

Minuscule Reverse Plane Partitions via Quiver Representations

Alexander Garver^{*}, Rebecca Patrias[†], and Hugh Thomas[‡]

LaCIM, Université du Québec à Montréal, Québec (Canada)

Abstract. The Hillman–Grassl correspondence is a well-known bijection between multisets of rim hooks of a partition shape λ and reverse plane partitions of λ . We use the tools of quiver representations to generalize Hillman–Grassl in type A and to define an analogue in all minuscule types.

Résumé. La correspondance de Hillman–Grassl est une bijection bien connue entre les multiensembles des équerres de bord (« rim hooks ») d'un partage λ et les partitions planes renversées (« reverse plane partitions ») de forme λ . Nous utilisons les représentations des carquois afin de généraliser la correspondance de Hillman–Grassl dans le type A et de définir un analogue dans tous les types minuscules.

Keywords: reverse plane partitions, quiver representations, minuscule posets

A reverse plane partition of a Young diagram is a filling of the boxes with $\{0, 1, 2, \dots\}$ such that the entries in rows and columns are weakly increasing. They are prominent combinatorial objects with connections to areas like symmetric functions and representation theory (see for example [12]). Their generating function was discovered by Stanley and is as follows, where $h(u)$ denotes the hook length of the cell u in partition λ and $|\rho|$ denotes the sum of the entries in reverse plane partition ρ .

Theorem 1. [12] *The generating function for reverse plane partitions of shape λ with respect to the sum of its entries is*

$$\sum_{\rho} q^{|\rho|} = \prod_{u \in \lambda} \frac{1}{1 - q^{h(u)}}.$$

The first bijective proof of this generating function was found by Hillman and Grassl in [7]. The authors give a bijection between nonnegative integer arrays of shape λ —which can be thought of as multisets of rim hooks of λ —and reverse plane partitions of λ , which is now known as the *Hillman–Grassl correspondence*. This correspondence has

^{*}alexander.garver@gmail.com. AG was supported by NSERC grant RGPIN/05999-2014 and the Canada Research Chairs program.

[†]patriasr@lacim.ca. BP was supported by NSERC, CRM-ISM, and the Canada Research Chairs program

[‡]hugh.ross.thomas@gmail.com. HT was supported by an NSERC Discovery Grant and the Canada Research Chairs program.

since been well studied, for example by Gansner in [4] and by Morales, Pak, and Panova in [9]. Another such bijection was given by Pak in [10] and rediscovered in a different form by Sulzgruber in [14].

More generally, one can define a reverse plane partition on any poset P to be an order-reversing map $\rho : P \rightarrow \{0, 1, 2, \dots\}$. In the case where P is a minuscule poset, Proctor [11] proves the following analogous generating function for reverse plane partitions of P with respect to the sum of its entries:

$$\sum_{\rho \in RPP(P)} q^{|\rho|} = \prod_{u \in P} \frac{1}{1 - q^{\text{rank}(u)}}.$$

In this paper, we use quiver representations to give a uniform bijective proof of this generating function for all minuscule types that generalizes the Hillman–Grassl and Pak correspondences in type A . Our bijection is between certain isomorphism classes of representations of a simply-laced Dynkin quiver and reverse plane partitions on the corresponding minuscule poset; see [Theorems 2](#) and [5](#). The interested reader may see [\[5\]](#) for a full version of this paper with proofs included.

In [Theorem 2](#), we describe our bijection as a sequence of piecewise-linear maps that involve an operation known as *toggling*. There is an action on the set of reverse plane partitions known as *promotion* defined using toggles. An important application of our work is that promotion applied h times to a reverse plane partition of a minuscule poset is the identity where h is the Coxeter number of the associated root system.

1 Quiver representations

We briefly review the basics of quiver representations; see [\[1\]](#) for more details. A *quiver* Q is a directed graph. In other words, Q is a 4-tuple (Q_0, Q_1, s, t) , where Q_0 is a set of *vertices*, Q_1 is a set of *arrows*, and $s, t : Q_1 \rightarrow Q_0$ are two functions defined so that for every $a \in Q_1$, we have $s(a) \xrightarrow{a} t(a)$. In all of our results, we will assume that Q is a *Dynkin quiver*, meaning that its underlying graph is a simply-laced Dynkin diagram.

A *representation* $V = ((V_i)_{i \in Q_0}, (f_a)_{a \in Q_1})$ of a quiver Q is an assignment of a finite-dimensional \mathbb{k} -vector space V_i to each vertex i and a \mathbb{k} -linear map $f_a : V_{s(a)} \rightarrow V_{t(a)}$ to each arrow a where \mathbb{k} is a field. The *dimension vector* of V is the vector $\mathbf{dim}(V) := (\dim V_i)_{i \in Q_0}$. In all of our results, we will assume that \mathbb{k} is algebraically closed.

