

Algebraic Structures on Typed Decorated Rooted Trees

Loïc FOISSY

Univ. Littoral Côte d'Opale, UR 2597 LMPA, Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville, F-62100 Calais, France

E-mail: foissy@univ-littoral.fr

URL: <https://loic.foissy.free.fr/pageperso/accueil.html>

Received February 02, 2021, in final form September 12, 2021; Published online September 21, 2021
<https://doi.org/10.3842/SIGMA.2021.086>

Abstract. Typed decorated trees are used by Bruned, Hairer and Zambotti to give a description of a renormalisation process on stochastic PDEs. We here study the algebraic structures on these objects: multiple pre-Lie algebras and related operads (generalizing a result by Chapoton and Livernet), noncommutative and cocommutative Hopf algebras (generalizing Grossman and Larson's construction), commutative and noncocommutative Hopf algebras (generalizing Connes and Kreimer's construction), bialgebras in cointeraction (generalizing Calaque, Ebrahimi-Fard and Manchon's result). We also define families of morphisms and in particular we prove that any Connes–Kreimer Hopf algebra of typed and decorated trees is isomorphic to a Connes–Kreimer Hopf algebra of non-typed and decorated trees (the set of decorations of vertices being bigger), through a contraction process, and finally obtain the Bruned–Hairer–Zambotti construction as a subquotient.

Key words: typed tree; combinatorial Hopf algebras; pre-Lie algebras; operads

2020 Mathematics Subject Classification: 05C05; 16T30; 18D50; 17D25

1 Introduction

Bruned, Hairer and Zambotti used in [3, 5] typed trees in an essential way to give a systematic description of a canonical renormalisation procedure of stochastic PDEs. Typed trees are rooted trees in which edges are decorated by elements of a fixed set \mathcal{T} of types. They also appear in a context of low dimension topology in [20] (there, described as nested parentheses) and for the description of combinatorial species in [1]. We here study several algebraic structures on these trees, generalizing results of Connes and Kreimer [10], Chapoton and Livernet [9], Grossman and Larson [14], Calaque, Ebrahimi-Fard and Manchon [7].

In the work of Bruned, Hairer and Zambotti, the considered trees are typed, with a finite set of types denoted by \mathfrak{L} , and labeled. We here forget about the labels and study the algebraic structures induced by types. We first define grafting products of trees, similar to the pre-Lie product of [8]. For any type t , we obtain a pre-Lie product \bullet_t on the space $\mathfrak{g}_{\mathcal{D},\mathcal{T}}$ of \mathcal{T} -typed trees whose vertices are decorated by elements of a set \mathcal{D} . For example, if $|$ and $\dot{|}$ are two types, if $a, b, c \in \mathcal{D}$, then

$$\begin{array}{c} \begin{array}{c} b \\ | \\ a \end{array} \bullet | \cdot c = \begin{array}{c} b \quad c \\ \diagdown \quad / \\ \quad a \end{array} + \begin{array}{c} \quad c \\ | \\ b \\ | \\ a \end{array}, \quad \begin{array}{c} \begin{array}{c} b \\ | \\ a \end{array} \bullet \dot{|} \cdot c = \begin{array}{c} b \quad c \\ \diagdown \quad / \\ \quad a \end{array} + \begin{array}{c} \quad c \\ | \\ b \\ | \\ a \end{array}. \end{array}$$

Then $\mathfrak{g}_{\mathcal{D},\mathcal{T}}$, equipped with all these products, is a \mathcal{T} -multiple pre-Lie algebra (Definition 3.1), also called matching pre-Lie algebras in [25]: for any types t and t' , for any $x, y, z \in \mathfrak{g}_{\mathcal{D},\mathcal{T}}$,

$$x \bullet_{t'} (y \bullet_t z) - (x \bullet_{t'} y) \bullet_t z = x \bullet_t (z \bullet_{t'} y) - (x \bullet_t z) \bullet_{t'} y.$$

This relation appears in [4]. We prove in Corollary 3.11 that it is the free \mathcal{T} -multiple pre-Lie algebra generated by \mathcal{D} , generalizing the result of [9], first mentioned in [4, Proposition 4.21]. Consequently, we obtain a combinatorial description of the operad of \mathcal{T} -multiple pre-Lie algebras in terms of \mathcal{T} -typed trees with indexed vertices (Theorem 3.17): for example,

$$\begin{array}{c} \downarrow_1^2 \\ \circlearrowleft_1 \end{array} \begin{array}{c} \downarrow_1^2 \\ \circlearrowright_1 \end{array} = \begin{array}{c} \downarrow_1^2 \\ \vee_1^3 \end{array} + \begin{array}{c} \downarrow_1^3 \\ \downarrow_1^2 \end{array}, \quad \begin{array}{c} \downarrow_1^2 \\ \circlearrowleft_1 \end{array} \begin{array}{c} \downarrow_1^2 \\ \circlearrowright_1 \end{array} = \begin{array}{c} \downarrow_1^3 \\ \downarrow_1^2 \end{array}.$$

We also give a description of the Koszul dual operad and of its free algebras in Propositions 3.19 and 3.20, generalizing a result of [8].

For any family $\lambda = (\lambda_t)_{t \in \mathcal{T}}$ with a finite support, the product $\bullet_\lambda = \sum \lambda_t \bullet_t$ is pre-Lie: using the Guin–Oudom construction [21, 22], we obtain a Hopf algebraic structure $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{GL}\lambda} = (S(\mathfrak{g}_{\mathcal{D},\mathcal{T}}), \star_\lambda, \Delta)$ on the symmetric algebra generated by \mathcal{T} -typed and \mathcal{D} -decorated trees, that is to say on the space of \mathcal{T} -typed and \mathcal{D} -decorated forests. The coproduct Δ is given by partitions of forests into two forests and the \star_λ product is given by grafting. For example,

$$\begin{array}{c} \downarrow_a^b \\ \star_\lambda \end{array} \begin{array}{c} \downarrow_a^c \\ \circlearrowright_a \end{array} = \begin{array}{c} \downarrow_a^b \\ \circlearrowleft_a \end{array} \begin{array}{c} \downarrow_a^c \\ \circlearrowright_a \end{array} + \lambda \left| \begin{array}{c} \downarrow_a^b \\ \vee_a^c \end{array} \right| + \lambda \left| \begin{array}{c} \downarrow_a^b \\ \downarrow_a^c \end{array} \right| + \lambda \left| \begin{array}{c} \downarrow_a^b \\ \vee_a^c \end{array} \right| + \lambda \left| \begin{array}{c} \downarrow_a^b \\ \downarrow_a^c \end{array} \right|.$$

In the non-typed case, we get back the Grossman–Larson Hopf algebra of trees [14]. Dually, we obtain Hopf algebras $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\lambda}$, generalizing the Connes–Kreimer Hopf algebra [10] of rooted trees. For example,

$$\begin{aligned} \Delta^{\text{CK}\lambda} \left(\begin{array}{c} \downarrow_a^b \\ \circlearrowleft_a \end{array} \right) &= \begin{array}{c} \downarrow_a^b \\ \circlearrowleft_a \end{array} \otimes 1 + 1 \otimes \begin{array}{c} \downarrow_a^b \\ \circlearrowright_a \end{array} + \lambda \left| \begin{array}{c} \downarrow_a^b \\ \circlearrowright_a \end{array} \right| \otimes \begin{array}{c} \downarrow_a^b \\ \circlearrowleft_a \end{array}, \\ \Delta^{\text{CK}\lambda} \left(\begin{array}{c} \downarrow_a^b \\ \vee_a^c \end{array} \right) &= \begin{array}{c} \downarrow_a^b \\ \vee_a^c \end{array} \otimes 1 + 1 \otimes \begin{array}{c} \downarrow_a^b \\ \vee_a^c \end{array} + \lambda \left| \begin{array}{c} \downarrow_a^b \\ \circlearrowleft_a \end{array} \right| \otimes \begin{array}{c} \downarrow_a^c \\ \circlearrowright_a \end{array} + \lambda \left| \begin{array}{c} \downarrow_a^b \\ \circlearrowright_a \end{array} \right| \otimes \begin{array}{c} \downarrow_a^c \\ \circlearrowleft_a \end{array} + \lambda^2 \left| \begin{array}{c} \downarrow_a^b \\ \circlearrowleft_a \end{array} \right| \otimes \begin{array}{c} \downarrow_a^c \\ \circlearrowright_a \end{array} \otimes \begin{array}{c} \downarrow_a^c \\ \circlearrowleft_a \end{array}, \\ \Delta^{\text{CK}\lambda} \left(\begin{array}{c} \downarrow_a^b \\ \vee_a^c \end{array} \right) &= \begin{array}{c} \downarrow_a^b \\ \vee_a^c \end{array} \otimes 1 + 1 \otimes \begin{array}{c} \downarrow_a^b \\ \vee_a^c \end{array} + \lambda \left| \begin{array}{c} \downarrow_a^b \\ \circlearrowleft_a \end{array} \right| \otimes \begin{array}{c} \downarrow_a^c \\ \circlearrowright_a \end{array} + \lambda \left| \begin{array}{c} \downarrow_a^b \\ \circlearrowright_a \end{array} \right| \otimes \begin{array}{c} \downarrow_a^c \\ \circlearrowleft_a \end{array} + \lambda \left| \lambda \right| \left| \begin{array}{c} \downarrow_a^b \\ \circlearrowleft_a \end{array} \right| \otimes \begin{array}{c} \downarrow_a^c \\ \circlearrowright_a \end{array} \otimes \begin{array}{c} \downarrow_a^c \\ \circlearrowleft_a \end{array}. \end{aligned}$$

This Hopf algebra satisfies a universal property in Hochschild cohomology, as does the Connes–Kreimer’s Hopf algebra. We describe it in the simpler case where \mathcal{T} is finite (Theorem 5.5). We finally give a second coproduct δ on $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\lambda}$, such that $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\lambda}$ is a Hopf algebra in the category of $(S(\mathfrak{g}_{\mathcal{D},\mathcal{T}}), m, \delta)$ -right comodules, generalizing the result of [7]. This coproduct δ is given by a contraction-extraction process. For example, in the non-decorated case,

$$\begin{aligned} \delta(\bullet) &= \bullet \otimes \bullet, \\ \delta(\downarrow) &= \downarrow \otimes \bullet + \bullet \otimes \downarrow, \\ \delta(\vee) &= \vee \otimes \bullet + 2 \downarrow \otimes \downarrow + \bullet \otimes \vee, \\ \delta(\vee) &= \vee \otimes \bullet + \downarrow \otimes \downarrow + \downarrow \otimes \downarrow + \bullet \otimes \vee. \end{aligned}$$

We are also interested in morphisms between these objects. We prove that if λ and μ are both nonzero, then the pre-Lie algebras $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_\lambda)$ and $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_\mu)$ are isomorphic (Corollary 4.8).

Consequently, if λ and μ are both nonzero, the Hopf algebras $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{GL}\lambda}$ and $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{GL}\mu}$ are isomorphic; dually, the Hopf algebras $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\lambda}$ and $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\mu}$ are isomorphic (Corollary 5.7). Using Livernet's rigidity theorem [16] and a nonassociative permutative coproduct defined in Proposition 4.1, we prove that if $\lambda \neq 0$, then $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_\lambda)$ is, as a pre-Lie algebra, freely generated by a family of typed trees $\mathcal{D}' = \mathcal{T}_{\mathcal{D},\mathcal{T}}^{(t_0)}$ satisfying a condition on the type of edges born from the root (Corollary 4.2). As a consequence, the Hopf algebra $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\lambda}$ of typed and decorated trees is isomorphic to a Connes–Kreimer Hopf algebra of non typed and decorated trees $\mathcal{H}_{\mathcal{D}'}$, and an explicit isomorphism is described with the help of contraction in Proposition 5.9.

This paper is organized as follows: the first section gives the basic definition of typed rooted trees and enumeration results, when the number of types and decorations are finite. The second section is about the \mathcal{T} -multiple pre-Lie algebra structures on these trees and the underlying operads. The freeness of the pre-Lie structures on typed decorated trees and its consequences are studied in the third section. In the last section, the dual Hopf algebras $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{GL}\lambda}$ and $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\lambda}$ are defined, studied and related to the constructions of Bruned, Hairer and Zambotti [3, 5]: forgetting the labels, the two coproducts they use on a family of typed and partially decorated trees are a subquotient of a the construction presented here.

Since the first version of this paper, written in 2018, multiple pre-Lie algebras have appeared in [4, 6] in a context of SPDEs. When the set of types has more structures (for example a product making it a commutative semigroup), then other possibilities for grafting products of typed trees are studied in [13, 18, 25], giving other interesting structures on these objects.

Notation 1.1.

- We denote by \mathbb{K} a commutative field of characteristic zero. All the objects (vector spaces, algebras, coalgebras, pre-Lie algebras, ...) in this text will be taken over \mathbb{K} .
- For any $n \in \mathbb{N}$, we denote by $[n]$ the set $\{1, \dots, n\}$.
- For any set \mathcal{T} , we denote by $\mathbb{K}^{\mathcal{T}}$ the set of family $\lambda = (\lambda_t)_{t \in \mathcal{T}}$ of elements of \mathbb{K} indexed by \mathcal{T} , and we denote by $\mathbb{K}^{(\mathcal{T})}$ the set of elements $\lambda \in \mathbb{K}^{\mathcal{T}}$ with a finite support. Note that if \mathcal{T} is finite, then $\mathbb{K}^{\mathcal{T}} = \mathbb{K}^{(\mathcal{T})}$.

2 Typed decorated trees

2.1 Definition

Definition 2.1. Let \mathcal{D} and \mathcal{T} be two nonempty sets.

1. A \mathcal{D} -decorated \mathcal{T} -typed forest is a triple $(F, \text{dec}, \text{type})$, where
 - F is a rooted forest. The set of its vertices is denoted by $V(F)$ and the set of its edges by $E(F)$.
 - $\text{dec}: V(F) \longrightarrow \mathcal{D}$ is a map.
 - $\text{type}: E(F) \longrightarrow \mathcal{T}$ is a map.

If the underlying rooted forest of F is connected, we shall say that F is a \mathcal{D} -decorated \mathcal{T} -typed tree.

2. If $(F, \text{dec}_F, \text{type}_F)$ and $(G, \text{dec}_G, \text{type}_G)$ are two \mathcal{D} -decorated \mathcal{T} -typed forests, they are isomorphic if there exists a rooted forest isomorphism f from F to G such that for any vertex v of F , $\text{dec}_G(f(v)) = \text{dec}_F(v)$ and for any edge e of F , $\text{type}_G(f(e)) = \text{type}_F(e)$
3. For any finite set A , we denote by $\mathbb{T}_{\mathcal{T}}(A)$ the set of A -decorated \mathcal{T} -typed trees T such that $V(T) = A$ and $\text{dec} = \text{Id}_A$, and by $\mathbb{F}_{\mathcal{T}}(A)$ the set of A -decorated \mathcal{T} -typed forests F such that $V(F) = A$ and $\text{dec} = \text{Id}_A$.

