Product Inequalities for $\mathbb{T}^{\rtimes}$-Stabilized Scalar Curvature

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Abstract. We study metric invariants of Riemannian manifolds X defined via the $\mathbb{T}^{\rtimes}$-*stabilized scalar curvatures* of manifolds Y mapped to X and prove in some cases additivity of these invariants under Riemannian products $X_1 \times X_2$.

Key words: scalar curvature; Riemannian manifold

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To the 75th birthday
of Jean-Pierre Bourguignon

1 $\mathbb{T}^{\rtimes}$-stabilization

A *"warped" $\mathbb{T}^N$-extension*, $N = 0, 1, \dots$, of a Riemannian manifold $X = (X, g)$, possibly with a boundary, is

$$X_N^{\rtimes} = X \rtimes \mathbb{T}^N = \left(X \times \mathbb{T}^N, g^{\rtimes}\right),$$

where $\mathbb{T}^N$ is the (flat split) N-torus and where $X_N^{\rtimes}$ is endowed with a *warped metric*,

$$g^{\rtimes} = g_N^{\rtimes} = g_{N,\{\varphi_i\}}^{\rtimes} = g + \sum_{i=1}^{N} \varphi_i^2 \, \mathrm{d}t_i^2,$$

where $\varphi_i(x) \geq 0$ are smooth positive functions, which are strictly positive (> 0) in the interior $X \setminus \partial X$ of X.

Assume g is smooth and let $\mathrm{Sc}_{\{\varphi_i\}}^{\rtimes}(X)$ be the scalar curvature of $g^{\rtimes}$, that is,

$$\mathrm{Sc}_{\{\varphi_i\}}^{\rtimes}(X) = \mathrm{Sc}(g_{N,\{\varphi_i\}}^{\rtimes}) = \mathrm{Sc}\left(g + \sum_{i=1}^{N} \varphi_i^2 \, \mathrm{d}t_i^2\right),$$

where $\mathrm{Sc}_{\{\varphi_i\}}^{\rtimes}(X) = \mathrm{Sc}_{\{\varphi_i\}}^{\rtimes}(X, x)$ is a function on X, since $g_N^{\rtimes}$ is invariant under the obvious action of $\mathbb{T}^N$ on $X_N^{\rtimes} = X \times \mathbb{T}^N$.

Let $\mathrm{Sc}^{\rtimes}(X)$, $X = (X, g)$, be the supremum of the numbers σ such that $\mathrm{Sc}_{N,\{\varphi_i\}}^{\rtimes}(X) > \sigma$ for some N and φ_i.

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⊃-Monotonicity. Clearly,

$$\mathrm{Sc}^{\rtimes}(Y) \geq \mathrm{Sc}^{\rtimes}(X)$$

for all smooth domains $Y \subset X$.

1.A. Formulas. A computation shows[1] that

$$\mathrm{Sc}\left(g + \sum_{i=1}^{N} \varphi_i^2 \, \mathrm{d}t_i^2\right) = \mathrm{Sc}(g) - 2\sum_{i=1}^{N} \frac{\Delta\varphi_i}{\varphi_i} - 2\sum_{i<j} \langle \nabla_g \log \varphi_i, \nabla_g \log \varphi_j \rangle.$$

For instance, if $N = 1$ and φ_1 is a positive eigenfunction of the operator $-\Delta + \frac{1}{2}\sigma$, for

$$\sigma(x) = \mathrm{Sc}(g, x),$$

that is,

$$\varphi(x) \mapsto -\Delta\varphi(x) + \frac{1}{2}\sigma(x)\varphi(x)$$

and

$$-\Delta\varphi_1 + \frac{1}{2}\sigma(x)\varphi_1 = \lambda_1\varphi_1,$$

then

$$\mathrm{Sc}\big(g + \varphi_1^2 \, \mathrm{d}t^2\big) = \sigma - 2\frac{\Delta\varphi_1}{\varphi_1} = \sigma - 2\frac{\frac{1}{2}\sigma\varphi_1 - \lambda\varphi_1}{\varphi_1} = 2\lambda_1.$$

Thus,

$$\mathrm{Sc}^{\rtimes}(X) \geq 2\lambda_1 = 2\lambda_1\left(\Delta + \frac{1}{2}\,\mathrm{Sc}(x)\right).$$

Recall at this point that if X is a compact connected manifold without boundary, then, for all functions $\sigma(x)$,

$$\begin{aligned}\lambda_1 &= \inf_{\varphi\neq 0}\left(\frac{-\int_X \varphi(x)\Delta\varphi(x)\mathrm{d}y}{\int_X \varphi(x)^2\mathrm{d}y} + \frac{1}{2}\frac{\int_X \sigma(x)\varphi^2(x)\mathrm{d}y}{\int_X \varphi(x)^2\mathrm{d}y}\right)\\ &\geq \inf_{\varphi\neq 0}\left(\frac{-\int_X \varphi(x)\Delta\varphi(x)\mathrm{d}y}{\int_X \varphi(x)^2\mathrm{d}y} + \frac{1}{2}\inf_{x\in X}\sigma(x)\right) \geq \frac{1}{2}\inf_{x\in X}\sigma(x),\end{aligned}$$

where the latter inequality follows from positivity of $-\Delta$ and this inequality is strict ($>$) with $\varphi = \varphi_1$, unless $\sigma(x)$ is constant.

Also, the strict inequality $\lambda_1 > \inf_{x\in X}\mathrm{Sc}(X, x)$ holds for the *Dirichlet eigenvalue* λ_1 on compact connected manifolds *with boundaries*, since the above relations are satisfied for functions $\varphi(x)$, which vanish on the boundary.

Next, now for all N and all φ_i, rewrite the above expression for $\mathrm{Sc}\big(g + \sum_{i=1}^{N}\varphi_i^2 \mathrm{d}t_i^2\big)$ with the function $\Phi(x) = \log(\varphi_1(x)\cdots\varphi(x)_N)$ as follows:

$$\mathrm{Sc}\left(g + \sum_{i=1}^{N} \varphi_i^2 \, \mathrm{d}t_i^2\right) = \mathrm{Sc}(g) - 2\Delta\Phi - ||\nabla\Phi||^2 - \sum_i ||\nabla \log \varphi_i||^2.$$

[1] See [10], [21, formulas (7.33) and (12.5)], also [46], and [17, Section 2.4.1].

This shows that $\mathrm{Sc}\big(g+\sum_{i=1}^{N}\varphi_i^2\,\mathrm{d}t_i^2\big)$ *increases* under replacing all φ_i by their *geometric mean*, $\varphi_i \rightsquigarrow \phi = \sqrt[N]{\prod_i \varphi_i}$, i.e.,

$$\mathrm{Sc}(g) - 2\Delta\Psi - \frac{N+1}{N}||\nabla\Psi||^2 \geq \mathrm{Sc}(g) - 2\Delta\Phi - ||\nabla\Phi||^2 - \sum_i ||\nabla \log\varphi_i||^2$$

for $\Psi = \log\phi^N$, where the equality holds only if all $\nabla\log\varphi_i$ are mutually equal. Hence,

$$\sup_{\varphi_i>0} \mathrm{Sc}\left(g + \sum_{i=1}^{N}\varphi_i^2\,\mathrm{d}t_i^2\right) = \sup_{\Psi}\mathrm{Sc}(g) - 2\Delta\Psi - \frac{N+1}{N}||\nabla\Psi||^2,$$

where this "sup" *increases* with N; thus, by letting $N\to\infty$, we see that

$$\begin{aligned}\mathrm{Sc}^{\rtimes}(X) &= \sup_{\Psi(x)}\inf_{x\in X}\mathrm{Sc}(X,x) - 2\Delta\Psi(x) - ||\nabla\Psi(x)||^2\\ &= \sup_{\psi(x)>0}\inf_{x\in X}\mathrm{Sc}(X,x) - 2\frac{\Delta\psi(x)}{\psi(x)} + \frac{||\nabla\psi(x)||^2}{\psi(x)^2}.\end{aligned}$$

Rewrite this equation with $\Psi = 2\Theta$ as follows:

$$\begin{aligned}\mathrm{Sc}^{\rtimes}(X) &= \sup_{\Theta}\inf_{x\in X}\mathrm{Sc}(X,x) - 4\big(\Delta\Theta(x) + ||\nabla\Theta(x)||^2\big)\\ &= \sup_{\theta}\inf_{x\in X}\mathrm{Sc}(X,x) - 4\frac{\Delta\theta(x)}{\theta(x)} \qquad \text{for} \quad \theta = \exp\Theta.\end{aligned}$$

Therefore, if X is compact, then

$$\mathrm{Sc}^{\rtimes}(X) \geq 4\lambda_1^{\rtimes}(X),$$

where $\lambda_1^{\rtimes}(X)$ is the lowest eigenvalue of the operator $-\Delta + \frac{1}{4}\mathrm{Sc}$ on X with the Dirichlet (vanishing on the boundary) condition.

(If a connected manifold X has no boundary and the scalar curvature of X is constant, then $\mathrm{Sc}^{\rtimes}(X) = \mathrm{Sc}(X)$; otherwise

$$\mathrm{Sc}^{\rtimes}(X) > \inf_{x\in X}\mathrm{Sc}(X,x).$$

Moreover,

$$\mathrm{Sc}^{\rtimes}(X) > \beta^{-1}\lambda_1\left(-\Delta + \beta\,\mathrm{Sc}(X)\right)$$

for all $\beta > 1/4$, since the operator $-\Delta$ is *strictly positive* on non constant functions on X.)

1.B. $\frac{1}{4}$-Proposition. Let $X = (X,g)$ be a compact Riemannian manifold with a boundary. Then

$$\mathrm{Sc}^{\rtimes}(X) = 4\lambda_1^{\rtimes}(X).$$

In fact, let $\theta_1(x) = \theta_1^{\rtimes}(x) \geq 0$, be the first Dirichlet eigenfunction of the operator $-\Delta + \frac{1}{4}\mathrm{Sc}$ and let $\theta(x) > 0$ be an arbitrary smooth strictly positive function on X.

Since θ_1 is strictly positive in the interior of X, the ratio $\frac{\theta_1(x)}{\theta(x)}$ assumes its maximum, call it a, at an interior point $x_0 \in X$, where

$$\theta_1(x_0) = a\theta(x), \qquad a\nabla\theta_1(x_0) = \nabla\theta(x) \qquad \text{and} \qquad \Delta\theta_1(x) \leq a\Delta\theta(x),$$

and consequently

$$-\frac{\Delta\theta_1(x_0)}{\theta_1(x_0)}+\frac{1}{4}\operatorname{Sc}(X,x_0)\geq-\frac{\Delta\theta(x_0)}{\theta(x_0)}+\frac{1}{4}\operatorname{Sc}(X,x_0).$$

Since the sum $-\frac{\Delta\theta_1(x)}{\theta_1(x)}+\frac{1}{4}\operatorname{Sc}(X,x)$ is constant ($=\lambda_1^{\rtimes}$), this inequality holds at the minimum point $x_o\in X$ of the function $-\frac{\Delta\theta(x)}{\theta(x)}+\frac{1}{4}\operatorname{Sc}(X,x)$; hence, $\inf_{x\in X}\operatorname{Sc}(X,x)-4\frac{\Delta\theta^{\rtimes}{}_1(x)}{\theta^{\rtimes}(x)}$ *majorizes* $\inf_{x\in X}\operatorname{Sc}(X,x)-4\frac{\Delta\theta_1(x)}{\theta(x)}$ for all smooth *strictly positive* functions on X.

Finally, by $\supset$-monotonicity of $\operatorname{Sc}^{\rtimes}$ and the continuity of the first eigenvalue $\lambda_1(-\Delta_g+\operatorname{Sc}(g)/4)$ in (the space of C^2-metrics) g, this majorization holds for functions θ which is strictly positive *only* in the interior of X.[2] Then the proof of the $\frac{1}{4}$-proposition follows.

$\frac{1}{4}$-Remark. This $\frac{1}{4}$ agrees with that in the Schrödinger–Lichnerowicz formula $\mathcal{D}^2=\nabla^*\nabla+\frac{1}{4}\operatorname{Sc}$ via the Kato inequality for the squared Dirac operator on $X^{\rtimes}=\big(X\times\mathbb{T}^N,g^{\rtimes}\big)$, which, to make it index-wise more interesting, may be twisted with the canonical N-parametric family of flat unitary complex line bundles over $X^{\rtimes}$. (Probably, there is much of what we do not understand about the relations between the two $\frac{1}{4}$.)