Let $V = ((V_i)_{i \in Q_0}, (f_a)_{a \in Q_1})$ and $W = ((W_i)_{i \in Q_0}, (g_a)_{a \in Q_1})$ be two representations of a quiver Q . A *morphism* $\theta : V \rightarrow W$ consists of a collection of linear maps $\theta_i : V_i \rightarrow W_i$ that are compatible with each of the linear maps in V and W . That is, for each arrow $a \in Q_1$, we have $\theta_{t(a)} \circ f_a = g_a \circ \theta_{s(a)}$. We let $\text{Hom}(V, W)$ denote the \mathbb{k} -vector space of all morphisms $\theta : V \rightarrow W$. Additionally, we say that a collection of linear maps $\{\theta_i : V_i \rightarrow W_i\}_{i \in Q_0}$ is *compatible* with a representation V if they define a morphism of V .

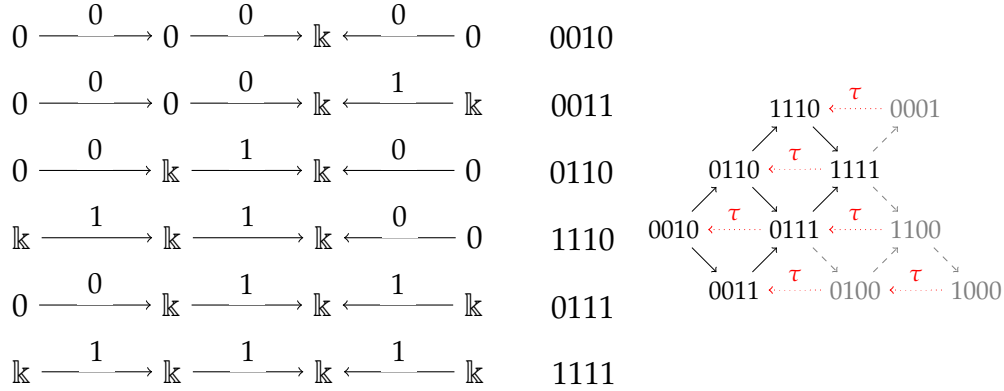


Figure 1: Let $Q = 1 \rightarrow 2 \rightarrow 3 \leftarrow 4$ with $m = 3$. On the left is the set of indecomposable representations (up to isomorphism) of Q that have support over vertex 3 and their dimension vectors. On the right is the minuscule poset (in black) inside of the AR quiver of Q . The action of τ is shown in red; the leftmost representations map to 0.

The representations of a quiver Q along with morphisms between them form an abelian category, which is equivalent to the category of finitely generated left modules over the path algebra of Q . It is therefore natural to ask for a classification of the indecomposable representations of Q , i.e., those which cannot be written as a direct sum of two non-zero representations. Since we will restrict to the case when Q is Dynkin, Gabriel’s Theorem [3] implies that the indecomposable representations of Q are in one-to-one correspondence with the positive roots of the root system corresponding to Q , under the map sending a representation to its dimension vector. Figure 1 shows examples of indecomposable representations of a Dynkin quiver of type A_4 with their dimension vectors.

One can understand a great deal about the category of representations of Q by constructing the Auslander–Reiten quiver (or AR quiver) of Q . In this paper, we use the AR quiver of Q to relate the representation theory of Q to the combinatorics of minuscule posets. By definition, the AR quiver of Q is the quiver whose vertices are indexed by the isomorphism classes of indecomposable representations of Q and whose arrows index a basis of the space of irreducible morphisms between the corresponding representations. In Figure 1, we show an example of an AR quiver.

There is a well-known endofunctor on the category of representations of Q , called the Auslander–Reiten translation and denoted by τ . We omit its homological definition, but we show how τ acts on indecomposables in Figure 1. For our work, the important fact is that τ partitions the indecomposable representations of Q into τ -orbits.

2 Minusculer posets

A vertex m of Q is called *minuscule* if every positive root in the root system of Q has the simple root corresponding to vertex m appearing in its simple root expansion with coefficient 0 or 1. In other words, a vertex m of Q is minuscule if every indecomposable representation of Q supported over m has dimension 1 at m . For example, every vertex of a type A Dynkin quiver is minuscule. [Figure 2](#) shows the possible choices of minuscule vertices.