4. For any $n \geq 0$, we denote by $\mathbb{T}_{\mathcal{D},\mathcal{T}}(n)$ the set of isomorphism classes of \mathcal{D} -decorated \mathcal{T} -typed trees T such that $|V(T)| = n$ and by $\mathbb{F}_{\mathcal{D},\mathcal{T}}(n)$ the set of isomorphism classes of \mathcal{D} -decorated \mathcal{T} -typed forests F such that $|V(F)| = n$. We also put

$$\mathbb{T}_{\mathcal{D},\mathcal{T}} = \bigsqcup_{n \geq 0} \mathbb{T}_{\mathcal{D},\mathcal{T}}(n), \quad \mathbb{F}_{\mathcal{D},\mathcal{T}} = \bigsqcup_{n \geq 0} \mathbb{F}_{\mathcal{D},\mathcal{T}}(n).$$

Example 2.2. We shall represent the decorations of the vertices by letters alongside them. If \mathcal{T} contains two elements, represented by $|$ and \vdots , then

$$\begin{aligned} \mathbb{F}_{\mathcal{D},\mathcal{T}}(1) &= \{ \bullet^a, d \in \mathcal{D} \}, \\ \mathbb{F}_{\mathcal{D},\mathcal{T}}(2) &= \left\{ \bullet^a \bullet^b, \downarrow_a^b, \downarrow_a^b, a, b \in \mathcal{D} \right\}, \\ \mathbb{F}_{\mathcal{D},\mathcal{T}}(3) &= \left\{ \bullet^a \bullet^b \bullet^c, \downarrow_a^b \bullet^c, \downarrow_a^b \bullet^c, \downarrow_a^b \bullet^c, \downarrow_a^b \downarrow_a^c, \downarrow_a^b \downarrow_a^c, \downarrow_a^b \downarrow_a^c, \downarrow_a^b \downarrow_a^c, \downarrow_a^b \downarrow_a^c, \downarrow_a^b \downarrow_a^c, \downarrow_a^b \downarrow_a^c, \downarrow_a^b \downarrow_a^c, \downarrow_a^b \downarrow_a^c, \downarrow_a^b \downarrow_a^c, \downarrow_a^b \downarrow_a^c, a, b, c \in \mathcal{D} \right\}. \end{aligned}$$

Note that for any $a, b, c \in \mathcal{D}$,

$$\bullet^a \bullet^b = \bullet^b \bullet^a, \quad \downarrow_a^b \downarrow_a^c = \downarrow_a^c \downarrow_a^b, \quad \downarrow_a^b \downarrow_a^c = \downarrow_a^c \downarrow_a^b, \quad \downarrow_a^b \downarrow_a^c = \downarrow_a^c \downarrow_a^b.$$

Moreover,

$$\begin{aligned} \mathbb{F}_{\mathcal{D},\mathcal{T}}([1]) &= \{ \bullet^1 \}, \\ \mathbb{F}_{\mathcal{D},\mathcal{T}}([2]) &= \left\{ \bullet^1 \bullet^2, \downarrow_1^2, \downarrow_1^2, \downarrow_1^2, \downarrow_1^2 \right\}, \\ \mathbb{F}_{\mathcal{D},\mathcal{T}}([3]) &= \left\{ \begin{aligned} &\bullet^1 \bullet^2 \bullet^3, \downarrow_1^2 \bullet^3, \downarrow_1^3 \bullet^2, \downarrow_1^3 \bullet^2, \downarrow_1^3 \bullet^2, \downarrow_1^3 \bullet^2, \downarrow_1^3 \bullet^2, \downarrow_1^3 \bullet^2, \\ &\downarrow_1^2 \bullet^3, \downarrow_1^3 \bullet^2, \downarrow_1^3 \bullet^2, \downarrow_1^3 \bullet^2, \downarrow_1^3 \bullet^2, \downarrow_1^3 \bullet^2, \downarrow_1^3 \bullet^2, \\ &\downarrow_1^2 \downarrow_1^3, \downarrow_1^2 \downarrow_1^3, \downarrow_1^2 \downarrow_1^3, \downarrow_1^2 \downarrow_1^3, \downarrow_1^2 \downarrow_1^3, \downarrow_1^2 \downarrow_1^3, \downarrow_1^2 \downarrow_1^3, \downarrow_1^2 \downarrow_1^3, \\ &\downarrow_1^2 \downarrow_1^3, \downarrow_1^2 \downarrow_1^3, \downarrow_1^2 \downarrow_1^3, \downarrow_1^2 \downarrow_1^3, \downarrow_1^2 \downarrow_1^3, \downarrow_1^2 \downarrow_1^3, \downarrow_1^2 \downarrow_1^3, \downarrow_1^2 \downarrow_1^3, \\ &\downarrow_1^2 \downarrow_1^3, \downarrow_1^2 \downarrow_1^3, \downarrow_1^2 \downarrow_1^3, \downarrow_1^2 \downarrow_1^3, \downarrow_1^2 \downarrow_1^3, \downarrow_1^2 \downarrow_1^3, \downarrow_1^2 \downarrow_1^3, \downarrow_1^2 \downarrow_1^3, \end{aligned} \right\}. \end{aligned}$$

Remark 2.3. If $|\mathcal{T}| = 1$, all the edges of elements of $\mathbb{F}_{\mathcal{D},\mathcal{T}}$ have the same type: we work with \mathcal{D} -decorated rooted forests. In this case, we shall omit \mathcal{T} in the indices describing the forests, trees, spaces we are considering.

2.2 Enumeration

We assume here that \mathcal{D} and \mathcal{T} are finite, of respective cardinality D and T . For all $n \geq 0$, we put

$$\begin{aligned} t_{D,T}(n) &= |\mathbb{T}_{\mathcal{D},\mathcal{T}}(n)|, & f_{D,T}(n) &= |\mathbb{F}_{\mathcal{D},\mathcal{T}}(n)|, \\ T_{D,T}(X) &= \sum_{n=1}^{\infty} t_{D,T}(n) X^n, & F_{D,T}(X) &= \sum_{n=0}^{\infty} f_{D,T}(n) X^n. \end{aligned}$$

As any element of $\mathbb{F}_{\mathcal{T},\mathcal{D}}$ can be uniquely decomposed as the disjoint union of its connected components, which are elements of $\mathbb{T}_{\mathcal{D},\mathcal{T}}$, we obtain

$$F_{D,T}(X) = \prod_{n=1}^{\infty} \frac{1}{(1 - X^n)^{t_{D,T}(n)}}. \quad (2.1)$$

We put $\mathcal{T} = \{t_1, \dots, t_T\}$. For any $d \in \mathcal{D}$, we consider

$$B_d: \begin{cases} (\mathbb{F}_{\mathcal{D},\mathcal{T}})^T \longrightarrow \mathbb{T}_{\mathcal{D},\mathcal{T}}, \\ (F_1, \dots, F_T) \longmapsto B_d(F_1, \dots, F_T), \end{cases}$$

where $B_d(F_1, \dots, F_T)$ is the tree obtained by grafting the forests F_1, \dots, F_n on a common root decorated by d ; the edges from this root to the roots of F_i are of type t_i for any $1 \leq i \leq T$. Then B_d is injective, homogeneous of degree 1, and moreover $\mathbb{T}_{\mathcal{D},\mathcal{T}}$ is the disjoint union of the $B_d((\mathbb{F}_{\mathcal{D},\mathcal{T}})^T)$, $d \in \mathcal{D}$. Hence,

$$T_{D,T}(X) = DX(F_{D,T})^T = DX \prod_{n=1}^{\infty} \frac{1}{(1 - X^n)^{t_{D,T}(n)T}}. \quad (2.2)$$

Note that (2.2) allows to compute $t_{D,T}(n)$ by induction on n , and (2.1) allows to deduce $f_{D,T}(n)$.

Lemma 2.4. *For any $n \in \mathbb{N}$,*

$$t_{D,T}(n) = \frac{t_{TD,1}(n)}{T}.$$

Proof. By induction on n . If $n = 1$, $t_{D,T}(1) = D$ and $t_{TD,1}(1) = TD$, which gives the result. Let us assume the result at all ranks $k < n$. Then $t_{D,T}(n)T$ is the coefficient of X^n in

$$TDX \prod_{k=1}^{n-1} \frac{1}{(1 - X^k)^{t_{D,T}(k)T}} = TDX \prod_{k=1}^{n-1} \frac{1}{(1 - X^k)^{t_{TD,1}(k)}},$$

which is precisely $t_{TD,1}(n)$. ■

Example 2.5. We obtain

$$\begin{aligned} t_{D,T}(1) &= D, \\ t_{D,T}(2) &= D^2T, \\ t_{D,T}(3) &= \frac{D^2T(3DT + 1)}{2}, \\ t_{D,T}(4) &= \frac{D^2T(8D^2T^2 + 3DT + 1)}{3}, \\ t_{D,T}(5) &= \frac{D^2T(125D^3T^3 + 54D^2T^2 + 31DT + 6)}{24}, \\ t_{D,T}(6) &= \frac{D^2T(162D^4T^4 + 80D^3T^3 + 45D^2T^2 + 10DT + 3)}{15}, \\ t_{D,T}(7) &= \frac{D^2T(16807D^5T^5 + 9375D^4T^4 + 5395D^3T^3 + 2025D^2T^2 + 838DT + 120)}{720}. \end{aligned}$$

We shall give tables of values of $t_{D,T}(k)$ in the appendix.

3 Multiple pre-Lie algebras

We here fix a nonempty set \mathcal{T} of edges types.

3.1 Definition

Definition 3.1. A \mathcal{T} -multiple pre-Lie algebra is a family $(V, (\bullet_t)_{t \in \mathcal{T}})$, where V is a vector space and for all $t \in \mathcal{T}$, \bullet_t is a bilinear product on V such that

$$\forall t, t' \in \mathcal{T}, \quad \forall x, y, z \in V, \quad x \bullet_{t'} (y \bullet_t z) - (x \bullet_{t'} y) \bullet_t z = x \bullet_t (z \bullet_{t'} y) - (x \bullet_t z) \bullet_{t'} y.$$

Remark 3.2. For any $t \in \mathcal{T}$, (V, \bullet_t) is a pre-Lie algebra. More generally, for any family $\lambda = (\lambda_t)_{t \in \mathcal{T}} \in \mathbb{K}^{(\mathcal{T})}$, putting $\bullet_\lambda = \sum \lambda_t \bullet_t$, (V, \bullet_λ) is a pre-Lie algebra.

Proposition 3.3. Let \mathcal{D} be any set; we denote by $\mathfrak{g}_{\mathcal{D}, \mathcal{T}}$ the vector space generated by $\mathbb{T}_{\mathcal{D}, \mathcal{T}}$. For any $T, T' \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$, $v \in V(T)$ and $t \in \mathcal{T}$, we denote by $T \bullet_t^{(v)} T'$ the \mathcal{D} -decorated \mathcal{T} -typed tree obtained by grafting T' on v (that is to say adding an edge between v and the root of T'), the created edge being of type t . We then define a product \bullet_t on $\mathfrak{g}_{\mathcal{D}, \mathcal{T}}$ by

$$\forall T, T' \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}, \quad T \bullet_t T' = \sum_{v \in V(T)} T \bullet_t^{(v)} T'.$$

Then $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, (\bullet_t)_{t \in \mathcal{T}})$ is a \mathcal{T} -multiple pre-Lie algebra.

Proof. Let T, T', T'' be elements of $\mathbb{T}_{\mathcal{D}, \mathcal{T}}$ and $t', t'' \in \mathcal{T}$.

$$\begin{aligned} & (T \bullet_{t'} T') \bullet_{t''} T'' - T \bullet_{t'} (T' \bullet_{t''} T'') \\ &= \sum_{v \in V(T), v' \in V(T) \sqcup V(T')} (T \bullet_{t'}^{(v)} T') \bullet_{t''}^{(v')} T'' - \sum_{v \in V(T), v' \in V(T')} T \bullet_{t'}^{(v)} (T' \bullet_{t''}^{(v')} T'') \\ &= \sum_{v \in V(T), v' \in V(T')} (T \bullet_{t'}^{(v)} T') \bullet_{t''}^{(v')} T'' + \sum_{v \in V(T), v' \in V(T')} (T \bullet_{t'}^{(v)} T') \bullet_{t''}^{(v')} T'' - T \bullet_{t'}^{(v)} (T' \bullet_{t''}^{(v')} T'') \\ &= \sum_{v \in V(T), v' \in V(T')} (T \bullet_{t'}^{(v)} T') \bullet_{t''}^{(v')} T'' \\ &= \sum_{v \in V(T), v' \in V(T')} (T \bullet_{t''}^{(v')} T'') \bullet_{t'}^{(v)} T' \\ &= (T \bullet_{t''} T'') \bullet_{t'} T' - T \bullet_{t''} (T' \bullet_{t'} T'). \end{aligned}$$

So $\mathfrak{g}_{\mathcal{D}, \mathcal{T}}$ is indeed a \mathcal{T} -multiple pre-Lie algebra. ■

Example 3.4. If $a, b, c \in \mathcal{D}$ and $|, \vdash \in \mathcal{T}$,

$$\begin{array}{c} \overset{b}{|} \\ \bullet_a \bullet | \end{array} \bullet \overset{c}{.} = \begin{array}{c} \overset{b}{\bullet} \overset{c}{\bullet} \\ \diagdown \quad \diagup \\ \bullet_a \end{array} + \begin{array}{c} \overset{c}{|} \\ \bullet_b \\ \bullet_a \end{array}, \quad \begin{array}{c} \overset{b}{|} \\ \bullet_a \bullet \vdash \end{array} \bullet \overset{c}{.} = \begin{array}{c} \overset{b}{\bullet} \overset{c}{\bullet} \\ \diagdown \quad \diagup \\ \bullet_a \end{array} + \begin{array}{c} \overset{c}{\vdash} \\ \bullet_b \\ \bullet_a \end{array}.$$

3.2 Guin–Oudom extension of the pre-Lie products

Notation 3.5. Let $(\partial_t)_{t \in \mathcal{T}}$ be a family of formal symbols indexed by \mathcal{T} . For any vector space V , we denote

$$V^{\oplus \mathcal{T}} = V \otimes \text{Vect}(\partial_t, t \in \mathcal{T}).$$

Then

$$V^{\oplus \mathcal{T}} = \bigoplus_{t \in \mathcal{T}} V \otimes \partial_t.$$

In order to enlighten the notations, we write $v\partial_t$ instead of $v \otimes \partial_t$ for any $v \in V$ and for any $t \in \mathcal{T}$.

Lemma 3.6. *If for any $t \in \mathcal{T}$, \bullet_t is a bilinear product on a vector space V , we define $\bullet : (V^{\oplus \mathcal{T}})^{\otimes 2} \rightarrow V^{\oplus \mathcal{T}}$ by*

$$x \partial_t \bullet x' \partial_{t'} = (x \bullet_{t'} y) \partial_t.$$

Then $(V, (\bullet_t)_{t \in \mathcal{T}})$ is a \mathcal{T} -multiple pre-Lie algebra if, and only if, $(V^{\oplus \mathcal{T}}, \bullet)$ is a pre-Lie algebra.

Proof. Let $x, x', x'' \in V$, $t, t', t'' \in \mathcal{T}$. Then

$$x \partial_t \bullet (x' \partial_{t'} \bullet x'' \partial_{t''}) - (x \partial_t \bullet x' \partial_{t'}) \bullet x'' \partial_{t''} = ((x \bullet_{t'} x') \bullet_{t''} x'' - x \bullet_{t'} (x' \bullet_{t''} x'')) \partial_t,$$

which implies the result. ■

Notation 3.7. The symmetric algebra $S(V)$ is given with its usual coproduct Δ , making it a bialgebra:

$$\forall x \in V, \quad \Delta(x) = x \otimes 1 + 1 \otimes x.$$

We shall use Sweedler's notation: for any $w \in S(V)$, $\Delta(w) = \sum w^{(1)} \otimes w^{(2)}$. The counit of this coproduct is denoted by ε : this is the unique algebra morphism from $S(V)$ to \mathbb{K} sending any $v \in V$ to 0.