Corollaries/Examples

1.B$_1$. If X is *non-compact*, then

$$\operatorname{Sc}^{\rtimes}(X)=\lim_{i\to\infty}\operatorname{Sc}^{\rtimes}(X_i)$$

for compact equidimensional submanifolds $X_1\subset\cdots\subset X_i\subset\cdots\subset X$, which *exhaust* X.

1.B$_2$. $\operatorname{Sc}^{\rtimes}$ is additive under Riemannian products:

$$\operatorname{Sc}^{\rtimes}(X_1\times X_2)=\operatorname{Sc}^{\rtimes}(X_1)+\operatorname{Sc}^{\rtimes}(X_2).$$

For instance, the rectangular solids satisfy

$$\operatorname{Sc}^{\rtimes}\left(\times_1^n[-a_i,b_i]\right)=4\sum_1^n\lambda_1[a_i,b_i]=\sum_1^n\frac{4\pi^2}{(b_i-a_i)^2}.$$

1.B$_3$. Manifolds X with constant scalar curvature σ satisfy

$$\operatorname{Sc}^{\rtimes}(X)=4\lambda_1(X)+\sigma$$

for the first eigenvalue λ_1 of the Laplace operator on X.

For instance, unit hemispheres satisfy

$$\operatorname{Sc}^{\rtimes}(S^n_+)=n(n-1)+4n=n(n+3)$$

and the unit balls $B^n=B^n(1)\subset\mathbb{R}^n$ satisfy

$$\operatorname{Sc}^{\rtimes}(B^n)=4j_\nu^2,$$

for the first zero of the Bessel function J_ν, $\nu=\frac{n}{2}-1$, where $j_{-1/2}=\frac{\pi}{2}$, $j_0=2.4042\ldots$, $j_{1/2}=\pi$ and if $\nu>1/2$, then

$$\nu+\frac{a\nu^{\frac{1}{3}}}{2^{\frac{1}{3}}}<j_\nu<\nu+\frac{a\nu^{\frac{1}{3}}}{2^{\frac{1}{3}}}+\frac{3}{20}\frac{2^{\frac{2}{3}}a^2}{\nu^{\frac{1}{2}}},$$

where $a=\left(\frac{9\pi}{8}\right)^{\frac{2}{3}}(1+\varepsilon)\approx2.32$ with $\varepsilon<0.13\left(\frac{8}{2.847\pi}\right)^2$ [36].

[2] One needs to be slightly careful here, since $\Delta\theta/\theta$ may, a priori, blow up at the boundary of X.

Specifically,

$$\begin{aligned}
&\mathrm{Sc}^{\rtimes}\left(B^2\right) = 4(2.404\dots)^2 = 23.116\dots > 10 = \mathrm{Sc}^{\rtimes}\left(S^2_+\right),\\
&\mathrm{Sc}^{\rtimes}\left(B^3\right) > 36 > 18 = \mathrm{Sc}^{\rtimes}\left(S^3_+\right),\\
&\mathrm{Sc}^{\rtimes}\left(B^4\right) = 4(3,817\dots)^2 = 52.727\dots > 28 = \mathrm{Sc}^{\rtimes}\left(S^4_+\right),\\
&\mathrm{Sc}^{\rtimes}\left(B^8\right) = 4(6.380\dots)^2 = 162.827\dots > 88 = \mathrm{Sc}^{\rtimes}\left(S^8_+\right).
\end{aligned}$$

1.B$_4$. Ricci comparison inequality. Let X be a (metrically) complete Riemannian manifold with a boundary such that $\mathrm{Ricci}(X) \geq (n-1)\kappa$ and $\mathrm{mean.curv}(\partial x) \geq \mu$. Then

$$\mathrm{Sc}^{\rtimes}(X) \geq \mathrm{Sc}^{\rtimes}(\underline{B}^n_{\kappa,\mu}),$$

where $\underline{B}^n_{\kappa,\mu}$ is the ball in the complete simply connected n-space $\underline{S}^n_\kappa$ with sectional curvature κ, and where the mean curvature of the boundary $\partial\underline{B}^n_{\kappa,\mu}$ is equal to μ.[3]

For instance, if $\mathrm{Ricci}(X) \geq 0$ and $\mathrm{mean.curv}(\partial X) \geq n-1$, then

$$\mathrm{Sc}^{\rtimes}(X) \geq \mathrm{Sc}^{\rtimes}(B^n = \underline{B}^n_{0,n-1}) = 4j_1^2.$$

In fact, let $\varphi(b) = \phi(d(b)) = \phi_{\kappa,\mu}(d)$ be the first Dirichlet eigenfunction in $B^n_{\rho,\mu}$ written as a function of $d = d(b) = \mathrm{dist}(b, \partial(\underline{B}^n_{\kappa,\mu}))$ and let $\varphi(x) = \phi(\mathrm{dist}(X, \partial x))$. Then, by the Bishop comparison inequality,

$$\frac{\Delta_X\varphi(x)}{\varphi(x)} \geq \frac{\Delta_{\underline{S}^n_\kappa}\phi(d(b))}{\phi(d(b))} = \lambda_1(\Delta_{\underline{B}^n_\kappa}), \qquad d(b) = \mathrm{dist}(X, \partial x),$$

and the proof follows.

1.B$_5$. If $B^n_{-1}(r)$ is the hyperbolic r-ball, then, clearly, $\mathrm{Sc}^{\rtimes}(B^n_{-1}(r))$ is monotonically decreasing in r, asymptotically to $4\frac{j_\nu^2}{r^2}$ for $r \to 0$ and $\mathrm{Sc}^{\rtimes}(B^n_{-1}(r)) \to -(n-1)$ for $r \to \infty$.

In fact, it follows from [1, Theorem 3.3] that

$$\mathrm{Sc}^{\rtimes}(B^n_{-1}(r)) = -n(n-1) + (n-1)^2\left(\frac{1}{r^2} + c(r)\right)$$

for a bounded positive function $c(r)$ such that $c(r) \to 1$ for $r \to \infty$, and

$$\frac{1}{6} \leq c(r) \leq 1 \qquad \text{for } r \geq 1 \text{ and } n \geq 2.$$

Thus,

$$\mathrm{Sc}^{\rtimes}(B^n_{-1}(r)) > 0 \qquad \text{for } r \leq \sqrt{\frac{6(n-1)}{5n+1}}, \qquad \mathrm{Sc}^{\rtimes}(B^n_{-1}(r)) < 0 \qquad \text{for } r \geq 3 \text{ and } n \geq 2$$

roughly.

1.C. General torical stabilizations. The most permissive torical "extension" of a Riemannian manifold X is a Riemannian manifold $X^\natural$ with an isometric $\mathbb{T}^N$-action and an isometry $X^\natural/\mathbb{T}^N \leftrightarrow X$. Here, as earlier, one defines the number $\mathrm{Sc}^\natural(X)$, which is clearly $\geq \mathrm{Sc}^{\rtimes}(X)$.

It seems, however – I did not honestly checked this – that the curvature formulas for Riemannian submersions [35] imply that $\mathrm{Sc}^\natural(X) \leq \mathrm{Sc}^{\rtimes}(X)$.

[3]The corresponding, comparison inequality for the Dirichlet (Schrödinger) $\lambda_1(\Delta)$ (compare with [7] and [1]) has, undoubtedly, been known for at least 45 years and the relation $\lambda_1 = -\sup_{\varphi>0}\inf_{x\in X}\frac{\Delta\varphi(x)}{\varphi(x)}$ must be dated to the 19th century. My apologies to the author(s), whose paper(s) I failed to find on the web.

Alternatively, if the fibration $X^\natural \to X^\natural/\mathbb{T}^N = X$ *admits a section*, then the $\rtimes$-rendition of the Schoen–Yau argument[4] implies the equality

$$\mathrm{Sc}^\natural(X) = \mathrm{Sc}^\rtimes(X)$$

for $\dim(X) = n \leq 8$,[5] while for all n this may follow from [43], where both arguments apply not only to Riemannian submersions but to all distance non-increasing maps $\big(X \times \mathbb{T}^N, G\big) \to (X, g)$.

1.D. $\boldsymbol{\lambda_1(\beta)}$-Remarks. The formal properties of the operators $-\Delta + \beta\,\mathrm{Sc}$ are similar for all β, e.g., if a Riemannian manifold X_0 admits a locally isometric equidimensional map to X, then

$$\lambda_1(-\Delta_{X_0} + \beta\,\mathrm{Sc}(X_0)) \geq \lambda_1(-\Delta_X + \beta\,\mathrm{Sc}(X));$$

the spectra $\{\lambda_1, \lambda_2, \dots\}$ of the operators $-\Delta + \beta\,\mathrm{Sc}$ are additive under Riemannian products,

$$\mathrm{spec}(-\Delta_{X_1\times X_2} + \beta\,\mathrm{Sc}(X_1 \times X_2)) = \mathrm{spec}(-\Delta_{X_1} + \beta\,\mathrm{Sc}(X_1)) + \mathrm{spec}(-\Delta_{X_2} + \beta\,\mathrm{Sc}(X_2)).$$

There are several special values of β:

- if $\beta = (N+1)/4N$ $(> 1/4)$, then the corresponding λ_1 is equal to the maximal constant scalar curvature of the warped metrics on $X \times \mathbb{T}^N$;
- if $\beta = (n-1)/4n$ $(< 1/4)$, $n = \dim(X)$, this implies positivity of the square of the Dirac operator by *refined Kato's inequality* [24].
- if $\beta = (n-2)(4(n-1))$ $(< (n-1)/4n)$, then the inequality $\lambda_1 > 0$ implies the existence of a conformal metric on X with $\mathrm{Sc} > 0$ by the Kazdan–Warner theorem.

The geometric meaning of other β, as well as of the higher eigenvalues $\lambda_i(X, \beta)$ of $-\Delta + \beta\,\mathrm{Sc}$ is unclear.[6]

2 $\mathrm{Sc}^{\rtimes\downarrow}$, $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{sp}}$, …, $\mathrm{Sc}^{\rtimes\downarrow}_{*}$ on homology

Let X be a metric space, e.g., a Riemannin manifold $\mathrm{Sc}^{\rtimes\downarrow}(h) = \mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{dist}}(h)$, $h \in H_m(X, \partial X)$, denote the supremum of the numbers σ such that the homology class h is representable by a *distance decreasing* map f from an oriented Riemannian m-manifold Y with $\mathrm{Sc}^\rtimes(Y) \geq \sigma$,

$$f\colon\ (Y, \partial Y) \to (X, \partial X), \qquad f_*[Y, \partial Y] = h.$$

Smoothness remark. If X is a smooth Riemannian manifold, then an obvious approximation argument shows that requiring maps f to be smooth does not change the value of $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{dist}}(h)$. (However, smoothness of a *distance non-increasing* map f in the *extremal case*, where $\mathrm{Sc}^\rtimes(Y) = \mathrm{Sc}^{\rtimes\downarrow}(h)$ is a delicate matter, see [5].)

Below the are several versions of this definition with the generic notation $\mathrm{Sc}^{\rtimes\downarrow}_*$.

I. Restrict/relax the topology of Y, e.g., by requiring that

$\bullet_{\mathrm{sp}}$ Y is spin;[7]

[4]See [18] and references therein.

[5]The case $n = 8$ depends on [40].

[6]The second eigenvalue of $\lambda_2(-\Delta + \frac{1}{2}\,\mathrm{Sc})$ is used by Marques and Neves in the proof of the S^3-min-max theorem, [33], but the role of $\lambda_i(X, \beta)$ remains problematic for $\dim(X) \geq 4$, $i \geq 2$ and all β.

[7]A referee suggested to constrain maps rather then manifolds Y, e.g., by allowing spin maps only (in the case, where X is a manifold), but I could not figure out what to do with it.

