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# On some factorization formulas of $K$-$k$-Schur functions

Motoki Takigiku*

*Graduate School of Mathematical Sciences, the University of Tokyo, Japan*

**Abstract.** We give some new formulas about factorizations of $K$-$k$-Schur functions $g_\lambda^{(k)}$, analogous to the $k$-rectangle factorization formula $s_{R_t \cup \lambda}^{(k)} = s_{R_t}^{(k)} s_\lambda^{(k)}$ of $k$-Schur functions, where $\lambda$ is any $k$-bounded partition and $R_t$ denotes the partition $(t^{k+1-t})$ called a *$k$-rectangle*. Although a formula of the same form does not hold for $K$-$k$-Schur functions, we can prove that $g_{R_t}^{(k)}$ divides $g_{R_t \cup \lambda}^{(k)}$, and in fact more generally that $g_P^{(k)}$ divides $g_{P \cup \lambda}^{(k)}$ for any multiple $k$-rectangles $P$ and any $k$-bounded partition $\lambda$. We give the factorization formula of such $g_P^{(k)}$ and the explicit formulas of $g_{P \cup \lambda}^{(k)} / g_P^{(k)}$ in some cases.

**Keywords:** $K$-$k$-Schur function, $k$-rectangle factorization, affine Schubert calculus

## 1 Introduction

Let $k$ be a positive integer. *$k$-Schur functions* $s_\lambda^{(k)}$ and their $K$-theoretic analogues $g_\lambda^{(k)}$, which are called *$K$-$k$-Schur functions*, are symmetric functions parametrized by $k$-bounded partitions $\lambda$, namely by the weakly decreasing strictly positive integer sequences $\lambda = (\lambda_1, \ldots, \lambda_l)$, $l \in \mathbb{Z}_{\geq 0}$, whose terms are all bounded by $k$.

Historically, $k$-Schur functions were first introduced by Lascoux, Lapointe and Morse [5], and subsequent studies led to several (conjecturally equivalent) characterizations of $s_\lambda^{(k)}$ such as the Pieri-like formula due to Lapointe and Morse [7], and Lam proved that $k$-Schur functions correspond to the Schubert basis of homology of the affine Grassmannian [1]. Moreover it was shown by Lam and Shimozono that $k$-Schur functions play a central role in the explicit description of the Peterson isomorphism between quantum cohomology of the Grassmannian and homology of the affine Grassmannian up to suitable localizations [3].

These developments have analogues in $K$-theory. Lam, Schilling and Shimozono [2] characterized the $K$-theoretic $k$-Schur functions as the Schubert basis of the $K$-homology of the affine Grassmannian, and Morse [9] investigated them from a combinatorial viewpoint, giving various properties including the Pieri-like formulas using affine set-valued strips (the form using cyclically decreasing words are also given in [2]).

*takigiku@ms.u-tokyo.ac.jp

In this paper we start from this combinatorial characterization (see Definition 6) and show certain new factorization formulas of $K$-$k$-Schur functions.

Among the $k$-bounded partitions, those of the form $(t^{k+1-t}) = (\underbrace{t,\dots,t}_{k+1-t} =: R_t)$, $1 \le t \le k$, called *k-rectangles*, play a special role. In particular, if a $k$-bounded partition has the form $R_t \cup \lambda$, where the symbol $\cup$ denotes the operation of concatenating the two sequences and reordering the terms in the weakly decreasing order, then the corresponding $k$-Schur function has the following factorization property [7, Theorem 40]:

$$s^{(k)}_{R_t \cup \lambda} = s^{(k)}_{R_t} s^{(k)}_{\lambda}. \tag{1.1}$$

It is suggested in [2, Remark 7.4] that the $K$-$k$-Schur functions should also possess similar properties, including the divisibility of $g^{(k)}_{R_t \cup \lambda}$ by $g^{(k)}_{R_t}$, and that it should be interesting to explore such properties. The present work is an attempt to materialize his suggestion.

We do show that $g^{(k)}_{R_t}$ divides $g^{(k)}_{R_t \cup \lambda}$ in the ring $\Lambda^{(k)} = \mathbb{Z}[h_1, \dots, h_k]$, where $h_i$ denotes the complete homogeneous symmetric functions of degree $i$, of which the $K$-$k$-Schur functions form a basis. However, unlike the case of $k$-Schur functions, the quotient $g^{(k)}_{R_t \cup \lambda} / g^{(k)}_{R_t}$ is not a single term $g^{(k)}_{\lambda}$ but, in general, a linear combination of $K$-$k$-Schur functions with leading term $g^{(k)}_{\lambda}$, namely in which $g^{(k)}_{\lambda}$ is the only highest degree term. Even the simplest case where $\lambda$ consists of a single part $(r)$, $1 \le r \le k$, displays this phenomenon: we show that

$$g^{(k)}_{R_t \cup (r)} = \begin{cases} g^{(k)}_{R_t} \cdot g^{(k)}_{(r)} & (\text{if } t < r), \\ g^{(k)}_{R_t} \cdot \left( g^{(k)}_{(r)} + g^{(k)}_{(r-1)} + \dots + g^{(k)}_{\varnothing} \right) & (\text{if } t \ge r) \end{cases} \tag{1.2}$$