Each choice of minuscule vertex gives rise to a minuscule poset. We now give explicit descriptions of the minuscule posets, which appear in the context of minuscule representations of complex semisimple Lie algebras and were classified up to isomorphism by Proctor in [\[11\]](#). Recall that for a poset P , we let $J(P)$ denote its poset of *order ideals* (i.e., subsets of $O \subset P$ where if $x \in O$ and $y \leq_P x$, one has that $y \in O$), ordered by inclusion. By [\[6, Theorem 8.3.10\]](#), there is a minuscule poset for each choice of a simply-laced Dynkin diagram other than E_8 with a chosen minuscule vertex. Their isomorphism types appear in [Table 1](#). There, we write $[n]$ for the poset that is a chain whose elements are $1, \dots, n$ in increasing order. [Figure 2](#) shows some examples of minuscule posets.

Type	m	minuscule poset
A_n	k	$[k] \times [n + 1 - k]$
D_n	1	$J^{n-3}([2] \times [2])$
D_n	$n - 1, n$	$J([2] \times [n - 2])$
E_6	1, 5	$J^2([2] \times [3])$
E_7	6	$J^3([2] \times [3])$

Table 1: The isomorphism types of the minuscule posets. Here we are referring to the vertex labeling of the Dynkin diagrams appearing in [Figure 2](#).

Our next result shows how the minuscule posets are related to the representation theory of quivers. Given a minuscule vertex $m \in Q_0$, let $P_{Q,m}$ denote the poset that is the transitive closure of the arrows in the full subquiver of the AR quiver of Q whose vertex set is the set of isomorphism classes of representations supported at m . Examples for type A quivers are shown in [Figures 1, 3](#) and [6](#). We let $\mathcal{C}_{Q,m}$ denote the category consisting of all direct sums of indecomposable representations of Q supported at m .

We will need the following lemma, which is not hard to verify but appears to be new.

Lemma 1. *The poset $P_{Q,m}$ is isomorphic to the minuscule poset determined by the underlying graph of Q and minuscule vertex m .*

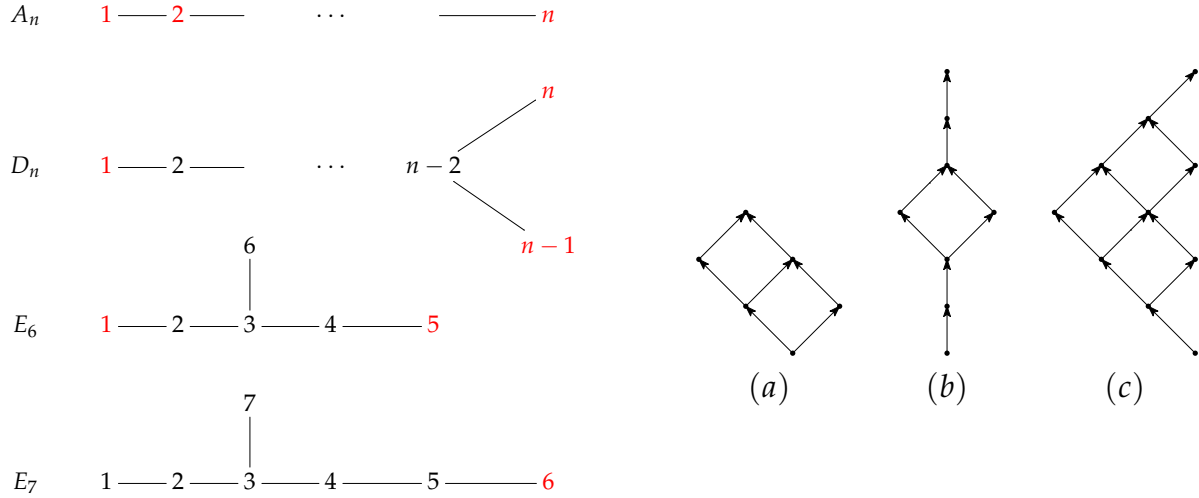


Figure 2: The possible choices of minuscule vertices of Dynkin diagrams are shown on the left in red. On the right, we have the minuscule poset of type A_4 with $m = 3$, of type D_5 with $m = 1$, and type D_5 with $m = 4$, respectively, in (a), (b), and (c).

3 Quiver representations and reverse plane partitions

Our main result is the following theorem.