Theorem 3.8. *Let V be a \mathcal{T} -multiple pre-Lie algebra. One can define a product*

$$\bullet : S(V) \otimes S(V^{\oplus \mathcal{T}}) \rightarrow S(V)$$

in the following way: for any $u, v \in S(V)$, $w \in S(V^{\oplus \mathcal{T}})$, $x \in V$, $t \in \mathcal{T}$,

$$\begin{aligned} 1 \bullet w &= \varepsilon(w), \\ u \bullet 1 &= u, \\ uv \bullet w &= \sum (u \bullet w^{(1)})(v \bullet w^{(2)}), \\ u \bullet w(x \partial_t) &= (u \bullet w) \bullet_t x - x \bullet (w \bullet_t x), \end{aligned}$$

where \bullet_t is extended to $S(V) \otimes V$ and $S(V^{\oplus \mathcal{T}}) \otimes V$ by the following: for any $x_1, \dots, x_k, x \in V$, for any $t_1, \dots, t_k \in \mathcal{T}$,

$$\begin{aligned} x_1 \cdots x_k \bullet_t x &= \sum_{i=1}^k x_1 \cdots (x_i \bullet_t x) \cdots x_k, \\ (x_1 \partial_{t_1}) \cdots (x_k \partial_{t_k}) \bullet_t x &= \sum_{i=1}^k (x_1 \partial_{t_1}) \cdots ((x_i \bullet_t x) \partial_{t_i}) \cdots (x_k \partial_{t_k}). \end{aligned}$$

Proof. Uniqueness. The last formula allows to compute $x \bullet w$ for any $x \in V$ and $w \in S(V^{\oplus \mathcal{T}})$ by induction on the length of w ; the other ones allow to compute $u \bullet w$ for any $u \in S(V)$ by induction on the length on u . So this product \bullet is unique.

Existence. Let us use the Guin–Oudom construction [21, 22] on the pre-Lie algebra $V^{\otimes \mathcal{T}}$. We obtain a product \bullet defined on $S(V^{\oplus \mathcal{T}})$ such that for any $u, v, w \in S(V^{\oplus \mathcal{T}})$, $x \in V^{\oplus \mathcal{T}}$:

$$\begin{aligned} 1 \bullet w &= \varepsilon(w), \\ u \bullet 1 &= u, \\ uv \bullet w &= \sum (u \bullet w^{(1)})(v \bullet w^{(2)}), \\ u \bullet wx &= (u \bullet w) \bullet x - x \bullet (w \bullet x). \end{aligned}$$

Let $f: \mathcal{T} \rightarrow \mathbb{K}$ be any nonzero map. We consider the surjective algebra morphism $F: S(V^{\oplus \mathcal{T}}) \rightarrow S(V)$, sending $x\partial_t$ to $f(t)x$ for any $x \in V$, $t \in \mathcal{T}$. Its kernel is generated by the elements $X_{t,t'}x = (f(t')\partial_t - f(t)\partial_{t'})x$, where $x \in V$ and $t, t' \in \mathcal{T}$. We denote by J the vector space generated by the elements $X_{t,t'}x$. Let us prove that for any $w \in S(V^{\oplus \mathcal{T}})$, $J \bullet w \subseteq J$ by induction on the length n of w . If $n = 0$, we can assume that $w = 1$ and this is obvious. If $n = 1$, we can assume that $w = x'\partial_{t'}$. Then

$$X_{t,t'}x \bullet w = (f(t')\partial_t - f(t)\partial_{t'})x \bullet_{t'} x' = X_{t,t'}x \bullet_{t'} x' \in J.$$

Let us assume the result at rank $n - 1$. We can assume that $w = w'x'\partial_t$, the length of w' being $n - 1$. For any $x \in J$,

$$x \bullet w = (x \bullet w') \bullet x' - x \bullet (w' \bullet x').$$

The length of w' and $w' \bullet x'$ is $n - 1$, so $x \bullet w'$ and $x \bullet (w' \bullet x')$ belong to J . From the case $n = 1$, $(x \bullet w') \bullet x' \in J$, so $x \bullet w \in J$.

For any $x \in J$, $u, v \in S(V^{\oplus \mathcal{T}})$,

$$xu \bullet v = \underbrace{(x \bullet v^{(1)})}_{\in J} (u \bullet v^{(2)}) \in \text{Ker}(F).$$

This proves that $\text{Ker}(F) \bullet S(V^{\oplus \mathcal{T}}) \subseteq \text{Ker}(F)$. Hence, \bullet induces a product also denoted by \bullet , defined from $S(V) \otimes S(V^{\otimes \mathcal{T}})$ to $S(V)$. It is not difficult to show that it does not depend on the choice of f and satisfies the required properties. \blacksquare

Definition 3.9. Let $d \in \mathcal{D}$, $T_1, \dots, T_k \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$, $t_1, \dots, t_k \in \mathcal{T}$. We denote by

$$B_d \left(\prod_{i \in [k]} T_i \partial_{t_i} \right)$$

the \mathcal{T} -typed \mathcal{D} -decorated tree obtained by grafting T_1, \dots, T_k on a common root decorated by d , the edge between this root and the root of T_i being of type t_i for any i . This defines a map $B_d: S(\text{Vect}(\mathbb{T}_{\mathcal{D}, \mathcal{T}})^{\oplus \mathcal{T}}) \rightarrow S(\text{Vect}(\mathbb{T}_{\mathcal{D}, \mathcal{T}}))$.

Lemma 3.10. For any $d \in \mathcal{D}$, $T_1, \dots, T_k \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$, $t_1, \dots, t_k \in \mathcal{T}$,

$$B_d \left(\prod_{i \in [k]} T_i \partial_{t_i} \right) = \bullet^d \bullet \prod_{i \in [k]} T_i \partial_{t_i}.$$

Proof. We write $F = \prod_{i \in [k]} T_i \partial_{t_i}$: the integer k is unique and denoted by $\ell(F)$. We proceed by induction on $\ell(F)$. If $\ell(F) = 0$, then $F = 1$ and $\bullet^d \bullet 1 = \bullet^d = B_d(1)$. let us assume the result for any forest F' with $\ell(F') = \ell(F) - 1$. Putting $k = \ell(F)$, we can write $F = F'T\partial_t$, with $\ell(F') = \ell(F) - 1$, $T = T_k$ and $t = t_k$. Then

$$\begin{aligned} \bullet^d \bullet F &= (\bullet^d \bullet F') \bullet T\partial_t - \bullet^d \bullet (F' \bullet T\partial_t) \\ &= B_d(F') \bullet_t T - B_d(F' \bullet_t T) \\ &= B_d(F'T\partial_t) + B_d(F' \bullet_t T) - B_d(F' \bullet_t T) \\ &= B_d(F). \end{aligned}$$

So the result holds for all forests. \blacksquare

Corollary 3.11. *Let A be a \mathcal{T} -multiple pre-Lie algebra and, for any $d \in \mathcal{D}$, $a_d \in A$. There exists a unique \mathcal{T} -multiple algebra morphism $\phi: \mathfrak{g}_{\mathcal{D}, \mathcal{T}} \rightarrow A$, such that for any $d \in \mathcal{D}$, $\phi(\bullet^d) = a_d$. In other words, $\mathfrak{g}_{\mathcal{T}, \mathcal{D}}$ is the free \mathcal{T} -multiple pre-Lie algebra generated by \mathcal{D} .*

Proof. *Uniqueness.* Using the Guin–Oudom product and lemma 3.10, ϕ is the unique linear map inductively defined by

$$\phi\left(B_d\left(\prod_{i \in [k]} T_i \partial_{t_i}\right)\right) = a_d \bullet \prod_{i \in [k]} \phi(T_i) \partial_{t_i}.$$

Existence. Let $T, T' \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$ and $t \in \mathcal{T}$. Let us prove that $\phi(T \bullet_t T') = \phi(T) \bullet_t \phi(T')$ by induction on $n = |T|$. If $n = 1$, we assume that $T = \bullet^d$. Then $T \bullet_t T' = B_d(T' \partial_t)$, so

$$\phi(T \bullet_t T') = a_d \bullet (\phi(T') \partial_t) = a_d \bullet_t \phi(T') = \phi(T) \bullet_t \phi(T').$$

Let us assume the result at all ranks $< |T|$. We put

$$T = B_d\left(\prod_{i=1}^k T_i \partial_{t_i}\right).$$

By definition of the pre-Lie product of $\mathfrak{g}_{\mathcal{D}, \mathcal{T}}$ in terms of grafting,

$$\begin{aligned} T \bullet T' &= B_d\left(\prod_{i=1}^k T_i \partial_{t_i} T' \partial_t\right) + \sum_{j=1}^k B_d\left(\prod_{i \neq j} T_i \partial_{t_i} (T_j \bullet_t T') \partial_{t_j}\right), \\ \phi(T \bullet T') &= a_d \bullet \prod_{i=1}^k \phi(T_i) \partial_{t_i} \phi(T') \partial_t + \sum_{j=1}^k a_d \bullet \prod_{i \neq j} \phi(T_i) \partial_{t_i} (\phi(T_j) \bullet_t \phi(T')) \partial_{t_j} \\ &= a_d \bullet \prod_{i=1}^k \phi(T_i) \partial_{t_i} \phi(T') \partial_t + \sum_{j=1}^k a_d \bullet \prod_{i \neq j} \phi(T_i) \partial_{t_i} (\phi(T_j) \bullet_t \phi(T')) \partial_{t_j} \\ &= a_d \bullet \prod_{i=1}^k \phi(T_i) \partial_{t_i} \phi(T') \partial_t + a_d \bullet \left(\left(\prod_{i=1}^k \phi(T_i) \partial_{t_i}\right) \bullet \phi(T') \partial_t\right) \\ &= \left(a_d \bullet \prod_{i=1}^k \phi(T_i) \partial_{t_i}\right) \bullet \phi(T') \partial_t \\ &= \phi(T) \bullet_t \phi(T'). \end{aligned}$$

So ϕ is a \mathcal{T} -multiple pre-Lie algebra morphism. ■

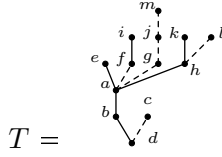
3.3 Operad of typed trees

We now describe an operad of typed trees, in the category of species. We refer to [2, 17, 19] for notations and definitions on operads.

Notation 3.12. Let A be a finite set. If $T \in \mathbb{T}_{\mathcal{T}}(A)$ and $a \in T$:

1. The subtrees formed by the connected components of the set of vertices, descendants of a (a excluded) are denoted by $T_1^{(a)}, \dots, T_{n_a}^{(a)}$. The type of the edge from a to the root of $T_i^{(a)}$ is denoted by t_i .
2. The tree formed by the vertices of T which are not in $T_1^{(a)}, \dots, T_{n_a}^{(a)}$, at the exception of a , is denoted by $T_0^{(a)}$.

Example 3.13. Let us consider the following tree:



with $a, \dots, m \in \mathcal{D}$, then

$$T_0^{(a)} = \begin{array}{c} b \quad c \\ \diagdown \quad \diagup \\ d \end{array}, \quad \{T_1^{(a)}, \dots, T_n^{(a)}\} = \left\{ \begin{array}{c} \cdot e, \downarrow f, \downarrow g, \downarrow h \\ \downarrow i, \downarrow j, \downarrow k, \downarrow l \\ \downarrow m \end{array} \right\}.$$

Proposition 3.14. For any nonempty finite set A , we denote by $\mathcal{P}_{\mathcal{T}}(A)$ the vector space generated by $\mathbb{T}_{\mathcal{T}}(A)$. We define a composition \circ on $\mathcal{P}_{\mathcal{T}}$ in the following way: for any $T \in \mathbb{T}_{\mathcal{T}}(A)$, $T' \in \mathbb{T}_{\mathcal{T}}(B)$ and $a \in A$,

$$T \circ_a T' = \sum_{v_1, \dots, v_{n_a} \in V(T')} \left(\dots \left((T_0^{(a)} \bullet_{a'}^{(t_0)} T') \bullet_{v_1}^{(t_1)} T_1^{(a)} \right) \dots \right) \bullet_{v_{n_a}}^{(t_{n_a})} T_{n_a}^{(a)},$$

where a' is the direct ascendant of a in T and t_0 is the type of the edge between a' and a . If a is the root of T , by convention $T_0^{(a)} \bullet_{a'}^{(t_0)} T' = T'$. With this composition, $\mathcal{P}_{\mathcal{T}}$ is an operad in the category of species.

Proof. Note that the tree $(\dots ((T_0^{(a)} \bullet_{a'}^{(t_0)} T') \bullet_{v_1}^{(t_1)} T_1^{(a)}) \dots) \bullet_{v_{n_a}}^{(t_{n_a})} T_{n_a}^{(a)}$, which is shortly denoted by $T \bullet_{\lambda}^{(v)} T'$, is obtained in the following process:

1. Delete the branches $T_1^{(a)}, \dots, T_{n_a}^{(a)}$ coming from a in T . One obtains a tree T'' , and a is a leaf of T'' .
2. Identify $a \in V(T'')$ with the root of T' . One obtains a tree T''' .
3. Graft $T_1^{(a)}$ on the vertex v_1 of T''' with an edge of type t_1, \dots , graft $T_{n_a}^{(a)}$ on the vertex v_{n_a} of T''' with an edge of type t_n . The obtained tree is $T \bullet_{\lambda}^{(v)} T'$.

This obviously does not depend on the choice of the indexation of $T_1^{(a)}, \dots, T_{n_a}^{(a)}$.

Let $T \in \mathbb{T}_{\mathcal{T}}(A)$, $T' \in \mathbb{T}_{\mathcal{T}}(B)$, $T'' \in \mathbb{T}_{\mathcal{T}}(C)$.

- If $a', a'' \in A$, with $a' \neq a''$, then

$$\begin{aligned} (T \circ_{a'} T') \circ_{a''} T'' &= \sum_{v' \in V(T')^{n_{a'}}, v'' \in V(T'')^{n_{a''}}} (T \bullet_{a'}^{(v')} T') \bullet_{a''}^{(v'')} T'' \\ &= \sum_{v' \in V(T')^{n_{a'}}, v'' \in V(T'')^{n_{a''}}} (T \bullet_{a''}^{(v'')} T'') \bullet_{a'}^{(v')} T' \\ &= (T \circ_{a''} T'') \circ_{a'} T'. \end{aligned}$$

- If $a' \in A$ and $b'' \in B$, then

$$\begin{aligned} (T \circ_{a'} T') \circ_{b''} T'' &= \sum_{v' \in V(T')^{n_{a'}}, v'' \in V(T'')^{n_{b''}}} (T \bullet_{a'}^{(v')} T') \bullet_{b''}^{(v'')} T'' \\ &= \sum_{v' \in V(T')^{n_{a'}}, v'' \in V(T'')^{n_{b''}}} T \bullet_{a'}^{(v')} (T' \bullet_{b''}^{(v'')} T'') \\ &= T \circ_{a'} (T' \circ_{b''} T''). \end{aligned}$$

Moreover, $\bullet^a \bullet_\lambda T = T$ for any tree T , and if $a \in V(T)$, $T \bullet_\lambda \bullet^a T$. So $\mathcal{P}_\mathbb{T}$ is indeed an operad in the category of species. \blacksquare

Consequently, the family $(\mathcal{P}_\mathcal{T}(n))_{n \geq 0}$ is an operad in the category of vector spaces, which we denote by $\mathcal{P}_\mathcal{T}$.

Example 3.15.

$$\begin{array}{c} \begin{array}{c} \color{red}{2} \\ \color{red}{\downarrow} \\ \color{red}{1} \end{array} \circ_1 \begin{array}{c} \color{red}{2} \\ \color{red}{\downarrow} \\ \color{red}{1} \end{array} = \begin{array}{c} \color{red}{2} \quad \color{red}{3} \\ \color{red}{\swarrow} \quad \color{red}{\searrow} \\ \color{red}{1} \end{array} + \begin{array}{c} \color{red}{3} \\ \color{red}{\downarrow} \\ \color{red}{2} \\ \color{red}{\downarrow} \\ \color{red}{1} \end{array}, \quad \begin{array}{c} \color{red}{2} \\ \color{red}{\downarrow} \\ \color{red}{1} \end{array} \circ_2 \begin{array}{c} \color{red}{2} \\ \color{red}{\downarrow} \\ \color{red}{1} \end{array} = \begin{array}{c} \color{red}{3} \\ \color{red}{\downarrow} \\ \color{red}{2} \\ \color{red}{\downarrow} \\ \color{red}{1} \end{array}.$$

Remark 3.16. Another operad on typed trees is introduced in [11]. It is a typed version of the operad of nonassociative, permutative operad of [16], and is different from ours.