(actually we have $g^{(k)}_{(s)} = h_s$ for $1 \le s \le k$, and $g^{(k)}_{\varnothing} = h_0 = 1$). So we may ask:

**Question 1.** *Which $g^{(k)}_{\mu}$, besides $g^{(k)}_{\lambda}$, appear in the quotient $g^{(k)}_{R_t \cup \lambda} / g^{(k)}_{R_t}$? With what coefficients?*

A $k$-bounded partition can always be written in the form $R_{t_1} \cup \dots \cup R_{t_m} \cup \lambda$ with $\lambda$ not having so many repetitions of any part as to form a $k$-rectangle. In such an expression we temporarily call $\lambda$ the remainder. Proceeding in the direction of Question 1, one ultimate goal may be to give a factorization formula in terms of the $k$-rectangles and the remainder. In the case of $k$-Schur functions, the straightforward factorization in (1.1) above leads to the formula $s^{(k)}_{R_{t_1} \cup \dots \cup R_{t_m} \cup \lambda} = s^{(k)}_{R_{t_1}} \dots s^{(k)}_{R_{t_m}} g^{(k)}_{\lambda}$. On the contrary, with $K$-$k$-Schur functions, the simplest case having a multiple $k$-rectangle gives

$$g^{(k)}_{R_t \cup R_t} = g^{(k)}_{R_t} \sum_{\lambda \subset R_t} g^{(k)}_{\lambda}. \tag{1.3}$$

Hence we cannot expect $g^{(k)}_{R_t \cup R_t}$ to be divisible by $g^{(k)}_{R_t}$ twice. Instead, upon organizing the part consisting of $k$-rectangles in the form $R^{a_1}_{t_1} \cup \cdots \cup R^{a_m}_{t_m}$ with $t_1 < \cdots < t_m$ and $a_i \geq 1$ $(1 \leq i \leq m)$, with $R^a_t = \underbrace{R_t \cup \cdots \cup R_t}_{a}$, actually we show that

$$g^{(k)}_{R^{a_1}_{t_1} \cup \cdots \cup R^{a_m}_{t_m} \cup \lambda} \text{ is divisible by } g^{(k)}_{R^{a_1}_{t_1} \cup \cdots \cup R^{a_m}_{t_m}},$$

which actually holds whether or not $\lambda$ is the remainder. Then we can subdivide our goal as follows:

**Question** $1'$**.** *Which $g^{(k)}_{\mu}$, besides $g^{(k)}_{\lambda}$, appear in the quotient $g^{(k)}_{P \cup \lambda} / g^{(k)}_{P}$ where $P = R^{a_1}_{t_1} \cup \cdots \cup R^{a_m}_{t_m}$, and with what coefficients?*

**Question** 2**.** *How can $g^{(k)}_{R^{a_1}_{t_1} \cup \cdots \cup R^{a_m}_{t_m}}$ be factorized?*

In this paper, we give a reasonably complete answer to Question 2 (Theorem 12), and partial answers to Question $1'$ (Theorems 13 to 15).

# 2 Preliminaries

In this section we review some requisite combinatorial backgrounds. First recall that the Pieri rule characterizes Schur functions. In the theory of $(K\text{-})k$-Schur functions, the underlying combinatorial objects are the set of *k-bounded partitions* (instead of partitions), which is isomorphic to the set of $(k+1)$-*cores*, and we have to consider *weak strips* instead of horizontal strips. For detailed definitions, see for instance [4, Chapter 2] or [8, Chapter I].

## 2.1 Partitions and Schur functions

Let $\mathcal{P}$ denote the set of partitions. A partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots) \in \mathcal{P}$ is identified with its *Young diagram* (or *shape*), for which we use the French notation here. We denote

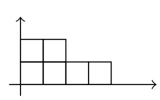

**Figure 1:** The Young diagram of $(4, 2)$.

the *size* of a partition $\lambda$ by $|\lambda|$, the *length* by $l(\lambda)$, and the *conjugate* by $\lambda'$. For partitions $\lambda, \mu$ we say $\lambda \subset \mu$ if $\lambda_i \leq \mu_i$ for all $i$. For a partition $\lambda$ and a cell $c = (i, j)$ in $\lambda$, we denote the *hook length* of $c$ in $\lambda$ by $\mathrm{hook}_c(\lambda) = \lambda_i + \lambda'_j - i - j + 1$.

