On the Convex Pfaff–Darboux Theorem of Ekeland and Nirenberg

Robert L. BRYANT

Department of Mathematics, Duke University,
PO Box 90320, Durham, NC 27708-0320, USA
E-mail: *bryant@math.duke.edu*
URL: `https://fds.duke.edu/db/aas/math/faculty/bryant`

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Abstract. The classical Pfaff–Darboux theorem, which provides local 'normal forms' for 1-forms on manifolds, has applications in the theory of certain economic models [Chiappori P.-A., Ekeland I., *Found. Trends Microecon.* **5** (2009), 1–151]. However, the normal forms needed in these models often come with an additional requirement of some type of *convexity*, which is not provided by the classical proofs of the Pfaff–Darboux theorem. (The appropriate notion of 'convexity' is a feature of the economic model. In the simplest case, when the economic model is formulated in a domain in $\mathbb{R}^n$, convexity has its usual meaning.) In [*Methods Appl. Anal.* **9** (2002), 329–344], Ekeland and Nirenberg were able to characterize necessary and sufficient conditions for a given 1-form ω to admit a convex local normal form (and to show that some earlier attempts [Chiappori P.-A., Ekeland I., *Ann. Scuola Norm. Sup. Pisa Cl. Sci. 4* **25** (1997), 287–297] and [Zakalyukin V.M., *C. R. Acad. Sci. Paris Sér. I Math.* **327** (1998), 633–638] at this characterization had been unsuccessful). In this article, after providing some necessary background, I prove a strengthened and generalized convex Pfaff–Darboux theorem, one that covers the case of a Legendrian foliation in which the notion of convexity is defined in terms of a torsion-free affine connection on the underlying manifold. (The main result of Ekeland and Nirenberg concerns the case in which the affine connection is flat.)

Key words: Pfaff–Darboux theorem; convexity; utility theory

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1 Introduction

The Pfaff–Darboux theorem provides a local 'normal form' for 1-forms on manifolds, assuming that certain constant rank conditions are met. A common version[1] of this classical theorem is the following: Let ω be a smooth 1-form on an n-manifold M and suppose that there is an integer $k > 0$ such that

$\omega \wedge (\mathrm{d}\omega)^k$ vanishes identically on M

while

$\omega \wedge (\mathrm{d}\omega)^{k-1}$ is nowhere vanishing on M.

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[1] There are many variants. See [2, Chapter II, Section 3].

Then each $m \in M$ has an open neighborhood $U \subset M$ on which there exist (smooth) functions $y^1, \ldots, y^k$, $p_2, \ldots, p_k$ and a nonvanishing function a such that[2]

$$U^*\omega = a(\mathrm{d}y^1 + p_2\,\mathrm{d}y^2 + \cdots + p_k\,\mathrm{d}y^k). \tag{1.1}$$

Since

$$U^*\big(\omega \wedge (\mathrm{d}\omega)^{k-1}\big) = (-1)^{k(k-1)/2}(k-1)!\,a^k\,\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^k \wedge \mathrm{d}p_2 \wedge \cdots \wedge \mathrm{d}p_k,$$

the functions $y^1, \ldots, y^k$, $p_2, \ldots, p_k$ in this representation must be independent on U.

The normal form (1.1) is often written more symmetrically as

$$U^*\omega = a_1\,\mathrm{d}u^1 + a_2\,\mathrm{d}u^2 + \cdots + a_k\,\mathrm{d}u^k, \tag{1.2}$$

where the a_i do not simultaneously vanish in U. In this representation, the independence of the functions $y^1, \ldots, y^k$, $p_2, \ldots, p_k$ translates into the condition that the mapping

$$(u,[a])\colon\ U \to \mathbb{R}^k \times \mathbb{RP}^{k-1} = \mathbb{P}\big(T^*\mathbb{R}^k\big)$$

be a submersion.

In fact, the representation (1.2) is more common in treatises on mathematical economics, where the Pfaff–Darboux theorem plays an important role [3]. Often, the normal forms needed in these models sometimes come with an additional requirement of *convexity*, i.e., the underlying manifold is $M = \mathbb{R}^n$ (or an open domain in $\mathbb{R}^n$), and one would like to arrange that the functions a_i be *positive* and the functions u^i be *strictly convex*, i.e., have positive definite Hessians.[3] A useful reference for the role of convexity in economic models is the book [4] and the article [1].

Now, it turns out that constructing such a *convex* Pfaff–Darboux representation is not always possible, which raises the question of how to determine when one exists. In [5], Ekeland and Nirenberg were able to provide necessary and sufficient conditions for a given 1-form $\omega \in \Omega^1(\mathbb{R}^n)$ to admit a local convex Pfaff–Darboux normal form. They also constructed examples that showed that some earlier attempts [3, 6] to find such conditions had been unsuccessful.

In this note, after providing some necessary background, I prove a generalization of the convex Pfaff–Darboux theorem of Ekeland and Nirenberg. This treatment has some notable features that make it of interest for the general problem.