In the non-typed case, this theorem is proved in [9]:

Theorem 3.17. *The operad of \mathcal{T} -multiple pre-Lie algebras is isomorphic to $\mathcal{P}_\mathcal{T}$, via the isomorphism Φ sending, for any $t \in \mathcal{T}$, \bullet_t to the tree $\begin{array}{c} \color{red}{2} \\ \color{red}{\downarrow} \\ \color{red}{1} \end{array}$, where the edge is of type t .*

Proof. The operad of \mathcal{T} -multiple pre-Lie algebras is generated by the binary elements \bullet_t , $t \in \mathcal{T}$, with the relations

$$\forall t, t' \in \mathcal{T}, \quad \bullet_{t'} \circ_2 \bullet_t - \bullet_t \circ_1 \bullet_{t'} = (\bullet_t \circ_2 \bullet_{t'} - \bullet_{t'} \circ_1 \bullet_t)^{(23)},$$

where in the right side we used the action of the symmetric group \mathfrak{S}_3 on $\mathcal{T}(3)$, and more specifically the action of the transposition (23). Firstly, if t and t' are elements of \mathcal{T} , symbolized by $\begin{array}{c} \color{red}{2} \\ \color{red}{\downarrow} \\ \color{red}{1} \end{array}$ and $\begin{array}{c} \color{red}{3} \\ \color{red}{\downarrow} \\ \color{red}{2} \\ \color{red}{\downarrow} \\ \color{red}{1} \end{array}$, by the preceding example:

$$\begin{array}{c} \color{red}{2} \\ \color{red}{\downarrow} \\ \color{red}{1} \end{array} \circ_1 \begin{array}{c} \color{red}{2} \\ \color{red}{\downarrow} \\ \color{red}{1} \end{array} - \begin{array}{c} \color{red}{2} \\ \color{red}{\downarrow} \\ \color{red}{1} \end{array} \circ_2 \begin{array}{c} \color{red}{2} \\ \color{red}{\downarrow} \\ \color{red}{1} \end{array} = \begin{array}{c} \color{red}{2} \quad \color{red}{3} \\ \color{red}{\swarrow} \quad \color{red}{\searrow} \\ \color{red}{1} \end{array} = \left(\begin{array}{c} \color{red}{2} \quad \color{red}{3} \\ \color{red}{\swarrow} \quad \color{red}{\searrow} \\ \color{red}{1} \end{array} \right)^{(23)} = \left(\begin{array}{c} \color{red}{2} \\ \color{red}{\downarrow} \\ \color{red}{1} \end{array} \circ_1 \begin{array}{c} \color{red}{2} \\ \color{red}{\downarrow} \\ \color{red}{1} \end{array} - \begin{array}{c} \color{red}{2} \\ \color{red}{\downarrow} \\ \color{red}{1} \end{array} \circ_2 \begin{array}{c} \color{red}{2} \\ \color{red}{\downarrow} \\ \color{red}{1} \end{array} \right)^{(23)}.$$

So the morphism ϕ exists. Let us prove that it is surjective: let $T \in \mathbb{T}_\mathcal{T}(n)$, we show that it belongs to $\text{Im}(\Phi)$ by induction on n . It is obvious if $n = 1$ or $n = 2$. Let us assume the result at all ranks $< n$. Up to a reindexation we assume that

$$T = B_1(T_1 \partial_{t_1} \cdots T_k \partial_{t_k}),$$

where for any $1 \leq i < j \leq k$, if $x \in V(T_i)$ and $y \in V(T_j)$, then $x < y$. We denote by T'_i the standardization of T_i . By the induction hypothesis on n , $T'_i \in \text{Im}(\Phi)$ for all i . We proceed by induction on k . The type t_k will be represented in red. If $k = 1$, then

$$T = \begin{array}{c} \color{red}{2} \\ \color{red}{\downarrow} \\ \color{red}{1} \end{array} \circ_2 T_1 \in \text{Im}(\Phi).$$

Let us assume the result at rank $k - 1$. We put $T' = B_1(T_1 \partial_{t_1} \cdots T_{k-1} \partial_{t_{k-1}})$. By the induction hypothesis on n , $T' \in \text{Im}(\Phi)$. Then

$$\begin{array}{c} \color{red}{2} \\ \color{red}{\downarrow} \\ \color{red}{1} \end{array} \circ_1 T' = T + x,$$

where x is a sum of trees with n vertices, such that the fertility of the root is $k - 1$. Hence, $x \in \text{Im}(\Phi)$, so $T \in \text{Im}(\Phi)$.

Let \mathcal{D} be a set. The morphism ϕ implies that the free $\mathcal{P}_{\mathbb{T}}$ -algebra generated by \mathcal{D} , that is to say $\mathfrak{g}_{\mathcal{D},\mathcal{T}}$, inherits a \mathcal{T} -multiple pre-Lie algebra structure defined by

$$\forall x, y \in \mathfrak{g}_{\mathcal{D},\mathcal{T}}, \quad \forall \dagger \in \mathcal{T}, \quad x \circ_{\dagger} y = \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array}^2 \cdot (x \otimes y),$$

where \cdot is the $\mathcal{P}_{\mathcal{T}}$ -algebra structure of $\mathfrak{g}_{\mathcal{D},\mathcal{T}}$. For any trees T, T' in $\mathbb{T}_{\mathcal{D},\mathcal{T}}$, by definition of the operadic composition of $\mathcal{P}_{\mathcal{T}}$,

$$T \circ_t T' = \sum_{v \in V(T)} T \bullet_t^{(v)} T',$$

so $\circ_t = \bullet_t$ for any t . As $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, (\bullet_t)_{t \in \mathcal{T}})$ is the free \mathcal{T} -multiple pre-Lie algebra generated by \mathcal{D} , Φ is an isomorphism. ■

Remark 3.18. Let us assume that \mathcal{T} is finite, of cardinality T . Then the components of $\mathcal{P}_{\mathcal{T}}$ are finite-dimensional. As the number of rooted trees which vertices are the elements of $[n]$ is n^{n-1} , for any $n \geq 0$ the dimension of $\mathcal{P}_{\mathcal{T}}(n)$ is $T^{n-1}n^{n-1}$, and the formal series of $\mathcal{P}_{\mathcal{T}}$ is

$$f_{\mathcal{T}}(X) = \sum_{n \geq 1} \frac{\dim(\mathcal{P}_{\mathcal{T}}(n))}{n!} X^n = \sum_{n \geq 1} \frac{(Tn)^{n-1}}{n!} X^n = \frac{f_1(TX)}{T}.$$

3.4 Koszul dual operad

If \mathcal{T} is finite, then $\mathcal{P}_{\mathcal{T}}$ is a quadratic operad. Its Koszul dual can be directly computed:

Proposition 3.19. *The Koszul dual operad $\mathcal{P}_{\mathcal{T}}^{\perp}$ of $\mathcal{P}_{\mathcal{T}}$ is generated by \diamond_t , $t \in \mathcal{T}$, with the relations*

$$\forall t, t' \in \mathcal{T}, \quad \diamond_{t'} \circ_1 \diamond_t = \diamond_t \circ_2 \diamond_{t'}, \quad \diamond_{t'} \circ_1 \diamond_t = (\diamond_t \circ_1 \diamond_{t'})^{(23)}.$$

The algebras on $\mathcal{P}_{\mathcal{T}}^{\perp}$ are called \mathcal{T} -multiple permutative algebras. Such an algebra A is given with the bilinear products \diamond_t , $t \in \mathcal{T}$, such that

$$\forall x, y, z \in A, \quad (x \diamond_t y) \diamond_{t'} z = x \diamond_t (y \diamond_{t'} z), \quad (x \diamond_t y) \diamond_{t'} z = (x \diamond_{t'} z) \diamond_t y.$$

In particular, for any t , \diamond_t is a permutative product.

Of course, the definition of \mathcal{T} -multiple permutative algebras makes sense even if \mathcal{T} is infinite. Permutative algebras are introduced in [8]. If A is a \mathcal{T} -multiple permutative algebra, then for any $(\lambda_t)_{t \in \mathcal{T}} \in \mathbb{K}^{(\mathcal{T})}$, $\diamond_a = \sum \lambda_t \diamond_t$ is a permutative product on A .

Proposition 3.20. *Let V be a vector space. Then $V \otimes S(V^{\oplus \mathcal{T}})$ is given with a \mathcal{T} -multiple permutative algebra structure:*

$$\forall t \in \mathcal{T}, \quad v, v' \in V, \quad w, w' \in S(V^{\oplus \mathcal{T}}), \quad (v \otimes w) \diamond_t (v' \otimes w') = v \otimes w w' (v' \partial_t).$$

This \mathcal{T} -multiple permutative algebra is denoted by $P_{\mathcal{T}}(V)$. For any \mathcal{T} -multiple permutative algebra V and any linear map $\phi: V \rightarrow A$, there exists a unique morphism $\Phi: P_{\mathcal{T}}(V) \rightarrow A$ such that for any $v \in V$, $\Phi(v \otimes 1) = \phi(v)$.

Proof. Let $t, t' \in \mathcal{T}$, $v, v', v'' \in V$, $w, w', w'' \in S(V^{\oplus \mathcal{T}})$.

$$\begin{aligned} (v \otimes w \diamond_t v' \otimes w') \diamond_{t'} v'' \otimes w'' &= v \otimes w \diamond_t (v' \otimes w' \diamond_{t'} v'' \otimes w'') \\ &= (v \otimes w \diamond_t v' \diamond_{t'} v'' \otimes w'') \otimes w' \\ &= v \otimes w w' w'' (v' \partial_t) (v'' \partial_{t'}), \end{aligned}$$

so $P_{\mathcal{T}}(V)$ is \mathcal{T} -multiple permutative.

Existence of Φ . Let $t_1, \dots, t_k \in \mathcal{T}$, $v, v_1, \dots, v_k \in V$. We inductively define $\Phi(v \otimes (v_1 \partial_{t_1}) \cdots (v_k \partial_{t_k}))$ by

$$\begin{aligned} \Phi(v \otimes 1) &= \phi(v), \\ \Phi(v \otimes (v_1 \partial_{t_1}) \cdots (v_k \partial_{t_k})) &= \Phi(v \otimes (v_1 \partial_{t_1}) \cdots (v_{k-1} \partial_{t_{k-1}})) \diamond_{t_k} \phi(v_k) \quad \text{if } k \geq 1. \end{aligned}$$

Let us prove that this does not depend on the order chosen on the factors $v_i \partial_{t_i}$ by induction on k . If $k = 0$ or 1 , there is nothing to prove. Otherwise, if $i < k$,

$$\begin{aligned} &\Phi(v \otimes (v_1 \partial_{t_1}) \cdots (v_{i-1} \partial_{t_{i-1}}) (v_{i+1} \partial_{t_{i+1}}) \cdots (v_k \partial_{t_k})) \diamond_{t_i} \phi(v_i) \\ &= (\Phi(v \otimes (v_1 \partial_{t_1}) \cdots (v_{i-1} \partial_{t_{i-1}}) (v_{i+1} \partial_{t_{i+1}}) \cdots (v_{k-1} \partial_{t_{k-1}})) \diamond_{t_k} \phi(v_k)) \diamond_{t_i} \phi(v_i) \\ &= (\Phi(v \otimes (v_1 \partial_{t_1}) \cdots (v_{i-1} \partial_{t_{i-1}}) (v_{i+1} \partial_{t_{i+1}}) \cdots (v_{k-1} \partial_{t_{k-1}})) \diamond_{t_i} \phi(v_i)) \diamond_{t_k} \phi(v_k) \\ &= \Phi(v \otimes (v_1 \partial_{t_1}) \cdots (v_{k-1} \partial_{t_{k-1}})) \diamond_{t_k} \phi(v_k) \\ &= \Phi(v \otimes (v_1 \partial_{t_1}) \cdots (v_k \partial_{t_k})). \end{aligned}$$

So Φ is well-defined. Let us prove that Φ is a \mathcal{T} -multiple permutative algebra morphism. Let $v, v' \in V$, $w, w' = (v_1 \partial_{t_1}) \cdots (v_k \partial_{t_k}) \in S(V^{\oplus \mathcal{T}})$, and $t \in \mathcal{T}$. Let us prove that $\Phi(v \otimes w \diamond_t v' \otimes w') = \Phi(v \otimes w) \diamond_t \Phi(v' \otimes w')$ by induction on k . If $k = 0$,

$$\begin{aligned} \Phi(v \otimes w \diamond_t v' \otimes 1) &= \Phi(v \otimes w (v' \partial_t)) \\ &= \Phi(v \otimes w) \diamond_t \phi(v') \\ &= \Phi(v \otimes w) \diamond_t \Phi(v' \otimes 1). \end{aligned}$$

Otherwise, we put $w'' = (v_1 \partial_{t_1}) \cdots (v_{k-1} \partial_{t_{k-1}})$. Then

$$\begin{aligned} \Phi(v \otimes w \diamond_t v' \otimes w') &= \Phi(v \otimes w w'' (v' \partial_t) (v_k \partial_{t_k})) \\ &= \Phi(v \otimes w w'' (v' \partial_t)) \diamond_{t_k} \phi(v_k) \\ &= \Phi(v \otimes w \diamond_t v' \otimes w'') \diamond_{t_k} \phi(v_k) \\ &= (\Phi(v \otimes w) \diamond_t \Phi(v' \otimes w'')) \diamond_{t_k} \phi(v_k) \\ &= \Phi(v \otimes w) \diamond_t (\Phi(v' \otimes w'') \diamond_{t_k} \phi(v_k)) \\ &= \Phi(v \otimes w') \diamond_t \Phi(v' \otimes w'). \end{aligned}$$

So Φ is a \mathcal{T} -multiple permutative algebra morphism.

Uniqueness. For any $v, v_1, \dots, v_k \in V$, $t_1, \dots, t_k \in \mathcal{T}$,

$$v \otimes (v_1 \partial_{t_1}) \cdots (v_k \partial_{t_k}) = (v \otimes (v_1 \partial_{t_1}) \cdots (v_{k-1} \partial_{t_{k-1}})) \diamond_{t_k} v_k.$$

It is then easy to prove that $P_{\mathcal{T}}(V)$ is generated by $V \otimes 1$ as a \mathcal{T} -multiple permutative algebra. Consequently, Φ is unique. ■

Remark 3.21.

1. We proved that $P_{\mathcal{T}}(V)$ is freely generated by V , identified with $V \otimes 1$. Consequently, $\mathcal{P}_{\mathcal{T}}^!(n)$ has the same dimension as the multilinear component of $V \otimes S(V^{\oplus \mathcal{T}})$ with $V = \text{Vect}(X_1, \dots, X_n)$, that is to say

$$\text{Vect}(X_i \otimes (X_1 \partial_{t_1}) \cdots (X_{i-1} \partial_{t_{i-1}}) (X_{i+1} \partial_{t_{i+1}}) \cdots (X_n \partial_{t_n}), 1 \leq i \leq n, t_j \in \mathcal{T}),$$

so

$$\dim(\mathcal{P}_{\mathcal{T}}^!(n)) = nT^{n-1}.$$

The formal series of $\mathcal{P}_{\mathcal{T}}^!$ is

$$f_{\mathcal{T}}^!(X) = \sum_{n \geq 1} \frac{\dim(\mathcal{P}_{\mathcal{T}}^!(n))}{n!} X^n = \sum_{n \geq 1} \frac{T^{n-1}}{(n-1)!} X^n = X \exp(TX) = \frac{f_1^!(TX)}{T}.$$

2. It is possible to prove that $\mathcal{P}_{\mathcal{T}}^!$ is a Koszul operad (and, hence, $\mathcal{P}_{\mathcal{T}}$ too) using the rewriting method of [17].

4 Structure of the pre-Lie products

4.1 A nonassociative permutative coproduct

For all $t \in \mathcal{T}$, we define a coproduct $\rho_t: \mathfrak{g}_{\mathcal{D}, \mathcal{T}} \longrightarrow \mathfrak{g}_{\mathcal{D}, \mathcal{T}}^{\otimes 2}$ by

$$\forall T = B_d \left(\prod_{i \in [k]} T_i \partial_{t_i} \right) \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}, \quad \rho_t(T) = \sum_{j \in [k]} \delta_{t, t_j} B_d \left(\prod_{i \in [k], i \neq j} T_i \partial_{t_i} \right) \otimes T_j \partial_t.$$

Proposition 4.1.

