

Quantum Groups for Restricted SOS Models

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Abstract. We introduce the notion of restricted dynamical quantum groups through their category of representations, which are monoidal categories with a forgetful functor to the category of π -graded vector spaces for a groupoid π .

Key words: elliptic quantum groups; dynamical R -matrices; groupoid grading; RSOS models

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*Dedicated to Vitaly Tarasov and Alexander Varchenko
on their round birthdays*

1 Introduction

The theory of quantum groups was designed in the 1980s to describe the algebraic structure underlying the theory of exactly solvable models of statistical mechanics and quantum mechanics. Since then the theory of quantum groups has entered diverse fields of mathematics and mathematical physics and the world of exactly solvable models is entirely explained by quantum groups, in the guise of Yangians, quantum loop algebras, and elliptic quantum groups. Well, not entirely . . . One small village of indomitable models, called the RSOS models still holds out against the invaders.¹ These Restricted Solid-On-Solid models, introduced in special cases by Baxter in his studies of the eight-vertex model and the hard hexagon model, and generalized by Andrews, Baxter and Forrester, are lattice models of two-dimensional statistical mechanics for which the technology of exact solutions has provided some of the most spectacular results. They play a central role also in conformal field theory (CFT) as their critical behaviour is (or is conjectured to be) given by the universality classes of minimal unitary CFT models. While the unrestricted SOS models, whose local degrees of freedom take values in an infinite set, are by now well-described by the representation theory of dynamical elliptic quantum groups, the RSOS models, with finitely many allowed states at every lattice point, are much less understood.

We propose a theory of dynamical quantum groups with discrete dynamical parameter with the goal to establish the representation theory underlying RSOS models and their higher rank generalizations. We introduce a new approach to this problem, based on groupoid-graded vector spaces, which may be of independent interest and applicability in representation theory.

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¹Actually there are also other exactly solvable models whose quantum group description is unknown, such as the Inozemtsev spin chain [28].

1.1 Quantum groups and exactly solvable models

The notion of quantum group [12] emerged in the Leningrad school in the 1980s as the algebraic structure underlying exactly solvable models of statistical mechanics in 2 dimensions and integrable quantum field theory in 1+1 dimensions, see [18]. While “groups” may be a misnomer for these Hopf algebras, quantum groups share with groups the fact that they have an interesting representation theory for which tensor products of representations are defined. Excellent textbooks on quantum groups are [8, 33, 39]. It soon appeared that quantum groups have a much wider scope of applications, ranging from low dimensional topology to conformal field theory, algebraic geometry, gauge theory, representation theory of affine Lie algebras etc.

Returning to the origin in quantum mechanics and statistical mechanics, the basic equation is the Yang–Baxter equation, which appeared in the 1960s in the works of J.B. McGuire and C.N. Yang on one-dimensional many-body quantum systems [40, 47] and later in R. Baxter’s work on statistical mechanics [4]. In its basic form it is an equation for a meromorphic function $z \mapsto R(z) \in \text{End}_{\mathbb{C}}(V \otimes V)$ of one complex variable (called the spectral parameter) with values in the linear endomorphisms of the tensor square of a finite dimensional complex vector space V . The Yang–Baxter equation is

$$R(z-w)^{(12)}R(z)^{(13)}R(w)^{(23)} = R(w)^{(23)}R(z)^{(13)}R(z-w)^{(12)}$$

in $\text{End}(V \otimes V \otimes V)$. Here the superscripts in the notation indicate the factors on which the endomorphisms act: for example $R(w)^{(12)}$ means $R(w) \otimes \text{id}$. One also requires that $R(z)$ is invertible for generic z . One of the simplest non-trivial solutions is the R -matrix $R(z) = \text{Id} + z^{-1}P_{VV}$ of McGuire and Yang where P_{VV} is the flip $v \otimes w \mapsto w \otimes v$. Here V is any finite-dimensional vector space.

As noted by Baxter, solutions of the Yang–Baxter equation give rise to families of commuting operators on n -fold tensor powers $W = V^{\otimes n}$: fix complex numbers z_1, \dots, z_n and consider the operator valued function $L(z) \in \text{End}(V \otimes W)$ given by the product $R(z-z_n)^{(0n)} \cdots R(z-z_2)^{(02)}R(z-z_1)^{(01)}$ (we number the factors from 0 to n where 0 refers to the “auxiliary space” V in $V \otimes W$). Then the (row-to-row) transfer matrices defined by the partial traces

$$T(z) = \text{tr}_V L(z)$$

are commuting endomorphisms of $W = V^{\otimes n}$:

$$T(z)T(w) = T(w)T(z),$$

as a consequence of the Yang–Baxter equation. The relation to statistical mechanics is that the trace of $T(z)^m$, say with all $z_i = 0$, when written out using matrix multiplication, is a sum of products of matrix entries of $R(z)$ over all the ways to assigning a basis vector to pairs of nearest neighbours of an $n \times m$ lattice with periodic boundary conditions. These assignments are configurations of local states of a system and the sum is the partition function of a statistical mechanics model.

The Bethe ansatz, invented by H. Bethe in 1931 [7] in the case of the Heisenberg spin chain, and further developed by E. Lieb, R. Baxter, B. Sutherland, C.N. Yang and others in the 1960s, is a technique to find simultaneous eigenvectors and eigenvalues of the $T(z)$. The Leningrad school, see [18] for a review of the results in the early phase, reformulated this technique under the name “algebraic Bethe ansatz” in the framework of the “quantum inverse scattering method” in terms of representation theory of an algebra with quadratic relations (called RLL or RTT relations) whose coefficients are matrix entries of a solution of the Yang–Baxter equation. For example the Yangian $Y(\mathfrak{gl}_N)$ corresponds to the McGuire–Yang R -matrix with $V = \mathbb{C}^N$. It can be defined as the algebra with generators $L_{ij;n}$, $i, j = 1, \dots, N$, $n = 1, 2, \dots$ with relations

$$R(z-w)L(z) \otimes L(w) = L(z) \otimes L(z)R(z-w), \quad R(z) = \text{Id} + \frac{1}{z}P_{VV},$$

where $L(z)$ is the $N \times N$ matrix with entries $\delta_{ij} + \sum_{n \geq 1} L_{ij,n} z^{-n}$. It is a Hopf algebra deformation of the universal enveloping algebra of the current Lie algebra $\mathfrak{gl}_N[t]$ and has a universal R -matrix \mathcal{R} in a completion of $Y(\mathfrak{gl}_N) \otimes Y(\mathfrak{gl}_N)$ relating the opposite coproduct Δ' to the coproduct via $\Delta'(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1}$. Evaluating \mathcal{R} in pairs $V_i \otimes V_j$ of finite dimensional representations of the Yangian yields solutions of the Yang–Baxter equation, in the generalized form

$$R_{V_1 V_2}^{(12)} R_{V_1 V_3}^{(13)} R_{V_2 V_3}^{(23)} = R_{V_2 V_3}^{(23)} R_{V_1 V_3}^{(13)} R_{V_1 V_2}^{(12)}. \quad (1.1)$$

The spectral parameter may be viewed as a parameter of the representations V_i , and in fact there is an issue of convergence of the action of \mathcal{R} on finite dimensional representations, resulting in the fact that R_{V_i, V_j} is a meromorphic function of the spectral parameters.

To such a system of R -matrices we can associate corresponding transfer matrices $T_i = \text{tr}_{V_i} R_{V_i V_3}$, $i = 1, 2$, acting on V_3 and the Yang–Baxter equation with an invertible $R_{V_1 V_2}$ implies that $T_1 T_2 = T_2 T_1$. Baxter’s transfer matrices are the special case $V_{1,2} = V$ and V_3 a tensor product of vector representations with equal spectral parameters.

This story extends to arbitrary semisimple (or reductive) Lie algebras \mathfrak{g} , and the Yangians $Y(\mathfrak{g})$ provide the algebraic structure underlying several integrable systems based on rational solutions of the Yang–Baxter equation such as the Heisenberg spin chain.

The theory admits a trigonometric version, leading to solutions of the Yang–Baxter equation with trigonometric coefficients. The corresponding quantum group is a Hopf algebra deformation of the loop Lie algebra $\mathfrak{g}[t, t^{-1}]$. It is (a subquotient of) the Drinfeld–Jimbo quantum enveloping algebra $U_q \hat{\mathfrak{g}}$ of the affine Kac–Moody Lie algebra $\hat{\mathfrak{g}}$. The corresponding solvable models are the six-vertex model, a special case of which is the two-dimensional ice model, and the XXZ spin chain.

1.2 Elliptic quantum groups and dynamical Yang–Baxter equation

The next level, after the rational and trigonometric functions, are the elliptic functions, which are meromorphic functions which are periodic with respect to two independent periods. Several solvable models have an elliptic version and the trigonometric and rational versions are obtained as degenerate limits as the periods tend to infinity. The relation to quantum groups is more tricky in the elliptic case. On one hand there is a solution of the Yang–Baxter equation with elliptic coefficients due to Baxter, corresponding to the XYZ spin chain and the eight-vertex model, whose underlying algebraic structure is the Sklyanin algebra (which is not a Hopf algebra). On the other hand there are the SOS (solid-on-solid) models also known as IRF (interaction-round-a-face) models. They are based on a variant of the Yang–Baxter equation, called the star-triangle relation. While the existence of the Baxter solution is special for $\mathfrak{g} = \mathfrak{sl}_N$, we now know that SOS models exist for all semisimple Lie algebras. Also the Baxter solution can be related to a solution of the star-triangle relation by the so-called vertex-IRF transformation, also due to Baxter.

The elliptic quantum groups introduced in [19, 20] provide a generalization of the theory of quantum groups that applies to elliptic SOS models. They are based on a modification of the Yang–Baxter equation, nowadays called the dynamical Yang–Baxter equation, see (1.2) below, which had previously been found by Gervais and Neveu in their study of the exchange relations of vertex operators in the Liouville conformal field theory [26]. The dynamical Yang–Baxter equation reappeared in various contexts since, e.g., [1, 9, 10, 14, 15, 29, 32, 42, 43]. A recent textbook on elliptic quantum groups is [36].

The unknown in the dynamical Yang–Baxter equation is a function $R(z, a) \in \text{End}_{\mathfrak{h}}(V \otimes V)$ of a second “dynamical” variable $a \in \mathfrak{h}^*$ with values in the dual vector space to an abelian Lie algebra \mathfrak{h} and $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_{\mu}$ is a finite dimensional semisimple \mathfrak{h} -module. The (quantum)

dynamical Yang–Baxter equation is

$$\begin{aligned} R(z-w, a+h^{(3)})^{(12)} R(z, a)^{(13)} R(w, a+h^{(1)})^{(23)} \\ = R(w, a)^{(23)} R(z, a+h^{(2)})^{(13)} R(z-w, a)^{(12)}. \end{aligned} \quad (1.2)$$

The “dynamical shift” notation is adopted here: for example $R(w, a+h^{(3)})^{(12)}$ acts as $R(w, a+\mu_3) \otimes \text{Id}$ on the product of weight subspaces $V_{\mu_1} \otimes V_{\mu_2} \otimes V_{\mu_3}$. More generally, we can consider dynamical Yang–Baxter equations in $\text{End}_{\mathfrak{h}}(V_1 \otimes V_2 \otimes V_3)$ for R -matrices $R_{V_i V_j}(z, a) \in \text{End}_{\mathfrak{h}}(V_i \otimes V_j)$ as in (1.1).