For a partition $\lambda$, a *removable corner* of $\lambda$ (or $\lambda$-*removable corner*) is a cell $(i, j) \in \lambda$ with $(i, j+1), (i+1, j) \notin \lambda$. $(i, j) \in (\mathbb{Z}_{>0})^2 \setminus \lambda$ is said to be an *addable corner* of $\lambda$ (or $\lambda$-*addable* *corner*) if $(i, j-1), (i-1, j) \in \lambda$ with the understanding that $(0, j), (j, 0) \in \lambda$.

Let $\Lambda = \mathbb{Z}[h_1, h_2, \dots]$ be the ring of symmetric functions, generated by the *complete symmetric functions* $h_r = \sum_{i_1 \le i_2 \le \cdots \le i_r} x_{i_1} \dots x_{i_r}$.

The *Schur functions* $\{s_\lambda\}_{\lambda \in \mathcal{P}}$ are the family of symmetric functions satisfying the *Pieri rule*: $h_r s_\lambda = \sum s_\mu$, summed over $\mu$ such that $\mu/\lambda$ is a horizontal $r$-strip.

## 2.2 Bounded partitions, cores and $k$-rectangles $R_t$

A partition $\lambda$ is called *k-bounded* if $\lambda_1 \le k$. Let $\mathcal{P}_k$ be the set of all $k$-bounded partitions. An *r-core* (or simply a *core* if no confusion can arise) is a partition none of whose cells have a hook length equal to $r$. We denote by $\mathcal{C}_r$ the set of all $r$-core partitions.

*Hereafter we fix a positive integer k.*

For a cell $c = (i, j)$, the *content* of $c$ is $j - i$ and the *residue* of $c$ is $\mathrm{res}(c) = j - i \mod (k+1) \in \mathbb{Z}/(k+1)$. For a set $X$ of cells, we write $\mathrm{Res}(X) = \{\, \mathrm{res}(c) \mid c \in X \,\}$. We will write a $\lambda$-removable corner of residue $i$ simply a $\lambda$-removable $i$-corner. For simplicity of notation, we may use an integer to represent a residue, omitting "mod $(k+1)$".

We denote by $R_t$ the partition $(t^{k+1-t}) = (t, t, \dots, t) \in \mathcal{P}_k$ for $1 \le t \le k$, which is called a *k-rectangle*. Naturally a $k$-rectangle is a $(k+1)$-core.

Now we recall the bijection between the $k$-bounded partitions in $\mathcal{P}_k$ and the $(k+1)$-cores in $\mathcal{C}_{k+1}$: The map $\mathfrak{p}\colon \mathcal{C}_{k+1} \longrightarrow \mathcal{P}_k; \kappa \mapsto \lambda$ is defined by $\lambda_i = \#\{j \mid (i, j) \in \kappa,\ \mathrm{hook}_{(i,j)}(\kappa) \le k\}$. Then in fact $\mathfrak{p}$ is bijective and we put $\mathfrak{c} = \mathfrak{p}^{-1}$. See [6, Theorem 7] for details. Note that if $\lambda$ is contained in a $k$-rectangle then $\lambda \in \mathcal{P}_k$ and $\lambda \in \mathcal{C}_{k+1}$, and besides $\mathfrak{p}(\lambda) = \lambda = \mathfrak{c}(\lambda)$.

For $i = 0, 1, \dots, k$, an action $s_i$ on $\mathcal{C}_{k+1}$ is defined as follows: For $\kappa \in \mathcal{C}_{k+1}$,

- if there is a $\kappa$-addable $i$-corner, then let $s_i \cdot \kappa$ be $\kappa$ with all $\kappa$-addable $i$-corners added,
- if there is a $\kappa$-removable $i$-corner, then let $s_i \cdot \kappa$ be $\kappa$ with all $\kappa$-removable $i$-corners removed,
- otherwise, let $s_i \cdot \kappa$ be $\kappa$.

In fact the first and second case never occur simultaneously and $s_i \cdot \kappa \in \mathcal{C}_{k+1}$.

## 2.3 Weak order and weak strips

We review the weak order on $\mathcal{C}_{k+1}$.

**Definition 1.** *The weak order $\prec$ on $\mathcal{C}_{k+1}$ is defined by the following covering relation:*

$$\tau \prec\!\cdot\, \kappa \iff \exists i \text{ such that } s_i\tau = \kappa,\ \tau \subsetneq \kappa.$$

**Definition 2.** *For $(k+1)$-cores $\tau \subset \kappa \in \mathcal{C}_{k+1}$, $\kappa/\tau$ is called a weak strip of size $r$ (or a weak $r$-strip) when*

$$\kappa/\tau \text{ is horizontal strip and } \tau \prec\cdot\ \exists\tau^{(1)} \prec\cdot\ \ldots \prec\cdot\ \exists\tau^{(r)} = \kappa.$$

## 2.4 $k$-Schur functions

We recall a characterization of $k$-Schur functions given in [7], since it is a model for and has a relationship with $K$-$k$-Schur functions.