1. For all $t, t' \in \mathcal{T}$, $(\rho_t \otimes \text{Id}) \circ \rho_{t'} = ((\rho_{t'} \otimes \text{Id}) \circ \rho_t)^{(23)}$, where in the right side we use the classical action of the symmetric group \mathfrak{S}_3 on $\mathfrak{g}_{\mathcal{D}, \mathcal{T}}^{\otimes 3}$ by permutations of the tensors, and more specifically the action of the transposition (23). With Sweedler's notation for ρ_t , this can be written, for any $x \in \mathfrak{g}_{\mathcal{D}, \mathcal{T}}$, as

$$\sum \sum (x^{(1)t'})^{(1)t} \otimes (x^{(1)t'})^{(2)t} \otimes x^{(2)t'} = \sum \sum (x^{(1)t})^{(1)t'} \otimes x^{(2)t} \otimes (x^{(1)t})^{(2)t'}.$$

2. For any $x, y \in \mathfrak{g}_{\mathcal{D}, \mathcal{T}}$, $t, t' \in \mathcal{T}$, with Sweedler's notations $\rho_t(x) = \sum x^{(1)t} \otimes x^{(2)t}$,

$$\rho_t(x \bullet_{t'} y) = \delta_{t, t'} x \otimes y + \sum x^{(1)t} \bullet_{t'} y \otimes x^{(2)t} + \sum x^{(1)t} \otimes x^{(2)t} \bullet_{t'} y.$$

3. For any $\mu = (\mu_t)_{t \in \mathcal{T}} \in \mathbb{K}^{\mathcal{T}}$, we put

$$\rho_{\mu} = \sum_{t \in \mathcal{T}} \mu_t \rho_t: \mathfrak{g}_{\mathcal{D}, \mathcal{T}} \longrightarrow \mathfrak{g}_{\mathcal{D}, \mathcal{T}}^{\otimes 2}.$$

This makes sense, as any tree in $\mathbb{T}_{\mathcal{D}, \mathcal{T}}$ does not vanish only under a finite number of ρ_t . Then ρ_{μ} is a nonassociative permutative (NAP) coproduct; for any $x, y \in \mathfrak{g}_{\mathcal{D}, \mathcal{T}}$, by the second point, using Sweedler's notation for ρ_{μ} ,

$$\rho_{\mu}(x \bullet_{\lambda} y) = \left(\sum_{t \in \mathcal{T}} \lambda_t \mu_t \right) x \otimes y + \sum x^{(1)\mu} \bullet_{\lambda} y \otimes x^{(2)\mu} + \sum x^{(1)\mu} \otimes x^{(2)\mu} \bullet_{\lambda} y.$$

In particular, if $\sum_{t \in \mathcal{T}} \lambda_t \mu_t = 1$, $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_{\lambda}, \rho_{\mu})$ is a NAP pre-Lie bialgebra in the sense of [16].

Proof. 1. For any tree T ,

$$(\rho_t \otimes \text{Id}) \circ \rho_{t'}(T) = \sum_{p,q \in [k], p \neq q} \delta_{t_p,t} \delta_{t_q,t'} B_d \left(\prod_{i \in [k], i \neq p,q} T_i \partial_{t_i} \right) \otimes T_p \partial_t \otimes T_q \partial_{t'},$$

which implies the result.

2. For any tree T, T' ,

$$\begin{aligned} \rho_t(T \bullet_{t'} T') &= \rho_t \left(B_d \left(\prod_{i \in [k]} T_i \partial_{t_i} T' \partial_{t'} \right) + \sum_{i \in [k]} B_d \left(\prod_{j \in [k], j \neq i} T_j \partial_{t_j} (T_i \bullet_{t'} T') \partial_{t_i} \right) \right) \\ &= \delta_{t,t'} B_d \left(\prod_{i \in [k]} T_i \partial_{t_i} \right) \otimes T' + \sum_{i \in [k]} \delta_{t_i,t} B_d \left(\prod_{j \in [k], j \neq i} T_j \partial_{t_j} T' \partial_{t'} \right) \otimes T_i \partial_t \\ &\quad + \sum_{i \in [k]} \delta_{t_i,t} B_d \left(\prod_{j \in [k], j \neq i} T_j \partial_{t_j} \right) \otimes (T_i \bullet_{t'} T') \partial_t \\ &\quad + \sum_{i \neq j \in [k]} \delta_{t_i,t} B_d \left(\prod_{p \in [k], p \neq i,j} T_p \partial_{t_p} (T_j \bullet_{t'} T') \partial_{t_j} \right) \otimes T_i \partial_t \\ &= \delta_{t,t'} T \otimes T' + \sum_{i \in [k]} \delta_{t_i,t} B_d \left(\prod_{j \in [k], j \neq i} T_j \partial_{t_j} \right) \bullet_{t'} T' \otimes T_i \partial_t \\ &\quad + \sum_{i \in [k]} \delta_{t_i,t} B_d \left(\prod_{j \in [k], j \neq i} T_j \partial_{t_j} \right) \otimes T_i \bullet_{t'} T' \partial_t \\ &= \delta_{t,t'} T \otimes T' + T^{(1)t} \bullet_{t'} T' \otimes T^{(2)t} + T^{(1)t} \otimes T^{(2)t} \bullet_{t'} T'. \end{aligned}$$

3. Obtained by summation. ■

Corollary 4.2. *If $\lambda \in \mathbb{K}^{(\mathcal{T})}$ is nonzero, let us choose $t_0 \in \mathcal{T}$ such that $\lambda_{t_0} \neq 0$. The pre-Lie algebra $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_\lambda)$ is freely generated by the set $\mathbb{T}_{\mathcal{D},\mathcal{T}}^{(t_0)}$ of \mathcal{T} -typed \mathcal{D} -decorated trees T such that there is no edge outgoing the root of T of type t_0 .*

Proof. For any tree T , we denote by α_T the number of edges outgoing the root of T of type T_0 . Our aim is to prove that $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_\lambda)$ is freely generated by the trees T such that $\alpha_T = 0$. We define a family of scalars $(\mu_t)_{t \in \mathcal{T}}$ by

$$\forall t \in \mathcal{T}, \quad \mu_t = \begin{cases} 0 & \text{if } t \neq t_0, \\ \frac{1}{\lambda_{t_0}} & \text{if } t = t_0. \end{cases}$$

Note that $\rho_\mu = \frac{1}{\lambda_{t_0}} \rho_{t_0}$. By Proposition 4.1, $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_\lambda, \rho_\mu)$ is a NAP pre-Lie bialgebra, so by Livernet's rigidity theorem [16], it is freely generated by $\text{Ker}(\rho_\mu) = \text{Ker}(\rho_{t_0})$. Obviously, if $\alpha_T = 0$, $T \in \text{Ker}(\rho_{t_0})$. Let us consider $x = \sum_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} x_T T \in \text{Ker}(\rho_{t_0})$. We consider the map

$$\Upsilon: \begin{cases} \mathfrak{g}_{\mathcal{D},\mathcal{T}} \otimes \mathfrak{g}_{\mathcal{D},\mathcal{T}} \longrightarrow \mathfrak{g}_{\mathcal{D},\mathcal{T}}, \\ T \otimes T' \longmapsto T \bullet_{t_0}^{\text{root}(T)} T'. \end{cases}$$

By definition of ρ_{t_0} , for any tree T , $\Upsilon \circ \rho_{t_0}(T) = \alpha_T T$. Consequently,

$$0 = \Upsilon \circ \rho_{t_0}(x) = \sum_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} x_T \alpha_T T.$$

So if $\alpha_T \neq 0$, $x_T = 0$, and x is a linear span of trees such that $\alpha_T = 0$: the set of trees T such that $\alpha_T = 0$ is a basis of $\text{Ker}(\rho_{t_0})$. ■

If $|\mathcal{D}| = D$ and $|\mathcal{T}| = T$, the number of elements of $\mathbb{T}_{\mathcal{D},\mathcal{T}}^{(t_0)}$ of degree n is denoted by $t'_{D,T}(n)$; it does not depend on t_0 . By direct computations,

$$\begin{aligned} t'_{D,T}(1) &= D, \\ t'_{D,T}(2) &= D^2(T-1), \\ t'_{D,T}(3) &= \frac{D^2(T-1)(3DT-D+1)}{2}, \\ t'_{D,T}(4) &= \frac{D^2(T-1)(16D^2T^2-8D^2T+D^2+6DT-3D+2)}{6}. \end{aligned}$$

In the particular case $D = 1$, $T = 2$, we recover sequence A005750 of the OEIS.

4.2 Pre-Lie algebra morphisms

Notation 4.3. Let \mathcal{T} and \mathcal{T}' be two sets of types. We denote by $\mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$ the space of matrices $M = (m_{t,t'})_{(t,t') \in \mathcal{T} \times \mathcal{T}'}$, such that for any $t' \in \mathcal{T}'$, $(m_{t,t'})_{t \in \mathcal{T}} \in \mathbb{K}^{(\mathcal{T})}$. If $\mathcal{T} = \mathcal{T}'$, we shall simply write $\mathcal{M}_{\mathcal{T}}(\mathbb{K})$. If $M \in \mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$ and $M' \in \mathcal{M}_{\mathcal{T}',\mathcal{T}''}(\mathbb{K})$, then

$$MM' = \left(\sum_{t' \in \mathcal{T}'} m_{t,t'} m'_{t',t''} \right)_{(t,t'') \in \mathcal{T} \times \mathcal{T}''} \in \mathcal{M}_{\mathcal{T},\mathcal{T}''}(\mathbb{K}).$$

If $\lambda \in \mathbb{K}^{(\mathcal{T}')}$ and $\mu \in \mathbb{K}^{\mathcal{T}}$, then

$$M\lambda = \left(\sum_{t' \in \mathcal{T}'} m_{t,t'} \lambda_{t'} \right)_{t \in \mathcal{T}} \in \mathbb{K}^{(\mathcal{T})}, \quad M^\top \mu = \left(\sum_{t \in \mathcal{T}} m_{t,t'} \mu_t \right)_{t' \in \mathcal{T}'} \in \mathbb{K}^{\mathcal{T}'}$$

In particular, $\mathcal{M}_{\mathcal{T}}(\mathbb{K})$ is an algebra, acting on $\mathbb{K}^{(\mathcal{T})}$ on the left and on $\mathbb{K}^{\mathcal{T}}$ on the right.

Definition 4.4. Let $M \in \mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$. We define a map $\Phi_M: \mathcal{H}_{\mathcal{D},\mathcal{T}'} \rightarrow \mathcal{H}_{\mathcal{D},\mathcal{T}}$, sending $F \in \mathbb{F}_{\mathcal{D},\mathcal{T}'}$ to the forest obtained by replacing $\text{type}(e)$ by $\sum_{t \in \mathcal{T}} m_{t,\text{type}(e)} t$ for any $e \in E(F)$, F being considered as linear in any of its edges. The restriction of Φ_M to $\mathfrak{g}_{\mathcal{D},\mathcal{T}'}$ is denoted by $\phi_M: \mathfrak{g}_{\mathcal{D},\mathcal{T}'} \rightarrow \mathfrak{g}_{\mathcal{D},\mathcal{T}}$.

Example 4.5. If \mathcal{T} contains two elements, the first one represented by $|$ and the second one by $\dot{!}$, if $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, for any $x, y, z \in \mathcal{D}$,

$$\begin{aligned} \phi_M(\dot{!}^y_x) &= \alpha \dot{!}^y_x + \gamma \dot{!}^y_x, \\ \phi_M(\dot{!}^y_x) &= \beta \dot{!}^y_x + \delta \dot{!}^y_x, \\ \phi_M(\begin{array}{c} y \quad z \\ \diagdown \quad \diagup \\ x \end{array}) &= \alpha\beta \begin{array}{c} y \quad z \\ \diagdown \quad \diagup \\ x \end{array} + \alpha\delta \begin{array}{c} y \quad z \\ \diagdown \quad \diagup \\ x \end{array} + \beta\gamma \begin{array}{c} y \quad z \\ \diagdown \quad \diagup \\ x \end{array} + \gamma\delta \begin{array}{c} y \quad z \\ \diagdown \quad \diagup \\ x \end{array}. \end{aligned}$$

Remark 4.6. For any $M \in \mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$, $M' \in \mathcal{M}_{\mathcal{T}',\mathcal{T}''}(\mathbb{K})$, $\Phi_M \circ \Phi_{M'} = \Phi_{MM'}$.

Proposition 4.7. Let $\lambda \in \mathbb{K}^{(\mathcal{T})}$, $\mu \in \mathbb{K}^{\mathcal{T}}$ and $M \in \mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$. Then ϕ_M is a pre-Lie morphism from $(\mathfrak{g}_{\mathcal{D},\mathcal{T}'}, \bullet_\lambda)$ to $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_{M\lambda})$ and a NAP coalgebra morphism from $(\mathfrak{g}_{\mathcal{D},\mathcal{T}'}, \rho_{M^\top \mu})$ to $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \rho_\mu)$.

Proof. Let $T, T' \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$. For any $t \in \mathcal{T}$, for any $v \in V(T)$,

$$\phi_M(T \bullet_t^{(v)} T') = \sum_{t' \in \mathcal{T}'} m_{t',t} \phi_M(T) \bullet_{t'} \phi_M(T'),$$

so

$$\phi_M(T \bullet_\lambda T') = \sum_{t,t' \in \mathcal{T}} m_{t',t} \lambda_t \phi_M(T) \bullet_{t'} \phi_M(T') = \phi_M(T) \bullet_{M\lambda} \phi_M(T').$$

We proved that ϕ_M is a pre-Lie algebra morphism from $(\mathfrak{g}_{\mathcal{D},\mathcal{T}'}, \bullet_\lambda)$ to $(\mathfrak{g}_{\mathcal{D},\mathcal{T}}, \bullet_{M\lambda})$.

For any $T \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$,

$$\rho_t \circ \phi_M(T) = \sum_{t' \in \mathcal{T}} m_{t,t'}(\phi_M \otimes \phi_M) \circ \rho_{t'}(T),$$

so

$$\rho_\mu \circ \phi_M(T) = \sum_{t,t' \in \mathcal{T}} m_{t,t'} \mu_t(\phi_M \otimes \phi_M) \circ \rho_{t'}(T) = (\phi_M \otimes \phi_M) \circ \rho_{M^\top \mu}(T).$$

So $\phi_M: (\mathfrak{g}_{\mathcal{D}, \mathcal{T}'}, \rho_{M^\top \mu}) \longrightarrow (\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \rho_\mu)$ is a NAP coalgebra morphism. \blacksquare

Corollary 4.8. *For any $\lambda \in \mathbb{K}^{(\mathcal{T})}$ and $\mu \in \mathbb{K}^{\mathcal{T}}$, such that $\sum_{t \in \mathcal{T}} \lambda_t \mu_t = 1$, for any $t_0 \in \mathcal{T}$, the NAP pre-Lie bialgebras $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_\lambda, \rho_\mu)$ and $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_{t_0}, \rho_{t_0})$ are isomorphic.*

Proof. Let us denote by $\lambda^{(0)}$ the element of $\mathbb{K}^{(\mathcal{T})}$ defined by

$$\lambda_t^{(0)} = \delta_{t,t_0}.$$

Note that for any $M \in \mathcal{M}_{\mathcal{T}}(\mathbb{K})$, invertible, $\phi_M: (\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_{\lambda^{(0)}}, \rho_{M^\top \mu}) \longrightarrow (\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_{M\lambda^{(0)}}, \rho_\mu)$ is an isomorphism. In particular, for a well-chosen M , $M\lambda^{(0)} = \lambda$; we can assume that $\lambda = \lambda^{(0)}$ without loss of generality. Then, by hypothesis, $\mu_{t_0} = 1$. We define a matrix $M \in \mathcal{M}_{\mathcal{T}}(\mathbb{K})$ in the following way:

$$m_{t,t'} = \begin{cases} \delta_{t,t_0} & \text{if } t' = t_0, \\ \delta_{t,t'} - \mu_{t'} \delta_{t,t_0} & \text{otherwise.} \end{cases}$$

Then M is invertible. Moreover, $M\lambda^{(0)} = \lambda^{(0)}$ and $M^\top \mu = \lambda^{(0)}$. So ϕ_M is an isomorphism from $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_{\lambda^{(0)}}, \rho_{\lambda^{(0)}})$ to $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_\lambda, \rho_\mu)$. \blacksquare

Proposition 4.9. *Let $\lambda \in \mathbb{K}^{(\mathcal{T})}$, and $t_0 \in \mathcal{T}$. We define a pre-Lie algebra morphism $\psi_{t_0}: (\mathfrak{g}_{\mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}}, \bullet) \longrightarrow (\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_\lambda)$, sending \cdot^x to T for any $T \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}$. Then ψ_{t_0} is a pre-Lie algebra isomorphism if, and only if, $\lambda_{t_0} \neq 0$.*

Proof. If $\lambda_{t_0} \neq 0$, then by corollary 4.2, $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_\lambda)$ is freely generated by $\mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}$, so ψ_{t_0} is an isomorphism. If $\lambda_{t_0} = 0$, then it is not difficult to show that any tree T with two vertices, with its unique edge of type t_0 , does not belong to $\text{Im}(\psi_{t_0})$. \blacksquare

5 Hopf algebraic structures

We here fix $\lambda \in \mathbb{K}^{(\mathcal{T})}$.