The elliptic quantum group associated with a solution of the dynamical Yang–Baxter equation and its tensor category of representations can be again defined by quadratic relations similar to those of the Yangian but with dynamical shifts at the appropriate places, see [19, 23, 25]. The main new feature is that the representations are vector spaces over the field of meromorphic functions of the dynamical variables and the elements of the elliptic quantum group act as difference operators in these variables. The underlying generalization of the notion of Hopf algebra was formalized by Etingof and Varchenko [16] who called it \mathfrak{h} -Hopf algebroid.

The transfer matrix construction generalizes to the dynamical setting [22]: suppose that we have invertible operators $R_{V_i V_j}(z, \lambda) \in \text{End}_{\mathfrak{h}}(V_i \otimes V_j)$, $i < j \in \{1, 2, 3\}$ obeying the dynamical Yang–Baxter equation (1.2) on $V_1 \otimes V_2 \otimes V_3$, depending meromorphically on $z \in \mathbb{C}$, $\lambda \in \mathfrak{h}^*$. Then the transfer matrix is defined as an operator acting on meromorphic functions of λ with values in the zero-weight subspace of V_3 :

$$T_i(z) = \sum_{\mu} \text{tr}_{V_{i\mu}} R_{V_i V_3}(z, \lambda) t_{\mu}. \quad (1.3)$$

Here the partial trace is over the weight- μ subspace of V_i , the R -matrix acts as a multiplication operator and $(t_{\mu} f)(\lambda) = f(\lambda + \mu)$. In the special case where $V_1 = V_2$, R_{V_1, V_3} has the interpretation of an L -operator obeying a dynamical version of the RLL -relations.

1.3 Restricted SOS models

The restricted solid-on-solid (RSOS) models introduced by Andrews, Baxter and Forrester [3], generalizing models previously considered by Baxter [4, 5] in his study of the eight-vertex model and of the hard hexagon model, form a general class of models of statistical mechanics in two dimensions. A configuration of a solid-on-solid model on a subset M of a square lattice in the plane or 2-dimensional torus is described by assigning an integer l_i (height) to each lattice site $i \in M$, with the restriction that $|l_i - l_j| = 1$ for neighbouring sites i, j . We can think of the graph of $i \mapsto l_i$ as a discrete random surface modeling the interface between a solid and a gas.

The probability of a configuration (l_i) is proportional to a product over the faces (unit squares with vertices in M) of Boltzmann weights $\mathcal{W}(l_i, l_j, l_k, l_m)$ depending on the heights on the corners i, j, k, l of the face. In the “solvable” SOS models the Boltzmann weights are part of a one-parameter family $\mathcal{W}(z; a, b, c, d)$ obeying the star-triangle relation

$$\begin{aligned} \sum_g \mathcal{W}(z-w; f, g, d, e) \mathcal{W}(z; a, b, g, f) \mathcal{W}(w; b, c, d, g) \\ = \sum_g \mathcal{W}(w; a, g, e, f) \mathcal{W}(z; c, d, e, g) \mathcal{W}(z-w; a, b, c, g), \end{aligned}$$

which is best understood graphically:

$$\sum_g a \begin{array}{c} f \quad e \\ \diagdown \quad \diagup \\ g \\ \diagup \quad \diagdown \\ b \quad c \end{array} d = \sum_g a \begin{array}{c} f \quad e \\ \diagdown \quad \diagup \\ g \\ \diagup \quad \diagdown \\ b \quad c \end{array} d$$

The star-triangle equation admits interesting families of solutions in terms of elliptic theta functions. Andrews, Baxter and Forrester considered a special limit of parameters so that the equation holds for the heights in a finite interval

$$l_i \in \{1, 2, \dots, r-1\}.$$

For these families of models (depending essentially on an elliptic curve and a point of order r on it) they were able to compute several quantities in the thermodynamic limit $M \rightarrow \mathbb{Z}^2$ (under some physically motivated assumptions on the asymptotic behaviour), including the probability distribution of the height at the origin as a function of the boundary conditions in the ordered phase. One interesting mathematical outcome of this calculation is that it involves for $r = 5$ (Baxter’s hard hexagon model) the celebrated Rogers–Ramanujan identities, which get generalized to arbitrary r . From the point of view of statistical mechanics and conformal field theory, these models are interesting since their scaling limit at the critical point are conjectured [27] to be the unitary A -series of minimal models of Belavin–Polyakov–Zamolodchikov and Friedan–Qiu–Shenker.

We will be mostly concerned with a generalization of the RSOS models in which the heights take values in the weight lattice of a simple Lie algebra, see [11, 30, 31]. The main difference is that in general the Boltzmann weights $\mathcal{W}(a, b, c, d)$ are no longer scalar-valued, but must be understood as linear operators.

The relation with the dynamical quantum groups comes from the simple observation that the above star-triangle equation is essentially a rewriting of the dynamical Yang–Baxter equation. The row-to-row transfer matrix of the RSOS model is the transfer matrix (1.3) for suitable representations of the elliptic quantum group associated with \mathfrak{gl}_2 , acting on functions with support on a finite set.

This restriction of a difference operator such as (1.3) with meromorphic coefficients to a finite set is rather subtle as one needs to avoid the poles and check that the support condition is preserved. This was done in the case of the RSOS model in [24], where it was also shown that the \mathfrak{gl}_2 -elliptic weight functions of [21] obey “resonance conditions”, guaranteeing that their restrictions to suitable discrete or finite subsets of the values of the dynamical variable provide, via the Bethe ansatz, well-defined eigenvectors of the row-to-row transfer matrix of the RSOS model.

1.4 Categories of representations

Instead of talking of quantum groups it is more convenient to talk about their tensor category of representations, and we take this approach in this paper. In the representation theory of elliptic quantum groups [13, 23], the representation space of a representation is defined as a graded vector space over the field of meromorphic functions of the dynamical variables, where the grading is by weights of the underlying Lie algebra. The representation structure is defined by \mathbb{C} -linear endomorphisms obeying quadratic relations and commutations relations with scalar multiplication by meromorphic functions.

For the application to RSOS models, where the dynamical variables take values in a discrete set, the approach with meromorphic functions is not suitable. In this paper we propose that the vector spaces underlying representations of quantum groups with discrete dynamical variables should be groupoid-graded vector spaces. More precisely we propose that representations of such quantum groups are monoidal categories equipped with a faithful monoidal functor to the category of π -graded vector spaces of finite type for a certain groupoid π (see Section 2 for the definitions). For applications to generalized RSOS models the groupoids are certain subgroupoids of the transformation groupoid for the translation action of the weight lattice of a semisimple Lie algebra. It turns out that in this approach the various shifts of dynamical variables appearing in the dynamical context appear naturally and one can immediately apply the standard technology of the quantum inverse scattering method (R -matrices, RLL relations, transfer matrices, Bethe ansatz). An instance of this is the fusion procedure, which consists in constructing solutions of the Yang–Baxter or star-triangle relations from known ones by taking subquotients of tensor products.

An interesting new feature in the groupoid-graded case is that the Grothendieck ring of the category of π -graded vector spaces is non-commutative in general, even in the case of action groupoids of abelian groups. Thus characters of representations of dynamical quantum groups live in a non-commutative ring. However if a collection of representations (V_i) admit R -matrices, which are isomorphisms $V_i \otimes V_j \cong V_j \otimes V_i$, then their characters generate a commutative subring of the Grothendieck ring of π -graded vector spaces. In the case of transformation groupoids these rings are realized as rings of commuting difference operators.

1.5 Outline of the paper

We introduce the category $\text{Vect}_k(\pi)$ of π -graded vector spaces of finite type over a field k in Section 2. It is a variant of a special case of the category of π -graded modules considered in [38]. It is an abelian monoidal category with duality. We discuss the notion of character of a π -graded vector space taking values in the convolution ring of π . In Section 3 we adapt the machinery of Yang–Baxter equations and transfer matrices to the case of π -graded vector spaces and explain the relation with the star-triangle relation. We introduce the notion of partial traces in this context and prove that solutions of the Yang–Baxter equation give rise to commuting transfer matrices. In the case of transformation groupoids and their subgroupoids, we show that the Yang–Baxter equation can be written as a dynamical Yang–Baxter equation, and that transfer matrices produce commuting difference operators. In Section 4 we consider in more detail the example of the elliptic quantum group of type A_{n-1} , which admits a dynamical R -matrix with restricted dynamical variables and thus a monoidal category with a forgetful functor to π -graded vector spaces for a finite groupoid π . We compute a few characters, in particular the characters of (analogues of the) exterior powers of the vector representation, obtained by the fusion procedure. Finally in Section 5 we consider the case of dynamical R -matrices arising from quantum groups at root of unity, which may be viewed as a toy model for restricted models, with R -matrices that are independent of the spectral parameters. The construction uses a semisimple rigid braided category $C_q(\mathfrak{g})$ of representations of quantum groups for each simple Lie algebra \mathfrak{g} and root of unity q . Technically it is a semisimple quotient of the category of tilting modules of the Lusztig quantum groups. It has finitely many isomorphism classes of simple objects. We construct a faithful monoidal functor from $C_q(\mathfrak{g})$ to the categories of π -graded vector spaces of finite type for a suitable finite groupoid π . The braiding in $C_q(\mathfrak{g})$ is then mapped to a system of dynamical R matrices with dynamical variable restricted to a finite set. This construction is a formalization of the “passage to the shadow world” of [34, 44] and is a version with discrete dynamical variable of [17]. The characters of simple modules define a representation of the Verlinde algebra by difference operators.

2 Grading by groupoids

2.1 Groupoids

A groupoid π on a set A is a small category with set of objects A whose morphisms, called arrows, are invertible. The set of morphisms from an object a to an object b is denoted by $\pi(a, b)$. The composition of arrows $\gamma \in \pi(a, b)$ and $\eta \in \pi(b, c)$ is denoted by $\eta \circ \gamma \in \pi(a, c)$ or by $\eta\gamma$ in case of typographical constraints. The inverse of $\gamma \in \pi(a, b)$ is $\gamma^{-1} \in \pi(b, a)$. We identify A with the subset of identity arrows and denote a groupoid by its set of arrows π when no confusion arises. The maps $s, t: \pi \rightarrow A$ sending $\gamma \in \pi(a, b)$ to a and b , respectively, are called source and target map, respectively.

A subgroupoid of a groupoid π is a subset of (the set of arrows of) π that is closed under composition and inversion. It is a groupoid on the set of its identity arrows. The *full subgroupoid* of π on a subset $B \subset A$ is the subgroupoid $s^{-1}B \cap t^{-1}B$ of arrows between objects of B .

2.2 The convolution ring of a groupoid

To a groupoid π we associate the *convolution ring* of π , which is a unital associative ring $\mathbb{Z}(\pi)$ with an involutive anti-automorphism.