- $\bullet_{\widetilde{\mathrm{sp}}}$ the universal covering of Y is spin;[8]
- $\bullet_{\pi_2=0}$ the second homotopy group of Y is zero;
- $\bullet_{\mathrm{st.par}}$ Y is stably parallelizable;
- $\bullet_{\odot}$ allow representation of h by *quasi-proper* maps from complete manifolds Y to X, where "*quasi-proper*" means *locally constant at infinity.*

Clearly,

$$\mathrm{Sc}^{\rtimes\downarrow}_{\odot} \geq \mathrm{Sc}^{\rtimes\downarrow} \geq \mathrm{Sc}^{\rtimes\downarrow}_{\widetilde{\mathrm{sp}}} \geq \mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{sp}} \geq \mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{st.par}} \qquad \text{and} \qquad \mathrm{Sc}^{\rtimes\downarrow}_{\widetilde{\mathrm{sp}}} \geq \mathrm{Sc}^{\rtimes\downarrow}_{\pi_2=0}\,.$$

II. Assuming X is a Riemannian manifold, relax the distance decreasing condition on f by the following

$\bullet_{\mathrm{area}}$ the map f *decreases the areas* of all surfaces in Y.[9]

Clearly, $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area}} \geq \mathrm{Sc}^{\rtimes\downarrow}$ and we show in §2.E below that the ratio $\frac{\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area}}}{\mathrm{Sc}^{\rtimes\downarrow}}$ can be arbitrarily large.

III. Replace the integer homology by the rational homology $H_*(X;\mathbb{Q})$, which is essentially (but not quite) the same as allowing maps $f\colon Y \to X$, where f_* sends the fundamental class of Y to a *non-zero multiple of* h.

IV. Instead of homology, use a bordism group of X, e.g., the spin bordism group, which is well adapted to $\mathrm{Sc} > 0$.

V. *Remarks on singular* Y.

(a) If $h_m \in H_m(X)$ is not representable as $f_*[Y]$ for a smooth manifold Y, it may be interesting to try pseudomanifolds Y with suitably defined singular Riemannian metrics g with $\mathrm{Sc}(g) \geq \sigma$.

(b) *Conical example.* Here Y has an isolated singularity at a point $y_0 \in Y$, where g is a smooth Riemannian metric on the complement to y_0 such that $\mathrm{Sc}(g) \geq \sigma$ and such that g is (approximately) *conical* at y_0.

This means that there exists an ε-neighbourhood (ball) $U = U_\varepsilon \subset Y$ of y_0, which topologically splits away from y_0,

$$U \setminus \{y_0\} = Z \times (0,\varepsilon],$$

where $Z = (Z, g_Z)$ is a compact smooth Riemannian manifold such that the metric g restricted to $U \setminus \{y_0\}$ is related to g_Z as follows:

$$g = a^2(t)t^2 g_Z + \mathrm{d}t^2,$$

where $a^2(t) > 0$ is a smooth positive function on the (now closed) interval $[0,\varepsilon]$. (One may assume, if one wishes, that $a(t)$ is constant near $t = 0$.)

(c) One may additionally assume that $\mathrm{Sc}(g_Z) \geq \mathrm{Sc}\big(S^{m-1}\big) = (m-1)(m-2)$ for $m = \dim(Y)$ and, to make the metric truly conical, to require $a(t)$ to be constant near $t = 0$.

But this is not truly needed, since it can always be achieved by a small deformation of our g near y_0.

(d) *Iterated conical singularities.* Next, following [8], define m-dimensional (roughly) *cone-singular spaces* Y with $\mathrm{Sc}(Y) \geq \sigma$ by induction on m, where (as in the above conical case) the metric (i.e., the distance function) on Y is defined by a smooth Riemannian metric g on the non-singular part $Y_0 \subset Y$, where the following conditions are satisfied:

(i) The singular locus $\Sigma = Y \setminus Y_0$ is a closed subset in Y with *codimension two* in Y.

(ii) $\mathrm{Sc}(g, y) \geq \sigma$ for all $y \in Y_0$.

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[8] This condition is satisfied in several interesting examples of *non-spin* manifolds X, e.g., where $\pi_2 = 0$. At the same time, much of the Dirac theoretic scalar curvature results apply to these X, see [13, Section $9\frac{1}{2}$].

[9] This makes sense for general metric spaces X with the "Hilbertian area" defined in [15].

(iii) Each $y_0 \in Y$ admits a neighbourhood U_0, which is topologically (but not metrically) cylindrically splits away from y_0,

$$U_0 \setminus \{y_0\} = Z_0 \times (0, \varepsilon_0],$$

where $Z_0 = (Z_0, g_{Z_0})$, is a compact $(m-1)$-dimensional cone-singular space and where the Riemannian metric g on the non-singular part of $U_0 \setminus \{y_0\}$, denoted $U_{00} \subset U_0 \setminus \{y_0\}$, is

$$g = a_0^2(t) t^2 g_{Z_0} + \mathrm{d}t^2 + \delta_0,$$

where $a_0^2(t) > 0$ is a smooth positive function on $[0, \varepsilon_0]$ and where $\delta_0 = \delta_0(u)$ is a (small) smooth quadratic differential form on U_0, which *converges to zero* for $u \to y_0$.

(e) Due to h_0, the above "conical" is slightly more general than how it is defined in (b) for an isolated singularity y_0.

(f) Similarly to the isolated singularity case, the requirement $\mathrm{Sc}(Z_0) \geq m-1)(m-2)$ does not significantly change the definition of $\mathrm{Sc}(Y) \geq \sigma$.

(g) One may also insist on the split-conical geometry at all points y_0: *if y_0 is contained in the interior of an l-dimensional strata $S \subset \Sigma$, then a small neighbourhood $U_0 \subset Y$ of y_0 metrically splits: $U_0 = S_0 \times N_0$, where $S_0 = U_0 \cap S$, and where N_0 is a con-singular manifold with an $(m-l-1)$-dimensional base.*

(h) Probably, as in the isolated singularity case, this additional condition can be achieved by a small deformation of g near Σ.

Question. How does the resulting $\mathrm{Sc}^{\rtimes\downarrow}(h)$, $h \in H_m(X)$, depend on the topology of the singular locus $\Sigma \subset Y$?

For instance, Y may be iterated conical space with initial cones based on products of complex projective spaces and/or other generators of the oriented bordisms groups with $\mathrm{Sc} > 0$ metrics (e.g., as in [21]), where we allow cones over l-dimensional Y only if they admit metrics g with "suitably defined" $\mathrm{Sc}(g) > 0$ and/or, which is probably equivalent with $\mathrm{Sc}^{\rtimes}(g) > 0$.

(Probably, stable minimal hypersurfaces and μ-bubbles in such Y, similarly to how it is in a smooth Y, enjoy necessary properties required for the study of the scalar curvature and this is also conceivable for the Dirac theoretic approach (compare with [2]).)

VI. If X is non-compact, allow classes h with infinite supports[10] and use proper (and quasi-proper) maps $f\colon Y \to X$.

Remark on Completeness of Y. Regardless of X being complete or not, the value $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area}}[X]$ defined with complete Y mapped to X may be very different without this completeness.

For instance, $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area,sp}}[\mathbb{R}^n]$ defined with non-complete Y is *infinite.*

In fact, if g_0 is a metric on $\mathbb{R}^2$ such that $\mathrm{Sc}(g_0) = 1$ and $\mathrm{area}(Y_0, g_0) = \infty$ (as in §2.A below), then $Y = \big(\mathbb{R}^{n-2} \times \mathbb{R}^2, g_{\mathrm{Eucl}} + g_0\big)$ admits an obvious area contracting diffeomorphism onto $\big(\mathbb{R}^N, g_{\mathrm{Eucl}}\big)$.

But if we limit to *complete spin* manifolds Y, then $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area,sp,compl}}[\mathbb{R}^n] = 0$.

(It is unknown for $n \geq 4$ whether $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area,compl}}[\mathbb{R}^n]$ is zero or infinity without the spin assumption on Y.)

2.A. Surface examples. Closed connected simply connected, i.e., *spherical* Riemann surfaces X satisfy

$$\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area}}[X] = \frac{8\pi}{\mathrm{area}(X)}.$$

[10] Sometimes referred to as "locally finite homology classes" as was pointed out to me by a referee.

Indeed, the inequality $Sc^{\rtimes\downarrow}_{area}[X] \geq \frac{8\pi}{\text{area}(X)}$ follows from the existence of a measure preserving diffeomorphism from the 2-sphere with constant scalar curvature $\sigma = \frac{8\pi}{\text{area}(X)}$ onto X; the opposite inequality follows from *Zhu's lemma* (see [46] and [17, Section 2.8]).

Similarly, one shows that $Sc^{\rtimes\downarrow}_{area}[X]$ for closed surfaces X of positive genera.

On the opposite end of the spectrum, non-compact connected surfaces X satisfy $Sc^{\rtimes\downarrow}_{area}[X] = \infty$, since all the surfaces X admit Riemannian metrics with $Sc = 1$, and with the given areas (including area $= \infty$ for non-compact X) and since connected mutually diffeomorphic Riemann surfaces X_1 and X_2 of equal areas admit area preserving diffeomorphisms $X_1 \leftrightarrow X_2$.

Problem with $Sc^{\rtimes\downarrow}[X] = Sc^{\rtimes\downarrow}_{dist}[X]$. Unlike $Sc^{\rtimes\downarrow}_{area}$, the geometric meaning of $Sc^{\rtimes\downarrow}[X]$ for spherical surfaces X remains obscure. All one knows besides Zhu's lemma for general X (see [17, Section 2.8]) is that

$$Sc^{\rtimes\downarrow}[X] < \frac{4\pi^2}{\text{diam}(X)^2}.$$

2.B. $H_{2m}(\mathbb{C}P^n)$-example. Let the complex projective space $\mathbb{C}P^n$ be endowed with the $U(n+1)$ invariant (Fubini–Study)-metric such that the projective lines have scalar curvatures equal 2 and let $\mathbb{C}P^m \hookrightarrow \mathbb{C}P^n$ be an m-plane.

If m is odd, then the manifold $\mathbb{C}P^m$ is spin, and both homology and the spin-bordism class of $\mathbb{C}P^m \hookrightarrow \mathbb{C}P^n$ satisfy

$$Sc^{\rtimes\downarrow}[\mathbb{C}P^m] \geq Sc^{\rtimes\downarrow}_{sp}(k[\mathbb{C}P^m]) \geq Sc^{\rtimes\downarrow}[\mathbb{C}P^m]_{sp.brd} \geq Sc(\mathbb{C}P^m) = m(m+1)$$

and the same holds for the multiples $k[\mathbb{C}P^m] \in H_{2m}(\mathbb{C}P^n)$, $k = \ldots, -1, 0, 1, 2, \ldots$.

If k is even, then if m is *even*, then the $Sc^{\rtimes\downarrow}\, k[\mathbb{C}P^m] \geq m(m+1)$ remains valid for all k. But if k is *odd*, then $Sc^{\rtimes\downarrow}_{sp}(k[\mathbb{C}P^m]) = 0$, since the classes $k[\mathbb{C}P^m] \in H_{2m}(\mathbb{C}P^n)$ are *not representable by the maps from spin manifolds* $Y \to \mathbb{C}P^n$.

In fact, if an oriented manifold Y^{2m} contains a smooth hypersurface H such that the m-fold self-intersection index $\underbrace{H \frown \cdots \frown H}_{m}$ is odd, then the $(m-1)$-fold intersection is an orientable surface $\Sigma \subset Y$, which for even m has non-trivial normal bundle; hence $w_2[\Sigma]_{\mathbb{Z}_2} \neq 0$.

And if k is *even*, then

$$Sc^{\rtimes\downarrow}_{sp}(k[\mathbb{C}P^m]) \geq m^2$$

since the class $2[\mathbb{C}P^m])$ is represented by the quadric $Q^m \subset \mathbb{C}P^{m+1} \subset \mathbb{C}P^n$ given by the equation $z_0^2 + z_1^2 + \cdots + z_m^2 = 0$, where this Q^m is spin and has scalar curvature m^2.

Finally,

$$Sc^{\rtimes\downarrow}_{st.par}(m!h_{2m}) \geq \text{const}_m \cdot Sc\big(\underbrace{S^2(1) \times \cdots \times S^2(1)}_{m}\big) = 2m \qquad \text{for all } m \text{ and } n \geq m,$$

since the quotient space

$$\big(S^2\big)^m/\Pi(m) \qquad \text{of} \quad \big(S^2\big)^m = \underbrace{S^2(1) \times \cdots \times S^2(1)}_{m}$$

by the permutation group $\Pi(m)$ admits a natural biholomorphic map $\psi\colon \big(S^2\big)^m \to \mathbb{C}P^m$, where $\text{const}_m > 0$ is the squared reciprocal to the minimal Lipschitz constant of maps in the homotopy class of this ψ.