First, the proof of Ekeland and Nirenberg does not assume the classical Pfaff–Darboux theorem; instead, it constructs the desired convex representation directly using the Frobenius theorem, essentially reproving the Pfaff–Darboux theorem but with the additional convexity condition imposed. The proof in this article assumes the classical Pfaff–Darboux theorem, so that the argument can more directly focus on choosing a Pfaff–Darboux representation that satisfies the convexity requirements. This results in a shorter proof, one that also brings the nature of the convexity requirements more sharply into focus.

Second, the notion of *strict convexity* turns out to be meaningful on any manifold endowed with a torsion-free affine connection, and the proof below covers this more general situation with no extra work.

Third, the proof yields a stronger result, in that it produces a local convex Pfaff–Darboux representation of ω adapted to any ω-Legendrian foliation that satisfies a certain geometrically natural positivity condition, one that is equivalent pointwise to the condition of Ekeland and Nirenberg.

[2] Throughout this article, I adopt the convention that, when $L \subset M$ is a submanifold and ψ is a differential form on M, then $L^*\psi$ denotes the *pullback* of ψ to L.

[3] Sometimes one only requires *weak convexity*, i.e., that the Hessian of u^i be positive definite on each of its level sets.

2 Classical Pfaff–Darboux theorems

Let ω be a smooth 1-form on an n-manifold M that, for some integer $k > 0$, satisfies

$$\omega \wedge (\mathrm{d}\omega)^k \text{ vanishes identically on } M \tag{2.1}$$

while

$$\omega \wedge (\mathrm{d}\omega)^{k-1} \text{ is nowhere vanishing on } M. \tag{2.2}$$

The integer $k{-}1$ is known as the *Pfaff rank* of ω [2, Chapter II, Section 3]. Note that $k \le \frac{1}{2}(n{+}1)$. When $k = \frac{1}{2}(n{+}1)$, ω is said to be a *contact form* on M.

When ω satisfies (2.1) and (2.2), so does $\tilde{\omega} = f\omega$ for any nonvanishing function f on M, since $\tilde{\omega} \wedge (\mathrm{d}\tilde{\omega})^{r-1} = f^r \omega \wedge (\mathrm{d}\omega)^{r-1}$ for all integers $r > 0$.

2.1 Canonical subbundles

An ω satisfying (2.1) and (2.2) defines a *kernel subbundle* $K = \omega^{-1}(0) \subset TM$ of corank 1 and a subbundle $A \subset K$ of corank $2(k{-}1)$ in K by the rule that, for each $m \in M$,

$$A_m = \{v \in K_m \mid \mathrm{d}\omega(v, w) = 0,\ \forall w \in K_m\}.$$

Replacing ω by $\tilde{\omega} = f\omega$ for any nonvanishing function f does not change K or A.

If $k > 1$, then $K \subset TM$ is not an integrable plane field, but the subbundle $A \subset TM$ is always integrable, since it is the Cauchy characteristic plane field of the differential ideal $\mathcal{I}$ generated by ω (see [2, Chapter II, Proposition 2.1]). In the contact case, i.e., when $n = 2k{-}1$ (which is, in some sense, generic), one has $A = (0)$.

There is a nondegenerate, skew-symmetric bilinear pairing $B_\omega\colon K/A \times K/A \to \mathbb{R}$ defined by

$$B_\omega\big(v{+}A_m, w{+}A_m\big) = \mathrm{d}\omega(v, w),$$

when $v, w \in K_m$. It satisfies $B_{f\omega} = fB_\omega$ for any nonvanishing function f on M.

Note that any subspace $W \subset T_m$ on which both ω and $\mathrm{d}\omega$ vanish, must, first of all, satisfy $W \subset K_m$ (since ω vanishes on W), and, second, must have codimension at least $k{-}1$ in K_m, since $\mathrm{d}\omega$, as a skew-symmetric form on K_m, has Pfaff rank $k{-}1$. Moreover, if W does have codimension $k{-}1$ in K_m, then it must contain A_m, so that W/A_m is a null subspace of B_ω.

2.2 Legendrian submanifolds and Grassmannians

Any submanifold $L \subset M$ that satisfies $L^*\omega = 0$, i.e., an *integral manifold of* ω, must also satisfy $L^*\mathrm{d}\omega = 0$ and hence, by the above linear algebra discussion, must have codimension at least k in M. When $L \subset M$ is an integral manifold of ω of codimension k, it is said to be an *ω-Legendrian submanifold.*

In particular, if $L \subset M$ is ω-Legendrian, then, for each $m \in L$, the tangent space T_mL satisfies $A_m \subset T_mL \subset K_m$ while B_ω vanishes identically on $T_mL/A_m \subset K_m/A_m$.

