 18q^{39} - 33q^{38} - 33q^{37} - 33q^{36} - 21q^{35} - 10q^{34} \\ & + 9q^{32} + 17q^{31} + 24q^{30} + 24q^{29} + 24q^{28} + 20q^{27} + 20q^{26} + 18q^{25} + 18q^{24} \\ & + 18q^{23} + 18q^{22} + 20q^{21} + 20q^{20} + 24q^{19} + 24q^{18} + 24q^{17} + 17q^{16} + 9q^{15} \end{aligned}$$

$$- 10q^{13} - 21q^{12} - 33q^{11} - 33q^{10} - 33q^9 - 18q^8 - 2q^7$$
$$+ 32q^6 + 50q^5 + 69q^4 + 69q^3 + 69q^2 + 47q + 24.$$

Note that there are 46 nonzero coefficients of $Q_{1,105}$ in the expansion of $ECyc_{105}(q, N_1)$, 14 of which have the incorrect sign.

The number $105 = 3{\cdot}5{\cdot}7$ is significant and interesting from a number theoretic point of view. This number is also the first $d$ such that ordinary cyclotomic polynomial $Cyc_d$ has a coefficient other than $-1, 0$, or 1.

$$\begin{aligned} Cyc_{105} = 1 + x + x^2 - x^5 - x^6 - 2x^7 - x^8 - x^9 + x^{12} + x^{13} + x^{14} \\ + x^{15} + x^{16} + x^{17} - x^{20} - x^{22} - x^{24} - x^{26} - x^{28} + x^{31} + x^{32} \\ + x^{33} + x^{34} + x^{35} + x^{36} - x^{39} - x^{40} - 2x^{41} - x^{42} - x^{43} \\ + x^{46} + x^{47} + x^{48}. \end{aligned}$$

Despite this counter-example, we still can prove that the coefficients of the $ECyc_d$'s alternate in sign for an infinite number of $d$'s. Specifically, we note that $ECyc_{2^m}$ resemble the coefficients of $N_{2^{m-1}}$, and moreover the pattern we find is given by the following proposition.

**Proposition 16.**

$$ECyc_{2^m} = 2Cyc_{2^{m-1}}(q) - N_{2^{m-1}}. \tag{27}$$

*In particular, for $i$ between 1 and $\phi(2^m) = 2^{m-1}$, we get*

$$Q_{i,2^m} = P_{i,2^{m-1}}, \tag{28}$$

*where the $P_{i,k}$ are the coefficients of $N_k$.*

Note that in our proof we use the fact that $ECyc_d$ can be written as

$$Cyc_d(1) \cdot Cyc_d(q) + \sum_{i=1}^{\phi(d)} (-1)^i Q_{i,d}(q) N_1^i,$$

where the $Q_{i,d}$'s are univariate polynomials with *possibly* negative coefficients. Therefore, our proof of Proposition 16 actually extends Proposition 15 to the case where $d$ is a power of 2 since we previously showed that the $P_{i,d}$'s alternate.

*Proof.* We note that $Cyc_{2^{m-1}} = 1 + q^{2^{m-1}}$ and that (28) follows from (27). Also, $ECyc_{2^m} = N_{2^m}/N_{2^{m-1}}$ and thus it suffices to prove

$$N_{2^m} = (2 + 2q^{2^{m-1}})N_{2^{m-1}} - N_{2^{m-1}}^2.$$

However, this is a special case of

$$N_2(q, N_1) = (2 + 2q)N_1(q, N_1) - N_1(q, N_1)^2$$

where we plug in $q^{2^{m-1}}$ in the place of $q$. ☐

Unfortunately, formulas for $Q_{i,d}$'s in terms of $P_{i,k}$'s when $d$ is not a power of 2 are not as simple. On the other hand, the last part of this proof highlights a principle that has the potential to open up a new direction. Namely, $N_k(q, N_1)$ is defined as the number of points on $C(\mathbb{F}_{q^k})$ where $q$ itself can also be a power of $p$. Consequently,

$$N_{m\cdot k}(q,\ N_1) = \#C(\mathbb{F}_{q^{m\cdot k}}) = N_m(q^k,\ N_k). \tag{29}$$

While this relation is immediate given our definition of $N_k = \#C(\mathbb{F}_{q^k})$, when we translate this relation in terms of spanning trees, the relation

$$\mathcal{W}_{mk}(q,t) = \mathcal{W}_m\big(q^k, \mathcal{W}_k(q,t)\big) \tag{30}$$

seems much more novel. Furthermore, in this case, this relation involves only positive integer coefficients and thus motivates exploration for a bijective proof. As noted in Section 4.1, such a compositional formula is indicative of the appearance of a linear transformation of $x^k$ or $T_k(x)$, which is also clear from the three-term recurrence satisfied by the $(1 + q^k - N_k)$'s.