Question. What are $Sc^{\rtimes\downarrow}\big[\big(S^2\big)^m/\Pi(m)\big]$ and of the symmetric powers $[(X)^m/\Pi(m)]$ for more general manifolds X?[11]

[11] These are among most attractive singular quasi-conical spaces discussed earlier.

2.C. Upper bounds and equalities. The ($\mathbb{T}^{\rtimes}$-stabilized and $\widetilde{\text{sp}}$-generalized) rigidity theorem by Min-Oo [34] and (the spin cobordims version of) Goette–Semmelmann's theorem from [11] imply that the class $h_{2m} = [\mathbb{C}P^m] \in H_{2m}(\mathbb{C}P^m)$ satisfy the following relations:

$$\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area,sp.brd}\mathbb{C}P^m}(h_{2m}) = m(m+1) \qquad \text{for all } m \text{ and } n \geq m,$$

where "sp.brd" indicates that this $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area}}$ defined with smooth maps $Y \to X$ which are spin-bordant to the embedding $\mathbb{C}P^m \hookrightarrow \mathbb{C}P^n$,[12]

$$\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area},\widetilde{\mathrm{sp}}}(h_{2m}) = m(m+1) \qquad \text{for odd } m,$$
$$\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area},\widetilde{\mathrm{sp}}}(h_{2m}) = m^2 \qquad \text{for even } m,$$
$$\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area,st.par}}(h_{2m})_{\mathrm{area}} = 2m \qquad \text{for all } m \text{ and } n \geq 2m-1.$$

2.D. Homological homogeneity conjecture. Let X be a compact symmetric space and $H \subset H_m(X, \mathbb{Q})$ be the linear subspace generated by the fundamental classes $[Y_i] \in H_m(X)$ of *homogeneous* (not necessarily totally geodesic) m-submanifolds $Y_i \subset X$.[13] Then all classes $h_m \in H$ can be represented by linear combinations of homogeneous Y_i such that $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area}}[Y_i] \geq \mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area}}(h)$. (This maybe overoptimistic in general, but the $\widetilde{\text{sp}}$-version of this can be, probably, proved with available means for products of spheres, complex and quaternionic projective spaces.)

2.E. Equivalence conjecture. All rational $h \in H_m(X)$ for compact Riemannian manifolds X without boundaries satisfy:[14]

$$\mathrm{Sc}^{\rtimes\downarrow}_{\odot}(h) = \mathrm{Sc}^{\rtimes\downarrow}(h) \qquad \text{and} \qquad \mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{sp},\odot}(h) = \mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{sp}}(h),$$
$$\mathrm{Sc}^{\rtimes\downarrow}(h) \leq A \cdot \mathrm{Sc}^{\rtimes\downarrow}_{\widetilde{\mathrm{sp}}}(h),$$
$$\mathrm{Sc}^{\rtimes\downarrow}_{\widetilde{sp}}(h) \leq B \cdot \mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{sp}}(h),$$
$$\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{sp}}(h) \leq C \cdot \mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{st.par}}(h),$$

where $A = A_n$, $B = B_n$ and $C = C_n$ are universal constants, and where the same relations are expected for the area version of these five $\mathrm{Sc}^{\rtimes\downarrow}$.

2.F. Positivity. Unlike $\mathrm{Sc}^{\rtimes}$, the values of all $\mathrm{Sc}^{\rtimes\downarrow} = \mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{dist}}$-invariants are *non-negative*, since all (compact or not) manifolds Y admit arbitrarily large Riemannian metrics with $\mathrm{Sc} > \varepsilon$.

Moreover, the fundamental classes of *compact* connected manifolds X with *non-empty boundaries* are *strictly positive* since such manifolds admit metrics with $\mathrm{Sc} > 0$. (For instance, the r-balls with hyperbolic metrics g, $\mathrm{sect.curv}(g) = -1$, admit (obvious radial) metrics $g_+ \geq g$ with $\mathrm{Sc}(g_+) \geq \exp -4r$.)

2.G. [$\exists\,\mathrm{Sc} > 0$]-Conjecture. If a rational homology class $h \in H_m(X; \mathbb{Q})$ vanishes under the classifying map $\beta\colon X \to \mathsf{B}(\pi_1(X))$

$$\beta_*(h) = 0,$$

then $\mathrm{Sc}^{\rtimes\downarrow}(h) > 0$.[15]

2.H. Finiteness. $\mathrm{Sc}^{\rtimes\downarrow}(h)$ may be, a priori, infinite. However, the finiteness of $\mathrm{Sc}^{\rtimes\downarrow} = \mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{dist}}$ easily follows from the $\square^m$-inequality (3.8) in [17], where the proof for $m \geq 9$ relies on Theorem 4.6 in [39] and where the finiteness of $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{sp}}(h)$ for all m follows from [43]. (Probably, the arguments used in [43] generalize to $\mathrm{Sc}^{\rtimes\downarrow}_{\widetilde{\mathrm{sp}}}$.)

[12] It is unclear what happens for $m \leq n \leq 2m-2$.
[13] $Y \subset X$ is *homogeneous* if an isometry group of X preserves Y and is transitive on Y.
[14] The dimension $m = 4$ may be special.
[15] This seems more realistic if β can be homotoped to the $(m-2)$-skeleton of (some cell decomposition of) $\mathsf{B}(\pi_1(X))$.

2.I. $\nexists$ Sc > 0-Problem. Does *non-vanishing* of a rational $h \in H_m(X;\mathbb{Q})$ under the above classifying map $\mathsf{B}(\pi_1(X))$ imply that $\mathrm{Sc}^{\rtimes\downarrow} = 0$? This is known for $m = 3$, and also for $\mathrm{Sc}^{\rtimes\downarrow}_{\widetilde{\mathrm{sp}}}(h)$ and all m if the *spinorial curvature* $\mathbb{S}\mathsf{p}.\mathrm{curv}^{\downarrow}(\beta_*(h) \in H_m(\mathsf{B}(\pi_1(X))$ defined in Section 7 *vanishes*, e.g., if our $\mathsf{B}(\pi_1)$ admits a complete metric with sect.curv ≤ 0, see [17] and references therein. (I am not certain if there are examples of non-zero rational homology classes $\underline{h}$ in aspherical, say, compact finite-dimensional spaces such that $\mathbb{S}\mathsf{p}.\mathrm{curv}^{\downarrow}(\underline{h}) \neq 0$.)

2.J. $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area}}$-finiteness question. Is $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area}}[X] < \infty$ for all compact Riemannian manifolds X without boundaries?

Remarks. (a) All metrics g_+ on a compact Riemannian manifold (X, g) such that $\mathrm{area}_{g_+}(\Sigma) \geq \mathrm{area}_g(\Sigma)$ for all surfaces $\Sigma \subset X$ satisfy

$$\mathrm{Sc}^{\rtimes}(g_+) \leq \mathrm{const} \cdot (X, g) < \infty.$$

In fact, this inequality holds for all (Y, g_+) what admit area decreasing *spin maps*[16] $f\colon Y \to X$ with non-zero degrees.

(b) Let $X = X_0 \times Y$, where Y is enlargeable[17] and $\dim(X_0) = 2$. Then the finiteness of $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area}}(X)$ for X follows from [47], where for $\dim(X) \geq 8$ one needs a version of Theorem 4.6 from [39].

Also the $\rtimes$-stabilized version of the area slicing theorem from [28] (this stabilization is likely to be true) delivers an effective finite bound on $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area}}(X_0 \times Y)$. For $\dim(X_0) = 3$, provided Y is enlargeable and $\dim(X) \leq 8$.

But the principal case, where $X = S^n$ remains problematic for all $n \geq 4$ and neither can one prove or disprove the existence of (necessarily non-spin) *complete* orientable n-manifolds Y, $n \geq 4$, with $\mathrm{Sc} \geq \sigma > 0$, which admit smooth proper area decreasing maps to $\mathbb{R}^n$, $n \geq 4$, with non-zero degrees.

2.K. Outline of construction for $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area}} / \mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{dist}} \to \infty$. Let g_0 be a metric on a manifold Y such that $\mathrm{Sc}(g_0) > 0$, then *there exists metrics g on Y with arbitrarily large ratios* $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area}}(g)/\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{dist}}(g)$. In fact, let $Y = (Y, g_0)$ be an arbitrary Riemannian manifold and $U \subset Y$ an open subset. Then for all $\varepsilon > 0$ and $\delta > 0$, there exists a Riemannian metric $g_{\varepsilon,\delta}$ on Y such that $\mathrm{Sc}(g_{\varepsilon,\delta}) \geq \mathrm{Sc}(g_0) - \varepsilon$ and

- $\bullet_{Y \setminus U}$ the metric $g_{\varepsilon,\delta}$ is equal to g_0 outside U;
- $\bullet_{\mathrm{area}}$ the metric $g_{\varepsilon,\delta}$ is area-wise smaller than g_0,

$$\mathrm{area}_{g_{\varepsilon,\delta}}(S) \leq \mathrm{area}_{g_0}(S) \qquad \text{for all smooth surfaces } S \subset Y;$$

- $\bullet_{\mathrm{dist}}$ all Riemannian manifolds X, which 1-Lipschitz dominate[18] $(Y, g_{\varepsilon,\delta})$, have $\mathrm{Sc}^{\rtimes}(X) \leq \delta$. The only non-trivial condition here is $\bullet_{\mathrm{dist}}$, which is achieved with the follow ing one.
- $\bullet_D$ There is an open subset $U_D \subset U$ with $D = 10\sqrt{\frac{1}{\delta}}$ such that $(U_D, g_{\varepsilon,\delta})$ is isometric to the product

$$\mathbb{T}^1(\varepsilon) \times \mathbb{T}^{n-2}(2\pi) \times [-D, D],$$

where $T^1(\varepsilon)$ is the circle of length ε and $\mathbb{T}^{n-2}(2\pi)$ is the standard flat torus.

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[16] A continuous map between orientable manifolds, $f\colon Y \to X$, is *spin* if $f^*(w_2(X)) = w_2(Y)$, where w_2 is the second Stiefel–Whitney class.

[17] A compact Riemannian n-manifold X is *enlargeable* if there exists a sequence of oriented coverings $\tilde{X}_i \to X$ and *distance decreasing* maps $f_i\colon \tilde{X}_i \to S^n(R_i)$, $R_i \to \infty$, which are constant at infinity and which have *non-zero degrees*, compare with [4, 20, 23, 38], in [17, §4.7], [18, §2.A].

[18] "Domination" is a map with non-zero degree, see [17, §1.5].

The construction of $g_{\varepsilon,\delta}$, which satisfies $\bullet_{Y\setminus U}$, $\bullet_{\text{area}}$ and $\bullet_D$ is elementary (left to the reader[19]) while the implication $\bullet_D \Rightarrow \bullet_{\text{dist}}$ follows the $\frac{2\pi}{n}$-inequality (see [17, §3.6] and references therein).[20]

Thus, we see that *if a non-torsion homology class h in a compact manifold X satisfies* $\mathrm{Sc}^{\rtimes\downarrow}_{\text{dist}}(h) > 0$*, then the ratio* $\mathrm{Sc}^{\rtimes\downarrow}_{\text{area,sp}}(h)/\,\mathrm{Sc}_{\text{dist,sp}}(h)^{\rtimes\downarrow}$ *can be made arbitrarily large with some Riemannian metric on X.*

2.L. Question on $\lambda_1^{\downarrow}(h,\beta)$. The definition of $\mathrm{Sc}^{\rtimes\downarrow}(h)$, which depends on $\lambda_1(Y, \beta = 1/4)$ (see $(N+1)/4N$-Remark 1.D) makes sense for all β and the arguments which depends on stable minimal hypersurfaces and μ-bubbles generalize to all β, e.g., as the $\square^{\exists\exists}(n, m, N)$-inequality, which is stated in [18, §2.B] for $\beta = N/(N+1)$. However, the geometric significance of this for $\beta \neq N/(N+1)$ is unclear.