This motivates defining the *Legendrian Grassmannian* $\mathrm{Leg}_m(\omega) \subset \mathrm{Gr}^k(T_mM)$ to be the set of subspaces $W \subset K_m$ that have codimension k in T_mM and on which both ω and $\mathrm{d}\omega$ vanish. By the above remarks, it follows that $\mathrm{Leg}_m(\omega)$ can be canonically identified with the *Lagrangian Grassmannian* $\mathrm{Lag}(K_m/A_m) \subset \mathrm{Gr}^{k-1}(K_m/A_m)$ consisting of the $(k{-}1)$-dimensional subspaces of K_m/A_m on which B_ω vanishes. Hence, $\mathrm{Leg}_m(\omega)$ is naturally a smooth manifold of dimension $\frac{1}{2}k(k{-}1)$. Moreover, the (disjoint) union

$$\mathrm{Leg}(\omega) = \bigcup_{m\in M} \mathrm{Leg}_m(\omega) \subset \mathrm{Gr}^k(TM)$$

is a smooth subbundle, and $\mathrm{Leg}(f\omega) = \mathrm{Leg}(\omega)$ for all nonvanishing f.

2.3 A local normal form

One version of the *Pfaff–Darboux theorem* [2, Chapter II, Theorem 3.1] states that, when $\omega \in \Omega^1(M)$ satisfies (2.1) and (2.2), each $m \in M$ has an open neighborhood $U \subset M$ on which there exist smooth functions $u^1, \ldots, u^k$ and $a_1, \ldots, a_k$ (with not all a_i simultaneously vanishing) so that

$$U^*\omega = a_1\,\mathrm{d}u^1 + \cdots + a_k\,\mathrm{d}u^k. \tag{2.3}$$

Moreover, the mapping $(u,[a])\colon U \to \mathbb{R}^k \times \mathbb{RP}^{k-1}$ is a submersion. (In fact, the kernel subbundle of the differential of this mapping is the restriction of A to U.)

Conversely, the existence of functions u^i and a_i for $1 \le i \le k$ on an open set $U \subset M$ satisfying (2.3) with the a_i not all simultaneously vanishing and having the property that $(u,[a])\colon U \to \mathbb{R}^k \times \mathbb{RP}^{k-1}$ be a submersion implies that both (2.1) and (2.2) hold on U.

2.4 Geometry of the normal form

It will be useful to have a geometric interpretation of the Pfaff–Darboux theorem. Now, in the representation (2.3), the functions u^i have independent differentials, i.e., $\mathrm{d}u^1 \wedge \cdots \wedge \mathrm{d}u^k$ does not vanish on U. Consequently, the simultaneous level sets of the functions u^i define a foliation $\mathcal{L}$ of $U \subset M$ by ω-Legendrian submanifolds, i.e., an ω-Legendrian foliation.

Conversely, given an ω-Legendrian foliation $\mathcal{L}$ on an open subset $V \subset M$, each point $m \in V$ will have an open neighborhood $U \subset V$ in which the leaves of $\mathcal{L}$ are the fibers of a submersion $u = (u^i)\colon U \to \mathbb{R}^k$. Since ω vanishes when pulled back to any fiber of u, it follows that there exists a mapping $a = (a_i)\colon U \to \mathbb{R}^k$ such that $U^*\omega = a_1\,\mathrm{d}u^1 + \cdots + a_k\,\mathrm{d}u^k$.

Thus, a geometric interpretation of the Pfaff–Darboux theorem is the statement that, when $\omega \in \Omega^1(M)$ satisfies (2.1) and (2.2), each point $m \in M$ has an open neighborhood $U \subset M$ on which there exists an ω-Legendrian foliation.

2.5 Variants and extensions

There are a number of variants and extensions of the classical Pfaff–Darboux theorem that can all be seen to be equivalent to the above versions by elementary arguments [2, Chapter II, Section 3]. In this article, two such variants will be important. For convenience of reference, they will be stated as propositions.

Proposition 2.1. *Suppose that $\omega \in \Omega^1(M)$ satisfies (2.1) and (2.2). Then for each $m \in M$ and $W \in \mathrm{Leg}_m(\omega)$, there exists a ω-Legendrian submanifold $L \subset M$ such that $m \in L$ and $W = T_mL$.*

Proposition 2.2. *Suppose that $\omega \in \Omega^1(M)$ satisfies (2.1) and (2.2) and that $L \subset M$ is an embedded ω-Legendrian submanifold. Then each $m \in L$ has an open neighborhood $U \subset M$ on which there exists an ω-Legendrian foliation $\mathcal{L}$ with the property that $L \cap U$ is a leaf of $\mathcal{L}$.*

3 Convexity and affine manifolds

3.1 Classical convexity

When $M = \mathbb{R}^n$, there is a notion of *strict convexity* of a function u, which is the condition that the Hessian quadratic form $H(u)$ be positive definite, where

$$H(u) = \frac{\partial^2 u}{\partial x^i \partial x^j}\,\mathrm{d}x^i \otimes \mathrm{d}x^j \tag{3.1}$$

and where $x^1, \ldots, x^n$ are the usual affine linear coordinates in $\mathbb{R}^n$. Note that strict convexity is an affine-invariant notion on $\mathbb{R}^n$.