Theorem 2. *Let Q be a Dynkin quiver with minuscule vertex m . There is a bijection between isomorphism classes of objects in $\mathcal{C}_{Q,m}$ and reverse plane partitions of the minuscule poset $P_{Q,m}$.*

Remark 3. *It is also possible to give a bijection between reverse plane partitions in any order filter of $P_{Q,m}$ and the isomorphism classes of objects in a suitable category. For simplicity, we only discuss the bijection for the full posets here.*

Remark 4. *Theorem 2 provides a proof of Proctor’s generating function identity from the Introduction. We may re-express that equation as follows, where the first equality is implied by Theorem 2 and the second holds since any representation in $\mathcal{C}_{Q,m}$ uniquely decomposes as a direct sum of indecomposable representations M_u belonging to $\mathcal{C}_{Q,m}$:*

$$\sum_{\rho \in RPP(P_{Q,m})} q^{|\rho|} = \sum_{V \in \mathcal{C}_{Q,m}} q^{\dim V} = \prod_{u \in P_{Q,m}} \frac{1}{1 - q^{\dim M_u}}.$$

In [5, Corollary 4.9], we show that there is a multivariate version of this generating function identity that recovers an identity of Gansner [4, Corollary 5.2].

In Section 3.2, we describe the bijection explicitly in terms of toggling, which we introduce in the next section. In Section 3.3, we give an alternative description of the bijection using the geometry of representation varieties.

3.1 Toggling

Let $\rho : P \rightarrow [0, N]$ be a reverse plane partition. For x an element of P , we define the *toggle* of ρ at $x \in P$ by

$$t_x \rho(y) = \begin{cases} \max_{y < y_1} \rho(y_1) + \min_{y_2 < y} \rho(y_2) - \rho(y) & : \text{ if } y = x \\ \rho(y) & : \text{ if } y \neq x, \end{cases}$$

where y is any element of P . If y is maximal, we interpret $\max_{y < y_1} \rho(y_1)$ as 0, and if y is minimal, we interpret $\min_{y_2 < y} \rho(y_2)$ as N . Since ρ is a reverse plane partition, so is $t_x \rho$. Additionally, observe that $t_x \circ t_x(\rho) = \rho$. This is the toggle operation considered by Einstein and Propp (up to rescaling, and restricted to lattice points) [2].

3.2 Combinatorial Description

We now give an explicit description of the bijection in [Theorem 2](#), which we will denote by $\rho_{Q,m}(-)$. We must first establish a linear order on the indecomposable representations of Q . Choose a linear order on the indecomposable representations of Q compatible with the opposite of the AR quiver order. Let M_1, \dots, M_s denote the indecomposable representations of Q in this order.

Let $M = \bigoplus_{j=1}^s M_j^{c_j} \in \mathcal{C}_{Q,m}$. The reverse plane partition $\rho_{Q,m}(M)$ corresponding to M is obtained by constructing a sequence of fillings of the minuscule poset $P_{Q,m}$, starting with ρ_0 , which is the zero filling. These fillings are defined by

$$\rho_k(x) := \begin{cases} \max_{x < y} \rho_{k-1}(y) + c_k & : \text{ if } x \text{ is the element of } P_{Q,m} \text{ corresponding to } M_k, \\ (t_x \rho_{k-1})(x) & : \text{ if } x \text{ corresponds to } \tau^\ell(M_k) \text{ for some } \ell < 0, \text{ and} \\ \rho_{k-1}(x) & : \text{ otherwise,} \end{cases}$$

where x is any element of $P_{Q,m}$. Then we have that $\rho_{Q,m}(M) = \rho_s$.

Note that using this description of the algorithm, the intermediate fillings ρ_k for $k < s$ are not reverse plane partitions of $P_{Q,m}$. However, by restricting ρ_k to the elements of $P_{Q,m}$ corresponding to M_1, \dots, M_k , we do obtain a reverse plane partition on the induced subposet of $P_{Q,m}$ whose elements correspond to M_1, \dots, M_k .

Observe that in the process of constructing ρ_s , we never toggle at a minimal element of one of these induced posets. Therefore, the result of the procedure does not depend on the choice of N used in the definition of toggling. See [Figures 3](#) and [6](#) for examples in type A worked out step-by-step using this explicit description.

3.3 Nilpotent endomorphisms of quiver representations

Let V be a representation of Q with dimension vector \mathbf{d} , over an algebraically closed ground field \mathbb{k} . Let ϕ be a nilpotent endomorphism of V . At each vertex i of Q , the