**Definition 3** ($k$-Schur function via "weak Pieri rule"). *$k$-Schur functions $\{s_\lambda^{(k)}\}_{\lambda\in\mathcal{P}_k}$ are the family of symmetric functions such that $s_\varnothing^{(k)} = 1$ and*

$$h_r s_\lambda^{(k)} = \sum_\mu s_\mu^{(k)} \quad \text{for } r \le k \text{ and } \mu \in \mathcal{P}_k,$$

*summed over $\mu \in \mathcal{P}_k$ such that $\mathfrak{c}(\mu)/\mathfrak{c}(\lambda)$ is a weak strip of size $r$.*

In fact $\{s_\lambda^{(k)}\}_{\lambda\in\mathcal{P}_k}$ forms a basis of $\Lambda^{(k)} = \mathbb{Z}[h_1,\ldots,h_k] \subset \Lambda$. In addition $s_\lambda^{(k)}$ is homogeneous of degree $|\lambda|$. It is proved in [7, Theorem 40] that

**Proposition 4** ($k$-rectangle property). *For $1 \le t \le k$ and $\lambda \in \mathcal{P}_k$, we have $s_{R_t\cup\lambda}^{(k)} = s_{R_t}^{(k)} s_\lambda^{(k)} (= s_{R_t} s_\lambda^{(k)})$.*

## 2.5 $K$-$k$-Schur functions $g_\lambda^{(k)}$

In [9] a combinatorial characterization of $K$-$k$-Schur functions is given via an analogue of the Pieri rule, using some kind of strips called *affine set-valued strips*.

For a partition $\lambda$, $(i,j) \in (\mathbb{Z}_{>0})^2$ is called *$\lambda$-blocked* if $(i+1,j) \in \lambda$.

**Definition 5** (affine set-valued strip). *For $r \le k$, $(\gamma/\beta,\rho)$ is called an affine set-valued strip of size $r$ (or an affine set-valued $r$-strip) if $\rho$ is a partition and $\beta \subset \gamma$ are cores both containing $\rho$ such that*

*(1) $\gamma/\beta$ is a weak $(r-m)$-strip where we put $m = \#\mathrm{Res}(\beta/\rho)$,*

*(2) $\beta/\rho$ is a subset of $\beta$-removable corners,*

*(3) $\gamma/\rho$ is a horizontal strip,*

*(4) For all $i \in \mathrm{Res}(\beta/\rho)$, all $\beta$-removable $i$-corners which are not $\gamma$-blocked are in $\beta/\rho$.*

In this paper we employ the following characterization [9, Theorem 48] of the $K$-$k$-Schur function as its definition.

**Definition 6** (*K*-*k*-Schur function via an "affine set-valued" Pieri rule)**.** *K-k-Schur functions* $\{g^{(k)}_\lambda\}_{\lambda\in\mathcal{P}_k}$ *are the family of symmetric functions such that* $g^{(k)}_\varnothing = 1$ *and for* $\lambda \in \mathcal{P}_k$ *and* $0 \le r \le k$,

$$h_r \cdot g^{(k)}_\lambda = \sum_{(\mu,\rho)} (-1)^{|\lambda|+r-|\mu|} g^{(k)}_\mu, \tag{2.1}$$

*summed over* $(\mu, \rho)$ *such that* $(\mathfrak{c}(\mu)/\mathfrak{c}(\lambda), \rho)$ *is an affine set-valued strip of size* $r$.

In fact $\{g^{(k)}_\lambda\}_{\lambda\in\mathcal{P}_k}$ forms a basis of $\Lambda^{(k)}$. Moreover, though $g^{(k)}_\lambda$ is an inhomogeneous symmetric function in general, the degree of $g^{(k)}_\lambda$ is $|\lambda|$ and its homogeneous part of highest degree is equal to $s^{(k)}_\lambda$.

# 3 Results

## 3.1 Possibility of factoring out $g^{(k)}_{R^{a_1}_{t_1}\cup\cdots\cup R^{a_m}_{t_m}}$ and some other general results

As discussed above, it does not hold that $g^{(k)}_{R_t\cup\lambda} = g^{(k)}_{R_t} g^{(k)}_\lambda$ for any $\lambda \in \mathcal{P}_k$. However, it holds that $g^{(k)}_{R_t}$ divides $g^{(k)}_{R_t\cup\lambda}$. We prove it in a slightly more general form.