Motivated by applications in economics, Ekeland and Nirenberg [5] asked what further conditions one must impose on an $\omega \in \Omega^1(\mathbb{R}^n)$ satisfying (2.1) and (2.2) in order to know that one can choose the functions u^j and a_j in the representation (2.3) so that the u^j be strictly convex and the a_j be positive. It is not hard to show, by example, that *some* further condition on ω is necessary to guarantee the existence of such a convex representation. (See the discussion at the beginning of Section 3.3.)

They showed that two earlier articles [3, 6] claiming to provide such necessary and sufficient conditions were flawed (indeed, they exhibited counterexamples to the claims of these articles) and then produced their own condition, which they showed to be necessary and sufficient.

In this note, I will show that their main result, properly formulated, holds good on an n-manifold M endowed with a torsion-free affine connection, not just on $\mathbb{R}^n$ endowed with the (flat) affine connection it inherits as a vector space.

3.2 Affine connections and convexity

Let ∇ be a torsion-free affine connection on an n-manifold M^n, i.e., ∇ is a first-order, linear differential operator

$$\nabla\colon\ \Omega^1(M) \to \Omega^1(M) \otimes \Omega^1(M)$$

that obeys the Leibnitz rule

$$\nabla(f\eta) = \mathrm{d}f \otimes \eta + f\nabla(\eta)$$

for all smooth functions f on M and smooth 1-forms η on M. The assumption that ∇ be torsion-free is the condition that the associated (second-order) *Hessian operator* $H(u) = \nabla(\mathrm{d}u)$ be a symmetric $(0,2)$-tensor for each smooth function u on M.

A smooth function u on M is said to be *strictly* ∇*-convex* if, as a quadratic form, $H(u)$ is positive definite at every point of M.

When $M = \mathbb{R}^n$ and ∇ is the standard (flat) connection, satisfying $\nabla(\mathrm{d}x^i) = 0$ for all of the coordinate functions x^i, then $H(u)$ is the usual Hessian tensor (3.1), and this notion of convexity is simply the classical one.

In the more general case, when $x = (x^i)\colon U \to \mathbb{R}^n$ is a local coordinate chart, one has

$$H\big(x^k\big) = \nabla\big(\mathrm{d}x^k\big) = \Gamma^k_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j,$$

where $\Gamma^k_{ij} = \Gamma^k_{ji} \in C^\infty(U)$ are the coefficients of the connection ∇ relative to the coordinate chart $x = (x^i)$. The general coordinate formula for H then becomes

$$H(u) = \left(\frac{\partial^2 u}{\partial x^i \partial x^j} + \Gamma^k_{ij}\frac{\partial u}{\partial x^k}\right)\mathrm{d}x^i \otimes \mathrm{d}x^j.$$

Thus, ∇-convexity of u is expressible in terms of a condition on the 2-jet of u, slightly more general than the condition for classical convexity.

Adopting the usual conventions

$$\alpha \wedge \beta = \tfrac{1}{2}(\alpha \otimes \beta - \beta \otimes \alpha), \qquad \alpha \circ \beta = \tfrac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha),$$

one sees that, for a 1-form ω of the form

$$\omega = a_1\,\mathrm{d}u^1 + \cdots + a_k\,\mathrm{d}u^k, \tag{3.2}$$

one has (using the summation convention)

$$\begin{aligned}\nabla\omega &= \mathrm{d}a_i \otimes \mathrm{d}u^i + a_i\nabla(u^i) = \mathrm{d}a_i \otimes \mathrm{d}u^i + a_i H(u^i)\\ &= \mathrm{d}a_i \wedge \mathrm{d}u^i + \mathrm{d}a_i \circ \mathrm{d}u^i + a_i H(u^i)\\ &= \mathrm{d}\omega + \mathsf{S}^\nabla\omega,\end{aligned}$$

where I have introduced the notation $\mathsf{S}^\nabla\omega$ to denote the *symmetrization* of $\nabla(\omega)$. Thus, $\mathsf{S}^\nabla\omega = \nabla\omega - \mathrm{d}\omega$ is a well-defined quadratic form on M. (Of course, the linear, first-order differential operator S^∇ depends on ∇.)

3.3 A positivity condition

The quadratic form $S^\nabla\omega$ provides some insight into the question of whether a ∇-convex Pfaff–Darboux representation of ω is possible.

Proposition 3.1. *Suppose that $\omega \in \Omega^1(M)$ satisfies (2.1) and (2.2). If there exist positive functions a_i and ∇-convex functions u^i for $1 \le i \le k$ such that (3.2) holds, then $S^\nabla\omega$ is positive definite on the leaves of the foliation $\mathcal{L}$ defined by $\mathrm{d}u^1 = \mathrm{d}u^2 = \cdots = \mathrm{d}u^k = 0$.*

Proof. Since $\mathsf{S}^\nabla\omega = \mathrm{d}a_i \circ \mathrm{d}u^i + a_i H(u^i)$, it follows that, when restricted to the plane field $N \subset TM$ defined by $\mathrm{d}u^1 = \mathrm{d}u^2 = \cdots = \mathrm{d}u^k = 0$, the terms $\mathrm{d}a_i \circ \mathrm{d}u^i$ in $\mathsf{S}^\nabla\omega$ vanish. Thus, $S^\nabla\omega = a_i H(u^i)$ as quadratic forms on N. By the positivity of the a_i and the ∇-convexity of the u^i, it follows that $S^\nabla\omega$ is positive definite on N.[4] ∎

This proposition provides necessary condition for the existence of a convex Pfaff–Darboux representation.