Probably, if X is simply connected, $\beta \leq \beta_m > 0$ and $m \geq 3$, then an integer multiples lh for some $l \neq 0$ and all $h \in H_m(X)$ are representable by a distance decreasing maps $Y \to X$, where $\lambda_1(Y, \beta) \geq C$ for a given $C > 0$.

Exercises

2.M. Let g be a Riemannian metric on an *open manifold*[21] X of dimension $\dim(X) = n \geq 2$. Show that there exists a Riemannian metric g_+ on X such that $\mathrm{Sc}(g_+) = 1$ and $\mathrm{area}_{g_+}(\Sigma) \geq \mathrm{area}_g(\Sigma)$ for all smooth surfaces $\Sigma \subset X$.

Hint: Observe that $[0,1] \times \mathbb{R}^{n-1}$ admits an area decreasing diffeomorphism onto $\mathbb{R}^n$ and use products of surfaces with constant curvatures by $\mathbb{R}^{n-2}$ as building blocks for (X, g_+).

Remark. If Y is a *complete* spin n-manifold with $\mathrm{Sc}^{\rtimes}(Y) \geq \sigma > 0$, then it admits no proper area decreasing map to $\mathbb{R}^n$ with non-zero degree [21].

2.N. Show that non-zero multiples of homology classes h in *simply connected* manifolds X have $\mathrm{Sc}^{\rtimes\downarrow}_{\text{st.par}}(ih) > 0$, for some $i \neq 0$.

Hint. Recall the Serre–Thom theorem on framed bordisms and apply Stolz' theorem on spin manifolds [41].

3 $\rtimes^{\downarrow}$-extremality and $\rtimes^{\downarrow}$-rigidity

3.A. Homological $\mathrm{Sc}^{\rtimes\downarrow}_*$-problems. Let X be a Riemannian manifold and $h \in H_m(X)$ a homology class, e.g., $m = n = \dim(X)$, and let h be the fundamental class $[X]$ of X, where X is assumed oriented.

Evaluate $\mathrm{Sc}^{\rtimes\downarrow}_*$ and/or find relations between $\mathrm{Sc}^{\rtimes\downarrow}_*$ and more accessible metric invariants of X.

Decide if $\mathrm{Sc}^{\rtimes\downarrow}_*(h)$ is represented by an $\mathrm{Sc}^{\rtimes\downarrow}_*$-*extremal*, or, for brevity, $\rtimes^{\downarrow}_*$-*extremal*, oriented Riemannian m-manifold mapped to X,

$$Y \xrightarrow{f} X \quad \text{such that } f_*(Y) = h \text{ and } \mathrm{Sc}^{\rtimes}(Y) = \mathrm{Sc}^{\rtimes\downarrow}_*(h),$$

where f is the distance or the area decreasing depending on "$*$" and where, ideally, f is an isometric immersion.

For instance, given a submanifold $Y \hookrightarrow X$, e.g., $Y = X$ decide if it is $\mathrm{Sc}^{\rtimes\downarrow}_*$-extremal, or, moreover, if it is *rigid*, that is *unique extremal* (compare with §3.D below).

Find examples of h, where there is no extremal manifold $Y \to X$ with $f_*[Y] = h$, but such a generalized Y, e.g., a singular extremal one does exist. (We saw some potential examples of such singular Y, and stable minimal singular hypersurfaces suggest further examples.)

[19] To get an insight, start with $Y = S^2$, then look at $Y = S^2 \times \mathbb{T}^{n-2}$.

[20] The proof of $\frac{2\pi}{n}$-inequality for $n \geq 9$ relies on Theorem 4.6 in [39], and if one is satisfied with $\mathrm{Sc}^{\rtimes\downarrow}_{\text{area}}(g)/\,\mathrm{Sc}^{\rtimes\downarrow}_{\text{dist,sp}}(g) \to \infty$, then one can use the spinorial version of $\frac{2\pi}{n}$ from [45].

[21] A manifold X is *open* if it contains no closed manifold connected component.

Determine which closed manifolds X admit $\rtimes^{\downarrow}$-extremal Riemannian metrics.

(Possibly, metrics g_0 with $\mathrm{Ricci}(g_0) > 0$ can be deformed to $g \geq g_0$ with $\mathrm{Sc}^{\rtimes}(g) = \mathrm{Sc}^{\rtimes_\downarrow}(g)$. But, for instance, metrics $g_0 = g_1 + g_2$ on $X = X_1 \times X_2$, where $\mathrm{sect.curv}(g_1) = 1$ and $\mathrm{sect.curv}(g_2) < 0$ admit no such deformations. However, the pointed Hausdorff limit manifolds $\lim_{\lambda\to\infty}(X, g_+ \lambda g_2)$, which are isometric to $X_1 \times \mathbb{R}^{\dim(X)}$, are extremal.

In general, the existence of a metric g_0 on X with $\mathrm{Ricci}(g_0) \geq 0$ might be necessary for the existence of an extremal metric g on X.

Examples

3.B$_{S^n}$. Complete manifolds with *constant sectional curvatures*, e.g., unit spheres, flat tori and Euclidean spaces are $\mathrm{Sc}^{\rtimes_\downarrow}_{\mathrm{area},\widetilde{\mathrm{sp}}}$*-extremal.*

This follows from the $\rtimes$-stabilized *Llarull's theorem* (see [17] and references therein).

3.B$_{\mathcal{R}>0}$. A compact spin manifold X with *non-negative* curvature operator, $\mathcal{R}(X) \geq 0$, e.g., a compact symmetric space is $\mathrm{Sc}^{\rtimes_\downarrow}_{\mathrm{area},\widetilde{\mathrm{sp}}}$-extremal, provided *scalar curvature* $\mathrm{Sc}(X)$ *is constant*[22] and *the Euler characteristic of the universal covering* $\tilde{X}$ *does not vanish.*

This follows by an elaboration on the proof of the *Goette–Semmelmann extremality theorem* [12]. (We say a few words about it in Section 5.)

Probably, the corresponding rigidity arguments (see [29] and references therein) also admit $\rtimes$-stabilization, but I did not check this carefully.

Also the condition $\chi(\tilde{X}) \neq 0$ seems redundant and $\rtimes^{\downarrow}_{\mathrm{area},\widetilde{\mathrm{sp}}}$-extremality can be strengthened, also conjecturally, to the $\rtimes^{\downarrow}_{\mathrm{area}}$-extremality.

3.B$_{\times[a_1,b_i]}$. The rectangular solids $\times_1^n[-a_i, b_i]$ are $\rtimes^{\downarrow}_{\mathrm{sp}}$-extremal and, if $n \leq 8$, they are $\rtimes^{\downarrow}$-extremal.

In fact, $\rtimes^{\downarrow}_{\mathrm{sp}}$-extremality follows by a slight generalization of the argument from [43], which, probably, can be adapted for the proof of the $\rtimes^{\downarrow}_{\widetilde{\mathrm{sp}}}$-extremality.

As for $\rtimes^{\downarrow}$-extremality for $n \leq 8$, this follows from the $\square^{\exists\exists}(n, m, N)$-inequality [18, §2.B].

Furthermore, the generic regularity theorem from [9] extended to μ-bubbles (I have not check this extension) yields the $\square^{\exists\exists}(n, m, N)$-inequality and thus $\rtimes^{\downarrow}$-extremality of solids for $n \leq 10$.

Moreover, granted a μ-bubble generalization of Theorem 4.6 from [39], the $\rtimes^{\downarrow}$-extremality (but not the $\square^{\exists\exists}(n, m, N)$-inequality) would follow for all n.

3.B$_{\times\times}$. Riemannian products of the manifolds from the above examples, e.g.,

$$X = \left(\times_1^{n-k}[-a_i, b_i]\right) \times S^k,$$

are $\rtimes^{\downarrow}_{\mathrm{sp}}$-extremal.

As above, this follows by a simple generalization of argument from [43] combined with the basic (algebraic) inequality in [12] for twisted Dirac operators on manifolds with $\mathcal{R} \geq 0$.

But the $\rtimes^{\downarrow}_{\mathrm{sp}}$-extremality remains problematic even for $n \leq 8$.

For instance, if $k \leq 4$ and $n \leq 8$ (probably $n \leq 10$ will do), then the $\square^{\exists\exists}(n, m, N)$-inequality combined with the warped product splitting argument in [17, §5.5] yield $\rtimes^{\downarrow}_{\mathrm{sp}}$-extremality of $X = \left(\times_1^{n-k}[-a_i, b_i]\right) \times S^k$.

Yet, there is no approach so far to non-spin extremality of the spheres S^k for $k \geq 5$.[23]

3.B$_{\mathrm{warp}}$. There are several classes of log-concave warped product manifolds, e.g., S^n minus a point, where the $\rtimes_{\mathrm{sp}}$-extremality (and $\rtimes$-extremality for $n = 4$) follow by $\rtimes$-stabilization of

[22] There are lots of metrics with $\mathcal{R} > 0$ on spheres S^n and if $n \geq 3$ many of these have constant scalar curvatures. On the other hand, it is possible that a closer look at the curvature term in the twisted Schrödinger–Lichnerowicz formula (see Section 5) would allow one to drop the constancy of the scalar curvature condition.

[23] The warped product splitting argument (combined with a stable version of [12]) applies to S^4, because 3-manifolds are spin.

the arguments in [17, §§5.5–5.7] and [6]. In fact, the $\rtimes$-extremality for warped manifolds is more common then non-stabilized extremality.

For instance, geodesic balls in spheres and in $\mathbb{R}^n$ are not non-stably extremal: one can increase their metrics without diminishing the scalar curvatures. But, probably, they are $\rtimes_\downarrow$-extremal.

3.C. Questions.

(i) Which convex subsets in $\mathbb{R}^n$ are $\rtimes^\downarrow$-extremal?

(ii) Which surfaces are $\rtimes^\downarrow$-extremal?

3.D. About rigidity. The proofs of extremality of the manifolds X in the above examples can be upgraded to *rigidity* that says in the present case that *if a smooth distance non-increasing positive degree map* $f\colon Y \to X$ *satisfies* $\mathrm{Sc}^{\rtimes}_*(Y) \geq \mathrm{Sc}^{\rtimes\downarrow}_*(X)$ (*where* $\mathrm{Sc}^{\rtimes\downarrow}_*(X) = \mathrm{Sc}^{\rtimes}_*(X)$ *by extremality*), *then* f *is homotopic to a local isometry, where one can drop "homotopic to" if* X *has no local scalar flat factors.*

This follows by combining the $\rtimes$-stabilized rigidity arguments in [12] and [29] with those in [17, §5.7] but to be honest, I did not check this in full generality.

4 $\mathrm{Sc}^{\rtimes\downarrow}$-product inequalities, conjectures and problems

4.A. Additivity for cylinders. Since, obviously,

$$\mathrm{Sc}^{\rtimes\downarrow}[X_1 \times X_2] \geq \mathrm{Sc}^{\rtimes\downarrow}[X_1] + \mathrm{Sc}^{\rtimes\downarrow}[X_2],$$

then, for all Riemannian manifolds X_1 and X_2, the inequality

$$\mathrm{Sc}^{\rtimes\downarrow}[X_1 \times X_2] \leq \mathrm{Sc}^{\rtimes\downarrow}[X_1] + \mathrm{Sc}^{\rtimes\downarrow}[X_2],$$

is equivalent to the equality

$$\mathrm{Sc}^{\rtimes\downarrow}[X_1 \times X_2] = \mathrm{Sc}^{\rtimes\downarrow}[X_1] + \mathrm{Sc}^{\rtimes\downarrow}[X_2].$$

Thus, in particular, the $\square^{\exists\exists}(n,m,N)$-*inequality* from [18] and/or *equivariant separation theorem* for stable μ-bubbles[24] along with the equality

$$\mathrm{Sc}^{\rtimes}[a,b] = 4\lambda_1[a,b] = \frac{4\pi^2}{(b-a)^2}.$$

imply the following.

Proposition. The fundamental homology classes of oriented Riemannian cylindrical manifolds $X = Y \times [a,b]$ of dimensions ≤ 8 satisfy

$$\mathrm{Sc}^{\rtimes\downarrow}[X] = \mathrm{Sc}^{\rtimes\downarrow}[Y] + \mathrm{Sc}^{\rtimes\downarrow}[a,b].$$

(This generalizes $\mathrm{Sc}^{\rtimes\downarrow}\left(\bigtimes_1^n [a_i,b_i]\right) = \sum_i \mathrm{Sc}^{\rtimes\downarrow}[a_i,b_i]$, that is §3.B$_{\times[a_1,b_i]}$ from the previous section.)