As an abelian group $\mathbb{Z}(\pi)$ consists of the maps $n: \pi \rightarrow \mathbb{Z}$ such that for all $a \in A$, the set of arrows $\alpha \in s^{-1}(a) \cup t^{-1}(a)$ with $n(\alpha) \neq 0$ is finite. The product is the convolution product

$$n * m: \gamma \mapsto \sum_{\beta \circ \alpha = \gamma} n(\alpha)m(\beta).$$

The sum has finitely many non-zero terms because of the finiteness assumption. The unit is the characteristic function on identity arrows and the involutive anti-automorphism σ sends n to $\sigma(n): \gamma \rightarrow n(\gamma^{-1})$. The assignment $\pi \mapsto \mathbb{Z}(\pi)$ is a contravariant functor from the category of groupoids to the category of unital involutive associative rings.

Remark 2.1. For any commutative ring R we have an R -algebra $R(\pi) = R \otimes_{\mathbb{Z}} \mathbb{Z}(\pi)$ obtained by extension of scalars. For transfer matrices we will need a more general construction where R is also π -graded, see Section 2.10 below.

2.3 Convolution rings of subgroupoids

It will be convenient to view the convolution ring of a subgroupoid $\pi' \subset \pi$ as a subring of $\mathbb{Z}(\pi)$.

Lemma 2.2. *The characteristic functions $\chi_{A'}$ of subsets $A' \subset A$ of the set of identity arrows are idempotents in $\mathbb{Z}(\pi)$.*

Proof. By definition $\chi_{A'}(\gamma) = 0$ unless γ is an identity arrow $a \in \pi(a, a)$ for $a \in A'$. In this case $\chi_{A'}(a) = 1$. Thus $\chi_{A'} * \chi_{A'}(\gamma)$ vanishes unless γ is an identity arrow $a \in A'$, in which case $\chi_{A'} * \chi_{A'}(a) = \chi_{A'}(a)\chi_{A'}(a) = 1$. ■

Lemma 2.3. *Let π be a groupoid on A and π' the full subgroupoid on $A' \subset A$. Then the induced morphism $\mathbb{Z}(\pi) \rightarrow \mathbb{Z}(\pi')$ restricts to a unital ring isomorphism*

$$\chi_{A'} * \mathbb{Z}(\pi) * \chi_{A'} \rightarrow \mathbb{Z}(\pi'),$$

where $\chi_{A'}$ is the unit element of the subring $\chi_{A'} * \mathbb{Z}(\pi) * \chi_{A'}$. Moreover the left-hand side is the subring of functions vanishing on the complement of π' .

Proof. The map $\mathbb{Z}(\pi) \rightarrow \mathbb{Z}(\pi')$ is the restriction map $r: n \mapsto n|_{\pi'}$. The extension by zero $\mathbb{Z}(\pi') \rightarrow \mathbb{Z}(\pi)$ is a right inverse. Its image consists of the functions vanishing outside π' . Thus r restricts to an isomorphism from the functions vanishing outside π' and $\mathbb{Z}(\pi)$. Now a function $n \in \mathbb{Z}(\pi)$ vanishes outside the full subgroupoid π' if and only if it vanishes everywhere except on arrows between elements of A' . But this is equivalent to $n = \chi_{A'} * n * \chi_{A'}$. Since $\chi_{A'}$ is an idempotent, $\chi_{A'} * \mathbb{Z}(\pi) * \chi_{A'}$ is a subring with unit element $\chi_{A'}$, which is sent to $1 \in \mathbb{Z}(\pi')$. ■

2.4 Action groupoids

The main examples of groupoids for our purpose are action groupoids and their subgroupoids. Let G be a group with identity element e and A be a set with a right action $A \times G \rightarrow A$. The action groupoid $A \rtimes G$ has set of objects A and an arrow $a \rightarrow a'$ for each $g \in G$ such that $a' = ag$. Thus an arrow is described by a pair $(a, g) \in A \times G$. The source and target are $s(a, g) = a$, $t(a, g) = ag$ and the composition is

$$(a', g') \circ (a, g) = (a, gg'), \quad \text{whenever } a' = ag.$$

The identity arrows are (a, e) , $a \in A$ and the inverse of (a, g) is (ag, g^{-1}) .

The convolution ring $\mathbb{Z}(A \rtimes G)$ contains the subring \mathbb{Z}^A of functions with support on the identity arrows as in the general case and a subring $\mathbb{Z}G$ isomorphic to the group ring of G via the injective ring homomorphism $t: \mathbb{Z}G \rightarrow \mathbb{Z}(A \rtimes G)$ sending $g \in G$ to

$$t_g: (a, h) \mapsto \delta_{g,h}.$$

The right action of G defines a group homomorphism $r: G \rightarrow \text{Aut}(\mathbb{Z}^A)$: for $g \in G$ and $f \in \mathbb{Z}^A$, $r_g f(a) = f(ag)$.

Proposition 2.4. *The convolution ring $\mathbb{Z}(A \rtimes G)$ is the crossed product $\mathbb{Z}^A \rtimes_r \mathbb{Z}G$ of its subrings \mathbb{Z}^A and $\mathbb{Z}G$. The involution acts trivially on \mathbb{Z}^A and as $t_g \mapsto t_{g^{-1}}$ on $\mathbb{Z}G$.*

This means that $\mathbb{Z}(A \rtimes G)$ is isomorphic to the algebra generated by \mathbb{Z}^A and elements t_g for $g \in G$ with relations

$$t_g t_h = t_{gh}, \quad t_g f = r_g(f) t_g, \quad g, h \in G, \quad f \in \mathbb{Z}^A.$$

Explicitly, a function $n \in \mathbb{Z}(A \rtimes G)$ corresponds to the element $\sum_{g \in G} n_g t_g$ where $n_g(a) = n(a, g)$.

Remark 2.5. In particular the convolution ring acts on the space of functions on A by difference operators (i.e., operators acting on functions by translations of the argument and multiplication by functions). This is the scalar case of the more general case of a convolution algebra acting on vector-valued functions on A by difference operators, which we construct in Section 3.7.

2.5 The category of π -graded vector spaces of finite type

Definition 2.6. Let π be a groupoid with set of objects A . A π -graded vector space of finite type over a field k is a collection $(V_\alpha)_{\alpha \in \pi}$ of finite-dimensional vector spaces indexed by the arrows of π such that for each $a \in A$ there are finitely many arrows α with source or target a and nonzero V_α .

The π -graded vector spaces over k form an abelian category $\text{Vect}_k(\pi)$: the k -vector space $\text{Hom}(V, W)$ of morphisms between objects V, W consists of families $(f_\alpha)_{\alpha \in \pi}$ of linear maps $f_\alpha: V_\alpha \rightarrow W_\alpha$ and the composition is defined componentwise.

2.6 Tensor product

The finite type condition allows us to define a monoidal structure (tensor product) on $\text{Vect}_k(\pi)$. The tensor product of objects is

$$(V \otimes W)_\gamma = \bigoplus_{\beta \circ \alpha = \gamma} V_\alpha \otimes W_\beta. \quad (2.1)$$

The direct sum is over all pairs of arrows whose composition is γ and has finitely many nonzero summands. Similarly the tensor product $f \otimes g$ of morphisms has components $\bigoplus_{\beta \circ \alpha = \gamma} f_\alpha \otimes g_\beta$. For any three objects U, V, W of $\text{Vect}_k(\pi)$ and $\delta \in \pi$,

$$\begin{aligned} ((U \otimes V) \otimes W)_\delta &= \bigoplus_{\alpha \circ \beta \circ \gamma = \delta} (U_\gamma \otimes V_\beta) \otimes W_\alpha, \\ (U \otimes (V \otimes W))_\delta &= \bigoplus_{\alpha \circ \beta \circ \gamma = \delta} U_\gamma \otimes (V_\beta \otimes W_\alpha). \end{aligned}$$

Therefore the associativity constraint in Vect_k defines an associativity constraint

$$\alpha_{UVW}: (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$$

in $\text{Vect}_k(\pi)$. The tensor unit in $\text{Vect}_k(\pi)$ is $\mathbf{1} = (\mathbf{1}_\gamma)_{\gamma \in \pi}$ with $\mathbf{1}_a = k$ for identity arrows $a \in A$ and $\mathbf{1}_\gamma = 0$ for all other arrows. Then for every object V of $\text{Vect}_k(\pi)$, $(\mathbf{1} \otimes V)_\gamma = k \otimes V_\gamma$ and $(V \otimes \mathbf{1})_\gamma = V_\gamma \otimes k$. Thus the structure isomorphisms $V_\gamma \cong V_\gamma \otimes k \cong k \otimes V_\gamma$ of Vect_k define natural isomorphisms $\lambda: V \cong V \otimes \mathbf{1}$, $\rho: V \cong \mathbf{1} \otimes V$ in $\text{Vect}_k(\pi)$.

2.7 Duality

The dual of a π -graded vector space $V \in \text{Vect}_k(\pi)$ is the π -graded vector space V^* with components $(V^*)_\gamma = \text{Hom}_k(V_{\gamma^{-1}}, k)$. Recall that an object V of a monoidal category admits a left dual of an object V if there is an object V^\vee , called left dual of V , together with morphisms $\delta: \mathbf{1} \rightarrow V \otimes V^\vee$, $\text{ev}: V^\vee \otimes V \rightarrow \mathbf{1}$ such that the compositions

$$\begin{aligned} V \otimes \mathbf{1} &\xrightarrow{\delta} (V \otimes V^\vee) \otimes V \cong V \otimes (V^\vee \otimes V) \rightarrow V \otimes \mathbf{1} \cong V, \\ V^\vee \otimes \mathbf{1} &\xrightarrow{\text{ev}} V^\vee \otimes (V \otimes V^\vee) \cong (V^\vee \otimes V) \otimes V^\vee \rightarrow \mathbf{1} \otimes V^\vee \end{aligned}$$

are equal to the identity morphism. Similarly one has the notion of right dual object ${}^\vee V$ with morphisms $\mathbf{1} \rightarrow {}^\vee V \otimes V$, $V \otimes {}^\vee V \rightarrow \mathbf{1}$. Right and left duals of finite dimensional vector spaces coincide.

Lemma 2.7. *Let $V \in \text{Vect}_k(\pi)$. Then V^\vee with components $(V^\vee)_\gamma = (V_{\gamma^{-1}})^* = \text{Hom}_k(V_{\gamma^{-1}}, k)$, and structure morphisms induced by those of the category of finite dimensional vector spaces is both left and right dual to V .*

For example the morphism $\delta: \mathbf{1} \rightarrow V \otimes V^\vee$ is the collection of maps $\mathbf{1}_a = k \rightarrow \bigoplus_{\gamma \in s^{-1}(a)} V_\gamma \otimes (V^\vee)_{\gamma^{-1}}$, defined by the canonical map $k \rightarrow V_\gamma \otimes (V_\gamma)^*$.

Monoidal categories admitting left and right duals for all objects, which are then uniquely determined up to unique isomorphism, are called rigid. Left and right dualities are monoidal functors to the opposite categories with opposite tensor product. Rigid monoidal categories with coinciding left and right dual functors are called pivotal, see [45, Sections 1.6 and 1.7] for more details.

Theorem 2.8. *The k -additive category $\text{Vect}_k(\pi)$ with the tensor product \otimes , the tensor unit $\mathbf{1}$, associativity constraint α , left and right multiplication by the tensor unit λ, ρ , and duality $(\)^\vee$ is an abelian pivotal monoidal category.*

This is an immediate consequence of the fact that Vect_k is a k -additive abelian monoidal category and Lemma 2.7.