Example 3.2 (an obstructed example)**.** Let $M = \mathbb{R}^n$ with standard coordinates $x = (x^i)$, and let ∇ be the (flat, torsion-free) connection such that $\nabla(\mathrm{d}x^i) = 0$ for $1 \le i \le n$. Let c_i, $f_{ij} = -f_{ji}$, and $g_{ij} = g_{ji}$ be constants and consider the 1-form

$$\omega = \big(c_i + (f_{ij} + g_{ij})x^j\big)\,\mathrm{d}x^i.$$

Then $\mathrm{d}\omega = f_{ij}\,\mathrm{d}x^j \wedge \mathrm{d}x^i = -f_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j$ and $\mathsf{S}^\nabla\omega = g_{ij}\,\mathrm{d}x^i \circ \mathrm{d}x^j$.

Now assume that the skew-symmetric matrix $f = (f_{ij})$ has rank $2(k{-}1) < n$, so that $(\mathrm{d}\omega)^{k-1} \ne 0$ but $(\mathrm{d}\omega)^k = 0$. Then, for generic choice of the constants c_i, $\omega \wedge (\mathrm{d}\omega)^{k-1}$ will be nonvanishing on an open neighborhood $U \subset \mathbb{R}^n$ of the origin $0 \in \mathbb{R}^n$, in which case, ω will satisfy the hypotheses (2.1) and (2.2) on U.

If the symmetric matrix $g = (g_{ij})$ does not have at least $n{-}k$ positive eigenvalues, then $\mathsf{S}^\nabla\omega$ cannot be positive definite on any $(n{-}k)$-dimensional subbundle $N \subset TU$, and, hence, by Proposition 3.1, ω cannot have a convex Pfaff–Darboux representation in any open subset of U.

It turns out that this necessary condition for a local 'convex' Pfaff–Darboux representation compatible with an ω-Legendrian foliation $\mathcal{L}$ is also sufficient.

Theorem 3.3. *Suppose ∇ be a torsion-free affine connection on M, that $\omega \in \Omega^1(M)$ satisfy (2.1) and (2.2) for some $k > 0$, and that $\mathcal{L}$ be an ω-Legendrian foliation on M with the property that $\mathsf{S}^\nabla\omega$ pulls back to each leaf of $\mathcal{L}$ to be positive definite. Then each $m \in M$ has an open neighborhood $U \subset M$ on which there exist strictly ∇-convex functions $u^1, \ldots, u^k$ that are constant on the leaves of $\mathcal{L}$ in U and positive functions $a_1, \ldots, a_k$ such that*

$$U^*\omega = a_1\,\mathrm{d}u^1 + \cdots + a_k\,\mathrm{d}u^k.$$

[4] It is worth pointing out that the same conclusion about the positive definiteness of $\mathsf{S}^\nabla\omega$ on the leaves of $\mathcal{L}$ would have followed if one had merely assumed that each u^i be only 'strictly ∇-*quasi*-convex', i.e., that $\mathrm{d}u^i$ be nonvanishing and $H(u^i)$ be positive definite when restricted to the hyperplane field $\mathrm{d}u^i = 0$. Compare [5, Lemma 1], and the preceding discussion about their Problem 2.

Before giving the proof of Theorem 3.3, I will state one of its corollaries, so that it can be compared with the main result of Ekeland and Nirenberg [5, Theorem 1].

First, some useful terminology. As always, assume that ω satisfies (2.1) and (2.2) for some $k > 0$.

Definition 3.4. An ω-Legendrian subspace $W \subset T_mM$ is ∇*-positive for* ω if the restriction of the quadratic form $\mathsf{S}^\nabla\omega$ to W is positive definite.

Let $\mathrm{Leg}^+(\omega,\nabla) \subseteq \mathrm{Leg}(\omega)$ denote the set of ω-Legendrian subspaces that are ∇-positive for ω. Then $\mathrm{Leg}^+(\omega,\nabla)$ is a (possibly empty) open subset of $\mathrm{Leg}(\omega)$. Consequently, the set of points $m \in M$ for which there exists a ∇-positive, ω-Legendrian subspace $W \subset T_mM$ is an open subset of M. Also, note that, since such a W contains A_m, it follows that $\mathsf{S}^\nabla\omega$ must be positive definite on A_m.