The following notation is often referred later:

(NP) Let $1 \le t_1, \dots, t_m \le k$ be distinct integers and $a_i \in \mathbb{Z}_{>0}$ $(1 \le i \le m)$, where $m \in \mathbb{Z}_{>0}$. Then we put

$$P = R^{a_1}_{t_1} \cup \cdots \cup R^{a_m}_{t_m},$$
$$\alpha_P(u) = \#\{t_v \mid 1 \le v \le m,\ t_v \ge u\} \quad \text{for each } u \in \mathbb{Z}_{>0}.$$

**Proposition 7.** *Let* $P$ *be as in the above* (NP)*. Then, for* $\lambda = (\lambda_1, \cdots, \lambda_l) \in \mathcal{P}_k$*, we have* $g^{(k)}_P | g^{(k)}_{\lambda\cup P}$ *in the ring* $\Lambda^{(k)}$.

*Remark.* Note that $\lambda$ may still have the form $\lambda = R_t \cup \mu$. Hereafter we will not repeat the same remark in similar statements.

Since the homogeneous part of highest degree of $g^{(k)}_\lambda$ is equal to $s^{(k)}_\lambda$ for any $\lambda$, it follows from Propositions 4 and 7 that

**Corollary 8.** *Let* $P$ *be as in* (NP)*. Then, for any* $\lambda \in \mathcal{P}_k$*, we can write*

$$g^{(k)}_{P\cup\lambda} = g^{(k)}_P \left( g^{(k)}_\lambda + \sum_\mu a_{\lambda\mu} g^{(k)}_\mu \right),$$

*summing over* $\mu \in \mathcal{P}_k$ *such that* $|\mu| < |\lambda|$*, for some coefficients* $a_{\lambda\mu}$ *(depending on* $P$*).*

Now we are interested in finding a explicit description of $g^{(k)}_{P\cup\lambda}/g^{(k)}_{P}$. Let us consider the case $P = R_t$ for simplicity.

As noted above, a linear map $\Theta$ extending $g^{(k)}_{\lambda} \mapsto g^{(k)}_{R_t\cup\lambda}$ $(\forall\lambda \in \mathcal{P}_k)$ does not coincide with the multiplication of $g^{(k)}_{R_t}$ because it does not commute with the multiplication by $h_r$ in the first place.

However, we can prove that the restriction of $\Theta$ to the subspace spanned by $\{g^{(k)}_{R_t\cup\mu}\}_{\mu\in\mathcal{P}_k}$ (in fact this is the principal ideal generated by $g^{(k)}_{R_t}$) commutes with the multiplication by $h_r$, and thus it coincides with the multiplication of $\Theta(g^{(k)}_{R_t})/g^{(k)}_{R_t} = g^{(k)}_{R_t\cup R_t}/g^{(k)}_{R_t}$ on that ideal (Proposition 9). Thus it is of interest to describe the value of $g^{(k)}_{R_t\cup R_t}/g^{(k)}_{R_t}$, which is shown to be $\sum_{\nu\subset R_t} g^{(k)}_{\nu}$ later.

**Proposition 9.** *For $\lambda \in \mathcal{P}_k$ and $1 \le t \le k$, we have* $g^{(k)}_{\lambda\cup R_t\cup R_t} = g^{(k)}_{\lambda\cup R_t} \cdot \dfrac{g^{(k)}_{R_t\cup R_t}}{g^{(k)}_{R_t}}$.

As a corollary, it turns out that the value of $g^{(k)}_{P\cup\lambda}/g^{(k)}_{P}$ is independent of $a_1, \dots, a_m$, the "multiplicities" of $k$-rectangles.

**Theorem 10.** *Let $P = R^{a_1}_{t_1} \cup \cdots \cup R^{a_m}_{t_m}$ be as in* (NP)*, and put $Q = R_{t_1} \cup \cdots \cup R_{t_m}$. Then, for $\lambda \in \mathcal{P}_k$ we have*

$$\frac{g^{(k)}_{P\cup\lambda}}{g^{(k)}_{P}} = \frac{g^{(k)}_{Q\cup\lambda}}{g^{(k)}_{Q}}.$$

Thus we can reduce Question 1′ to the case where the $k$-rectangles are of all different sizes.