Remark 2.9. Contrary to the case of finite dimensional vector spaces, the monoidal category $\text{Vect}_k(\pi)$ is not symmetric or braided, so that $V \otimes W$ is not isomorphic to $W \otimes V$ in general. As we will see presently, the Grothendieck ring is not commutative in general.

Remark 2.10. The above construction works for any k -additive rigid monoidal category C over a commutative ring k instead of Vect_k . The resulting category of π -graded objects of C of finite type is a k -additive monoidal category. For example, if we view a ring as a monoidal category with one object, the convolution ring $\mathbb{Z}(\pi)$ is the category of π -graded objects of finite type of \mathbb{Z} . One can also replace π by a general small category, at the cost of giving up duality.

2.8 Characters

The *character* $\text{ch}_V \in \mathbb{Z}(\pi)$ of $V \in \text{Vect}_k(\pi)$ is the map $\gamma \mapsto \dim(V_\gamma)$.

Lemma 2.11. *Let $V, W \in \text{Vect}_k(\pi)$. Then*

$$\begin{aligned} \text{ch}_1 &= 1, & \text{ch}_{V^\vee} &= \sigma(\text{ch}_V), \\ \text{ch}_{V \oplus W} &= \text{ch}_V + \text{ch}_W, \\ \text{ch}_{V \otimes W} &= \text{ch}_V * \text{ch}_W \end{aligned}$$

Since exact sequences of vector spaces split, the character map $\text{ch}: V \mapsto \text{ch}_V$ descends to a ring homomorphism from the Grothendieck ring $K(\text{Vect}_k(\pi))$ to the convolution ring $\mathbb{Z}(\pi)$.

Proposition 2.12. *The map $K(\text{Vect}_k(\pi)) \rightarrow \mathbb{Z}(\pi)$ is an isomorphism of involutive unital rings.*

The inverse map sends $n = n_+ - n_-$ with $n_\pm(\gamma) \geq 0$ for all γ to the formal difference $[(k^{n_+(\gamma)})_{\gamma \in \pi}] - [(k^{n_-(\gamma)})_{\gamma \in \pi}]$.

2.9 Subgroupoids

Let $i: \pi' \hookrightarrow \pi$ be a subgroupoid. Then we have an exact fully faithful functor $i_*: \text{Vect}_k(\pi') \rightarrow \text{Vect}_k(\pi)$ so that

$$(i_*V)_\gamma = \begin{cases} V_\gamma, & \text{if } \gamma \in \pi', \\ 0, & \text{otherwise.} \end{cases}$$

We can thus view $\text{Vect}_k(\pi')$ as a full subcategory of $\text{Vect}_k(\pi)$ of π -graded vector spaces V such that $V_\gamma = 0$ for $\gamma \notin \pi'$.

2.10 Convolution algebras with coefficients in π -graded algebras

In the setting of π -graded vector spaces the natural home for transfer matrices is convolution algebras with coefficients in π -graded algebras over a field (or commutative ring) k .

Definition 2.13. Let π be a groupoid. A π -graded algebra R over k is a collection $(R_\gamma)_{\gamma \in \pi}$ of k -vector spaces labeled by arrows of π with bilinear products $R_\alpha \times R_\beta \rightarrow R_{\beta \circ \alpha}$, $(x, y) \mapsto xy$, defined for composable arrows α, β and units $1_a \in R_a$, for $a \in A$ such that (i) $(xy)z = x(yz)$ whenever defined and (ii) $x1_b = x = 1_ax$ for all $x \in R_\alpha$ of degree $\alpha \in \pi(a, b)$.

Remark 2.14. An algebra object in Vect_π defines a π -graded algebra, but we will need to consider more general examples which do not necessarily fulfill the finite type condition, such as [End 1](#) below.

Example 2.15. Let $V \in \text{Vect}_k(\pi)$ and let $\underline{\text{End}} V$ be the π -graded vector space with $(\underline{\text{End}} V)_\alpha = \bigoplus_{\gamma \in \pi(a,a)} \text{Hom}_k(V_{\alpha \circ \gamma \circ \alpha^{-1}}, V_\gamma)$, where $a = s(\alpha)$. Then $\underline{\text{End}} V$ with the product given by the composition of linear maps

$$\text{Hom}_k(V_{\alpha\gamma\alpha^{-1}}, V_\gamma) \otimes \text{Hom}_k(V_{\beta\alpha\gamma\alpha^{-1}\beta^{-1}}, V_{\alpha\gamma\alpha^{-1}}) \rightarrow \text{Hom}_k(V_{\beta\alpha\gamma(\beta\alpha)^{-1}}, V_\gamma)$$

and unit $1_a = \bigoplus_{\gamma \in \pi(a,a)} \text{Id}_{V_\gamma}$ is a π -graded algebra.

Definition 2.16. Let R be a π -graded algebra. The *convolution algebra* $\Gamma(\pi, R)$ with coefficients in R is the k -algebra of maps $f: \pi \rightarrow \sqcup_{\alpha \in \pi} R_\alpha$ such that

- (i) $f(\alpha) \in R_\alpha$ for all arrows $\alpha \in \pi$,
- (ii) for every $a \in A$, there are finitely many $\alpha \in s^{-1}(a) \cup t^{-1}(a)$ such that $f(\alpha) \neq 0$.

The product is the convolution product

$$f * g(\gamma) = \sum_{\beta \circ \alpha = \gamma} f(\alpha)g(\beta).$$

Example 2.17. Let $R = \underline{\text{End}} \mathbf{1}$ be the π -graded algebra of Example 2.15 for the tensor unit $\mathbf{1}$. Then $R_\alpha = k$ for all arrows α and $\Gamma(\pi, R) = k(\pi) = k \otimes \mathbb{Z}(\pi)$ is the extension of scalars of the convolution ring of π , see Remark 2.1.

Lemma 2.18. Let π be a groupoid with object set A . The convolution algebra $\Gamma(\pi, R)$ with coefficients in a π -graded algebra R is an associative unital k -algebra. The unit is the map $a \mapsto 1_a$ for identity arrows $a \in A$ and $\alpha \mapsto 0$ for other arrows.

Let $\varphi: R \rightarrow R'$ be a morphism of π -graded algebras. Then $\varphi_*: \Gamma(\pi, R) \rightarrow \Gamma(\pi, R')$ given $\varphi_*(f) = \varphi \circ f$ is an algebra homomorphism. This defines a functor from the category of π -graded k -algebras to associative unital algebras. In particular we have a morphism of algebras $k(\pi) \rightarrow \Gamma(\pi, R)$ for any R .

3 Yang–Baxter equation and RLL relations

3.1 Yang–Baxter equation

Let $k = \mathbb{C}$. A *Yang–Baxter operator* on $V \in \text{Vect}_k(\pi)$ is a meromorphic function $z \mapsto \check{R}(z) \in \text{End}(V \otimes V)$ of the *spectral parameter* $z \in \mathbb{C}$ with values in the endomorphisms of $V \otimes V$, obeying the Yang–Baxter equation²

$$\check{R}(z-w)^{(23)} \check{R}(z)^{(12)} \check{R}(w)^{(23)} = \check{R}(w)^{(12)} \check{R}(z)^{(23)} \check{R}(z-w)^{(12)} \quad (3.1)$$

in $\text{End}(V \otimes V \otimes V)$ for all generic values of the spectral parameters z, w , and the inversion (also called unitarity) relation

$$\check{R}(z) \check{R}(-z) = \text{id}_{V \otimes V},$$

for generic z . The restriction of $\check{R}(z)$ to $V_\alpha \otimes V_\beta$ for composable arrows α, β has components in each direct summand of the decomposition (2.1):

$$\check{R}(z)|_{V_\alpha \otimes V_\beta} = \bigoplus_{\gamma, \delta} \mathcal{W}(z; \alpha, \beta, \gamma, \delta).$$

²The Yang–Baxter equation is usually formulated for the operator $R(z) = p \circ \check{R}(z)$ obtained by composition with the flip $p: u \otimes v \mapsto v \otimes u$. Since p is not a morphism of π -graded vector spaces in general, it is better to use $\check{R}(z)$.

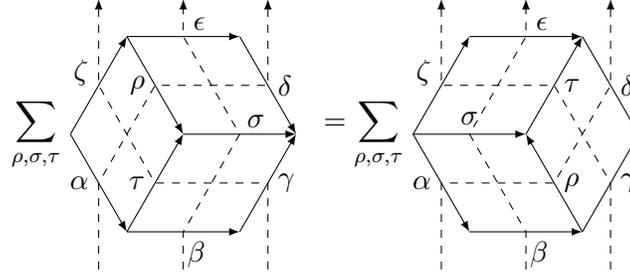


Figure 1. The Yang–Baxter equation. The arrows α, \dots, ζ form a commutative hexagon and the sum is over arrows ρ, σ, τ making the squares commutative.

The sum is over γ, δ such that $\beta \circ \alpha = \delta \circ \gamma$, and \mathcal{W} is the component

$$\mathcal{W}(z; \alpha, \beta, \gamma, \delta) \in \text{Hom}_k(V_\alpha \otimes V_\beta, V_\gamma \otimes V_\delta).$$

The Yang–Baxter equation translates to its IRF (interaction-round-a-face) version, called the star-triangle relation

$$\begin{aligned} & \sum_{\rho, \sigma, \tau} \mathcal{W}(z - w; \rho, \sigma, \epsilon, \delta)^{(23)} \mathcal{W}(z; \alpha, \tau, \zeta, \rho)^{(12)} \mathcal{W}(w; \beta, \gamma, \tau, \sigma)^{(23)} \\ &= \sum_{\rho, \sigma, \tau} \mathcal{W}(w; \sigma, \tau, \zeta, \epsilon)^{(12)} \mathcal{W}(z; \rho, \gamma, \tau, \delta)^{(23)} \mathcal{W}(z - w; \alpha, \beta, \sigma, \rho)^{(12)}. \end{aligned}$$

There is one such equation for all α, \dots, ζ so that $\gamma \circ \beta \circ \alpha = \delta \circ \epsilon \circ \zeta$ and the sum is over arrows ρ, σ, τ for which all factors are defined, namely such that the diagrams are commutative in π .

It is convenient to have a graphical representation for these morphisms:

$$\mathcal{W}(z; \alpha, \beta, \gamma, \delta) = \begin{array}{ccc} & \uparrow & \\ & \gamma & \\ a & \square & d \\ & \downarrow & \\ & \beta & \\ b & \square & c \\ & \downarrow & \\ & & \end{array}$$

The dashed lines are associated with the vector spaces between which \mathcal{W} acts: moving from the southwest to the northeast according to the orientation of the dashed lines we move from $V_\alpha \otimes V_\beta$ to $V_\gamma \otimes V_\delta$. We have also displayed in the corners the objects between which the arrows α, \dots, δ are defined. For example α is an arrow from a to b . In the literature one often considers the case where there is at most one arrow from one object to any other object, as is the case in the original Andrews–Baxter–Forrester RSOS models, and it is then customary to label \mathcal{W} by the four objects a, b, c, d instead of the morphisms.