Corollary 3.5. *Suppose that ∇ be a torsion-free affine connection on M, that $\omega \in \Omega^1(M)$ satisfy (2.1) and (2.2) for some $k > 0$, and that there exist a $W \in \mathrm{Leg}^+(\omega,\nabla)$ with $W \subset T_mM$. Then $m \in M$ has an open neighborhood $U \subset M$ on which there exist strictly ∇-convex functions $u^1,\dots,u^k$ and positive functions $a_1,\dots,a_k$ such that*

$$U^*\omega = a_1\,\mathrm{d}u^1 + \cdots + a_k\,\mathrm{d}u^k.$$

The proof of Corollary 3.5 follows by applying Propositions 2.1 and 2.2 to produce an ω-Legendrian foliation $\mathcal{L}$ on an open neighborhood V of m whose leaf through m has W as a tangent space. Since $\mathsf{S}^\nabla\omega$ is positive definite on W, it follows that it is positive definite on all the tangent spaces to the leaves of $\mathcal{L}$ in some (possibly) smaller m-neighborhood $V' \subset V$. Now apply Theorem 3.3 to $\mathcal{L}$ on V'.

Remark 3.6. In the special case in which $M = \mathbb{R}^n$ and ∇ is the flat connection satisfying $\nabla(\mathrm{d}x^i) = 0$ for x^i the standard coordinates on $\mathbb{R}^n$, Corollary 3.5 simply becomes Theorem 1 of Ekeland and Nirenberg [5], since their Condition 3 turns out to be equivalent to the existence of a $W \in \mathrm{Leg}^+(\omega,\nabla)$ with $W \subset T_mM$ in this case.

Proof of Theorem 3.3. There exists an m-neighborhood $V_0 \subset M$ on which there exist smooth functions $y^1,\dots,y^k$ vanishing at m so that the leaves of $\mathrm{d}y^1 = \cdots = \mathrm{d}y^k = 0$ are intersections of the leaves of $\mathcal{L}$ with V_0 as well as functions $p_2,\dots,p_k$, also vanishing at m, and a nonvanishing function a so that

$$V_0^*\omega = a\bigl(\mathrm{d}y^1 + p_2\,\mathrm{d}y^2 + \cdots + p_k\,\mathrm{d}y^k\bigr)$$

By reversing the signs of a and the y^i, if necessary, one can assume that $a(m) > 0$. Let $W \subset T_mM$ be the tangent to the leaf of $\mathcal{L}$ that passes through m, so that W is the common kernel of the $\mathrm{d}y^i$ evaluated at m.

Set $\bar\omega = a^{-1}\omega$ and note that, since $\mathrm{d}\bar\omega \equiv a^{-1}\,\mathrm{d}\omega \bmod \omega$, it follows that $\mathcal{L}$ is also $\bar\omega$-Legendrian. Moreover, since

$$\mathsf{S}^\nabla\bar\omega = \mathrm{d}\bigl(a^{-1}\bigr)\circ\omega + a^{-1}\mathsf{S}^\nabla\omega,$$

it follows that the tangent spaces of $\mathcal{L}$ (which, of course, satisfy $\omega = 0$) are also ∇-positive for $\bar\omega$. Since $\omega = a\bar\omega$ and $a > 0$, finding the desired convex representation for $\bar\omega$ will also yield one for ω. Thus, it suffices to prove the theorem with $\bar\omega$ in the place of ω, i.e., to assume that $a = 1$, so I will do that from now on. Thus,

$$V_0^*\omega = \mathrm{d}y^1 + p_2\,\mathrm{d}y^2 + \cdots + p_k\,\mathrm{d}y^k.$$

Since $\omega\wedge(\mathrm{d}\omega)^{k-1}\neq 0$, the functions $y^1,\ldots,y^k$, $p_2,\ldots,p_k$ have linearly independent differentials at m.

Restricting to V_0, i.e., setting $M=V_0$, one has

$$\mathsf{S}^\nabla\omega = H\bigl(y^1\bigr)+\mathrm{d}p_2\circ\mathrm{d}y^2+\cdots+\mathrm{d}p_k\circ\mathrm{d}y^k+p_2H\bigl(y^2\bigr)+\cdots+p_kH\bigl(y^k\bigr).$$

Since the p_j vanish at m, it follows that, when restricted to $W\subset T_mM$, the two quadratic forms $H\bigl(y^1\bigr)$ and $\mathsf{S}^\nabla\omega$ are equal. Thus $H\bigl(y^1\bigr)$ is positive definite on W, and so there is a constant $c>0$ so that $H\bigl(y^1\bigr)+c\bigl(\mathrm{d}y^2\bigr)^2+\cdots+c\bigl(\mathrm{d}y^k\bigr)^2$ is positive definite on $K_m=\{v\in T_mM \mid \mathrm{d}y^1(v)=0\}$. Writing

$$\omega=\mathrm{d}\bigl(y^1+\tfrac12c\bigl(y^2\bigr)^2+\cdots+\tfrac12c\bigl(y^k\bigr)^2\bigr)+\bigl(p_2-cy^2\bigr)\,\mathrm{d}y^2+\cdots+\bigl(p_k-cy^k\bigr)\,\mathrm{d}y^k$$

and observing that

$$\begin{aligned}&H\bigl(y^1+\tfrac12c\bigl(y^2\bigr)^2+\cdots+\tfrac12c\bigl(y^k\bigr)^2\bigr)\\&\qquad=H\bigl(y^1\bigr)+c\bigl(\mathrm{d}y^2\bigr)^2+\cdots+c\bigl(\mathrm{d}y^k\bigr)^2+cy^2H\bigl(y^2\bigr)+\cdots+cy^kH\bigl(y^k\bigr)\end{aligned}$$

shows that, setting

$$\bar y^1=y^1+\tfrac12c\bigl(y^2\bigr)^2+\cdots+\tfrac12c\bigl(y^k\bigr)^2,\qquad \bar y^i=y^i,\qquad\text{and}\qquad \bar p_i=p_i-cy^i,$$

one has $\omega=\mathrm{d}\bar y^1+\bar p_2\,\mathrm{d}\bar y^2+\cdots+\bar p_k\,\mathrm{d}\bar y^k$.