4.5. **Geometric interpretation of elliptic cyclotomic polynomials.** Despite the fact that the above expressions of elliptic cyclotomic polynomials do not have positive coefficients nor coefficients with alternating signs, we can nonetheless describe a set of geometric objects which the elliptic cyclotomic polynomials enumerate.

**Theorem 7.** *We have*

$$ECyc_d = \big|Ker\big(Cyc_d(\pi)\big) :\ C(\overline{\mathbb{F}_q}) \to C(\overline{\mathbb{F}_q})\big|,$$

*where $\pi$ denotes the Frobenius map, and $Cyc_d(\pi)$ is an element of $End(C) = End(C(\overline{\mathbb{F}_q}))$.*

*Proof.* One of the key properties of the Frobenius map is the fact that $C(\mathbb{F}_{q^k}) = Ker(1-\pi^k)$, where $1-\pi^k$ is an element of $End(C)$. See [17] for example. The map $(1-\pi^k)$ factors into cyclotomic polynomials in $End(C)$ since the endomorphism ring contains both integers and powers of $\pi$. Since the maps $Cyc_d(\pi)$ are each group homomorphisms, it follows that the cardinality of $\big|\text{Ker }\big(Cyc_{d_1}Cyc_{d_2}(\pi)\big)\big|$ equals $\big|\text{Ker } Cyc_{d_1}(\pi)\big| \cdot \big|\text{Ker } Cyc_{d_2}(\pi)\big|$ . Thus

$$\prod_{d|k} ECyc_d = N_k = \big|\text{Ker } (1-\pi^k)\big| = \big|\text{Ker}\prod_{d|k} Cyc_d(\pi)\big| = \prod_{d|k}\big|\text{Ker } Cyc_d(\pi)\big|,$$

and since the last equation is true for all $k \geq 1$, we must have the relations

$$ECyc_d = \big|\text{Ker } Cyc_d(\pi)\big| \tag{31}$$

for all $d \geq 1$. ∎

Since

$$N_k = \prod_{d|k} ECyc_d(q, N_1)$$

and $\mathcal{W}_k(q,t) = -N_k\big|_{N_1 \to -t}$, it also makes sense to consider the decomposition

$$\mathcal{W}_k(q,t) = \prod_{d|k} WCyc_d(q,t),$$

where $WCyc_d(q,t) = -ECyc_d\big|_{N_1 \to -t}$.

This motivates the analogous question, namely does there exist a combinatorial or geometric interpretation of these polynomials? We in fact can answer this in the affirmative and do so in [15, Chap. 6] as well as in a forthcoming paper.

*Remark* 8. The coefficients of the $WCyc_d$'s are always integers, but not necessarily positive, as seen in the constant coefficient, as well as in the counter-example $WCyc_{105}$. Nonetheless, plugging in specific integers $q \geq 0$ and $t \geq 1$ do in fact result in positive expressions, which factor $\mathcal{W}_k(q,t)$. It is these values that we are interested in understanding.

## 5. Conclusions and open problems

The new combinatorial formula for $N_k$ presented in this write-up appears fruitful. It leads one to ask how spanning trees of the wheel graph are related to points on elliptic curves. For instance, is there a reciprocity that explains combinatorially why the bivariate integral polynomial formulas for counting points on elliptic curves and counting spanning trees of the wheel graph are equivalent except for the appearance of alternating signs? Such reciprocities occur frequently in combinatorics. For example given the chromatic polynomial $\chi(\lambda)$ of a graph $G = (V,E)$, the expression $(-1)^{|V|}\chi(-1)$ provides a formula for the number of acyclic orientations of $G$ [19].

The fact that the Fibonacci and Lucas numbers also enter the picture is also exciting since these numbers have so many different combinatorial interpretations, and there is such an extensive literature about them. Perhaps these combinatorial interpretations will lend insight into why $N_k$ depends only on the finite data of $N_1$ and $q$ for an elliptic curve, and how we can associate points over higher extension fields to points on $C(\mathbb{F}_q)$.

The elliptic cyclotomic polynomials provide an additional source of new questions. What is the spanning tree interpretation of $\mathcal{W}_k(q,N_1)$'s factorization? Is there a combinatorial interpretation of $\mathcal{W}_{mk}(q,t) = \mathcal{W}_m(q^k, \mathcal{W}_k(q,t))$? What is a combinatorial interpretation of the integral polynomials $Q_{i,d}$, and what does the fact their coefficients are almost all positive mean? We will tackle some of these problems in a forthcoming paper in which we compare more thoroughly the structures of elliptic curves and spanning trees.

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Mathematics Department, University of California, San Diego
*E-mail address*: `gmusiker@math.ucsd.edu`