3.2 RLL relations

The machinery of the quantum inverse scattering method [18] can be applied: given a solution $\tilde{R}(z) \in \text{End}(V \otimes V)$ of the dynamical Yang–Baxter equation for a π -graded vector space V , an L -operator on $W \in \text{Vect}_k$ is a meromorphic function $z \mapsto L(z) \in \text{Hom}(V \otimes W, W \otimes V)$ such that

- (i) $L(z)$ is invertible for generic z ,

(ii) L obeys the RLL relations

$$\check{R}(z-w)^{(23)}L(z)^{(12)}L(w)^{(23)} = L(w)^{(12)}L(z)^{(23)}\check{R}(z-w)^{(12)},$$

in $\text{Hom}(V \otimes V \otimes W, W \otimes V \otimes V)$.

For example $\check{R}(z)$ is an L -operator on V thanks to the Yang–Baxter equation.

Given a basis of V the RLL relations may be written as relations for the matrix entries $L_{ij}(z) \in \text{End}(W)$. Thus L -operators may be understood as π -graded meromorphic representations of the quadratic algebra A_R with generators $L_{ij}(z)$, and RLL relations. Here meromorphic refers to the required meromorphic dependence on $z \in \mathbb{C}$.

3.3 The monoidal category $M(R, \pi)$

The L -operators form an abelian monoidal category $M(R, \pi)$ of π -graded meromorphic representations of A_R : an object (W, L_W) is a π -graded vector space $W \in \text{Vect}_k(\pi)$ endowed with an L -operator L_W on W . A morphism from (W, L_W) to (Z, L_Z) is a morphism $f: W \rightarrow Z$ of π -graded vector spaces such that

$$(f \otimes \text{id}_V)L_W(z) = L_Z(z)(\text{id}_V \otimes f),$$

for all z .

The tensor product $(W \otimes Z, L_{W \otimes Z})$ is the tensor product in $\text{Vect}_k(\pi)$ endowed with the composition

$$L_{W \otimes Z}(z): V \otimes W \otimes Z \xrightarrow{L_W(z) \otimes \text{id}_Z} W \otimes V \otimes Z \xrightarrow{\text{id}_W \otimes L_Z(z)} W \otimes Z \otimes V.$$

The fact that $L_{W \otimes Z}$ is an L -operator is a straightforward consequence of the definitions. We have an action of \mathbb{C} on the category $M(R, \pi)$: for each $u \in \mathbb{C}$ let t_u be the endofunctor sending an object (W, L_W) to $(W, L_W(\cdot + u))$ and a morphism f to f . Clearly t_0 is the identity endofunctor and $t_u t_v = t_{u+v}$. Moreover t_u is a monoidal functor: the obvious map $t_u(W) \otimes t_u(Z) \rightarrow t_u(W \otimes Z)$ is a natural isomorphism.

Example 3.1. Let $V \in M(R, \pi)$ be the representation with L -operator \check{R} . Then for each $u \in \mathbb{C}$, the representation $V(u) = t_u V$ has L operator $L_{V(u)}(z) = \check{R}(z + u)$. This object of $M(R, \pi)$ is called the vector representation with evaluation point u .

Example 3.2. Let $\mathbf{1} \in \text{Vect}_k(\pi)$ be the tensor unit, see Section 2.6 and let $L_{\mathbf{1}}(z)$ be the composition $\rho\lambda^{-1}: V \otimes \mathbf{1} \rightarrow V \rightarrow \mathbf{1} \otimes V$ of the structure isomorphisms. Then $\mathbf{1}$ with this L -operator is a representation, called the trivial representation. It is fixed by the action of t_u .

Example 3.3. The dual representation of a representation (W, L_W) admitting a dual is the representation (W^\vee, L_{W^\vee}) on the π -graded dual vector space W^\vee (see Section 2.7). Its L -operator $L_{W^\vee}(z) = \check{L}_W(z)^{-1}$ is the inverse of the dual operator $\check{L}_W(z): W^\vee \otimes V \rightarrow V \otimes W^\vee$ defined as the composition

$$W^\vee \otimes V \rightarrow W^\vee \otimes V \otimes W \otimes W^\vee \xrightarrow{L_W(z)^{(23)}} W^\vee \otimes W \otimes V \otimes W^\vee \rightarrow V \otimes W^\vee$$

with the structure maps defining the duality in the category of π -graded vector spaces, see Section 2.7. It exists whenever $\check{L}_W(z)$ is invertible for generic z .

3.4 R -matrices

Let $W, Z \in M(R, \pi)$. An isomorphism

$$\check{R}_{W,Z}: W \otimes Z \rightarrow Z \otimes W$$

in $M(R, \pi)$ is called an R -matrix. This means that $\check{R}_{W,Z}$ is a morphism of π -graded vector spaces obeying the intertwining relation

$$\check{R}_{W,Z}^{(12)} L_Z(z)^{(23)} L_W(z)^{(12)} = L_W(z)^{(23)} L_Z(z)^{(12)} \check{R}_{W,Z}^{(23)}$$

in $\text{Hom}(V \otimes W \otimes Z, Z \otimes W \otimes V)$.

Example 3.4. Let $V(u)$ be the vector representation with evaluation point u . Then the Yang–Baxter equation implies that $\check{R}(u-v)$ is a morphism $V(u) \otimes V(v) \rightarrow V(v) \otimes V(u)$. It is an R -matrix except if $u-v$ or $v-u$ is a pole of \check{R} .

Proposition 3.5.

- (i) If $\check{R}_{W,Z}$ is an R -matrix for $W, Z \in M(R, \pi)$ and $u \in \mathbb{C}$, then the same isomorphism $\check{R}_{W,Z}$ of π -graded vector spaces is an R -matrix for $t_u V, t_u W$.
- (ii) If $\check{R}_{W,Z}, \check{R}_{W,Z'}$ are R -matrices then $\check{R}_{W,Z \otimes Z'} = \check{R}_{W,Z'}^{(23)} \check{R}_{W,Z}^{(12)}$ is an R -matrix for $W, Z \otimes Z'$.
- (iii) If $\check{R}_{W,Z}, \check{R}_{W',Z}$ are R -matrices then $\check{R}_{W \otimes W', Z} = \check{R}_{W,Z}^{(12)} \check{R}_{W',Z}^{(23)}$ is an R -matrix for $W \otimes W', Z$.

3.5 Partial traces and transfer matrices

The partial trace over V is the map

$$\text{tr}_V: \text{Hom}_{\text{Vect}_k(\pi)}(V \otimes W, W \otimes V) \rightarrow \Gamma(\pi, \underline{\text{End}} W)$$

defined as follows.

For $f \in \text{Hom}(V \otimes W, W \otimes V)$ and $\alpha \in \pi(a, b), \gamma \in \pi(a, a)$, let $f(\alpha, \gamma)$ be the component of f mapping

$$f(\alpha, \gamma): V_\alpha \otimes W_{\alpha\gamma\alpha^{-1}} \rightarrow W_\gamma \otimes V_\alpha.$$

Define

$$\text{tr}_{V_\alpha} f(\alpha, \gamma) = \sum_i (\text{id} \otimes e_i^*) f(\alpha, \gamma) (e_i \otimes \text{id}) \in \text{Hom}(W_{\alpha\gamma\alpha^{-1}}, W_\gamma).$$

for any basis e_i of V_α and dual basis e_i^* of the dual vector space $(V_\alpha)^*$.

Definition 3.6. The *partial trace* $\text{tr}_V f \in \Gamma(\pi, \underline{\text{End}} W)$ of $f \in \text{Hom}_{\text{Vect}_\pi}(V \otimes W, W \otimes V)$ over V is the section

$$\text{tr}_V f: \alpha \mapsto \bigoplus_{\gamma \in \pi(a, a)} \text{tr}_{V_\alpha} f(\alpha, \gamma) \in (\underline{\text{End}} W)_\alpha.$$

Example 3.7. Let $W = \mathbf{1}$ be the tensor unit, with nonzero components $W_a = k$, indexed by identity arrows $a \in A$. For $\alpha \in \pi(a, b)$, we have $(\underline{\text{End}} W)_\alpha = \text{Hom}_k(W_a, W_b) = k$, see Example 2.17.

The convolution algebra $\Gamma(\pi, \underline{\text{End}} \mathbf{1})$ is the extension of scalars of the convolution ring of π . The partial trace of the identity $V \otimes \mathbf{1} \cong V \rightarrow V \cong \mathbf{1} \otimes V$ is

$$\text{tr}_V(\text{id}): \alpha \mapsto \dim V_\alpha,$$

which is the (image in k of the) character ch_V of V .

Lemma 3.8.

(i) If $\varphi: V \rightarrow V'$ is an isomorphism of π -graded vector spaces then

$$\mathrm{tr}_{V'}((\mathrm{id} \otimes \varphi)f(\varphi^{-1} \otimes \mathrm{id})) = \mathrm{tr}_V f.$$

(ii) Let $f_i \in \mathrm{Hom}(V_i \otimes W, W \otimes V_i)$, $i = 1, 2$, be homomorphisms of π -graded vector spaces and let $f_1^{(12)} f_2^{(23)}$ be the composition

$$V_1 \otimes V_2 \otimes W \xrightarrow{\mathrm{id} \otimes f_2} V_1 \otimes W \otimes V_2 \xrightarrow{f_1 \otimes \mathrm{id}} W \otimes V_1 \otimes V_2.$$

Then

$$\mathrm{tr}_{V_1 \otimes V_2} f_1^{(12)} f_2^{(23)} = \mathrm{tr}_{V_1} f_1 \mathrm{tr}_{V_2} f_2.$$

Proof. Recall that a morphism φ is a collection of linear maps $\varphi_\alpha: V_\alpha \rightarrow V'_\alpha$, so (i) is the standard property of the trace on each V_α .

As for (ii) to compute $\mathrm{tr}_{V_1 \otimes V_2}$ we need to select for each $\gamma \in \pi(a, b)$ and $\mu \in \pi(b, b)$ the component $g(\alpha)$ of $g = f_1^{(12)} f_2^{(23)}$ sending $(V_1 \otimes V_2)_\gamma \otimes W_\mu$ to $W_{\mu'} \otimes (V_1 \otimes V_2)_\gamma$ with $\mu' = \gamma^{-1} \circ \mu \circ \gamma$. A basis of $(V_1 \otimes V_2)_\gamma$ is given by choosing a basis of each component $V_{1\alpha} \otimes V_{2\beta}$ with $\gamma = \beta\alpha$ with $\alpha \in \pi(a, c)$ and $\beta \in \pi(c, b)$. Thus the trace is non-trivial on the components of f mapping $V_{1\alpha} \otimes V_{2\beta} \otimes W_\mu$ to $W_{\mu'} \otimes V_{1\alpha} \otimes V_{2\beta}$. These components factor as

$$V_{1\alpha} \otimes V_{2\beta} \otimes W_\mu \xrightarrow{\mathrm{id} \otimes f_2(\beta)} V_{1\alpha} \otimes W_{\beta^{-1} \circ \mu \circ \beta} \otimes V_{2\beta} \xrightarrow{f_1(\alpha) \otimes \mathrm{id}} W_{\mu'} \otimes V_{1\alpha} \otimes V_{2\beta}.$$

The claim follows by taking the tensor product of bases of $V_{1\alpha}$ and $V_{2\beta}$. ■

Corollary 3.9. Suppose that $\check{R}_{V_i V_j} \in \mathrm{Hom}(V_i \otimes V_j, V_j \otimes V_i)$, ($1 \leq i < j \leq 3$) are a solution of the Yang–Baxter equation

$$\check{R}_{V_1 V_2}^{(23)} \check{R}_{V_1 V_3}^{(12)} \check{R}_{V_2 V_3}^{(23)} = \check{R}_{V_2 V_3}^{(12)} \check{R}_{V_1 V_3}^{(23)} \check{R}_{V_1 V_2}^{(12)}$$

with invertible $\check{R}_{V_1 V_2}$. Then the transfer matrices

$$T_i = \mathrm{tr}_{V_i} \check{R}_{V_i V_3} \in \Gamma(\pi, \underline{\mathrm{End}}(V_3)), \quad i = 1, 2.$$

commute: $T_1 T_2 = T_2 T_1$.