Thus, one could have chosen the functions $y^1,\ldots,y^k$, $p_2,\ldots,p_k$ with $H\bigl(y^1\bigr)$ being positive definite on the hyperplane K_m. Assume now that this was done.

It still needs to be shown that one can choose the functions $y^1,\ldots,y^k$, $p_2,\ldots,p_k$ with $H\bigl(y^1\bigr)$ being positive definite on all of T_mM, not just on K_m, which is the kernel of $\mathrm{d}y^1$ at m. To do this, note that, if ϕ is any smooth function on a neighborhood of the origin in $\mathbb{R}$, then

$$H\bigl(\phi\bigl(y^1\bigr)\bigr)=\phi'\bigl(y^1\bigr)H\bigl(y^1\bigr)+\phi''\bigl(y^1\bigr)\bigl(\mathrm{d}y^1\bigr)^2.$$

Hence, by choosing a ϕ with $\phi(0)=0$, $\phi'(0)=1$ and $\phi''(0)>0$ sufficiently large, I can arrange that $\phi\bigl(y^1\bigr)$ be strictly ∇-convex at m. Since

$$\omega=\frac{1}{\phi'(y_1)}\bigl(\mathrm{d}\bigl(\phi\bigl(y^1\bigr)\bigr)+\phi'(y_1)p_2\,\mathrm{d}y^2+\cdots+\phi'(y_1)p_k\,\mathrm{d}y^k\bigr),$$

one sees that the functions $\bar y^1,\ldots,\bar y^k$, $\bar p_2,\ldots,\bar p_k$, where

$$\bar y^1=\phi\bigl(y^1\bigr),\qquad\text{and}\qquad \bar y^i=y^i,\qquad \bar p_i=\phi'(y_1)p_i,\qquad 2\le i\le k,$$

(with $a=1/\phi'(y_1)>0$), give a Pfaff–Darboux representation for ω that is compatible with the foliation $\mathcal{L}$ and for which $\bar y^1$ is strictly ∇-convex.[5]

Thus, one can assume henceforth that, on an open m-neighborhood $V_1\subset M$, one has a representation of the form

$$\omega=\mathrm{d}y^1+p_2\,\mathrm{d}y^2+\cdots+p_k\,\mathrm{d}y^k,$$

where the functions $y^1,\ldots,y^k,p_2,\ldots,p_k\in C^\infty(V_1)$ all vanish at m, the equations $\mathrm{d}y^i=0$ define the tangents to the leaves of $\mathcal{L}$ in V_1, and y^1 is strictly ∇-convex.

[5] This is the same idea that Ekeland and Nirenberg [5] used in their generalization of their Theorem 1 to cover the quasi-convex case.

Under these assumptions, there is a constant $b > 0$ sufficiently large so that $H\big(y^i + by^1\big) = H\big(y^i\big) + bH\big(y^1\big)$ is positive definite at m for $2 \le i \le k$. Thus, writing

$$\omega = \big(1 - b(p_2 + \cdots + p_k)\big)\,\mathrm{d}y^1 + p_2\,\mathrm{d}\big(y^2 + by^1\big) + \cdots + p_k\,\mathrm{d}\big(y^k + by^1\big),$$

it follows that I can, after restricting to an m-neighborhood $V_2 \subset V_1$ on which the function $a = \big(1 - b(p_2 + \cdots + p_k)\big)$ is positive, dividing by $a > 0$, and replacing y^j by $y^j + by^1$ and p_j by p_j/a for $2 \le j \le k$, assume that I have a representation

$$\omega = \mathrm{d}y^1 + p_2\,\mathrm{d}y^2 + \cdots + p_k\,\mathrm{d}y^k,$$

in which all of the $H\big(y^j\big)$ are positive definite at m, i.e., the y^j are strictly ∇-convex on some neighborhood of m and the p_i all vanish at m.