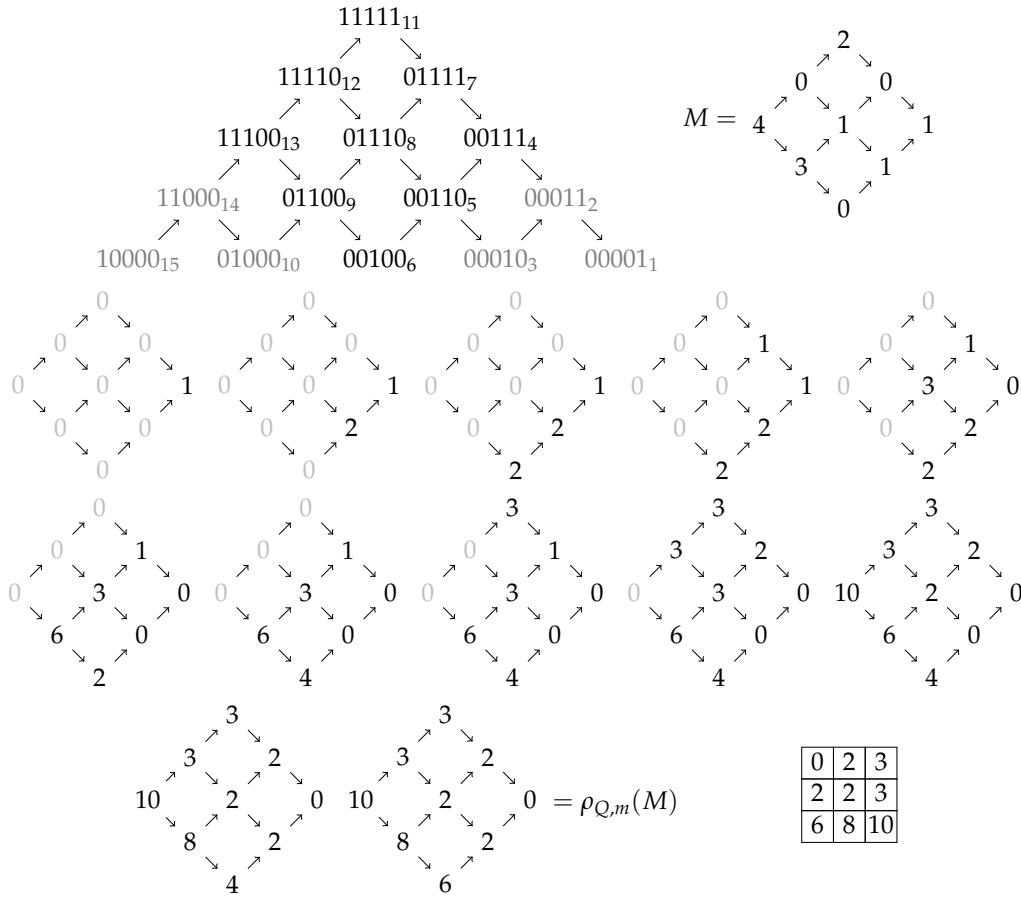


Figure 3: The top left image is the AR quiver associated with the quiver $Q = 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \leftarrow 5$ with chosen minuscule vertex 3. The dimension vectors with support on vertex 3 are black, while the others are gray. The top right image represents a representation M whose indecomposable summands all have support on vertex 3. The images below show the steps in computing $\rho_{Q,m}(M)$, starting at ρ_4 , and the bottom right shows the corresponding reverse plane partition of a Young diagram. Note that these posets are oriented from left to right. The computation of ρ_8 is the first time in the process that we apply a toggle. The computation of ρ_{10} is the first time in the process that we apply only a toggle.

endomorphism ϕ induces an endomorphism ϕ_i of V_i . We can consider the Jordan form of each of these vector space endomorphisms, which gives us a sequence of partitions $\lambda^i \vdash d_i$. We show that for a generic choice of ϕ , the n -tuple $\lambda = (\lambda^1, \dots, \lambda^n)$ is well-defined. We refer to this as the *Jordan data* of V , and we write it as $\text{GenJF}(V)$. The following is an alternate description of the bijection $\rho_{Q,m}(-)$.

Theorem 5. *For $V \in \mathcal{C}_{Q,m}$, define a map $\rho_V : P_{Q,m} \rightarrow \mathbb{N}$ as follows: The values of ρ_V restricted to the τ -orbit corresponding to vertex j are the entries of $\text{GenJF}(V)^j$, padded with extra zeros if necessary, and ordered so that, restricted to this τ -orbit, the function is order-reversing. Then ρ_V is a reverse plane partition of $P_{Q,m}$. The map from isomorphism classes in $\mathcal{C}_{Q,m}$ to reverse plane partitions of $P_{Q,m}$, sending V to ρ_V , is a bijection that agrees with $\rho_{Q,m}(-)$.*

Example 6. *Let $Q = 1 \rightarrow 2 \leftarrow 3$ and $m = 2$. By identifying indecomposable representations of Q with their dimension vectors, each $V \in \mathcal{C}_{Q,2}$ is isomorphic to $010^a \oplus 011^b \oplus 110^c \oplus 111^d$ for some $a, b, c, d \in \mathbb{Z}_{\geq 0}$. By direct calculation, $\text{GenJF}(V) = ((c + d), (\max(b, c) + a + d, \min(b, c)), (b + d))$. We obtain the corresponding reverse plane partition shown in [Figure 4](#).*