Proof. We can write the Yang–Baxter equation as

$$(\mathrm{id} \otimes \varphi) \check{R}_{V_2 V_3}^{(12)} \check{R}_{V_1 V_3}^{(23)} (\varphi^{-1} \otimes \mathrm{id}) = \check{R}_{V_1 V_3}^{(12)} \check{R}_{V_2 V_3}^{(23)}$$

with $\varphi = \check{R}_{V_1 V_2}^{-1}$. By Lemma 3.8 (i) we deduce that

$$\mathrm{tr}_{V_2 \otimes V_1} \check{R}_{V_2 V_3}^{(12)} \check{R}_{V_1 V_3}^{(23)} = \mathrm{tr}_{V_1 \otimes V_2} \check{R}_{V_1 V_3}^{(12)} \check{R}_{V_2 V_3}^{(23)}.$$

The claim then follows from Lemma 3.8(ii). ■

Remark 3.10. For $W = \mathbf{1}$ the R -matrix $R_{V, \mathbf{1}}$ is the tautological map $V \otimes \mathbf{1} \rightarrow \mathbf{1} \otimes V$ of Example 3.7. Thus the transfer matrix generalizes the notion of character.

Remark 3.11. In particular Corollary 3.9 applies to the case of the category $M(R, \pi)$: for each object (W, L_W) we can interpret the RLL relations of Section 3.2 as a Yang–Baxter equation with $V_1 = V(z)$, $V_2 = V(w)$ (see Example 3.1), $V_3 = W$ and $\check{R}_{VW} = L_W(z)$. One gets the basic statement of the quantum inverse scattering method that the transfer matrices $\mathrm{tr}_V L_W(z)$ commute for different values of the spectral parameter z . In that language Corollary is the generalization of this statement to the case of arbitrary “auxiliary spaces” V_1, V_2 .

3.6 Action groupoids and dynamical Yang–Baxter equation

If $\pi = A \rtimes G$ is an action groupoid, R -matrices for π -graded vector spaces are expressed in terms of the graded components as dynamical R -matrices. Then the tensor product of π -graded modules is

$$(V \otimes W)_{(a,g)} = \sum_{h \in G} V_{(a,h)} \otimes W_{(ah, h^{-1}g)}.$$

Let $\check{R}(z)$ be a solution of the Yang–Baxter equation (3.1) and let

$$\check{R}(z, a) \in \bigoplus_{g \in G} \text{End}_k((V \otimes V)_{(a,g)})$$

be the restriction of $\check{R}(z)$ to the graded components with fixed $a \in A$. Then the Yang–Baxter equation can be written as

$$\check{R}^{(23)}(z - w, ah^{(1)})\check{R}^{(12)}(z, a)\check{R}^{(23)}(w, ah^{(1)}) = \check{R}^{(12)}(w, a)\check{R}^{(23)}(z, ah^{(1)})\check{R}^{(12)}(z - w, a).$$

Here we use the “dynamical” notation with the placeholder $h^{(i)}$:

$$\check{R}^{(23)}(ah^{(1)})(u \otimes v \otimes w) = u \otimes \check{R}(ag)(v \otimes w) \quad \text{if } u \in V_{(a,g)}.$$

If we compose on the left with the product $p^{(23)}p^{(13)}p^{(12)} = p^{(12)}p^{(13)}p^{(23)}$ of flips $p: v \otimes w \mapsto w \otimes v$ we get the YBE for $R = p \circ \check{R}$ in the form

$$R^{(12)}(z - w, ah^{(3)})R^{(13)}(z, a)R^{(23)}(w, ah^{(1)}) = R^{(23)}(w, a)R^{(13)}(z, ah^{(2)})R^{(12)}(z - w, a).$$

3.7 Transfer matrices in the case of action groupoids

In the case of action groupoid $\pi = A \rtimes G$ we can identify convolution algebras with coefficients in π -graded endomorphisms with algebras of difference operators (or discrete connections) acting on the sections of sheaves over A . Let $G(a) = \{g \in G \mid ag = a\}$ denote the stabilizer subgroup of an object $a \in A$ and for $W \in \text{Vect}_k(\pi)$ let $W_{G(a)} = \bigoplus_{g \in G(a)} W_{(a,g)}$. Let $\Gamma(A, W)$ be the space of maps $\psi: A \rightarrow \bigsqcup_{a \in A} W_{G(a)}$ such that $\psi(a) \in W_{G(a)}$ for all $a \in A$. Then $\Gamma(A, W)$ is naturally a module over $\Gamma(\pi, \underline{\text{End}} W)$ and the transfer matrices can be realized as linear operators on $\Gamma(A, W)$.

Explicitly, we have $(\underline{\text{End}} W)_{(a,g)} = \bigoplus_{h \in G(a)} \text{Hom}(W_{(ag, g^{-1}hg)}, W_{(a,h)})$. For $\psi \in \Gamma(A, W)$ let $(t_g \psi)(a) = \psi(ag) \in W_{G(ag)}$. Then $f \in \Gamma(\pi, \underline{\text{End}} W)$ acts on $\psi \in \Gamma(A, W)$ as

$$(f\psi)(a) = \sum_{g \in G} f(a, g)(t_g \psi)(a).$$

Let $\check{R}_{UW}[a, g]$ be the component $U_{(a,g)} \otimes W_{G(ag)} \rightarrow W_{G(a)} \otimes U_{(a,g)}$ of a dynamical R -matrix $U \otimes W \rightarrow W \otimes U$. Then $r_g(a) = \text{tr}_{U_{(a,g)}} \check{R}_{UW}[a, g]$ maps $W_{G(ga)}$ to $W_{G(a)}$. The transfer matrix is

$$\text{tr}_U \check{R}_{UW} = \sum_{g \in G} r_g t_g,$$

where r_g is understood as a multiplication operator.

4 Elliptic quantum groups

As a class of examples of the above construction let us work out the case of the dynamical R -matrix defining the elliptic quantum group in the \mathfrak{gl}_n -case, see [19]. Here $k = \mathbb{C}$.

4.1 Action groupoids of the weight lattice

Let $\mathfrak{h} \cong \mathbb{C}^n$ be the Lie subalgebra of diagonal matrices in \mathfrak{gl}_n and $\mathfrak{h}_0 \subset \mathfrak{h}$ the subalgebra of traceless diagonal matrices. The weight lattice is the lattice $P = \sum_{i=1}^n \mathbb{Z}\epsilon_i \subset \mathfrak{h}^*$ spanned by the coordinate functions $\epsilon_i: x \mapsto x_i$. It acts on $\mathfrak{h}_0^* = \mathfrak{h}^*/\mathbb{C}(1, \dots, 1)$ by translations on \mathfrak{h}^* composed with the canonical projection $\mathfrak{h}^* \rightarrow \mathfrak{h}_0^*$. For each orbit $O_b = b + P$, $b \in \mathfrak{h}_0^*$ we have a groupoid $O_b \times P$, consisting of pairs $(a, \mu) \in O_b \times P$ with composition $(a_1, \mu_1) \circ (a_2, \mu_2) = (a_2, \mu_1 + \mu_2)$, defined if $a_1 = a_2 + \mu_2$.

We consider the following standard subsets of \mathfrak{h}_0^* :

- The set of \mathfrak{sl}_n -dominant weights $P_+ = \{\lambda \in \mathbb{Z}^n/\mathbb{Z}(1, \dots, 1) \mid \lambda_1 \geq \dots \geq \lambda_n\}$.
- The set of regular dominant weights $P_{++} = \{\lambda \in P_+ \mid \lambda_1 > \dots > \lambda_n\}$.
- The set of dominant affine weights $P_+^r = \{\lambda \in P_+ \mid \lambda_1 - \lambda_n \leq r\}$, of level $r \in \mathbb{Z}_{\geq 0}$.
- The set of regular dominant affine weights $P_{++}^r = \{\lambda \in P_{++} \mid \lambda_1 - \lambda_n < r\}$, of level $r \in \mathbb{Z}_{\geq n}$.

We have a bijection $P_+^r \rightarrow P_+^{r+n}$ given by $\lambda \mapsto \lambda + \rho$, $\rho = (n-1, \dots, 1, 0)$.

4.2 Dynamical R -matrix

Fix two complex numbers τ, γ such that $\text{Im } \tau > 0$ and $\gamma \notin \mathbb{Z} + \tau\mathbb{Z}$. Let

$$\theta(z, \tau) = - \sum_{n \in \mathbb{Z}} e^{i\pi(n+\frac{1}{2})^2\tau + 2\pi i(n+\frac{1}{2})(z+\frac{1}{2})}$$

be the odd Jacobi theta function and $[z] = \theta(\gamma z, \tau)/(\gamma\theta'(0, \tau))$ is normalized to have derivative 1 at $z = 0$. The function $z \mapsto [z]$ of one complex variable is an odd entire function with first order zeros on the lattice

$$\Lambda = \mathbb{Z}\frac{1}{\gamma} + \mathbb{Z}\tau.$$

The defining representation $\bar{V} = \mathbb{C}^n$ of \mathfrak{gl}_n has a weight decomposition $\bar{V} = \bigoplus_{i=1}^n \bar{V}_{\epsilon_i}$ where $\bar{V}_{\epsilon_i} = \mathbb{C}e_i$ is the span of the i -th standard basis vector.

Let E_{ij} be the $n \times n$ matrix such that $E_{ij}e_k = \delta_{jk}e_i$ for all $k \in \{1, \dots, n\}$. The (unnormalized) elliptic dynamical R -matrix with spectral parameter $z \in \mathbb{C}$ is

$$\begin{aligned} \check{R}(z, a) &= \sum_{i=1}^n E_{ii} \otimes E_{ii} - \sum_{i \neq j=1}^n \frac{[a_i - a_j + 1][z]}{[a_i - a_j][1 - z]} E_{ij} \otimes E_{ji} \\ &+ \sum_{i \neq j=1}^n \frac{[a_i - a_j + z][1]}{[a_i - a_j][1 - z]} E_{ii} \otimes E_{jj}. \end{aligned} \quad (4.1)$$

It is a meromorphic function of $z \in \mathbb{C}$, $a \in \mathbb{C}^n$ and solves the dynamical Yang–Baxter equation in the additive form

$$\begin{aligned} \check{R}(z-w, a+h^{(1)})^{(23)} \check{R}(z, a)^{(12)} \check{R}(w, a+h^{(1)})^{(23)} \\ = \check{R}(w, a)^{(12)} \check{R}(z, a+h^{(1)})^{(23)} \check{R}(z-w, a)^{(12)}, \end{aligned}$$

and the inversion relation

$$\check{R}(z, a) \check{R}(-z, a) = \text{id}_{\mathbb{C}^n \otimes \mathbb{C}^n},$$

valid for generic z, a . The relation to the R matrix presented in [19] is $\check{R}(z, a) = PR(z, \lambda)$ where $\lambda = \gamma a$. By restricting a to take values in an orbit O_b we can construct R -matrices acting

on groupoid-graded vector spaces. To do this we need to avoid the poles on the hyperplanes $a_i - a_j \equiv 0 \pmod{\Lambda}$, ($i \neq j$) of the R -matrix.