Finally, for $\varepsilon > 0$ and sufficiently small, write

$$\omega = \mathrm{d}\big(y^1 - \varepsilon\big(y^2 + \cdots + y^k\big)\big) + (p_2 + \varepsilon)\,\mathrm{d}y^2 + \cdots + (p_k + \varepsilon)\,\mathrm{d}y^k.$$

Then, setting $u^1 = y^1 - \varepsilon\big(y^2 + \cdots + y^k\big)$ and $u^j = y^j$ for $j > 1$ and setting $a_1 = 1$ and $a_j = \varepsilon + p_j$ for $j > 1$, one achieves the desired convex Pfaff–Darboux representation on an open m-neighborhood $U \subset V_2$. ■

Example 3.7 (dependence on ∇)**.** Let ω be defined on $\mathbb{R}^n$ as in Example 3.2 with the constants c_i, f_{ij} and g_{ij} as specified there so that $\omega \wedge (\mathrm{d}\omega)^k = 0$ and $\omega \wedge (\mathrm{d}\omega)^{k-1}$ is non-vanishing on an open neighborhood $U \subset \mathbb{R}^n$ of $0 \in \mathbb{R}^n$. Now, however, suppose that ∇ is the torsion-free connection that satisfies $\nabla(\mathrm{d}x^i) = \Gamma^i_{jk}\,\mathrm{d}x^j \otimes \mathrm{d}x^k$, where $\Gamma^i_{jk} = \Gamma^i_{kj}$ are constants. Then calculation yields

$$\mathsf{S}^\nabla\omega = \big(g_{ij} + \big(c_k + (f_{kl} + g_{kl})x^l\big)\Gamma^k_{ij}\big)\,\mathrm{d}x^i \circ \mathrm{d}x^j.$$

Consequently, at the origin $x = 0$, one finds $\big(\mathsf{S}^\nabla\omega\big)_0 = \big(g_{ij} + c_k\Gamma^k_{ij}\big)\,\mathrm{d}x^i \circ \mathrm{d}x^j$. It follows that, no matter the value of $g = (g_{ij})$, one can always choose Γ^i_{jk} so that $\big(\mathsf{S}^\nabla\omega\big)_0$ be positive definite. Thus, by Theorem 3.3 there always exists a torsion-free connection ∇ for which a ∇-convex Pfaff–Darboux representation of ω exists on a neighborhood of $0 \in \mathbb{R}^n$.

This highlights the significance of the choice of background connection for the convex representability of a given ω. In turn, this makes clear how the choice of coordinates in which a given problem is described affects the existence of convex representability.

Remark 3.8 (global considerations)**.** While Theorem 3.3 gives necessary and sufficient conditions for the existence of local ∇-convex Pfaff–Darboux representations, for applications one would like to know something about how large an open set in the model M one can cover with such a representation, and this seems to be a subtle problem.

Even in the simplest case of a 3-manifold M endowed with a contact 1-form ω and a torsion-free affine connection ∇ for which $\mathsf{S}^\nabla\omega$ is positive definite on the 2-plane bundle $K \subset TM$, it is not clear how to characterize the domains $U \subset M$ that support a ∇-convex Pfaff–Darboux representation for ω.

Example 3.9 (global non-existence)**.** Let $S^3 \subset \mathbb{R}^4 \simeq \mathbb{H}$ be the unit sphere regarded as the Lie group of unit quaternions. Let ω_i for $i = 1, 2, 3$ be a basis for the left-invariant 1-forms on S^3, which obey the formulae $\mathrm{d}\omega_i = \epsilon_{ijk}\omega_j \wedge \omega_k$ where ϵ is the fully skew-symmetric symbol satisfying $\epsilon_{123} = 1$. In particular, $\omega_i \wedge (\mathrm{d}\omega_i) = 2\omega_1 \wedge \omega_2 \wedge \omega_3 \neq 0$ but $\omega_i \wedge (\mathrm{d}\omega_i)^2 = 0$, so that ω_i satisfies the usual hypotheses with $k = 1$, i.e., it is a contact form on S^3.

Now let $\Gamma_{ijk} = \Gamma_{ikj}$ be constants for $1 \le i, j, k \le 3$ and let ∇ be the affine connection (necessarily torsion-free) that satisfies

$$\nabla(\omega_i) = (\epsilon_{ijk} + \Gamma_{ijk})\omega_j \otimes \omega_k.$$

Then

$$S^{\nabla}(\omega_i) = \Gamma_{ijk}\omega_j \circ \omega_k.$$

Note that, if we choose $\Gamma_{ijk} = 0$, so that ∇ is the Levi-Civita connection of the constant curvature metric $g = {\omega_1}^2 + {\omega_2}^2 + {\omega_3}^2$ on S^3, then $S^{\nabla}(\omega_i) \equiv 0$, so ω_i cannot have a ∇-convex Pfaff–Darboux representation on any open set in S^3.

However, with appropriate choice of constants Γ_{ijk}, we could arrange that $S^{\nabla}(\omega_i)$ be positive definite for each i. In this case, by [5, Theorem 1], each point $m \in S^3$ will have an open neighborhood on which ω_i has a ∇-convex Pfaff–Darboux representation.

However, because S^3 is compact, any smooth function u on S^3 must have a local maximum, and $H(u)$ will be negative semidefinite there, independent of the choice of ∇. Thus, there cannot be any global functions u on S^3 that are ∇-convex. *A fortiori,* there cannot be a global ∇-convex Pfaff–Darboux representation for ω_i for any i.

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