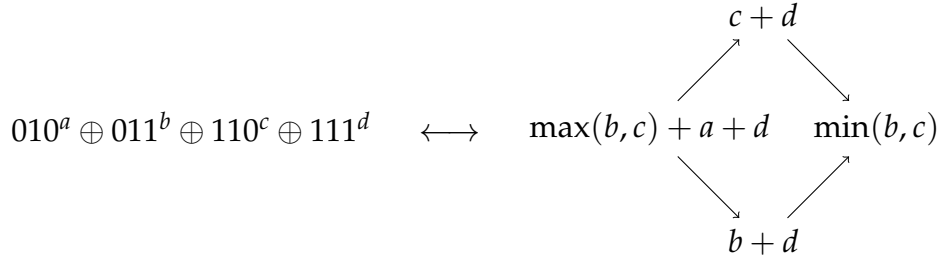


Figure 4: The correspondence between isomorphism classes of representations of $Q = 1 \rightarrow 2 \leftarrow 3$ belonging to $\mathcal{C}_{Q,2}$ and reverse plane partitions of $P_{Q,2}$.

3.4 Periodicity

For this section, we assume that the vertices of Q are labeled with the numbers $1, \dots, n$ so that the arrows of Q are oriented from vertices with larger labels to vertices with smaller labels.

Note that t_x and t_y commute unless x and y are related by a cover. For $P_{Q,m}$, and $i \in Q_0$, the elements of τ -orbit corresponding to i are never related by a cover, so we can define $T_i = \prod_x t_x$, where we toggle at each element x of the τ -orbit containing the projective at i , without worrying about the order in which the composition is taken.

Define $\text{pro}_Q = T_n \circ \dots \circ T_1$. Define h to be the Coxeter number of Q : by definition, this is the order of the product of the simple reflections in the Coxeter group, or, equivalently, the largest degree of the root system. [Figure 5](#) illustrates the following theorem.

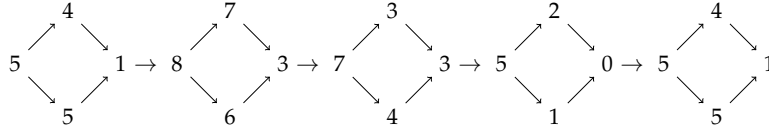


Figure 5: An example of periodicity, where $N = 8$. In each step, we first toggle the elements in the middle τ -orbit, and then we toggle the top and bottom τ -orbits.

Theorem 7. For Q a Dynkin quiver and m a minuscule vertex, pro_Q^h is the identity transformation on RPPs on $P_{Q,m}$ with entries in $[0, N]$.

4 Type A examples

Suppose we start with the type A quiver Q shown in Figure 3, where 3 is the chosen minuscule vertex. The corresponding AR quiver is also shown in the figure. The type A minuscule poset $P_{Q,m}$ associated with vertex 3 is shown in black in the figure.

We denote a representation M as a labeling of the poset $P_{Q,m}$, where the label at a vertex denotes how many copies of the corresponding indecomposable are in M . The representation M in Figure 3 contains 4 copies of the indecomposable with dimension vector 11100 and 3 copies of the indecomposable with dimension vector 01100.

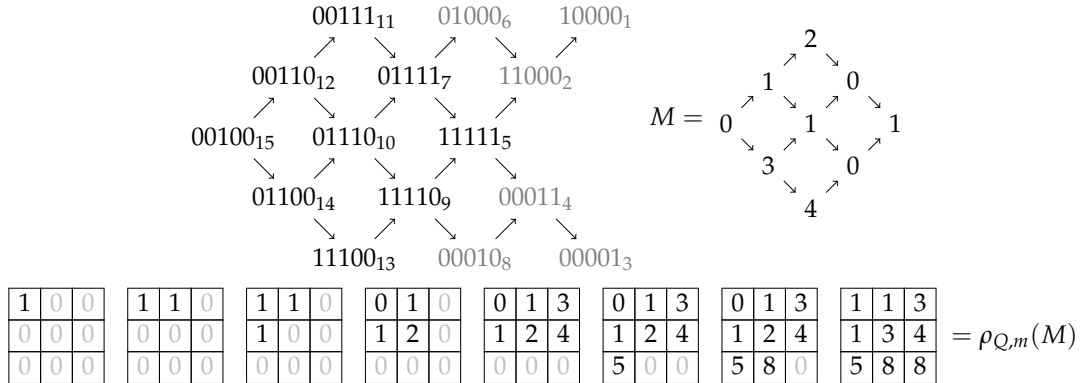


Figure 6: The top left image is the AR quiver associated with the quiver $Q = 1 \rightarrow 2 \rightarrow 3 \leftarrow 4 \leftarrow 5$ with chosen minuscule vertex 3. The dimension vectors with support on vertex 3 are black, while the others are gray. The top right image represents a representation M whose indecomposable summands all have support on vertex 3. The images below show the steps in computing $\rho_{Q,m}(M)$, shown on the corresponding Young diagram.