5.1 Enveloping algebra of $\mathfrak{g}_{\mathcal{D}, \mathcal{T}}$

Using again the Guin–Oudom construction, we obtain the enveloping algebra of $(\mathfrak{g}_{\mathcal{D}, \mathcal{T}}, \bullet_\lambda)$. We first identify the symmetric coalgebra $S(\mathfrak{g}_{\mathcal{D}, \mathcal{T}})$ with the vector space generated by $\mathbb{F}_{\mathcal{D}, \mathcal{T}}$, which we denote by $\mathcal{H}_{\mathcal{D}, \mathcal{T}}$. Its product m is given by disjoint union of forests, its coproduct by

$$\forall T_1, \dots, T_k \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}, \quad \Delta(T_1 \cdots T_n) = \sum_{I \subseteq [n]} \prod_{i \in I} T_i \otimes \prod_{i \notin I} T_i.$$

We denote by \bullet_λ the Guin–Oudom extension of \bullet_λ to $\mathcal{H}_{\mathcal{D}, \mathcal{T}}$ and \star_λ the associated associative product.

Theorem 5.1. For any $F \in \mathbb{F}_{\mathcal{D},\mathcal{T}}$, $T_1, \dots, T_n \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$,

$$F \bullet_{\lambda} T_1 \cdots T_n = \sum_{\substack{v_1, \dots, v_n \in V(F), \\ t_1, \dots, t_n \in \mathcal{T}}} \left(\prod_{i \in [n]} \lambda_{t_i} \right) (\cdots (F \bullet_{t_1}^{(v_1)} T_1) \cdots) \bullet_{t_n}^{(v_n)} T_n,$$

$$F \star_{\lambda} T_1 \cdots T_n = \sum_{I \subseteq [n]} \left(F \bullet_{\lambda} \prod_{i \in I} T_i \right) \prod_{i \notin I} T_i.$$

The Hopf algebra $(\mathcal{H}_{\mathcal{D},\mathcal{T}}, \star_{\lambda}, \Delta)$ is denoted by $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{GL}\lambda}$. Moreover, for any $M \in \mathcal{M}_{\mathcal{T},\mathcal{T}'}(\mathbb{K})$, for any $\lambda \in \mathbb{K}^{(\mathcal{T}')}$, Φ_M is a Hopf algebra morphism from $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{GL}\lambda}$ to $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{GL}M\lambda}$. The extension of ψ_{t_0} as a Hopf algebra morphism from $\mathcal{H}_{\mathbb{T}_{\mathcal{D},\mathcal{T}}(t_0)}^{\text{GL}\lambda}$ to $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{GL}\lambda}$ is denoted by Ψ_{t_0} ; it is an isomorphism if, and only if, $\lambda_{t_0} \neq 0$.

In particular, if $\mathcal{T} = \{t\}$ and $\lambda_t = 1$, we recover the Grossman–Larson Hopf algebra [14].

5.2 Dual construction

Proposition 5.2. Let $T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$.

1. A cut c of T is a nonempty subset of $E(T)$; it is said to be admissible if any path in the tree from the root to a leaf meets at most one edge in c . The set of admissible cuts of T is denoted by $\text{Adm}(T)$.
2. If c is admissible, one of the connected components of $T \setminus c$ contains the root of c : we denote it by $R^c(T)$. The product of the other connected components of $T \setminus c$ is denoted by $P^c(T)$.

Let $\lambda \in \mathbb{K}^{\mathcal{T}}$. We define a multiplicative coproduct $\Delta^{\text{CK}\lambda}$ on the algebra $(\mathcal{H}_{\mathcal{D},\mathcal{T}}, m)$ by

$$\forall T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}, \quad \Delta^{\text{CK}\lambda}(T) = T \otimes 1 + 1 \otimes T + \sum_{c \in \text{Adm}(T)} \left(\prod_{e \in c} \lambda_{\text{type}(e)} \right) R^c(T) \otimes P^c(T).$$

Then $(\mathcal{H}_{\mathcal{D},\mathcal{T}}, m, \Delta^{\text{CK}\lambda})$ is a Hopf algebra, which we denote by $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\lambda}$.

Proof. We first assume that $\lambda \in \mathbb{K}^{(\mathcal{T})}$. Let us define a nondegenerate pairing $\langle -, - \rangle$ on $\mathcal{H}_{\mathcal{D},\mathcal{T}}$ by

$$\forall F, F' \in \mathbb{F}_{\mathcal{D},\mathcal{T}}, \quad \langle F, F' \rangle = \delta_{F, F'} s_F,$$

where s_F is the number of symmetries of F . Let us consider three forests F, F', F'' . We put

$$F = \prod_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} T^{\lambda_t}, \quad F' = \prod_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} T^{a'_T}, \quad F'' = \prod_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} T^{a''_T}.$$

Then

$$\begin{aligned} \langle \Delta(F), F' \otimes F'' \rangle &= \sum_{a=b+c} \prod_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} \frac{\lambda_t!}{\mu_t! c_T!} \left\langle \prod_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} T^{\mu_t}, F' \right\rangle \left\langle \prod_{T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}} T^{c_T}, F'' \right\rangle \\ &= \sum_{a=b+c} \delta_{b, a'} \delta_{c, a''} \frac{\lambda_t!}{a'_T! a''_T!} s_{F'} s_{F''} = \delta_{a, a' + a''} \frac{\lambda_t!}{a'_T! a''_T!} a'_T! a''_T! s_T^{a'_T + a''_T} \\ &= \delta_{a, a' + a''} \lambda_t! s_T^{\lambda_t} = \delta_{F, F' F''} s_F \\ &= \langle F, F' F'' \rangle. \end{aligned}$$

Therefore,

$$\forall x, y, z \in \mathcal{H}_{\mathcal{D}, \mathcal{T}}, \quad \langle \Delta(x), y \otimes z \rangle = \langle x, yz \rangle.$$

Let F, G be two forests and T be a tree. Observe that if F is a forest with at least two trees, then $F \star_{\lambda} G$ does not contain any tree, so $\langle F \star_{\lambda} G, T \rangle = 0$. If $F = 1$, then $\langle F \star_{\lambda} G, T \rangle \neq 0$ if, and only if, $G = T$; moreover, $\langle 1 \star_{\lambda} T, T \rangle = 1$. If F is a tree, then

$$\langle F \star_{\lambda} G, T \rangle = \langle F \bullet_{\lambda} G, T \rangle.$$

Moreover, if $F = B_d(F')$ and $G = T_1 \cdots T_k$,

$$F \bullet_{\lambda} G = \sum_{I \subseteq [k]} \sum_{(t_i) \in \mathcal{T}^k} \left(\prod_{i \in [k]} \lambda_{t_i} \right) B_d \left(\prod_{i \in I} T_i \partial_{t_i} F' \bullet \prod_{i \notin I} T_i \partial_{t_i} \right),$$

where \bullet is the pre-Lie product on $\mathfrak{g}_{\mathcal{D}, \mathcal{T}}^{\mathcal{T}}$ induced by the \mathcal{T} -multiple pre-Lie structure. Consequently, we can inductively define a coproduct $\Delta^{\text{CK}\lambda}: \mathcal{H}_{\mathcal{D}, \mathcal{T}} \rightarrow \mathcal{H}_{\mathcal{D}, \mathcal{T}} \otimes \mathcal{H}_{\mathcal{D}, \mathcal{T}}$, multiplicative for the product m , such that, if we denote for any tree T , $\overline{\Delta}_{\text{CK}}(T) = \Delta(T) - 1 \otimes T$, for any tree $T = B_d(T_1 \partial_{t_1} \cdots T_k \partial_{t_k})$,

$$\overline{\Delta}_{\lambda}^{\text{CK}}(T) = (B_d \otimes \text{Id}) \left(\prod_{i \in [k]} \left(\overline{\Delta}_{\lambda}^{\text{CK}}(T_i) \partial_{t_i} \otimes 1 + \lambda_{t_i} 1 \otimes T_i \right) \right).$$

Then, for any $x, y, z \in \mathcal{H}_{\mathcal{D}, \mathcal{T}}$,

$$\langle x \star_{\lambda} y, z \rangle = \langle x \otimes y, \Delta^{\text{CK}\lambda}(z) \rangle.$$

A quite easy induction on the number of vertices of trees proves that this coproduct is indeed the one we define in the statement of the proposition. As $\langle -, - \rangle$ is nondegenerate, $(\mathcal{H}_{\mathcal{D}, \mathcal{T}}, m, \Delta^{\text{CK}\lambda})$ is a Hopf algebra, dual to $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{GL}\lambda}$.

In the general case, for any $x \in \mathcal{H}_{\mathcal{D}, \mathcal{T}}$, there exists a finite subset \mathcal{T}' of \mathcal{T} such that $x \in \mathcal{H}_{\mathcal{D}, \mathcal{T}'}$. Putting $\lambda' = \lambda|_{\mathcal{T}'}$, $\lambda' \in \mathbb{K}^{\mathcal{T}'} = \mathbb{K}^{(\mathcal{T}'})$, so

$$\begin{aligned} (\Delta^{\text{CK}\lambda} \otimes \text{Id}) \circ \Delta^{\text{CK}\lambda}(x) &= (\Delta^{\text{CK}\lambda'} \otimes \text{Id}) \circ \Delta^{\text{CK}\lambda'}(x) \\ &= (\text{Id} \otimes \Delta^{\text{CK}\lambda'}) \circ \Delta^{\text{CK}\lambda'}(x) \\ &= (\text{Id} \otimes \Delta^{\text{CK}\lambda}) \circ \Delta^{\text{CK}\lambda}(x). \end{aligned}$$

Hence, Δ_{λ} is coassociative, and $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}\lambda}$ is a Hopf algebra. ■

Example 5.3. Let us fix a subset \mathcal{T}' of \mathcal{T} and choose $(\lambda_t)_{t \in \mathcal{T}}$ such that

$$\lambda_t = \begin{cases} 1 & \text{if } t \in \mathcal{T}', \\ 0 & \text{otherwise.} \end{cases}$$

For any tree $T \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$, let us denote by $\text{Adm}_{\mathcal{T}'}(T)$ the set of admissible cuts c of T such that the type of any edge in c belongs to \mathcal{T}' . Then

$$\Delta^{\text{CK}\lambda}(T) = T \otimes 1 + 1 \otimes T + \sum_{c \in \text{Adm}_{\mathcal{T}'}(T)} R^c(T) \otimes P^c(T).$$

Remark 5.4.

1. If $\mathcal{T} = \{t\}$ and $\lambda_t = 1$, we recover the usual Connes–Kreimer Hopf algebra of \mathcal{D} -decorated rooted trees, which we denote by $\mathcal{H}_{\mathcal{D}}^{\text{CK}}$, and its duality with the Grossman–Larson Hopf algebra [10, 15, 23].
2. If \mathcal{T} and \mathcal{D} are finite, for any $\lambda \in \mathbb{K}^{\mathcal{T}}$, both $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}\lambda}$ and $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{GL}\lambda}$ are graded Hopf algebras (by the number of vertices), and their homogeneous components are finite-dimensional. Via the pairing $\langle -, - \rangle$, each one is the graded dual of the other.

5.3 Hochschild cohomology of coalgebras

For the sake of simplicity, we assume that the set of types \mathcal{T} is finite and we put $\mathcal{T} = \{t_1, \dots, t_N\}$.

Let (C, Δ) be a coalgebra and let (M, δ_L, δ_R) be a C -bicomodule. We define a complex, dual to the Hochschild complex for algebras, in the following way:

1. For any $n \geq 0$, $H_n = \mathcal{L}(M, C^{\otimes n})$.
2. For any $L \in H_n$,

$$b_n(L) = (\text{Id} \otimes L) \circ \delta_L + \sum_{i=1}^n (-1)^i (\text{Id}^{\otimes(i-1)} \otimes \Delta \otimes \text{Id}^{\otimes(n-i)}) \circ L \\ + (-1)^{n+1} (L \otimes \text{Id}) \circ \delta_R.$$

In particular, one-cocycles are maps $L: M \rightarrow C$ such that

$$\Delta \circ L = (\text{Id} \otimes L) \circ \delta_L + (L \otimes \text{Id}) \circ \delta_R.$$

We shall consider in particular the bicomodule (M, δ_L, δ_R) such that

$$\forall x \in C, \quad \begin{cases} \delta_L(x) = 1 \otimes x, \\ \delta_R(x) = \Delta(x). \end{cases}$$

If C is a bialgebra, then $M^{\otimes N}$ is also a bicomodule:

$$\forall x_t \in C, \quad \begin{cases} \delta_L \left(\bigotimes_{1 \leq i \leq N} x_i \right) = 1 \otimes \bigotimes_{1 \leq i \leq N} x_i, \\ \delta_R \left(\bigotimes_{1 \leq i \leq N} x_i \right) = \bigotimes_{1 \leq i \leq N} x_i^{(1)} \otimes \prod_{1 \leq i \leq N} x_i^{(2)}. \end{cases}$$

We denote by $\underline{1} = (1)_{t \in \mathcal{T}} \in \mathbb{K}^{\mathcal{T}}$, and we take $C = \mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_1}$. One can identify $S(\text{Vect}(\mathbb{T}_{\mathcal{D}, \mathcal{T}})^{\oplus \mathcal{T}})$ and $C^{\otimes N}$, $x \delta_{T_i}$ being identified with $1^{\otimes(i-1)} \otimes x \otimes 1^{\otimes(n-i)}$ for any $x \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$ and $1 \leq i \leq N$. Then for any d , $B_d: C^{\otimes N} \rightarrow C$ is a 1-cocycle. Moreover, there is a universal property, proved in the same way as for the Connes–Kreimer's one [10]:

Theorem 5.5. *Let B be a commutative bialgebra and, for any $d \in \mathcal{D}$, let $L_d: C^{\otimes N} \rightarrow C$ be a 1-cocycle:*

$$\forall d \in \mathcal{D}, \quad \forall x_t \in B, \quad \Delta \circ L_d \left(\bigotimes_{1 \leq i \leq N} x_i \right) = 1 \otimes \bigotimes_{1 \leq i \leq N} x_i + L_d \left(\bigotimes_{1 \leq i \leq N} x_i^{(1)} \right) \otimes \prod_{1 \leq i \leq N} x_i^{(2)}.$$

There exists a unique bialgebra morphism $\phi: \mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_1} \rightarrow C$ such that for any $d \in \mathcal{D}$, $\phi \circ L_d = B_d \circ \phi^{\otimes N}$.