We consider the case where $r = 1/\gamma$ is an integer $> n$. Then for $a \in P$ the poles are at

$$a_i - a_j = mr, \quad i \neq j \in \{1, \dots, n\}, \quad m \in \mathbb{Z}.$$

Let $\Delta \subset P$ be the union of these hyperplanes. The affine Weyl group W_r is the group generated by orthogonal reflections at these hyperplane in the Euclidean space $P \otimes \mathbb{R} = \mathbb{R}^n$. It acts freely and transitively on the complement of $\Delta \otimes \mathbb{R}$ and P_{++}^r is the set of weights in a connected component of the complement and is a fundamental domain for the action of W_r on $P \setminus \Delta$.

We distinguish two cases, named after the corresponding models of statistical mechanics.

1. **Generalized SOS model.** Let $b \in \mathfrak{h}^*$, $\pi = O_b \rtimes P$, $V^b = \bigoplus_{\gamma \in \pi} V_\gamma^b$ with

$$V_{(a,\mu)}^b = \begin{cases} \bar{V}_{\epsilon_i} = \mathbb{C}e_i & \text{if } \mu = \epsilon_i, i = 1, \dots, n \text{ and } a \in O_b, \\ 0, & \text{if } \mu \notin \{\epsilon_1, \dots, \epsilon_n\}. \end{cases}$$

If b does not lie in Δ then $R(z)$ is a well-defined endomorphism of $V^b \otimes V^b$ and obeys the Yang–Baxter equation.

2. **Generalized RSOS model.** Let $b = 0$. The groupoid π is the full subgroupoid of $O_0 \rtimes P$ on P_{++}^r : it consists of pairs $(a, \mu) \in O_0 \rtimes P$ such that both a and $a + \mu$ lie in P_{++}^r . The non-zero components are

$$V_{(a,\epsilon_i)}^{\text{RSOS}} = \bar{V}_{\epsilon_i} = \mathbb{C}e_i, \quad a, a + \epsilon_i \in P_{++}^r.$$

To define $\check{R}(z)$ in the RSOS case we view V^{RSOS} as an $O_0 \rtimes P$ -graded subspace of $V^{b=0}$ such that $V_\gamma^{\text{RSOS}} = 0$ for $\gamma \notin \pi$, see Section 2.9.

Proposition 4.1. *Let $V = V^{b=0}$ and let $V^{\text{RSOS}} \subset V$ be viewed as an $O_0 \rtimes P$ -graded subspace of V . Then $\check{R}(z)$ is well-defined on $V^{\text{RSOS}} \otimes V^{\text{RSOS}}$ and maps $V^{\text{RSOS}} \otimes V^{\text{RSOS}}$ to itself.*

Proof. While $\check{R}(z)$ is not defined on all vectors in $V \otimes V$, it is well-defined on $V^{\text{RSOS}} \otimes V^{\text{RSOS}}$ since by construction the denominators $[a_i - a_j]$ don't vanish for $a \in P_{++}^r$. Suppose $(a + \epsilon_j, \epsilon_i)$ and (a, ϵ_j) are composable arrows in the subgroupoid π , meaning that $a, a + \epsilon_j$ and $a + \epsilon_i + \epsilon_j$ belong to P_{++}^r . Then $\check{R}(z)$ maps $V_{(a,\epsilon_j)} \otimes V_{(a+\epsilon_j,\epsilon_i)}$ to itself if $i = j$ and to

$$V_{(a,\epsilon_j)} \otimes V_{(a+\epsilon_j,\epsilon_i)} \oplus V_{(a,\epsilon_i)} \otimes V_{(a+\epsilon_i,\epsilon_j)}$$

if $i \neq j$. The first summand is indexed by a pair of arrows in the subgroupoids but the second is not if $a + \epsilon_i \notin P_{++}^r$. Thus $\check{R}(z)$ preserves V^{RSOS} if and only if the component of the image of $\check{R}(z)$ in $V_{(a,\epsilon_i)} \otimes V_{(a+\epsilon_i,\epsilon_j)}$ for $a + \epsilon_i \notin P_{++}^r$ vanishes. So we need to check the vanishing of the component

$$-\frac{[a_i - a_j + 1][z]}{[a_i - a_j][1 - z]} E_{ij} \otimes E_{ji} : V_{(a,\epsilon_j)} \otimes V_{(a+\epsilon_j,\epsilon_i)} \rightarrow V_{(a,\epsilon_i)} \otimes V_{(a+\epsilon_i,\epsilon_j)}, \quad i \neq j, \quad (4.2)$$

of $\check{R}(z, a)$ in the case where $a + \epsilon_i \notin P_{++}^r$ and $a, a + \epsilon_j, a + \epsilon_i + \epsilon_j \in P_{++}^r$. The condition for a being in P_{++}^r is $a_1 > \dots > a_n > a_1 - r$. For $a \in P_{++}^r$, $a + \epsilon_i$ violates an inequality if and only if $i \geq 2$ and $a_{i-1} = a_i + 1$ or $i = 1$ and $a_n = a_1 - r + 1$. The condition that $a + \epsilon_i + \epsilon_j \in P_{++}^r$ implies that $j = i - 1$ if $i \geq 2$ and $j = n$ if $i = 1$. In both cases $a_j \equiv a_i + 1 \pmod{r}$ and thus (4.2) vanishes. ■

Corollary 4.2. *The restriction of $\check{R}(z)$ to $V^{\text{RSOS}} \otimes V^{\text{RSOS}}$ obeys the dynamical Yang–Baxter equation.*

Example 4.3. The case $n = 2$ is the original RSOS model of [3]. In this case we have a bijection $O_0 = \mathbb{Z} \times \mathbb{Z}/(\mathbb{Z}(1, 1)) \rightarrow \mathbb{Z}$ sending the class of (a_1, a_2) , to $l = a_1 - a_2$ under this identification $A = P_{++}^r = \{1, \dots, r - 1\}$.

The dynamical R -matrix in the basis $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$ is

$$\check{R}(z, a) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{[l+z][1]}{[l][1-z]} & -\frac{[l+1][z]}{[l][1-z]} & 0 \\ 0 & -\frac{[l-1][z]}{[l][1-z]} & \frac{[l-z][1]}{[l][1-z]} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} W_1 & 0 & 0 & 0 \\ 0 & W_5 & W_4 & 0 \\ 0 & W_3 & W_6 & 0 \\ 0 & 0 & 0 & W_2 \end{pmatrix}.$$

The Boltzmann weights W_j correspond to the local configurations of Fig. 2.

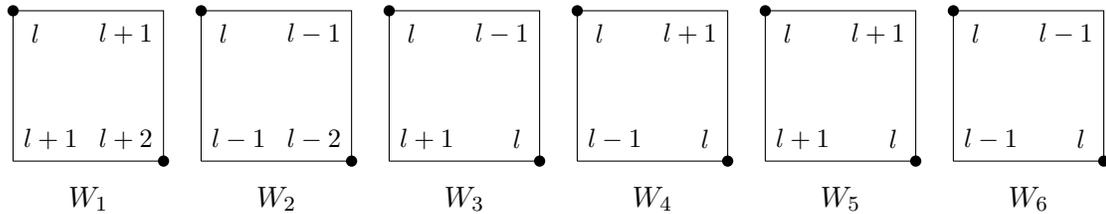


Figure 2. Six possible types of Boltzmann weight in the eight-vertex SOS model.

The hard hexagon model is part of the family of RSOS models with $r = 5$. In this case the allowed values of l are 1, 2, 3, 4 and, because of the identity $[u] = [5 - u]$, $u \in \mathbb{C}$, the Boltzmann weights are invariant under $l \mapsto 5 - l$. We can map the model to a lattice gas model on a square lattice. A configuration of the RSOS model, i.e., a map from the vertices of a square lattice to $\{1, 2, 3, 4\}$ is mapped to a configuration of particles on the lattice: there is a particle at the vertices with value 1 or 4 and no particle at the vertices with value 2 or 3. To each configuration of particles there correspond two RSOS configuration related by $l \mapsto 5 - l$ and thus with the same Boltzmann weight. The rule that the value at neighbouring lattice sites differs by 1 translate to the rule that no two particles of the lattice gas can sit at nearest neighbouring sites. We can think of this model as a hard square model: the squares whose vertices are the nearest neighbours of the positions of the particles are required not to overlap (i.e., to have disjoint interiors). The row-to-row transfer matrices of these models commute among themselves and in particular with the transfer matrix with spectral parameter $z = -1$, which is the transfer matrix of the hard hexagon model. In this case W_5 vanishes for $l = 1$ and W_6 vanishes for $l = 4$, implying that in a configuration with non-zero Boltzmann weight no two particles can sit at the endpoints of a NW-SE diagonal. We can translate these rules by thinking of the particles as midpoints of lattice hexagons which are not allowed to overlap.

4.3 Characters

Here we consider the RSOS case and write V instead of V^{RSOS} for the vector representation.

Out of the vector evaluation representation $V(z)$ one can construct several new representations by the fusion or reproduction method [35, 37], which admit pairwise R -matrices for generic values of the evaluation parameters. It follows that their characters form a unital commutative algebra of difference operators with integer coefficients. The unit element is the character of the trivial representation.

Here are some examples of character calculations. The groupoid $\pi_r(P)$ of the restricted elliptic quantum group with weight lattice P is the full subgroupoid of $O_0 \rtimes P$ on the set $A = P_{++}^r$. Its convolution ring is isomorphic to the subring $\chi_A D_P(O_0) \chi_A$ of the crossed product $D_P(O_0) = \mathbb{Z}^{O_0} \rtimes \mathbb{Z}P$, see Lemma 2.3 and Proposition 2.4. The subring $\mathbb{Z}P = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ is the ring of Laurent polynomials with generators $t_i = t_{\epsilon_i}$, $i = 1, \dots, n$.