The order we use to compute $\rho_{Q,m}(M)$ is indicated in the subscripts on the AR quiver.

The procedure is shown step by step in the 12 fillings in [Figure 3](#). In future sections, it will be useful to realize the resulting reverse plane partition as a reverse plane partition for the Young diagram of shape $(3, 3, 3)$, as shown.

[Figure 6](#) shows another example using a different orientation Q and a representation that assigns the same multiplicities to the representations with the same dimension vectors as above. In this figure, we show how to carry out the algorithm by identifying the intermediate fillings with fillings of a Young diagram.

A reader who is familiar with the known bijections between multisets of rim hooks of a Young diagram and reverse plane partitions of the same Young diagram can check that the example in [Figure 3](#) agrees with the well-known Hillman–Grassl correspondence and the example in [Figure 6](#) agrees with the bijection first described by Pak [\[10\]](#) and later by Sulzgruber [\[14\]](#) and Hopkins [\[8\]](#). In fact, this is not a coincidence, and we make the correspondence precise in [Sections 5](#) and [6](#).

5 The Hillman–Grassl Correspondence

A *rim hook* of the Young diagram λ is a connected strip of border boxes in λ such that the result of removing these boxes is again a Young diagram. The Hillman–Grassl correspondence [\[7\]](#) is a bijection between multisets of rim hooks of λ and the set of reverse plane partitions of λ . See, for example, [\[13\]](#).

Let M be a representation in $\mathcal{C}_{Q,m}$, where Q is a type A Dynkin quiver. Then as shown in [Figure 3](#), $\rho_{Q,m}(M)$ can be viewed as a reverse plane partition of the Young diagram of shape $m^{(n+1-m)}$. We may identify the indecomposable summands of M as rim hooks of $m^{(n+1-m)}$ by reading through the southeast border of $m^{(n+1-m)}$ from southwest to northeast and including a box in the rim hook exactly when the corresponding entry in the dimension vector of the indecomposable is 1. See [Figure 7](#). In this way, we may identify M with a multiset of rim hooks of $m^{(n+1-m)}$. Given a multiset M of rim hooks of $m^{(n+1-m)}$ (i.e., an M in $\mathcal{C}_{Q,m}$), let $HG(M)$ denote the reverse plane partition of shape $m^{(n+1-m)}$ obtained using the Hillman–Grassl correspondence.

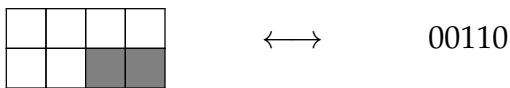


Figure 7: A rim hook and the corresponding dimension vector.

Theorem 8. *Let Q be a type A_n quiver with chosen minuscule vertex m and the following orientation.*

$$Q = 1 \leftarrow 2 \leftarrow \cdots \leftarrow n - 1 \leftarrow n$$

Then for any $M \in \mathcal{C}_{Q,m}$, $\rho_{Q,m}(M) = HG(M)$.

Remark 9. We can, in fact, recover the Hillman–Grassl correspondence for any shape λ —not just for rectangles—using $\rho_{Q,m}(-)$.

6 The Pak correspondence

In [10], Pak gives a different bijection between multisets of rim hooks of a partition λ and reverse plane partitions of λ , which involves toggles. We again use the identification between rim hooks of a rectangle and dimension vectors of indecomposable representations in $\mathcal{C}_{Q,m}$ from Section 5. Let $\text{Pak}(M)$ denote the reverse plane partition obtained from Pak’s correspondence using the multiset of rim hooks determined by M . The following result is illustrated in Figure 6.

Theorem 10. Let Q be a type A_n quiver with chosen minuscule vertex m and the following orientation.

$$Q = 1 \rightarrow \cdots \rightarrow m \leftarrow \cdots \leftarrow n$$

Then for any $M \in \mathcal{C}_{Q,m}$, $\rho_{Q,m}(M) = \text{Pak}(M)$.