## 3.2 Answer to Question 2

For Question 2, we first show that multiple $k$-rectangles of different sizes entirely split, namely,

**Theorem 11.** *For $1 \le t_1 < \cdots < t_m \le k$ and $a_1, \dots, a_m > 0$,*

$$g^{(k)}_{R^{a_1}_{t_1}\cup\cdots\cup R^{a_m}_{t_m}} = g^{(k)}_{R^{a_1}_{t_1}} \cdots g^{(k)}_{R^{a_m}_{t_m}}.$$

Then we show that for each $1 \le t \le k$ and $a > 1$, we have a nice factorization generalizing the formula (1.3):

**Theorem 12.** *For $1 \le t \le k$ and $a > 0$, we have*

$$g^{(k)}_{R_t^a} = g^{(k)}_{R_t} \left( \sum_{\lambda \subset R_t} g^{(k)}_{\lambda} \right)^{a-1}.$$

*Thus, substituting this into Theorem 11, we have*

$$g^{(k)}_{R^{a_1}_{t_1} \cup \cdots \cup R^{a_n}_{t_n}} = g^{(k)}_{R_{t_1}} \left( \sum_{\lambda^{(1)} \subset R_{t_1}} g^{(k)}_{\lambda^{(1)}} \right)^{a_1-1} \cdots g^{(k)}_{R_{t_n}} \left( \sum_{\lambda^{(n)} \subset R_{t_n}} g^{(k)}_{\lambda^{(n)}} \right)^{a_n-1}.$$

### 3.3 (Partial) Answer to Question $1'$

An easiest case is where $\lambda = (r)$ consists of a single part, which generalizes the case (1.2) in Introduction. Namely we show that

**Theorem 13.** *Let $P, \alpha_P(u)$ be as in $(\mathsf{NP})$ and $1 \le r \le k$. Then we have*

$$\frac{g^{(k)}_{P \cup (r)}}{g^{(k)}_P} = \sum_{s=0}^{r} \binom{\alpha_P(r) + r - s - 1}{r - s} h_s.$$

*In particular, if $t_m < r$, which means $\alpha_P(r) = 0$, we have*

$$\frac{g^{(k)}_{P \cup (r)}}{g^{(k)}_P} = h_r = g^{(k)}_{(r)}$$

*On the other hand, when $m = 1$,*

$$\frac{g^{(k)}_{R_t \cup (r)}}{g^{(k)}_{R_t}} = \begin{cases} h_r & (\text{if } r > t), \\ h_r + h_{r-1} + \cdots + h_0 & (\text{if } r \le t). \end{cases}$$

Then generalizing this case, we derive explicit formulas in the cases where $\lambda = (\lambda_1, \ldots, \lambda_l)$ satisfies the following condition $(\mathsf{N}\lambda)$ and that the parts of $\lambda$ except for $\lambda_l$ are all larger than the widths of the $k$-rectangles.

$(\mathsf{N}\lambda)$ Let $(\varnothing \neq)\lambda \in \mathcal{P}_k$ with satisfying $\bar{\lambda} \subset R'_{\bar{l}}$, where we write $\bar{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)-1})$, $l = l(\lambda)$ and $\bar{l} = l(\bar{\lambda}) = l - 1$. (Here we consider $R_t$ to be $\varnothing$ unless $1 \le t \le k$)

(*Note*: when $l(\lambda) = 1$, we have $\bar{l} = 0$ and $\bar{\lambda} = \varnothing = R'_{\bar{l}}$ thus $\lambda$ satisfies $(\mathsf{N}\lambda)$. When $l(\lambda) > k + 1$, we have $\bar{l} > k$ and $\bar{\lambda} \neq \varnothing = R'_{\bar{l}}$ thus $\lambda$ does not satisfy $(\mathsf{N}\lambda)$.)

Namely, we prove that

**Theorem 14.** *Let $P$ and $\alpha_P(u)$ (for $u \in \mathbb{Z}_{>0}$) be as in* (NP) *in Section 3.1, before Proposition 7. Let $\lambda, l, \bar{\lambda}, \bar{l}$ be as in* (N$\lambda$) *above. Assume $\max_i\{t_i\} < \bar{\lambda}_{\bar{l}}$. Then we have*

$$g_P^{(k)} g_\lambda^{(k)} = \sum_{s=0}^{\lambda_l} (-1)^s \binom{\alpha_P(\lambda_l + 1 - s)}{s} g^{(k)}_{P \cup \bar{\lambda} \cup (\lambda_l - s)}. \tag{1}$$

$$g^{(k)}_{P \cup \lambda} = g_P^{(k)} \sum_{s=0}^{\lambda_l} \binom{\alpha_P(\lambda_l) + s - 1}{s} g^{(k)}_{\bar{\lambda} \cup (\lambda_l - s)}. \tag{2}$$

*In particular, if $t_n < \lambda_l$ then $\alpha_P(\lambda_l) = 0$ and*

$$g^{(k)}_{P \cup \lambda} = g_P^{(k)} g_\lambda^{(k)}.$$

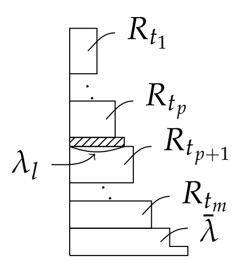


**Figure 2:** In this figure $p = m - \alpha_P(\lambda_l)$ and $a_i = 1$ for all $i$.

Moreover, we show a formula in a slightly different case where $P$ is a single $k$-rectangle $R_t$ and $\lambda = (\lambda_1, \ldots, \lambda_l)$ satisfies (N$\lambda$) and that the parts of $\lambda$ except for $\lambda_l$ are all larger than *or equal to* the widths of the $k$-rectangles.

