A NOTE ON THE DISTRIBUTION OF THE THREE TYPES OF NODES IN UNIFORM BINARY TREES

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Abstract. We use Zeilberger’s algorithm to compute some sums that came up in Mahmoud’s analysis of the distribution of types of nodes in binary trees.

In [4], Mahmoud has considered uniform binary trees (counted by Catalan numbers $b_n = \frac{1}{n+1}{2n \choose n}$) and the three statistics $X_n^{(0)}$, $X_n^{(1)}$, $X_n^{(2)}$, counting the numbers of internal nodes having 0, 1, 2 internal nodes as successors, respectively. He obtained the following theorem (with one typo removed):

**Theorem 1.** [Mahmoud]

\[
\Pr \left\{ X_n^{(0)} = j \right\} = \frac{1}{b_n} \sum_{i=0}^{n-j} (-1)^{n-j-i} b_i \binom{i+1}{n-i} \binom{n-i}{j},
\]

\[
\Pr \left\{ X_n^{(1)} = j \right\} = \frac{1}{b_n} \sum_{i=0}^{n-j} (-1)^{n-j-i} 2^{n-i} b_i \binom{n-1}{i-1} \binom{n-i}{j},
\]

\[
\Pr \left\{ X_n^{(2)} = j \right\} = \frac{1}{b_n} \sum_{k=0}^{j-i} (-1)^{j-k} b_k \frac{n-k}{j-k} \sum_{i=0}^{n-k} 2^{n-k-i} \binom{n-1}{k+i-1} \binom{k+1}{i}.
\]

Mahmoud used generating functions to get these results. For convenience, we sketch how to do it.

The generating function $B(z)$ of the binary trees is normally obtained via the equation $B = 1 + z B^2$. However, we can also get it via the equation $B = 1 + z + 2z(B - 1) + z(B - 1)^2$. Observe that $B(z) - 1$ is the generating function for the nonempty trees, and the recursion now distinguishes between zero, one, or two successors. Now, we can use additional variables $u$, $v$, $w$ in order to count nodes with zero, one, or two successors. Then the equation is

\[ B = 1 + z u + 2 z v (B - 1) + z w (B - 1)^2, \]

with the solution

\[ B(z; u, v, w) = \frac{1 - 2z(v - w) - \sqrt{1 - 4zv + 4z^2(v^2 - uw)}}{2zw}. \]

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Now the probabilities from Theorem 1 are obtained by reading off appropriate coefficients:

\[
\begin{align*}
\Pr \{ X_n^{(0)} = j \} & = \frac{1}{b_n} [z^n u^j] B(z; u, 1, 1), \\
\Pr \{ X_n^{(1)} = j \} & = \frac{1}{b_n} [z^n v^j] B(z; 1, v, 1), \\
\Pr \{ X_n^{(2)} = j \} & = \frac{1}{b_n} [z^n w^j] B(z; 1, 1, w).
\end{align*}
\]

With Zeilberger’s algorithm EKHAD (see [5]), I found the following explicit formulæ:

**Theorem 2.**

\[
\begin{align*}
\Pr \{ X_n^{(0)} = j \} & = \frac{2^{n+1-2j}(n + 1)! n!(n - 1)!}{j!(j - 1)!(n + 1 - 2j)!(2n)!}, \\
\Pr \{ X_{2n+1}^{(1)} = 2j \} & = \frac{2^{2j-1}(2n + 2)!(2n)!}{(2j)!(n - j + 1)!(n - j)!(4n + 1)!}, \\
\Pr \{ X_{2n}^{(1)} = 2j + 1 \} & = \frac{2^{2j+1}(2n + 1)!(2n)!}{(2j + 1)!(n - j)!(n - j - 1)!(4n)!}, \\
\Pr \{ X_n^{(1)} = j \} & = 0 \quad \text{for } n + j \text{ even}, \\
\Pr \{ X_n^{(2)} = j \} & = \frac{2^{n-1-2j}(n + 1)! n!(n - 1)!}{(j + 1)! j!(n - 1 - 2j)!(2n)!}.
\end{align*}
\]

The quantity

\[
\Pr \{ X_n^{(0)} = j \} b_n = \frac{2^{n+1-2j}(n - 1)!}{j!(j - 1)!(n + 1 - 2j)!}
\]

is the number of binary trees of size $n$ and $j$ leaves. It is interesting to compare this quantity with the corresponding one for planted plane trees; the number of planted plane trees of size $n$ with $j$ leaves is the Narayana number

\[
\frac{1}{n} \binom{n}{j} \binom{n - 2}{j - 1} = \frac{(n - 1)!(n - 2)!}{j!(j - 1)!(n - 1 - j)!^2},
\]

see [3].