5.4 Hopf algebra morphisms

Our aim is, firstly, to construct Hopf algebra morphisms between $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\lambda}$ and $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\mu}$; secondly, to construct Hopf algebra isomorphisms between $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\lambda}$ and $\mathcal{H}_{\mathcal{D}', \mathcal{T}}^{\text{CK}}$ for a well-chosen \mathcal{D}' .

Proposition 5.6. *Let $M \in \mathcal{M}_{\mathcal{T}, \mathcal{T}'}(\mathbb{K})$, $\lambda \in \mathbb{K}^{\mathcal{T}}$. Then $\Phi_M: \mathcal{H}_{\mathcal{D}, \mathcal{T}'}^{\text{CK}_M \tau \lambda} \rightarrow \mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\lambda}$ is a Hopf algebra morphism.*

Proof. Φ_M is obviously an algebra morphism. Let $T \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$.

$$\begin{aligned} \Delta_\lambda \circ \Phi_M(T) &= \Phi_M(T) \otimes 1 + 1 \otimes \Phi_M(T) \\ &+ \sum_{c \in \text{Adm}(T)} \prod_{e \in \mathcal{E}^c} \left(\sum_{t \in \mathcal{T}} m_{t, \text{type}(e)} \lambda_t \right) \Phi_M(R^c(T)) \otimes \Phi_M(P^c(T)) \\ &= \Phi_M(T) \otimes 1 + 1 \otimes \Phi_M(T) \\ &+ \sum_{c \in \text{Adm}(T)} \prod_{e \in \mathcal{E}^c} (M^\top \lambda)_{\text{type}(e)} \Phi_M(R^c(T)) \otimes \Phi_M(P^c(T)) \\ &= (\Phi_M \otimes \Phi_M) \circ \Delta_{M^\top a}(T). \end{aligned}$$

So Φ_M is a coalgebra morphism from $\mathcal{H}_{\mathcal{D}, \mathcal{T}'}^{\text{CK}_{M^\top \lambda}}$ to $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\lambda}$. \blacksquare

Corollary 5.7. Let $\lambda, \mu \in \mathbb{K}^{\mathcal{T}}$, both nonzero. Then $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\lambda}$ and $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\mu}$ are isomorphic Hopf algebras.

Proof. There exists $M \in \mathcal{M}_{\mathcal{T}}(\mathbb{K})$, invertible, such that $M^\top \lambda = \mu$. Then Φ_M is an isomorphism between $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\mu}$ and $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\lambda}$. \blacksquare

Definition 5.8. Let us fix $t_0 \in \mathcal{T}$. For any $F \in \mathbb{F}_{\mathcal{D}, \mathcal{T}}$, we shall say that $\{T_1, \dots, T_k\} \triangleleft_{t_0} F$ if the following conditions hold:

- $\{T_1, \dots, T_k\}$ is a partition of $V(F)$. Consequently, for any $i \in [k]$, $T_i \in \mathbb{F}_{\mathcal{D}, \mathcal{T}}$, by restriction.
- For any $i \in [k]$, $T_i \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}$.

If $\{T_1, \dots, T_k\} \triangleleft_{t_0} F$, we denote by $F/\{T_1, \dots, T_k\}$ the forest obtained by contracting T_i to a single vertex for any $i \in [k]$, decorating this vertex by T_i , and forgetting the type of the remaining edges. Then $F/\{T_1, \dots, T_k\}$ is a $\mathcal{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}$ -decorated forest.

Proposition 5.9. Let $\lambda \in \mathbb{K}^{\mathcal{T}}$, $t_0 \in \mathcal{T}$. Let us consider the map

$$\Psi_{t_0}^* : \begin{cases} \mathcal{H}_{\mathcal{D}, \mathcal{T}} \longrightarrow \mathcal{H}_{\mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}}, \\ F \in \mathbb{F}_{\mathcal{D}, \mathcal{T}} \longmapsto \sum_{\{T_1, \dots, T_k\} \triangleleft_{t_0} F} \left(\prod_{e \in E(F) \setminus \sqcup E(T_i)} \lambda_{\text{type}(e)} \right) F/\{T_1, \dots, T_k\}. \end{cases}$$

Then $\Psi_{t_0}^*$ is a Hopf algebra morphism from $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\lambda}$ to $\mathcal{H}_{\mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}}^{\text{CK}_{\lambda_{t_0}}}$. It is an isomorphism if, and only if, $\lambda_{t_0} \neq 0$.

Proof. *First case.* We first assume that \mathcal{D} and \mathcal{T} are finite. In this case, $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}_\lambda}$ is the graded dual of $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{GL}_\lambda}$, with the Hopf pairing $\langle -, - \rangle$; grading $\mathcal{H}_{\mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}}$ by the number of vertices of the decorations, $\mathcal{H}_{\mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}}^{\text{CK}_{\lambda_{t_0}}}$ is the graded dual of $\mathcal{H}_{\mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}}^{\text{GL}_{\lambda_{t_0}}}$. Moreover, $\Psi_{t_0}^*$ is the transpose of Ψ_{t_0} of Proposition 4.9, so is a Hopf algebra morphism. If $\lambda_{t_0} \neq 0$, Ψ_{t_0} is an isomorphism, so $\Psi_{t_0}^*$ also is.

General case. Let $x, y \in \mathcal{H}_{\mathcal{D}, \mathcal{T}}$. There exist finite $\mathcal{D}', \mathcal{T}'$, such that $x, y \in \mathcal{H}_{\mathcal{D}', \mathcal{T}'}$; we can assume that $t_0 \in \mathcal{T}'$. We denote by $\lambda' = \lambda|_{\mathcal{T}'}$. Then, by the preceding case, denoting by Ψ'_{t_0} the restriction of $\Psi_{t_0}^*$ to $\mathcal{H}_{\mathcal{D}', \mathcal{T}'}$,

$$\begin{aligned} \Psi_{t_0}^*(xy) &= \Psi'_{t_0}(xy) = \Psi'_{t_0}(x)\Psi'_{t_0}(y) = \Psi_{t_0}^*(x)\Psi_{t_0}^*(y), \\ \Delta^{\text{CK}_\lambda} \circ \Psi_{t_0}^*(x) &= \Delta^{\text{CK}_{\lambda'}} \circ \Psi'_{t_0}(x) = (\Psi'_{t_0} \otimes \Psi'_{t_0}) \circ \Delta^{\text{CK}_{\lambda'}}(x) = (\Psi_{t_0}^* \otimes \Psi_{t_0}^*) \circ \Delta^{\text{CK}_\lambda}(x), \end{aligned}$$

so Ψ is a Hopf algebra morphism.

Let us assume that $\lambda_{t_0} \neq 0$. If $\Psi_{t_0}^*(x) = 0$, then $\Psi'_{t_0}(x) = 0$. As $a'_{t_0} \neq 0$, by the first case, $x = 0$, so $\Psi_{t_0}^*$ is injective. Moreover, there exists $z \in \mathcal{H}_{\mathcal{D}', \mathcal{T}'}$, such that $\Psi'_{t_0}(z) = y$; so $\Psi_{t_0}^*(z) = y$, and $\Psi_{t_0}^*$ is surjective.

Let us assume that $\lambda_{t_0} = 0$. Let T be a tree with two vertices, such that its unique edge is of type t_0 . As $T \notin \mathbb{T}_{\mathcal{D}, \mathcal{T}}^{(t_0)}$, $\Phi_{t_0}(T)$ has a unique term, given by the partition $X = \{\{x_1\}, \{x_2\}\}$, where x_1 and x_2 are the vertices of T . Hence,

$$\Psi_{t_0}^*(T) = \lambda_{t_0} T' = 0,$$

so $\Psi_{t_0}^*$ is not injective. ■

Example 5.10. Here, \mathcal{T} contains two elements, $|$ and $\dot{|}$. In order to simplify, we omit the decorations of vertices. We put

$$x = \bullet, \quad y = \dot{|}, \quad z = \vee, \quad u = \dot{|}, \quad v = \ddot{|}.$$

Applying Ψ^* :

$$\begin{aligned} \Psi^*_{|}(\bullet) &= \bullet^x, & \Psi^*_{|}(\vee) &= \lambda^2_{|} \vee^x, \\ \Psi^*_{|}(\dot{|}) &= \lambda_{|} \dot{|}^x, & \Psi^*_{|}(\dot{|}) &= \lambda^2_{|} \dot{|}^x, \\ \Psi^*_{\dot{|}}(\dot{|}) &= \lambda_{\dot{|}} \dot{|}^x + \bullet^y, & \Psi^*_{\dot{|}}(\dot{|}) &= \lambda_{|} \lambda_{\dot{|}} \dot{|}^x + \lambda_{|} \dot{|}^y, \\ \Psi^*_{|}(\vee) &= \lambda_{|} \lambda_{\dot{|}} \vee^x + \lambda_{|} \dot{|}^y, & \Psi^*_{\dot{|}}(\dot{|}) &= \lambda_{|} \lambda_{\dot{|}} \dot{|}^x + \lambda_{|} \dot{|}^y + \bullet^u, \\ \Psi^*_{|}(\vee) &= \lambda^2_{|} \vee^x + 2\lambda_{\dot{|}} \dot{|}^y + \bullet^z, & \Psi^*_{\dot{|}}(\dot{|}) &= \lambda^2_{\dot{|}} \dot{|}^x + \lambda_{\dot{|}} \dot{|}^y + \lambda_{|} \dot{|}^x + \bullet^v. \end{aligned}$$

Remark 5.11. Although it is not indicated, Ψ_{t_0} and $\Psi_{t_0}^*$ depend on λ .

5.5 Bialgebras in cointeraction

By [12], for any $\lambda \in \mathbb{K}^{(\mathcal{T})}$, the operad morphism $\theta_\lambda: \mathbf{PreLie} \rightarrow \mathcal{P}_{\mathcal{T}}$, which send \bullet to \bullet_λ , where \mathbf{PreLie} is the operad of pre-Lie algebras, induces a pair of cointeracting bialgebras for any finite set \mathcal{D} . By construction, the first bialgebra of the pair is $\mathcal{H}_{\mathcal{D}, \mathcal{T}}^{\text{CK}\lambda}$. Let us describe the second one.

Definition 5.12. Let $F \in \mathbb{F}_{\mathcal{T}, \mathcal{D}}$. We shall say that $\{T_1, \dots, T_k\} \triangleleft F$ if

1. $\{T_1, \dots, T_k\}$ is a partition of $V(F)$. Consequently, for any $i \in [k]$, $T_i \in \mathbb{F}_{\mathcal{D}, \mathcal{T}}$, by restriction.
2. For any $i \in [k]$, $T_i \in \mathbb{T}_{\mathcal{D}, \mathcal{T}}$.

If $\{T_1, \dots, T_k\} \triangleleft F$ and $\text{dec}: [k] \rightarrow \mathcal{D}$, we denote by $(F/\{T_1, \dots, T_k\}, \text{dec})$ the forest obtained by contracting T_i to a single vertex, and decorating this vertex by $\text{dec}(i)$, for all $i \in [k]$. This is an element of $\mathbb{F}_{\mathcal{D}, \mathcal{T}}$.

Proposition 5.13. If \mathcal{D} is finite, $\mathcal{H}_{\mathcal{D}, \mathcal{T}}'$ is the free commutative algebra generated by pairs (T, d) , where $T \in \mathbb{T}_{\mathcal{T}, \mathcal{D}}$ and $d \in \mathcal{D}$. The coproduct is given, for any $F \in \mathbb{F}_{\mathcal{D}, \mathcal{T}}$, $d \in \mathcal{D}$, by

$$\delta(F, d) = \sum_{\{T_1, \dots, T_k\} \triangleleft F} \sum_{\text{dec}: [k] \rightarrow \mathcal{D}} ((F/\{T_1, \dots, T_k\}, \text{dec}), d) \otimes (T_1, \text{dec}(1)) \cdots (T_k, \text{dec}(k)).$$

Then $(\mathcal{H}'_{\mathcal{D},\mathcal{T}}, m, \delta)$ is a bialgebra, and $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\lambda}$ is a coalgebra in the category of $\mathcal{H}'_{\mathcal{D},\mathcal{T}}$ -comodules via the coaction given, for any $T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$, by

$$\bar{\delta}(T) = \sum_{\{T_1, \dots, T_k\} \triangleleft T} \sum_{\text{dec}: [k] \rightarrow \mathcal{D}} ((T/\{T_1, \dots, T_k\}, \text{dec}) \otimes (T_1, \text{dec}(1)) \cdots (T_k, \text{dec}(k))).$$

Corollary 5.14. *Let us assume that \mathcal{D} is given a semigroup law denoted by $+$. If $F \in \mathbb{F}_{\mathcal{T},\mathcal{D}}$, and $\{T_1, \dots, T_k\} \triangleleft F$, then naturally $T_i \in \mathbb{T}_{\mathcal{T},\mathcal{D}}$ for any i and the \mathcal{T} -typed forest $F/\{T_1, \dots, T_k\}$ is given a \mathcal{D} -decoration, decorating the vertex obtained in the contradiction of T_i by the sum of the decorations of the vertices of T_i . Then $\mathcal{H}_{\mathcal{D},\mathcal{T}}$ is given a second coproduct δ such that for any $F \in \mathbb{F}_{\mathcal{D},\mathcal{T}}$,*

$$\delta(F) = \sum_{\{T_1, \dots, T_k\} \triangleleft F} F/\{T_1, \dots, T_k\} \otimes T_1 \cdots T_k.$$

Then $(\mathcal{H}_{\mathcal{D},\mathcal{T}}, m, \delta)$ is a bialgebra and $\mathcal{H}_{\mathcal{D},\mathcal{T}}^{\text{CK}\lambda}$ is a coalgebra in the category of $\mathcal{H}_{\mathcal{D},\mathcal{T}}$ -comodules via the coaction δ .

Proof. We denote by I the ideal of $\mathcal{H}'_{\mathcal{D},\mathcal{T}}$ generated by pairs (T, d) such that $T \in \mathbb{T}_{\mathcal{D},\mathcal{T}}$ and $d \in \mathcal{D}$, with

$$d \neq \sum_{v \in V(T)} \text{dec}(v).$$

The quotient $\mathcal{H}'_{\mathcal{D},\mathcal{T}}/I$ is identified with $\mathcal{H}_{\mathcal{D},\mathcal{T}}$, through the surjective algebra morphism

$$\varpi: \begin{cases} \mathcal{H}'_{\mathcal{D},\mathcal{T}} \longrightarrow \mathcal{H}_{\mathcal{D},\mathcal{T}}, \\ (F, d) \in \mathbb{F}_{\mathcal{D},\mathcal{T}} \times \mathcal{D} \longmapsto \begin{cases} F & \text{if } d = \sum_{v \in V(F)} \text{dec}(v), \\ 0 & \text{otherwise.} \end{cases} \end{cases}$$

Let us prove that I is a coideal. Let $T \in \mathbb{T}_{\mathcal{T},\mathcal{D}}$, $d \in \mathcal{D}$, $\{T_1, \dots, T_k\} \triangleleft F$, $\text{dec}: [k] \rightarrow \mathcal{D}$ such that $((T/\{T_1, \dots, T_k\}, \text{dec}), d) \notin I$ and for any i , $(T_i, \text{dec}(i)) \notin I$. Then

$$\forall i \in [k], \quad \sum_{v \in V(T_i)} \text{dec}(v) = \text{dec}(i), \quad \sum_{i=1}^k \text{dec}(i) = d.$$

Hence,

$$\sum_{v \in V(T)} \text{dec}(v) = \sum_{i=1}^k \sum_{v \in V(T_i)} \text{dec}(v) = \sum_{i=1}^k \text{dec}(i) = d,$$

so $(T, d) \notin I$. Consequently, if $T \in I$, then $((T/\{T_1, \dots, T_k\}, \text{dec}), d) \in I$ or at least one of the $(T_i, \text{dec}(i))$ belongs to I . Hence,

$$\delta(I) \subseteq I \otimes \mathcal{H}'_{\mathcal{D},\mathcal{T}} + \mathcal{H}'_{\mathcal{D},\mathcal{T}} \otimes I.$$

So I is a coideal. The coproduct induced on $\mathcal{H}_{\mathcal{D},\mathcal{T}}$ by the morphism ϖ is precisely the one given in the setting of this corollary. \blacksquare

In particular, if \mathcal{D} is reduced to a single element, denoted by $*$, if we give it its unique semigroup structure ($* + * = *$), We obtain again the result of [7].