*Notation.* For any partition $\lambda$, let $\lambda^\circ = (\lambda_1, \ldots, \lambda_i)$ if $\lambda_i > t \geq \lambda_{i+1}$ (we set $\lambda^\circ = \varnothing$ if $t \geq \lambda_1$).

**Theorem 15.** *Let $\lambda, l, \bar{\lambda}, \bar{l}$ be as in* (N$\lambda$). *Assume $\bar{\lambda}_{\bar{l}} \geq t \geq \lambda_l$. Then we have*

$$g^{(k)}_{R_t \cup \lambda} = g^{(k)}_{R_t} \sum_{\mathfrak{c}(\lambda^\circ) \subset \mathfrak{c}(\nu) \subset \mathfrak{c}(\lambda)} g_\nu^{(k)}.$$

## 3.4 Example

Let us illustrate the sketch of the proof of Theorem 14 with a small example.

Consider the case $P = R_t$: we shall show that $g^{(k)}_{R_t \cup \lambda} = g^{(k)}_{R_t} g^{(k)}_\lambda$ if $\lambda_l > t$ and $\lambda_1 + l \leq k + 2$. Let us assume Theorem 13 (the case $l = 1$) and consider the case $l = 2$. Set $\lambda = (a, b)$ with $k \geq a \geq b > t$.

Step (A): Expand $g^{(k)}_{(a,b)}$ into a linear combination of products of complete symmetric functions and $K$-$k$-Schur functions labeled by partitions with fewer rows:

By using the Pieri rule (2.1) we have

$$\begin{aligned} g^{(k)}_{(a)}h_i = &\left(g^{(k)}_{(a,i)} - g^{(k)}_{(a,i-1)}\right) \\ &+ \left(g^{(k)}_{(a+1,i-1)} - g^{(k)}_{(a+1,i-2)}\right) \\ &+ \dots \\ &\begin{cases} \dots + \left(g^{(k)}_{(a+i-1,1)} - g^{(k)}_{(a+i-1,0)}\right) & \text{(if } a+i \le k\text{)} \\ +g^{(k)}_{(a+i,0)} & \\ \dots + \left(g^{(k)}_{(k-1,a+i-k+1)} - g^{(k)}_{(k-1,a+i-k)}\right) & \text{(if } a+i > k\text{)} \\ + \left(g^{(k)}_{(k,a+i-k)} - g^{(k)}_{(k,a+i-k-1)}\right) & \end{cases} \end{aligned}$$

for $i \le a$, and summing this over $0 \le i \le b$, we have

$$\begin{aligned} g^{(k)}_{(a)}\left(h_b + \dots + h_0\right) &= g^{(k)}_{(a,b)} + g^{(k)}_{(a+1,b-1)} + \dots \begin{cases} g^{(k)}_{(a+b,0)} & \text{(if } a+b \le k\text{)} \\ g^{(k)}_{(k,a+b-k)} & \text{(if } a+b \ge k\text{)} \end{cases} \\ &= \sum_{\substack{\mu/(a)\text{:horizontal strip} \\ |\mu|=a+b \\ \mu_1 \le k}} g^{(k)}_{\mu}. \end{aligned} \tag{3.1}$$

Similarly we have

$$g^{(k)}_{(a+1)}\left(h_{b-1} + \dots + h_0\right) = g^{(k)}_{(a+1,b-1)} + g^{(k)}_{(a+2,b-2)} + \dots = \sum_{\substack{\mu/(a+1)\text{:horizontal strip} \\ |\mu|=a+b \\ \mu_1 \le k}} g^{(k)}_{\mu},$$

hence

$$g^{(k)}_{(a,b)} = g^{(k)}_{(a)}\left(h_b + \dots + h_0\right) - g^{(k)}_{(a+1)}\left(h_{b-1} + \dots + h_0\right).$$

Step (B): Multiply $g^{(k)}_{(a,b)}$ by $g^{(k)}_{R_t}$. Then we have

$$\begin{aligned} g^{(k)}_{R_t}g^{(k)}_{(a,b)} &= g^{(k)}_{R_t}g^{(k)}_{(a)}\left(h_b + \dots + h_0\right) - g^{(k)}_{R_t}g^{(k)}_{(a+1)}\left(h_{b-1} + \dots + h_0\right) \\ &= g^{(k)}_{R_t \cup (a)}\left(h_b + \dots + h_0\right) - g^{(k)}_{R_t \cup (a+1)}\left(h_{b-1} + \dots + h_0\right) \end{aligned}$$

because $g^{(k)}_{R_t}g^{(k)}_{(a)} = g^{(k)}_{R_t \cup (a)}$ since $t < a$. Then carry out calculations similar to Step (A).