4.B. The spin case. This additivity formula remains *problematic* for $n \geq 9$,[25] but *the spin cube inequality* from [43] (proved with an index theorem for deformed Dirac operators on manifolds with boundaries) implies, as we stated earlier, that

$$\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{sp}}\left(\bigtimes_1^n[-a_i,b_i]\right) = \mathrm{Sc}^{\rtimes}\left(\bigtimes_1^n[-a_i,b_i]\right) = \sum_1^n \frac{4\pi^2}{(b_i-a_i)^2} = \sum_1^n \mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{sp}}[a_i,b_i]$$

for all n.

[24] See [17, §5.4] and compare with [22, 37] and with [18, the proof of §2.B].

[25] The dimensions $n = 9, 10$, probably, can be taken care by the argument in [9].

Yet, as far as I can see, the present day Dirac theoretic argument does not yield the general Sc_{sp}-inequality

$$Sc^{\rtimes\downarrow}_{sp}[Y\times[a,b]] \leq Sc^{\rtimes\downarrow}_{sp}[Y] + Sc^{\rtimes\downarrow}_{sp}[a,b].$$

However, this argument does apply, if Y is a special (extremal) manifold as in §3.$B_{\times\times}$, e.g., a product of spheres.

4.C. [sect.curv $\leq$ 0]-Remark. Let Y and Z be compact Riemannian manifolds, where Z has no boundary and the sectional curvature sect.curv$(Y)\leq 0$. Then, similarly as above, one can prove additivity in the following two cases:

(i) If $\dim(Y\times Z)=n\leq 8$,[26] then

$$Sc^{\rtimes\downarrow}[Y\times Z] = Sc^{\rtimes\downarrow}[Y] = Sc^{\rtimes\downarrow}[Y] + Sc^{\rtimes\downarrow}[Z] = Sc^{\rtimes\downarrow}[Y].$$

(ii) If Y is as in §3.$B_{\times\times}$, then

$$Sc^{\rtimes\downarrow}_{sp}[Y\times Z] = Sc^{\rtimes\downarrow}_{sp}[Y] = Sc^{\rtimes\downarrow}_{sp}[Y] + Sc^{\rtimes\downarrow}_{sp}[Z] = Sc^{\rtimes\downarrow}[Y]$$

for all n.

4.D. Riemannian additivity conjecture. Riemannian products of all oriented Riemannian manifolds satisfy

$$Sc^{\rtimes\downarrow}[X_1\times X_2] = Sc^{\rtimes\downarrow}[X_1] + Sc^{\rtimes\downarrow}[X_2].$$

In fact, the following stronger inequality might be true.

4.E. Sup-metric product conjecture. Let X_i, $i=1,\dots,k$, be metric spaces (e.g., closed oriented Riemannian manifolds) and let

$$X = \bigtimes_{\sup_i} X_i = (X_1\times\dots\times X_k, \mathrm{dist}_{\sup})$$

be their product endowed with the *sup-metric*

$$\mathrm{dist}((x_1,\dots,x_k),(x'_1,\dots,x'_k)) = \max_{i=1,\dots,k}\mathrm{dist}(x_i,x'_i).$$

Then rational homology classes $h_i\in H_{m_i}(X_i;\mathbb{Q})$ (e.g., the rational fundamental classes $[X_i]$[27]) satisfy

$$Sc^{\rtimes\downarrow}(\otimes_i h_i) \leq \sum_{i=1,\dots,k} Sc^{\rtimes\downarrow}(h_i), \qquad \text{e.g.,} \qquad Sc^{\rtimes\downarrow}[\bigtimes_{\sup_i} X_i]_{\mathbb{Q}} \leq \sum_{i=1,\dots,k} Sc^{\rtimes\downarrow}[X_i]_{\mathbb{Q}}, \tag{4.1}$$

where the opposite inequality

$$Sc^{\rtimes\downarrow}(\otimes_i h_i) \geq \sum_{i=1,\dots,k} Sc^{\rtimes\downarrow}(h_i)$$

follows from additivity of the scalar curvature; hence, (4.1) implies the equality

$$Sc^{\rtimes\downarrow}(\otimes_i h_i) = \sum_{i=1,\dots,k} Sc^{\rtimes\downarrow}(h_i).$$

[26] In view of [9], the inequality $n\leq 10$ may suffice.

[27] "Rational" in the case of compact locally contractible spaces means "*a non-zero integer multiple of*", that is,

$$Sc^{\rtimes\downarrow}(h_{\mathbb{Q}}) \overset{\text{def}}{=} \sup_{N\neq 0} Sc^{\rtimes\downarrow}(N\cdot h_{\mathbb{Q}}).$$

4.F. $\bigtimes_{\sup_i}[a_i, b_i]$-Example. The above indicated proofs of §4.A and §4.B actually show that the rectangular solids $\bigtimes_1^n[a_i, b_i]$ with the Riemannian product and the sup-product metrics have the same $\mathrm{Sc}^{\rtimes\downarrow}$ for $n \leq 8$ and have the same $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{sp}}$ for all n. This *confirms the validity of* (4.1) *for rectangular solids.*

4.F′. $\bigtimes[0, d_i]$-Sub-Example. Let $Y \subset \mathbb{R}^n$ be a diffeomorphic image of the n-cube and let d_i, $i = 1, \dots, n$, be the distances between the images in Y of the pairs of the opposite $(n-1)$-faces of the cube. Then the first Dirichlet eigenvalue of the Laplacian $-\Delta_Y$ is bounded by that of the solid $\bigtimes_i[0, d_i]$,

$$\lambda_1(-\Delta_Y) \leq \sum_i \frac{\pi^2}{d_i^2}.$$

Exercise. Find a direct elementary proof of this inequality.[28]

Sup-distance, sup-area and $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{sup.area}}$. The Riemannian product metric, that is the Pythagorean one

$$\mathrm{dist}((x_1, \dots, x_k), (x'_1, \dots, x'_k)) = \sqrt{\sum_i \mathrm{dist}(x_i, x'_i)^2},$$

is greater than the sup-metric but only by a factor $\sqrt{k}$,

$$1 \leq \frac{\mathrm{dist}((x_1, \dots, x_k), (x'_1, \dots, x'_k))}{\mathrm{dist}_{\sup}((x_1, \dots, x_k), (x'_1, \dots, x'_k))} \leq \sqrt{k}.$$

The situation is somewhat different with areas. Namely, let $X = \bigtimes_i X_i$ be the product of Riemannian manifolds and let $\sup_i$-area(Σ) for a smooth surface $\Sigma \subset X$ be the maximum of the areas of the projections $\Sigma \to X_i$. Here again

$$\underset{i}{\sup}\text{-area}(\Sigma) \leq \mathrm{area}(\Sigma)$$

but now, unlike to how it is with the distances, the ratio

$$\frac{\mathrm{area}(\Sigma)}{\sup_i\text{-area}(\Sigma)}$$

may be infinite. Accordingly, the corresponding $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{sup.area}}(h)$, $h \in H_m(\bigtimes_i X_i)$, defined with smooth maps $f\colon Y^m \to \bigtimes_i X_i$, $f_*[Y] = h$ such that the corresponding $f_i\colon Y^m \to X_i$ are area decreasing, can be *significantly greater* than $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{sup.area}}(h)$, where the maps f must be area decreasing themselves.

Thus, the area version of (4.1),

$$\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{sup.area}}(\otimes_i h_i) \leq \sum_{i=1,\dots,k} \mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area}}(h_i), \tag{4.2}$$

e.g.,

$$\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{sup.area}}[\bigtimes_{\sup_i} X_i]_{\mathbb{Q}} \leq \sum_{i=1,\dots,k} \mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area}}[X_i]_{\mathbb{Q}},$$

is qualitatively stronger than corresponding inequality for $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area}}(\otimes_i h_i)$.

[28] To my shame, I could not solve it.

Although we have no known means for bounding $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area}}$ and even less for $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{sup.area}}$ in most cases, we shall do this in the next section for $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{sup.area},\widetilde{\mathrm{sp}}}$ and thus prove the $\widetilde{\mathrm{sp}}$-version of (4.2) in some cases.

4.G. Semiadditivity problem. Let $X = X^n$ and $Z = Z^k$, $k \leq n-2$, be compact Riemannian manifolds, possibly with boundaries, and let $f\colon X \to Z$ be a smooth distance decreasing map such that $\partial X \overset{f}{\to} \partial Z$, and let $h_m = \left[f^{-1}(z)\right] \in H_m(X)$, $m = n-k$, be the homology class of the pullback of a generic $z \in Z$.

Identify the cases, where

$$\mathrm{Sc}^{\rtimes\downarrow}(h_m)_{\mathbb{Q}} \geq \mathrm{Sc}^{\rtimes\downarrow}[X]_{\mathbb{Q}} - \mathrm{Sc}^{\rtimes\downarrow}[Y]_{\mathbb{Q}},$$

at least for "simple" manifolds Z, e.g., compact convex domains in $\mathbb{R}^k$ and in S^k and, in general, evaluate the difference

$$\mathrm{Sc}^{\rtimes\downarrow}[X]_{\mathbb{Q}} - \mathrm{Sc}^{\rtimes\downarrow}[Y]_{\mathbb{Q}} - \mathrm{Sc}^{\rtimes\downarrow}(h_m)_{\mathbb{Q}}$$

in terms of the geometry of Z, for instance, where Z is the product of balls $Z = \bigtimes_i B^{k_i}(R_i)$ or product of spheres $S^{k_i}(R_i)$.

If $n = m + k \leq 8$, a satisfactory lower bound on $\mathrm{Sc}^{\rtimes\downarrow}(h_m)$ for rectangular solids Z follows from §4.A. Also [18, §2.B] yields similar bounds for products of 2-discs and 2-spheres (compare [22]). But it is unclear, for instance, how large the difference $\mathrm{Sc}^{\rtimes\downarrow}[X]_{\mathbb{Q}} - \mathrm{Sc}^{\rtimes\downarrow}[Y]_{\mathbb{Q}} - \mathrm{Sc}^{\rtimes\downarrow}(h_m)_{\mathbb{Q}}$ can be for the balls $B^k \subset \mathbb{R}^k$ and spheres S^k for large k.

5 Additivity of the twisted SLWB-formula and applications

Let Y be a Riemannian spin n-manifold and $V \to X$ be a complex vector bundle with a unitary connection ∇ and let $\mathcal{D}_{\otimes V}$ denote the Dirac operator on spinors $\mathbb{S}$ on Y tensored with V. Then the square of $\mathcal{D}_{\otimes V}$ satisfies the following *Schrödinger–Lichnerowicz–Weitzenböck–Bochner formula* (see [27])

$$\mathcal{D}^2_{\otimes V} = \nabla^2_{\otimes V} + \frac{1}{4}\mathrm{Sc}(Y) + \mathcal{K}_{\otimes V},$$

where $\mathcal{K}_{\otimes V}$ is an endomorphism $\mathbb{S} \otimes V \to \mathbb{S} \otimes V$ such that

$$\mathcal{K}_{\otimes V}(s \otimes v) = \frac{1}{2}\sum_{i,j}(e_i \circ e_j \circ s) \otimes R^V_{e_i \wedge e_j}(v),$$

where $e_i \in T_y(Y)$, $i = 1, \dots, m = \dim(Y)$, are orthonormal tangent vectors at $y \in Y$, where $\circ$ is the Clifford multiplication and $R^V_{e_i\wedge e_j}\colon V \to V$ is the curvature operator of ∇.