Remark 11. We can, in fact, recover the Pak correspondence for any shape λ using $\rho_{Q,m}(-)$.

Fix n and $m \in \{1, \dots, n\}$. For each type A_n Dynkin quiver, we obtain a bijection $\rho_{Q,m}(-)$ that we can think of as a map between multisets of rim hooks of $m^{(n+1-m)}$ and reverse plane partitions of $m^{(n+1-m)}$. We have shown that for two particular orientations, we recover the Hillman–Grassl and Pak correspondences. For other orientations, we obtain different bijections. We demonstrate this with an example.

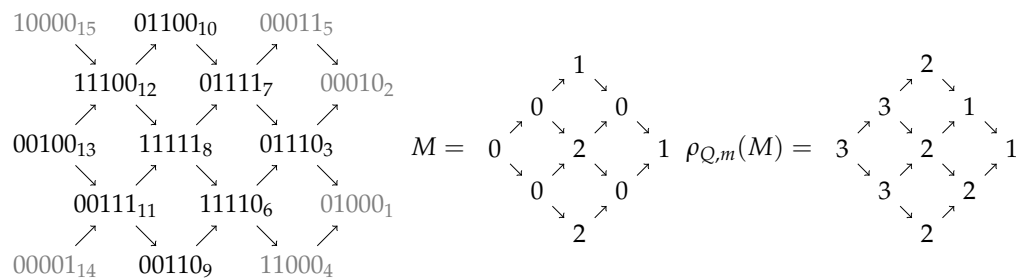
We begin with the quiver

$$Q = 1 \leftarrow 2 \rightarrow \mathbf{3} \leftarrow 4 \rightarrow 5$$

with minuscule vertex 3 and use the following multiset of rim hooks of the 3-by-3 rectangle.

$$01100, 01110, 11111, 11111, 00110, 00110$$

This multiset of rim hooks determines the representation M of the quiver Q shown below. We then obtain $\rho_{Q,m}(M)$ as shown. We leave it to the reader to check that both Hillman–Grassl and the Pak correspondences applied to this same multiset of rim hooks yield a reverse plane partition where the maximal element is labeled with 0.



References

- [1] I. Assem, D. Simson, and A. Skowroński. *Elements of the Representation Theory of Associative Algebras. Vol. 1: Techniques of Representation Theory*. London Mathematical Society Student Texts 65. Cambridge University Press, Cambridge, 2006.
- [2] D. Einstein and J. Propp. “Combinatorial, piecewise-linear, and birational homomesy for products of two chains”. 2018. [arXiv:1310.5294](https://arxiv.org/abs/1310.5294).
- [3] P. Gabriel. “Unzerlegbare Darstellungen I”. *Manuscripta mathematica* 6.1 (1972), pp. 71–103.
- [4] E. R. Gansner. “The Hillman–Grassl correspondence and the enumeration of reverse plane partitions”. *J. Combin. Theory Ser. A* 30.1 (1981), pp. 71–89. [Link](#).
- [5] A. Garver, R. Patrias, and H. Thomas. “Minuscule reverse plane partitions via quiver representations”. 2019. [arXiv:1812.08345](https://arxiv.org/abs/1812.08345).
- [6] R. M. Green. *Combinatorics of Minuscule Representations*. Camb. Tracts Math. 199. Cambridge University Press, 2013.
- [7] A. P. Hillman and R. M. Grassl. “Reverse plane partitions and tableau hook numbers”. *J. Combin. Theory Ser. A* 21.2 (1976), pp. 216–221. [Link](#).
- [8] S. Hopkins. “RSK via local transformations”. Unpublished notes. 2014. [Link](#).
- [9] A. H. Morales, I. Pak, and G. Panova. “Hook formulas for skew shapes I. q-analogues and bijections”. *J. Combin. Theory Ser. A* 154 (2018), pp. 350–405. [Link](#).
- [10] I. Pak. “Hook length formula and geometric combinatorics”. *Sém. Loth. Comb.* 46 (2001), Art. B46f, 13 pp. [Link](#).
- [11] R. A. Proctor. “Bruhat lattices, plane partition generating functions, and minuscule representations”. *European J. Comb.* 5.4 (1984), pp. 331–350. [Link](#).
- [12] R. P. Stanley. “Theory and application of plane partitions: Part 1”. *Studies in Appl. Math.* 50.2 (1971), pp. 167–188. [Link](#).
- [13] R. P. Stanley. *Enumerative Combinatorics. Volume 2*. Cambridge University Press, 1999.
- [14] R. Sulzgruber. “Inserting rim-hooks into reverse plane partitions”. 2017. [arXiv:1710.09695](https://arxiv.org/abs/1710.09695).