We provide a glimpse of what Zeilberger’s algorithm does. Dealing with the first sum in Theorem 1, and setting

\[
F(n, i) := \frac{1}{b_n} (-1)^{n-j-i} b_i \binom{i + 1}{n - i} \binom{n - i}{j},
\]
where we treat \( j \) as a parameter, the algorithm computes

\[
G(n, i) := \frac{(n + 2)(n - 2i)(n - 2i - 1)}{2(n + 1 - i - j)} F(n, i)
\]

such that

\[
(n - 2j + 2)(2n + 1) F(n + 1, i) - (n + 2)n F(n, i) =
G(n, i + 1) - G(n, i) .
\]

Denoting the sum on \( i \) by \( F(n) \) and summing up, the right hand side telescopes, and we get

\[
(n - 2j + 2)(2n + 1) F(n + 1) = (n + 2)n F(n) .
\]

Such a first order recursion can always be solved by iteration, leading to the announced formula.

If we do the same thing for the second sum, we find the recursion

\[
(2n + 3)(2n + 1)(n - j + 3)(n - j + 1) F(n + 2) =
(n + 3)(n + 2)(n + 1)n F(n)
\]

which is not of first order, but can, because of the lack of the term \( F(n + 1) \), still be solved by iteration. It turns out that four cases \((n, j)\) even or odd have to be distinguished.

The third entry is a double sum and thus not so easily treatable. I recently showed it to F. Chyzak, and he is able to handle it with his package MGFUN (see [1]). However, because of the following argument, the double sum is reducible to the other instances. We will show that the three statistics are basically the same. Assume that there are \( i \) nodes with 0 successors, \( j \) nodes with 1 successor, and \( k \) nodes with 2 successors. Then we have by an elementary argument the equations

\[
i + j + k = n ,
\]

\[
j + 2k = n + 1 ,
\]

whence \( k = i + 1 \) and furthermore \( j = n - 2i - 1 \). Thus, with

\[
H(n, j) := \frac{2^{n+1-2i}(n + 1)!n!(n - 1)!}{j!(j - 1)!(n + 1 - 2j)!(2n)!} ,
\]
we get alternatively
\[
\begin{align*}
\Pr\{X_n^{(0)} = j\} &= H(n, j), \\
\Pr\{X_n^{(1)} = j\} &= H(n, \frac{n-j+1}{2}), \quad \text{for } n+j \text{ is odd,} \\
\Pr\{X_n^{(1)} = j\} &= 0, \quad \text{for } n+j \text{ even,} \\
\Pr\{X_n^{(2)} = j\} &= H(n, j+1).
\end{align*}
\]

The fact, that the covariance matrix has rank 1 (see [4]) becomes perhaps a little bit more transparent by this combinatorial argument.

With Zeilberger’s algorithm we can do even more: We can compute the \(s\)-th factorial moments explicitly. Let \(n^k = n(n-1)\ldots(n-k+1)\), and
\[
M_{n,s}^{(i)} := \sum_j \Pr\{X_n^{(i)} = j\} j^s
\]
denote the \(s\)-th factorial moments for \(i = 0, 1, 2\), then:

**Theorem 3.**
\[
\begin{align*}
M_{n,s}^{(0)} &= \frac{(n+1)!n!(2n-2s)!}{(2n)!(n+1-2s)!(n-s)!}, \\
M_{n,s}^{(1)} &= \frac{2^s(n+1)!n!(n-1)!(2n-2s)!}{(2n)!(n-s+1)!(n-s)!(n-s-1)!}, \\
M_{n,s}^{(2)} &= \frac{n!(n-1)!(2n-2s)!}{(2n)!(n-1-2s)!(n-s)!}.
\end{align*}
\]

We end this little note by citing two other papers ([2], [6]) dealing with binary trees and statistics on the leaves.

**References**


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