Next, recall that the curvature of the tensor product of two bundles with connections satisfies

$$R^{V_1 \otimes V_2} = 1^{V_1} \otimes R^{V_2} + R^{V_1} \otimes 1^{V_2},$$

where $1^V\colon L \to V$ is the identity operator, and observe that the operators on $\mathbb{S}\otimes V_1 \otimes V_2$ defined by

$$s \otimes v_1 \otimes v_2 \mapsto \frac{1}{2}\sum_{i,j}(e_i \circ e_j \circ s) \otimes R^{V_1}_{e_i\wedge e_j}(v_1) \otimes v_2$$

and by

$$s \otimes v_1 \otimes v_2 \mapsto \frac{1}{2}\sum_{i,j}(e_i \circ e_j \circ s) \otimes v_1 \otimes R^{V_2}_{e_i\wedge e_j}(v_2)$$

have the same spectra up to multiplicity as

$$s \otimes v_1 \mapsto \frac{1}{2} \sum_{i,j} (e_i \circ e_j \circ s) \otimes R^{V_1}_{e_i \wedge e_j}(v_1)$$

and

$$s \otimes v_2 \mapsto \frac{1}{2} \sum_{i,j} (e_i \circ e_j \circ s) \otimes R^{V_2}_{e_i \wedge e_j}(v_2)$$

correspondingly. Therefore, the lowest eigenvalue $\lambda_{\otimes 1 \otimes 2}$ (often negative) of the (self-adjoint) operator $\mathcal{R}_{\otimes(V_1 \otimes V_2)}$ is bounded from below by the sum of these for $\mathcal{R}_{\otimes V_1}$ and $\mathcal{R}_{\otimes V_2}$,[29]

$$\lambda_{\otimes 1 \otimes 2} \geq \lambda_{\otimes 1} + \lambda_{\otimes 2}.$$

This yields the following.

5.A. Theorem.[30] Let $X = \bigtimes_k X_k$, $k = 1, \ldots, l$, be an orientable Riemannian n-manifold split into Riemannian product, where the factors $X_k = (X_k, \underline{g}_k)$ are either

(a) compact n_k-manifolds with *non-negative curvature operators*, $\mathcal{R}^{X_k} \geq 0$ (e.g., closed convex hypersurfaces in $\mathbb{R}^{n_k+1}$) and with *non-vanishing Euler characteristics* $\chi(X_k) \neq 0$ (hence of even dimensions n_k), or

(b) spheres S^{n_k} with constant sectional curvatures (possibly of odd dimension n_k).

Let $\underline{g}^\natural_k = \mathrm{Sc}(\underline{g}_k) \cdot \underline{g}_k$. Let $Y = (Y, g)$ be a smooth complete orientable Riemannian $(n + N)$-manifold with $\mathrm{Sc}(g) > 0$, and let $g^\natural = \mathrm{Sc}(g) \cdot g$. Let Z be an orientable enlargeable N-manifold, e.g., $Z = \mathbb{R}^N$, and let $f \colon Y \to X \times Z$ be a smooth proper (quasi-proper will do) map such that the corresponding maps $f_k \colon Y \to X_k$ are *strictly sum-wise area decreasing* with respect to $g^\natural$ in Y and $\underline{g}^\natural_k$ in X_k.[31]

This means that the norms of the exterior squares of the differentials of f_k with respect the $\natural$-metrics satisfies

$$\sum_k || \wedge^2 \mathrm{d} f_k || < 1. \tag{5.1}$$

(Notice that $\sum_k || \wedge^2 \mathrm{d} f_k || = 1$ if $X = Y$ and f is the identity map.)

If either the map f is spin or the universal covering of Y is spin, then the topological degree of f is zero.

Proof. First, let X and Y be spin, let $f \colon Y \to X$ be a smooth map and let $V \to Y$ be the f-pullback of the spin bundle $\mathcal{S}(X) \to X$ to Y. Then, if $\mathcal{R}^X \geq 0$ and $f \colon Y \to X$ is $\natural$-area decreasing at point $y \in Y$, i.e., $\|\wedge^2 \mathrm{d} f(y)\| \leq 1$, then according to [12] (also see [29]) the lowest eigenvalue of the operator $\mathcal{R}_{\otimes V}$ at $y \in Y$ satisfies

$$\lambda_{\otimes V} \geq -\frac{\mathrm{Sc}(Y, y)}{4},$$

where this inequality is strict if f is strictly $\natural$-area decreasing at y.

Next, let X_k be spin, let $X = \bigtimes_k X_k$ and let $V \to Y$ be the tensor product $V = \bigotimes_k V_k$ of the pullbacks $V_k = f_k^*(\mathbb{S}_k) \to Y$ of $f_k^*(\mathbb{S}_k) = \mathbb{S}(X_k) \to X_k$ to Y for $f_k \colon Y \to X_k$.

[29] A referee pointed out to me that $\lambda_{\otimes 1 \otimes 2} = \lambda_{\otimes 1} + \lambda_{\otimes 2}$.

[30] This is a refinement of the Llarull–Goette–Semmelmann–Listing rigidity theorem.

[31] The quadratic forms $\underline{g}^\natural_k$ on X_k may vanish but "area" makes sense anyway.

Then, if the maps f_k are $\natural$-area decreasing and at least one of f_k is strictly $\natural$-area decreasing at a point $y \in Y$ then, assuming Y is connected and spin, the Dirac operator $\mathcal{D}_{\otimes V}$ on Y has *index zero.*

On the other hand, if $\chi(X_k) \neq 0$, if $\dim(Y) = \dim(X)$ and $\deg(f) \neq 0$,[32] then $\mathrm{ind}(\mathcal{D}_{\otimes V}) \neq 0$ by the Atiyah–Singer theorem (compare with [12, 29, 30]).

This proves §5.A in the case where the manifold X is spin and it contains neither a Z-factor, nor an odd spherical factor.

To pass to the general case we argue as follows:

1. Odd dimensional spheres are suspended to even dimensional ones $S^{n_k} \rightsquigarrow S^{n_k+1}$, where these suspensions are accompanied by multiplying Y by a long circles and a suspending $[f_k\colon Y \to S^{n_k} \rightsquigarrow [Y \times S^1 \to S^{n_k+1}]$ as in [30], also see [17, §3.4.1] and [18].

2. If a Z, which may be assumed even-dimensional, is enlargeable, it supports an almost flat bundle, say $W \to Z$ with non-zero top-dimensional Chern class and the above $V \to Y$ is tensored by the pullback $f_Z^*(W) \to Y$ of W to Y.

3. If neither X nor Y are spin but the map f is spin, then the Dirac operator $\mathcal{D}_{\otimes V}$ is defined (this is explained in the present context in [34] and in [12]) and the above applies.

5.B. Spherical trace and symplectic remarks. The $|| \wedge^2 \mathrm{d}f_k||$ contribution of each spherical factor X_k with constant sectional curvature can be replaced in the formula (5.1) by an a priori smaller entity, that is, $\frac{2||\wedge^2 \mathrm{d}f||_{\mathrm{trace}}}{n_k(n_k-1)}$, where

$$|| \wedge^2 \mathrm{d}f_k(y)||_{\mathrm{trace}} = \sum_{1\leq i<j\leq n+N} \lambda_{i,k}(y)\lambda_{j,k}(y),$$

and where the numbers $\lambda_{j,k}(y) \geq 0$ are defined by diagonalizing the differential $\mathrm{d}f_k\colon T_y(Y) \to T_{f_k}(X_k)$ with an *orthonormal* frame $e_{i,k} \in T(y)(Y)$, which is sent by $\mathrm{d}f_k$ to an *orthogonal* frame in $T_{f_k}(X_k)$ with the vectors of lengths $\lambda_{j,k}(y)$.

In fact, this follows from [30, formula (4.6)] (also [29] and [17, §3.4]).

The S^2 factors in X contribute to complex line bundles as $\otimes$-factors in $V \to Y$. This, in view of Schrödinger–Hitchin (see [25]) formula for $\mathcal{D}_{\otimes L}$ allows one to replace the product of these S^2 by a single (quasi)symplectic manifold (compare with [17, §2.7 and §3.4.4(4)]).

5.C. $\mathrm{Sc}^{\rtimes\downarrow}_{\mathbf{area},\widetilde{\mathbf{sp}}}$-additivity corollary. Let X_k be manifolds as in 5.A, where we additionally assume that they are spin and have constant scalar curvatures. Then the fundamental classes $[X_k]$ satisfy the $\widetilde{\mathrm{sp}}$-version of the $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{sup.area}}$-additivity (4.2) in §4.F:

$$\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{sup.area},\widetilde{\mathrm{sp}}}(\bigtimes_k X_k) = \sum_k \mathrm{Sc}^{\rtimes\downarrow}_{\widetilde{\mathrm{sp}}}(X_k) = \sum_k \mathrm{Sc}^{\rtimes}(X_k) = \sum_k \mathrm{Sc}(X_k).$$

Consequently,

$$\mathrm{Sc}^{\rtimes\downarrow}_{\widetilde{\mathrm{sp}}}(\bigtimes_{\sup_k} X_k) = \sum_k \mathrm{Sc}(X_k).$$

5.D. Questions. (i) Does vanishing of $\mathrm{Sc}^{\rtimes\downarrow}[X_k]_{\mathbb{Q}}$ for closed manifolds X_k (this is a homotopy invariant of X) implies vanishing of $\mathrm{Sc}^{\rtimes\downarrow}[\bigtimes_k X_k]_{\mathbb{Q}}$?

There are examples of manifolds X_k, where $\mathrm{Sc}^{\rtimes\downarrow}[X_i] = 0$ and where their products admit metrics with $\mathrm{Sc} > 0$; hence, $\mathrm{Sc}^{\rtimes\downarrow}[\bigtimes_i X_i] > 0$ for these X_i, see [19].

(ii) Do products of spheres $X = S^{n_1} \times S^{n_2}$, $n_1, n_2 \geq 2$, admit Riemannian metrics g_ε, for all $\varepsilon > 0$, with $\mathrm{Sc}(g_\varepsilon) \geq 1$ and such that *all non-zero* homology classes h in $H_{n_1}(X)$ and in $H_{n_2}(X)$ satisfy $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area}_{g_\varepsilon}}(h) \leq \varepsilon$?

[32] If $\dim(Y) = \dim(X) + 4m$, then instead of $\deg(f) \neq 0$ one assumes that the f-pullback of a generic point $x \in X$ has non-zero hat A-genus.

The existence of such a g_ε, for $n_1 = n_2 = 2$, would imply the absence of the lower bounds on the 2-*systoles* of manifolds (X, g) in terms of $\sigma(g) = \inf_{x\in X} \mathrm{Sc}(X, g, x) > 0$,[33]

$$\sup_{\sigma(g)\geq 1} \mathrm{syst}_2(X, g) = \infty.$$

Recall, that the 2-*systole* is the *infimum of the areas of all non-zero classes* $h \in H_2(X)$, for $\mathrm{area}(h) = \inf_{[c]\in h} \mathrm{area}(c)$ for the 2-cycles $c \subset Y$ that represent h.[34]

(iii) Let X be a compact symmetric space. What is the *minimal* seminorm on the linear maps $\lambda^2\mathrm{d}\colon \wedge^2 R^n \to \wedge^2 T(X)$, say $||\lambda^2\mathrm{d}||_{\min}$ such that the $\natural$-normalized inequality $|| \wedge^2 \mathrm{d}f||_{\min} < 1$ for smooth equidimensional spin maps $f\colon Y \to X$ would imply that $\deg(f) = 0$?[35]

(If X is the products of spheres, this seminorm is equal to the sum of the mean trace norms (as in §5.B) for maps $\mathbb{R}^n \overset{\mathrm{d}_k}{\to} X_k = S^{n_k}$ and for all symmetric spaces X of dimension ≥ 4 with $\chi(X) \neq 0$ this norm is, probably, strictly smaller then the sup-norm $|| \wedge^2 \mathrm{d}||$ from the Goette–Semmelmann theorem.)

6 *P*-families of maps to product of spheres

Let $Y = (Y, g)$ be an n-dimensional Riemannian manifold with $\mathrm{Sc}(X) > 0$, where as earlier $g^\natural = \mathrm{Sc}(Y) \cdot g$, let $h_m \in H_m(Y)$ be a homology class and let P be a locally contractible topological space, e.g., a manifold and $h_K \in H_K(P)$ be a homology class.

Let X be a product of spheres of variable radii,

$$X = \times_k S^{n_k}(R_k),$$

where $\dim(X) = \sum_k n_k = m + K$, and where the spheres are endowed with the usual metrics with sectional curvatures $1/R_k^2$.

Let $F\colon Y \times P \to X$ be a continuous map such that the maps $F_p = F_{|Y\times p}\colon Y \to X$ are smooth and C^1-continuous in $p \in P$.