5.6 The Bruned–Hairer–Zambotti construction

We now consider the coproducts on typed trees in [3, Theorem 2.2.16], the first one with $\overline{\mathfrak{A}}(F) = \mathfrak{A}(F)$ and the second one with $\overline{\mathfrak{A}}(F) = \mathfrak{A}^+(F)$ of [3, Definition 2.4.1]. By definition [3, Definition 2.26] of admissible subtrees, according to the notations we choose in this paper:

- Let \mathcal{L} be a finite set of types. The considered trees are \mathcal{L} -typed and the leaves are \mathcal{L} -decorated. Considering that the internal vertices of such a tree are in fact decorated by an element $0 \notin \mathcal{L}$, these trees form a subset of $\mathbb{T}_{\mathcal{L} \sqcup \{0\}, \mathcal{L}}$ which we denote by $\mathbb{T}'_{\mathcal{L}}$.
- The first coproduct Δ_+ is given on any tree $T \in \mathbb{T}'_{\mathcal{L}}$ by

$$\Delta_+(T) = \sum_{c \in \text{Adm}'(T)} R^c(T) \otimes P^c(T),$$

where $\text{Adm}'(T)$ is the set of admissible cuts of T such that the set of leaves of $R^c(T)$ is a subset of leaves of T (note that automatically, the leaves of $P^c(T)$ are also leaves of T).

- The second coproduct is given on any tree $T \in \mathbb{T}'_{\mathcal{L}}$ by

$$\overline{\Delta}(T) = \sum_{\{T_1, \dots, T_k\} \triangleleft' T} T / \{T_1, \dots, T_k\} \otimes T_1 \cdots T_k,$$

where the sum runs over all $\{T_1, \dots, T_k\} \triangleleft T$ such that the leaves of $T / \{T_1, \dots, T_k\}$ are leaves of T .

We denote by $\mathcal{H}'_{\mathcal{L}}$ the subalgebra of $\mathcal{H}_{\mathcal{L} \sqcup \{0\}, \mathcal{L}}$ generated by $\mathbb{T}'_{\mathcal{L}}$. Then $(\mathcal{H}'_{\mathcal{L}}, m, \Delta_+)$ and $(\mathcal{H}'_{\mathcal{L}}, m, \delta)$ are bialgebras.

Let us give to \mathcal{L} any product \times making it a commutative associative semigroup. We extend this structure to $\mathcal{L} \sqcup \{0\}$ by

$$\forall t \in \mathcal{L} \sqcup \{0\}, \quad 0 \times t = t \times 0 = 0.$$

We take $\lambda_t = 1$ for any $t \in \mathcal{L} \sqcup \{0\}$, and obtain with this data two coproducts Δ and δ on $\mathcal{H}_{\mathcal{L} \sqcup \{0\}, \mathcal{L}}$, the first one given by admissible cuts and the second one by contractions of subtrees. The subalgebra $\mathcal{H}''_{\mathcal{L}}$ of $\mathcal{H}_{\mathcal{L} \sqcup \{0\}, \mathcal{L}}$ generated by trees such that any internal vertex is decorated by 0 (note that the leaves of such a tree are decorated by $\mathcal{L} \sqcup \{0\}$) is a subbialgebra for both coproducts. We denote by I the ideal of $\mathcal{H}''_{\mathcal{L}}$ generated by trees with at least one leaf decorated by 0. Then it is a coideal for both coproducts, so the quotient algebra $\mathcal{H}''_{\mathcal{L}}/I$ inherits two coproducts, still denoted by Δ and δ . This algebra is trivially identified with the algebra generated by $\mathbb{T}'_{\mathcal{L}}$, that is to say with $\mathcal{H}'_{\mathcal{L}}$. By construction of the different coproducts, this identification is an isomorphism from the Hopf algebra $(\mathcal{H}''_{\mathcal{L}}/I, \Delta)$ to $(\mathcal{H}'_{\mathcal{L}}/I, \Delta_+)$ and from the Hopf algebra $(\mathcal{H}''_{\mathcal{L}}/I, \delta)$ to $(\mathcal{H}'_{\mathcal{L}}/I, \overline{\Delta})$. In other words, the Bruned–Hairer–Zambotti construction of cointeracting bialgebras on typed trees is a subquotient of the construction presented here.

6 Appendix

6.1 Values of $t_{D,T}(\mathbf{k})$

For $D = 1$,

$T \setminus k$	1	2	3	4	5	6	7	8
1	1	1	2	4	9	20	48	115
2	1	2	7	26	107	458	2058	9498
3	1	3	15	82	495	3144	20875	142773
4	1	4	26	188	1499	12628	111064	1006840
5	1	5	40	360	3570	37476	410490	4635330
6	1	6	57	614	7284	91566	1200705	16232820
7	1	7	77	966	13342	195384	2984142	46990952
8	1	8	100	1432	22570	377320	6578116	118238600
9	1	9	126	2028	35919	674964	13225632	267188229
10	1	10	155	2770	54465	1136402	24723000	554540590

For $D = 2$,

$T \setminus k$	1	2	3	4	5	6	7	8
1	2	4	14	52	214	916	4116	18996
2	2	8	52	376	2998	25256	222128	2013680
3	2	12	114	1228	14568	183132	2401410	32465640
4	2	16	200	2864	45140	754640	13156232	236477200
5	2	20	310	5540	108930	2272804	49446000	1109081180
6	2	24	444	9512	224154	5606520	146204792	3930863232
7	2	28	602	15036	413028	12043500	366122190	11475005616
8	2	32	784	22368	701768	23373216	811575408	29052861280
9	2	36	990	31764	1120590	41969844	1638712716	65965167108
10	2	40	1220	43480	1703710	70875208	3073688160	137426005200

For $D = 3$,

$T \setminus k$	1	2	3	4	5	6	7	8
1	3	9	45	246	1485	9432	62625	428319
2	3	18	171	1842	21852	274698	3602115	48698460
3	3	27	378	6084	107757	2024892	39676896	801564687
4	3	36	666	14268	336231	8409780	219307188	5896294848
5	3	45	1035	27690	814680	25444584	828506340	27812997990
6	3	54	1485	47646	1680885	62954766	2458069074	98947750662
7	3	63	2016	75432	3103002	135520812	6170116638	289616448690
8	3	72	2628	112344	5279562	263423016	13701398868	734709311208
9	3	81	3321	159678	8439471	473586264	27703353159	1670715963729
10	3	90	4095	218730	12842010	800524818	52018920345	3484841027040

For $D = 4$,

$T \setminus k$	1	2	3	4	5	6	7	8
1	4	16	104	752	5996	50512	444256	4027360
2	4	32	400	5728	90280	1509280	26312464	472954400
3	4	48	888	19024	448308	11213040	292409584	7861726464
4	4	64	1568	44736	1403536	46746432	1623150816	58105722560
5	4	80	2440	86960	3407420	141750416	6147376320	274852010400
6	4	96	3504	149792	7039416	351230688	18268531824	979612414944
7	4	112	4760	237328	13006980	756866096	45910215120	2871018269632
8	4	128	6208	353664	22145568	1472317056	102037088448	7290356719488
9	4	144	7848	502896	35418636	2648533968	206451156768	16590568445280
10	4	160	9680	689120	53917640	4479065632	387863411920	34625886677920

we find the following sequences of the OEIS [24]:

$T \setminus D$	1	2	3	4
1	A0081	A038055	A038059	A136793
2	A00151	A136794		
3	A006964			
4	A052763			
5	A052788			
6	A246235			
7	A246236			
8	A246237			
9	A246238			
10	A246239			

6.2 Values of $t'_{D,T}(k)$

For $D = 1$,

$T \setminus k$	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0	0
2	1	1	3	10	39	160	702	3177
3	1	2	9	46	268	1660	10845	73270
4	1	3	18	124	963	7968	69236	621999
5	1	4	30	260	2525	26136	283528	3178696
6	1	5	45	470	5480	68096	885805	11904160
7	1	6	63	770	10479	151956	2304974	36110880
8	1	7	84	1176	18298	303296	5255964	94051770
9	1	8	108	1704	29838	556464	10845732	218239560
10	1	9	135	2370	46125	955872	20696076	462558987

For $D = 2$,

$T \setminus k$	1	2	3	4	5	6	7	8
1	2	0	0	0	0	0	0	0
2	2	4	22	144	1090	8864	76162	678532
3	2	8	68	688	7886	96896	1250780	16713504
4	2	12	138	1888	29004	476736	8213588	146342376
5	2	16	232	4000	77060	1586304	34185344	761389360
6	2	20	350	7280	168670	4171744	107932710	2884827980
7	2	24	492	11984	324450	9370368	282934428	8822987856
8	2	28	658	18368	569016	18793600	648698792	23119514576
9	2	32	848	26688	930984	34609920	1344232416	53898191520
10	2	36	1062	37200	1442970	59627808	2573660298	114661732500

For $D = 3$,

$T \setminus k$	1	2	3	4	5	6	7	8
1	3	0	0	0	0	0	0	0
2	3	9	72	705	7947	96588	1237728	16450389
3	3	18	225	3408	58347	1072224	20685195	413084610
4	3	27	459	9405	216081	5315112	136987407	3650993163
5	3	36	774	19992	576405	17763984	572991726	19100718828
6	3	45	1170	36465	1264950	46852884	1815034140	72635168880
7	3	54	1647	60120	2437722	105455952	4768982442	222723271080
8	3	63	2205	92253	4281102	211832208	10953036318	584744300226
9	3	72	2844	134160	7011846	390570336	22727284344	1365242802048
10	3	81	3564	187137	10877085	673533468	43560017892	2907844041231

For $D = 4$,

$T \setminus k$	1	2	3	4	5	6	7	8
1	4	0	0	0	0	0	0	0
2	4	16	168	2192	32844	531200	9051376	159962784
3	4	32	528	10656	242792	5939968	152518064	4053650976
4	4	48	1080	29488	902100	29551104	1014147872	35989518528
5	4	64	1824	62784	2411024	98976256	4252211232	188790415552
6	4	80	2760	114640	5297820	261422336	13491005840	719200139360
7	4	96	3888	189152	10218744	588999936	35487727184	2208096700896
8	4	112	5208	290416	17958052	1184031744	81574704960	5802692175744
9	4	128	6720	422528	29428000	2184360960	169377005376	13557899008896
10	4	144	8424	589584	45668844	3768659712	324805399344	28894042642464

References

- [1] Bergeron F., Labelle G., Leroux P., Combinatorial species and tree-like structures, *Encyclopedia of Mathematics and its Applications*, Vol. 67, Cambridge University Press, Cambridge, 1998.
- [2] Bremner M.R., Dotsenko V., Algebraic operads. An algorithmic companion, CRC Press, Boca Raton, FL, 2016.

- [3] Bruned Y., Singular KPZ type equations, available at <https://tel.archives-ouvertes.fr/tel-01306427>.
- [4] Bruned Y., Chandra A., Chevyrev I., Hairer M., Renormalising SPDEs in regularity structures, *J. Eur. Math. Soc. (JEMS)* **23** (2021), 869–947, [arXiv:1711.10239](https://arxiv.org/abs/1711.10239).
- [5] Bruned Y., Hairer M., Zambotti L., Algebraic renormalisation of regularity structures, *Invent. Math.* **215** (2019), 1039–1156, [arXiv:1610.08468](https://arxiv.org/abs/1610.08468).
- [6] Bruned Y., Manchon D., Algebraic deformation for SPDEs, [arXiv:2011.05907](https://arxiv.org/abs/2011.05907).
- [7] Calaque D., Ebrahimi-Fard K., Manchon D., Two interacting Hopf algebras of trees: a Hopf-algebraic approach to composition and substitution of B-series, *Adv. in Appl. Math.* **47** (2011), 282–308, [arXiv:0806.2238](https://arxiv.org/abs/0806.2238).
- [8] Chapoton F., Un endofoncteur de la catégorie des opérades, in Dialgebras and Related Operads, *Lecture Notes in Math.*, Vol. 1763, Springer, Berlin, 2001, 105–110.
- [9] Chapoton F., Livernet M., Pre-Lie algebras and the rooted trees operad, *Int. Math. Res. Not.* **2001** (2001), 395–408, [arXiv:math.QA/0002069](https://arxiv.org/abs/math/0002069).
- [10] Connes A., Kreimer D., Hopf algebras, renormalization and noncommutative geometry, *Comm. Math. Phys.* **199** (1998), 203–242, [arXiv:hep-th/9808042](https://arxiv.org/abs/hep-th/9808042).
- [11] Dotsenko V., Griffin J., Cacti and filtered distributive laws, *Algebr. Geom. Topol.* **14** (2014), 3185–3225, [arXiv:1109.5345](https://arxiv.org/abs/1109.5345).
- [12] Foissy L., Algebraic structures associated to operads, [arXiv:1702.05344](https://arxiv.org/abs/1702.05344).
- [13] Foissy L., Generalized prelie and permutative algebras, [arXiv:2104.00909](https://arxiv.org/abs/2104.00909).
- [14] Grossman R., Larson R.G., Hopf-algebraic structure of combinatorial objects and differential operators, *Israel J. Math.* **72** (1990), 109–117.
- [15] Hoffman M.E., Combinatorics of rooted trees and Hopf algebras, *Trans. Amer. Math. Soc.* **355** (2003), 3795–3811, [arXiv:math.CO/0201253](https://arxiv.org/abs/math/0201253).
- [16] Livernet M., A rigidity theorem for pre-Lie algebras, *J. Pure Appl. Algebra* **207** (2006), 1–18, [arXiv:math.QA/0504296](https://arxiv.org/abs/math/0504296).
- [17] Loday J.-L., Vallette B., Algebraic operads, *Grundlehren der Mathematischen Wissenschaften*, Vol. 346, Springer, Heidelberg, 2012.
- [18] Manchon D., Zhang Y., Free pre-Lie family algebras, [arXiv:2003.00917](https://arxiv.org/abs/2003.00917).
- [19] Markl M., Shnider S., Stasheff J., Operads in algebra, topology and physics, *Mathematical Surveys and Monographs*, Vol. 96, Amer. Math. Soc., Providence, RI, 2002.
- [20] Mathar R.J., Topologically distinct sets of non-intersecting circles in the plane, [arXiv:1603.00077](https://arxiv.org/abs/1603.00077).
- [21] Oudom J.-M., Guin D., Sur l’algèbre enveloppante d’une algèbre pré-Lie, *C. R. Math. Acad. Sci. Paris* **340** (2005), 331–336.
- [22] Oudom J.-M., Guin D., On the Lie enveloping algebra of a pre-Lie algebra, *J. K-Theory* **2** (2008), 147–167, [arXiv:math.QA/0404457](https://arxiv.org/abs/math/0404457).
- [23] Panaite F., Relating the Connes–Kreimer and Grossman–Larson Hopf algebras built on rooted trees, *Lett. Math. Phys.* **51** (2000), 211–219, [arXiv:math.QA/0003074](https://arxiv.org/abs/math/0003074).
- [24] Sloane N.J.A., The on-line encyclopedia of integer sequences, <http://oeis.org/>.
- [25] Zhang Y., Gao X., Guo L., Matching Rota–Baxter algebras, matching dendriform algebras and matching pre-Lie algebras, *J. Algebra* **552** (2020), 134–170, [arXiv:1909.10577](https://arxiv.org/abs/1909.10577).