*Notation.* For a proposition $P$, we shall write $\delta[P] = 1$ if $P$ is true and $\delta[P] = 0$ if $P$ is false.

Since the number of residues of $\mathfrak{c}(R_t \cup (a,j))$-nonblocked $\mathfrak{c}(R_t \cup (a))$-removable corners is $1 + \delta[t > j]$,

$$\begin{aligned} g^{(k)}_{R_t\cup(a)} h_i = &\left( g^{(k)}_{R_t\cup(a,i)} - \binom{1+\delta[t>i-1]}{1} g^{(k)}_{R_t\cup(a,i-1)} + \binom{1+\delta[t>i-2]}{2} g^{(k)}_{R_t\cup(a,i-2)} \right) \\ &+ \left( g^{(k)}_{R_t\cup(a+1,i-1)} - \binom{1+\delta[t>i-2]}{1} g^{(k)}_{R_t\cup(a+1,i-2)} + \binom{1+\delta[t>i-3]}{2} g^{(k)}_{R_t\cup(a+1,i-3)} \right) \\ &+ \dots . \end{aligned}$$

Summing this over $0 \le i \le b$, we have

$$\begin{aligned} g^{(k)}_{R_t\cup(a)} (h_b + \dots + h_0) = &\left( g^{(k)}_{R_t\cup(a,b)} - \delta[t > b-1]\, g^{(k)}_{R_t\cup(a,b-1)} \right) \\ &+ \left( g^{(k)}_{R_t\cup(a+1,b-1)} - \delta[t > b-2]\, g^{(k)}_{R_t\cup(a+1,b-2)} \right) + \dots \end{aligned}$$

Similarly we have

$$\begin{aligned} g^{(k)}_{R_t\cup(a+1)} (h_{b-1} + \dots + h_0) = &\left( g^{(k)}_{R_t\cup(a+1,b-1)} - \delta[t > b-2]\, g^{(k)}_{R_t\cup(a+1,b-2)} \right) \\ &+ \left( g^{(k)}_{R_t\cup(a+2,b-2)} - \delta[t > b-3]\, g^{(k)}_{R_t\cup(a+2,b-3)} \right) + \dots, \end{aligned}$$

hence we have

$$g^{(k)}_{R_t} g^{(k)}_{(a,b)} = g^{(k)}_{R_t\cup(a,b)} - \delta[t > b-1]\, g^{(k)}_{R_t\cup(a,b-1)} = g^{(k)}_{R_t\cup(a,b)}$$

since we have assumed $b > t$.

# 4 Discussions

It is worth noting that, in all cases we have seen, $g^{(k)}_{P\cup\lambda} / g^{(k)}_P$ is a linear combination of $K$-$k$-Schur functions with *positive coefficients*:

**Conjecture 16.** *For all $\lambda \in \mathcal{P}_k$ and $P = R^{a_1}_{t_1} \cup \dots \cup R^{a_m}_{t_m}$, write*

$$g^{(k)}_{P\cup\lambda} = g^{(k)}_P \sum_\mu a_{P,\lambda,\mu} g^{(k)}_\mu .$$

*Then $a_{P,\lambda,\mu} \ge 0$ for any $\mu$.*

In the case $P = R_t$, it is observed that $a_{R_t,\lambda,\mu} = 0$ or 1. Moreover, the set of $\mu$ such that $a_{R_t,\lambda,\mu} = 1$ is expected to be an "interval", but we have to consider the *strong order* on $\mathcal{P}_k \simeq \mathcal{C}_{k+1}$, which can be seen as just inclusion as shapes in the poset of cores. Namely, the strong order $\lambda \le \mu$ on $\mathcal{P}_k$ is defined by $\mathfrak{c}(\lambda) \subset \mathfrak{c}(\mu)$. Notice that $\lambda \preceq \mu \implies \lambda \subset \mu \implies \lambda \le \mu$ for $\lambda, \mu \in \mathcal{P}_k$. Then,

**Conjecture 17.** *For all $\lambda \in \mathcal{P}_k$ and $1 \le t \le k$, there exists $\mu \in \mathcal{P}_k$ such that*

$$g^{(k)}_{R_t \cup \lambda} = g^{(k)}_{R_t} \sum_{\mu \le \nu \le \lambda} g^{(k)}_{\nu}.$$

It will be interesting to study the geometric meaning of these results and conjectures.

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