Let the universal covering of Y be spin and let h_m be equal to the homology class of the pullback of a genetic point under a smooth map $\phi\colon Y \to Z$, where Z is a smooth enlargeable manifold of dimension $\dim(Y) - m$. For instance, $m = n$ and $h_m = [Y]$ or $Y = Y_0^m \times \mathbb{T}^{n-m}$ and $h_m = [Y_0]$.

6.A. Theorem. Let the norms of the exterior squares of the differentials of the maps $f_k\colon Y \to S^{n_k}(R_k)$ with respect to the $\natural$-metrics in Y and in $S^{n_k}(R_k)$ satisfy

$$\sum_k || \wedge^2 \mathrm{d}f_k|| < 1.$$

Then, in the following two cases, the F-image $F_*(h_m \otimes h_K) \in H_{m-K}(X) = \mathbb{Z}$ vanishes:

(1) The ranks of the (differentials of the) maps $f_p\colon Y \to X$ are everywhere $\leq m$, e.g., $\dim(Y) = m$ and $h_m = [Y]$.

(2) The dimension of Y is bounded by $n \leq 8$.

[33] Such a counter example would undermine (but not disprove) the conjectural bound $\mathrm{waist}_2(X) \leq \frac{\mathrm{const}_n}{\sigma}$ for compact Riemannian n-manifolds with $\mathrm{Sc}(X) \geq \sigma > 0$. Thus, it may be safer to assume $n_1, n_2 \geq 3$.

[34] There are bounds on the 2-systoles of manifolds X with $\mathrm{Sc}^{\rtimes}(X) \geq \sigma$ in terms of their $\tilde{\square}^{\perp}$-spreads (see [37, 44]) which are proved as $\square^{\exists\exists}(n, m)$-inequality in §2.B with a use of minimal hypersurfaces and μ-bubbles. Also there are similar bounds on the *stable systoles* of spin manifolds obtained with Dirac operators twisted with line bundles, where, recall, $\mathrm{st.syst}_2(X) = \liminf_{N\to\infty} \frac{\mathrm{area}(Nh)}{N}$.

[35] This norm must be invariant under isometries of X.

Proof. Case 1. If ranks$(f_p) \leq m$, then the Llarull (Listing) trace inequality (4.6) in [30] together withe the above $\lambda_{\otimes V}$-additivity show that index of the family of the Dirac operators on Y, twisted with the pullbacks of $\bigotimes_k \mathbb{S}_k$ as in §5.A, vanishes and the Atiyah–Singer theorem for families shows that $F_*(h_m \otimes h_K) = 0$. (See in [17, §4] and references therein.)

Case 2. If $\dim(Y) \leq 8$, then, at last generically, the homology class h_m can be realized by an m-submanifold $Y_0 \subset Y$ such that the product $Y_0 \times \mathbb{T}^{\dim(Y)-m}$ admits a warped product metric $g^{\rtimes}$ such that $\mathrm{Sc}(g^{\rtimes}, y) \geq \mathrm{Sc}(Y, y)$ for all $y \in Y_0$ (see [18, §3] and references therein). Now the case 1 applies to $Y_0 \times \mathbb{T}^{\dim(Y)-m}$ and the proof follows.

Remarks/Problems. (a) For all we know, the spin and $\dim(Y) \leq 8$ condition are redundant and there is a fair chance that a further study of singularities of minimal hypersurfaces in he spirit of [39] and/or [31, 32] will allow one to remove the latter. But removing the spin condition needs a new idea.

The argument in Case 1 can be extended to maps of foliated manifolds to $\bigtimes_k S^{n_k}$ as in [42], but a foliated version of Case 2 is problematic.

7 Spinorial curvature

Given a closed orientable even dimensional Riemannian manifold Y let $\mathbb{S}\mathsf{p}.\mathrm{curv}(Y)$ be the infimum of the numbers $\kappa \geq 0$ such that there exist a complex vector bundle $V \to X$ with a unitary connection such that

$$\big(\mathrm{Ch}(V) \smile \hat{A}\big)[Y] \neq 0$$

and the lowest eigenvalue of the operators $\mathcal{K}_{\otimes V}$ on $(\mathcal{S} \otimes V)_y$ (see Section 5) satisfies

$$\lambda_{\otimes V} \geq -\kappa$$

at all points $y \in Y$.[36]

Observe that

$$\mathbb{S}\mathsf{p}.\mathrm{curv}(Y_1 \times Y_2) \leq \mathbb{S}\mathsf{p}.\mathrm{curv}(Y_1) + \mathbb{S}\mathsf{p}.\mathrm{curv}(Y_2)$$

by the inequality $\lambda_{\otimes 1 \otimes 2} \geq \lambda_{\otimes 1} + \lambda_{\otimes 2}$ from Section 5, that

$$\mathbb{S}\mathsf{p}.\mathrm{curv}(Y) = 0$$

for enlargeable manifolds Y and that if the universal coverings of Y is *spin*, then

$$\mathrm{Sc}^{\rtimes}(Y) \leq 4\kappa$$

by the ($\mathbb{T}^{\rtimes}$-stabilized) index theorem, SLWB-formula and Kato's inequality.[37]

Remarks on $\lambda_1(X, \beta)$ for $\beta < 1/4$. (a) The refined Kato inequality strengthens the above to

$$\lambda_1(X, \beta) \leq 4\kappa \qquad \text{for} \quad \beta = (n-1)/4n,$$

where $\lambda_1(X, \beta)$ is the lowest eigenvalue of $-\Delta + \beta\,\mathrm{Sc}(X)$ (see §1.D).

(b) The Kazdan–Warner conformal change theorem [26] and conformal invariance of harmonic spinors [25] show that if $\lambda_1(X, \beta) > 0$ for $\beta = (n-2)(4(n-2)$, then X supports no non-zero harmonic spinors.

[36] Although the spin bundle $\mathcal{S} \to Y$ is defined only for spin manifolds, this definition, being local, makes sense for all Y, since $\lambda_{\otimes V}$ does not depend on the spin structure.

[37] Instead of the $\mathbb{T}^{\rtimes}$-stabilization and Kato's inequality one may use Kazdan–Warner conformal change theorem [26] and conformal invariance of harmonic spinors [25].

However, it is unclear how to extract further geometric, rather than topological information from the inequality $\lambda_1(X,\beta) > \sigma$ for $\beta < (n-1)/4n$ and $\sigma > 0$.

Next, let X be a Riemannian manifold, let $h_m \in H_m(X)$ be a homology class and let $\mathcal{Y}$ be a class of smooth closed orientable m-manifolds Y along with maps $f\colon Y \to X$.

Define $\mathbb{S}\mathsf{p}.\mathrm{curv}^{\downarrow}_{\mathcal{Y}}(X)$ via smooth maps $F\colon Y \times \mathbb{T}^N \to X \times \mathbb{T}^N$ and Riemannian metrics G on $Y \times \mathbb{T}^N$ as the infimum

$$\mathbb{S}\mathsf{p}.\mathrm{curv}^{\downarrow}_{\mathcal{Y}}(X)\mathsf{p} \inf_{Y,N,G,F} \mathbb{S}\mathsf{p}.\mathrm{curv}\big(Y \times \mathbb{T}^N, G\big),$$

where the infimum is taken over N such that $m+N$ is *even*, where F is *area decreasing* with respect to the metric G and where

$$F_*\big[Y \times \mathbb{T}^N\big] = h_m \otimes \big[\mathbb{T}^N\big] \in H_{m+N}\big(X \times \mathbb{T}^N\big) \qquad \text{and} \qquad (Y, F_{|Y\times 0}) \in \mathcal{Y}.$$

Clearly, by the above, if the universal coverings of manifolds $Y \in \mathcal{Y}$ are spin,[38] then

$$\mathbb{S}\mathsf{p}.\mathrm{curv}^{\downarrow}_{\mathcal{Y}}(h_m) \geq \frac{1}{4}\big(\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area},\mathcal{Y}}(h_m)\big).$$

Remark. If the universal coverings of the manifolds $Y \in \mathcal{Y}$ are spin, then the fundamental classes $[X]$ of compact symmetric spaces X with $\chi(X) \neq 0$, satisfy the equally

$$\mathbb{S}\mathsf{p}.\mathrm{curv}^{\downarrow}_{\mathcal{Y}}(X) = \frac{1}{4}\big(\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area},\mathcal{Y}}(h)\big)$$

by the $\mathbb{T}^{\rtimes}$-stabilized Goette–Semmelmann theorem and this equally applies to products $X = \bigtimes X_i$, where X_i are as in §5.A.

Possibly, (a version of) this equality holds true for all symmetric spaces but it seems unlikely in general, even for rational homology classes h, that the Dirac operator is the only source of bounds on $\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area},\mathcal{Y}}(h)$.

Power stabilization. Let

$$X^M = \underbrace{X \times X \times \cdots \times X}_{M},$$

$$\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area},\mathcal{Y}}\big(\otimes^{[\infty/\infty]} h_m\big) = \sup_{M=1,2,\dots} \frac{1}{M}\,\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area},\mathcal{Y}}(h_{Mm}), \qquad h_{Mm} = \otimes^M h_m \in H_{Mm}\big(X^M\big)$$

and

$$\mathbb{S}\mathsf{c}.\mathrm{curv}^{\rtimes\downarrow}_{\mathrm{area},\mathcal{Y}}\big(\otimes^{[\infty/\infty]} h_m\big) = \inf_{M=1,2,\dots} \frac{1}{M}\mathbb{S}\mathsf{c}.\mathrm{curv}^{\rtimes\downarrow}_{\mathrm{area},\mathcal{Y}}(h_{Mm}),$$
$$h_{Mm} = \otimes^M h_m \in H_{Mm}\big(X^M\big).$$

Questions. **I.** What are further instances (besides the above $h = [X]$) of the equality

$$\mathbb{S}\mathsf{p}.\mathrm{curv}^{\downarrow}_{\mathcal{Y}}\big(\otimes^{[\infty/\infty]} h_m\big) = \frac{1}{4}\big(\mathrm{Sc}^{\rtimes\downarrow}_{\mathrm{area},\mathcal{Y}}\big(\otimes^{[\infty/\infty]} h_m\big)\big),$$

and what are examples where this fails to be true?

II. Can one pass to the limit, set $M = \infty$ and prove scalar curvature bounds for "Riemannian metrics" G on infinite dimensional manifolds X, e.g., where such a G differs from the infinite sum of Riemannian metrics, $\sum_1^\infty g_i$, on $X = \bigtimes_1^\infty (X_i, g_i)$ (and/or on Y mapped to X) by a "fast decaying in i" error term Δ?

[38]This condition is necessary but its $\mathbb{Q}$-version may be redundant.

Remarks. (a) If $\Delta = \Delta_{i,j}$ decays very fast, for i and/or j tending to infinity, then finite products $X_M = \bigtimes_1^M X_i$ embed to $X = \bigtimes_1^\infty (X_i, g_i)$ with small relative curvatures and a bound on "Sc(X)" may be derived in some cases from such a bound on X_M, but it would be more interesting to develop a truly infinite dimensional argument for bounds on "Sc(X)" and/or to find applications of such bounds.

Test question. Let $X = \{x_i\}_{\sum_i x_i^2 \leq \infty}$ be the Hilbert space and $G = G_{ij}$ be a smooth Riemannian metric on X, which is *greater* than the background Hilbertian metric,

$$G(\tau, \tau) \geq ||\tau||^2$$

for all tangent vectors $\tau \in T(X)$ and let $M = 3, 4, \dots$ be an integer.

Can the M*-scalar curvature* of G (defined below) *be strictly positive*, say $\mathrm{Sc}_M(G) \geq 1$? Here Sc_M is the function on the tangent M-planes $P^M \in T_x(X)$, $x \in X$, which is equal to the scalar curvature at zero in P^M $\left(=\mathbb{R}^M\right)$ of the Riemannian metric induced by the exponential map $\exp\colon P^M \to X$ from G. (It may be worthwhile to compare Sc_M with with the m*-intermediate curvature* from [3].)

(b) A natural approach to these problems is by a finite-dimensional approximation as in (a) but this seems that uncomfortably restrictive conditions on G are needed (compare with [14]).

(c) Basic features of positive scalar curvature have their counterparts for *mean convex* hypersurfaces (see [16]), where the infinite dimensional geometry is a bit more transparent than that of the scalar curvature.

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