- The character of the trivial representation is the multiplication operator by the characteristic function of $A = P_{++}^r$:

$$\text{ch}_{\mathbf{1}} = \chi_A.$$

- The character of the vector representation $V(z)$.

$$\text{ch}_{V(z)} = \sum_{i=1}^n \chi_A t_i \chi_A = \sum_{i=1}^n \chi_{A \cap (A - \epsilon_i)} t_i.$$

- The R -matrix (4.1) has a pole at $z = 1$ and is not invertible at $z = -1$. We set $\check{R}_{\text{reg}}(1, a) = \text{res}_{z=1} \check{R}(z, a)$. Then we have an exact sequence (see Appendix A)

$$\dots \xrightarrow{\check{R}(-1)} V(z+1) \otimes V(z) \xrightarrow{\check{R}_{\text{reg}}(1)} V(z) \otimes V(z+1) \xrightarrow{\check{R}(-1)} \dots$$

The analogue of the symmetric square of the vector representation, is $S^2 V(z) = \text{Ker } \check{R}_{\text{reg}}(1) \cong \text{Coker } \check{R}(-1)$. Its character can be computed from the explicit basis of Lemma A.1

$$\text{ch}_{S^2 V(z)} = \sum_{i=1}^n \chi_{A \cap (A - 2\epsilon_i)} t_i^2 + \sum_{1 \leq i < j \leq n} \chi_{A \cap (A - \epsilon_i) \cap (A - \epsilon_j)} t_i t_j.$$

Similarly, the second exterior power $\Lambda^2 V(z) = \text{Coker } \check{R}_{\text{reg}}(1) \cong \text{Ker } \check{R}(-1)$ has character

$$\text{ch}_{\Lambda^2 V(z)} = \sum_{1 \leq i < j \leq n} \chi_{A \cap (A - \epsilon_i - \epsilon_j)} t_i t_j.$$

From the exact sequence $0 \rightarrow S^2 V(z) \rightarrow V(z+1) \otimes V(z) \rightarrow \Lambda^2 V(z) \rightarrow 0$, follows the identity

$$\text{ch}_{V(z)}^2 = \text{ch}_{\Lambda^2 V(z)} + \text{ch}_{S^2 V(z)}.$$

- For $n = 2$ we have the bijection $O_0 = \mathbb{Z} \times \mathbb{Z} / \mathbb{Z}(1, 1) \rightarrow \mathbb{Z}$ sending the class of (a_1, a_2) to $a_1 - a_2$. Under this identification, $A = [1, r-1] = \{1, 2, \dots, r-1\}$. Then $V(z+p-1) \otimes \dots \otimes V(z+1) \otimes V(z)$ has a subrepresentation $S^p V(z) = \cap_{i=1}^{p-1} \text{Ker } \check{R}_{\text{reg}}(1)^{(i, i+1)}$ with character

$$\text{ch}_{S^p V(z)} = \chi_{[1, r-p-1]} t_1^p + \chi_{[2, r-p]} t_1^{p-1} t_2 + \dots + \chi_{[p+1, r-1]} t_2^p,$$

$p = 0, \dots, r-2$. The characters $L_p = \text{ch}_{S^p V(z)}$ obey the fusion rules

$$L_p L_q = \sum_{s \equiv p+q \pmod{2}} N_{pq}^s u^{\frac{p+q-s}{2}} L_s,$$

where u is the central element $t_1 t_2$ and

$$N_{pq}^s = \begin{cases} 1, & |p-q| \leq s \leq \min(p+q, 2r-4-p-q), \\ 0, & \text{otherwise.} \end{cases}$$

These are the famous fusion rules that first appeared in conformal field theory [6, 46]. The algebra with generators L_p and relations above is called the Verlinde algebra.

4.4 Exterior powers

The k th exterior power $\bigwedge^k V(z)$ is defined as the quotient of $V(z+k-1) \otimes \cdots \otimes V(z+1) \otimes V(z)$ by the sum of the images of $\check{R}_{\text{reg}}(1)^{(j,j+1)}$ for $j = 1, \dots, k-1$. Its character is

$$\text{ch} \bigwedge^k V(z) = \chi_A e_k(t_1, \dots, t_n) \chi_A,$$

with $A = P_{++}^r$. Here $e_k(t_1, \dots, t_n) = \sum_{1 \leq i_1 < \dots < i_k \leq n} t_{i_1} \cdots t_{i_k}$ is the k th elementary symmetric polynomial. It follows from the existence of R -matrices for pairs of exterior powers that these characters commute.

The convolution ring $\mathbb{Z}[\pi_r(P)] \cong \chi_A D_P(O_0) \chi_A$ acts naturally on functions on $A = P_{++}^r$. The characters can be simultaneously diagonalized.

Theorem 4.4. *Let $q = e^{2\pi i/r}$ and pick any n -th root $q^{1/n}$ of q . For each $\lambda \in A = P_{++}^r$ let ψ_λ be the function on A given by*

$$\psi_\lambda(a) = q^{-\frac{1}{n} \sum_i a_i \sum_i \lambda_i} \det(q^{\lambda_i a_j}).$$

Then

$$\text{ch} \bigwedge^k V(z) \psi_\lambda = e_k(q^{\bar{\lambda}_1}, \dots, q^{\bar{\lambda}_n}) \psi_\lambda, \quad \bar{\lambda}_i = \lambda_i - \frac{1}{n} \sum_j \lambda_j.$$

Proof. Let us first ignore the characteristic functions χ_A and consider ψ_λ as a function on the whole weight lattice. Then for any symmetric polynomial $P(t)$ in t_1, \dots, t_n ,

$$P(t) \psi_\lambda = P(q^{\bar{\lambda}_1}, \dots, q^{\bar{\lambda}_n}) \psi_\lambda.$$

The function $\psi_\lambda(a)$ is a skew-symmetric r -periodic function of a_1, \dots, a_n . It thus vanishes on the hyperplanes $a_i - a_j \equiv 0 \pmod{r}$ and in particular on the walls $a_{i+1} - a_i = 0$, $a_1 - a_n = r$ forming the complement of $A = P_{++}^r$ in $\bar{A} = P_+^r$. Thus $\chi_A \psi_\lambda = \chi_{\bar{A}} \psi_\lambda$. The monomials $t_I = t_{i_1} \cdots t_{i_k}$ with $i_1 < \dots < i_k$ map P_{++}^r to P_+^r . Thus

$$\begin{aligned} \chi_A e_k(t) \chi_A \psi_\lambda &= \chi_A e_k(t) \chi_{\bar{A}} \psi_\lambda = \sum_{I \subset \{1, \dots, n\}, |I|=k} \chi_{A \cap t_I^{-1} \bar{A}} e_k(t) \psi_\lambda \\ &= \sum_{I \subset \{1, \dots, n\}, |I|=k} \chi_A e_k(t) \psi_\lambda = e_k(q^{\bar{\lambda}}) \chi_A \psi_\lambda. \quad \blacksquare \end{aligned}$$

5 Quantum enveloping algebras at roots of unity

We are mainly concerned with dynamical R -matrices with non-trivial dependence on the spectral parameter, but it is instructive to consider the case of constant dynamical R -matrices arising from the representation theory of semisimple Lie algebras and their quantum versions. The main simplification in this case is that the category of finite dimensional modules is braided, namely for each pair of objects V, W there is an isomorphism $\tau_{V,W}: V \otimes W \rightarrow W \otimes V$, obeying compatibility conditions with the structure of a monoidal category, given by the evaluation of the universal R -matrix composed with the permutation of factors. This property fails for some pairs of objects in the case of quantum affine Lie algebras or Yangians, which is the case when the R -matrices have a non-trivial dependence on the spectral parameter.

We focus on the case of quantum groups at root of unity which is a toy model for restricted models. Strictly speaking the above has to be corrected in this case and one has to be more careful in the definition of the category of finite dimensional modules. We consider the semisimple quotient of the category of tilting modules [8, Section 11.3], [2, 41]. It is an abelian monoidal

\mathbb{C} -linear ribbon category $C_q(\mathfrak{g})$ depending on a simple Lie algebra \mathfrak{g} and a primitive ℓ -th root of unity q . It has finitely many equivalence classes of simple objects L_λ labeled by dominant weights in a scaled Weyl alcove $P_+^\ell(\mathfrak{g})$, and any object is isomorphic to a direct sum of simple modules. The alcove $P_+^\ell(\mathfrak{g})$ is a finite subset of the cone P_+ of dominant weight, bounded by a hyperplane as in Section 4.1. See [41] for a description in the most general case of a Lie algebra and any root of unity.

Let P be the weight lattice and π be the full subgroupoid of $P \rtimes P$ on $P_+^\ell(\mathfrak{g})$. Its objects are dominant weights and there is exactly one arrow $a \rightarrow b$ for any two weights $a, b \in P_+^\ell(\mathfrak{g})$. As usual we denote by $(a, b - a)$ this arrow.

Theorem 5.1. *There is a faithful exact monoidal functor $U_q\mathfrak{g}\text{-mod} \rightarrow \text{Vect}_k(\pi)$ sending an object W to $\hat{W} = \bigoplus_{(a,\lambda) \in \pi} \hat{W}_{a,\lambda}$ with*

$$\hat{W}_{a,\lambda} = \text{Hom}_{C_q(\mathfrak{g})}(L_a, W \otimes L_{a+\lambda}).$$

Proof. A morphism $f: W \rightarrow W'$ in $U_q\mathfrak{g}\text{-mod}$ induces a morphism $\hat{f}: \varphi \mapsto (f \otimes \text{id}) \circ \varphi$ from \hat{W} to \hat{W}' and this assignment is compatible with compositions so that we get a well-defined functor. The trivial module k which is the tensor unit in $U_q\mathfrak{g}$ is mapped to the tensor unit $\hat{k} = \mathbf{1}$ in $\text{Vect}_k(\pi)$, see Section 2.6. Moreover we have a natural transformation

$$\hat{W} \otimes \hat{Z} \rightarrow \widehat{W \otimes Z},$$

whose restriction to $\hat{W}_{a,\mu} \otimes \hat{Z}_{a+\mu,\lambda-\mu}$ is the composition

$$\begin{aligned} \text{Hom}(L_a, W \otimes L_{a+\mu}) \otimes \text{Hom}(L_{a+\mu}, Z \otimes L_{a+\lambda}) &\rightarrow \text{Hom}(L_a, W \otimes Z \otimes L_{a+\lambda}), \\ \varphi \otimes \psi &\mapsto (\text{id} \otimes \psi) \circ \varphi. \end{aligned} \quad \blacksquare$$

Since $C_q(\mathfrak{g})$ is semisimple, by taking the direct sum over μ we get an isomorphism $(\hat{W} \otimes \hat{Z})_{a,\lambda} \rightarrow \widehat{(W \otimes Z)}_{a,\lambda}$ on each graded component.

Theorem 5.2. *There is a system of isomorphisms (dynamical R-matrices)*

$$\check{R}_{W,Z}: \hat{W} \otimes \hat{Z} \rightarrow \hat{Z} \otimes \hat{W}$$

obeying the quasitriangularity relations

$$\check{R}_{W \otimes W', Z} = \check{R}_{W,Z}^{(12)} \check{R}_{W',Z}^{(23)}, \quad \check{R}_{W,Z \otimes Z'} = \check{R}_{W,Z'}^{(23)} \check{R}_{W,Z}^{(12)}$$

and the dynamical Yang–Baxter equation

$$\check{R}_{W,Z}^{(12)} \check{R}_{Y,Z}^{(23)} \check{R}_{Y,W}^{(12)} = \check{R}_{Y,W}^{(23)} \check{R}_{Y,Z}^{(12)} \check{R}_{W,Z}^{(23)},$$

for any objects W, W', Z, Z', Y .

Let us compute the character of \hat{W} . Let $N_{ab}^c = \dim \text{Hom}(L_c, L_a \otimes L_b)$ be the multiplicity of L_c in the decomposition of $L_a \otimes L_b$ as a direct sum of simple objects. The numbers N_{ab}^c are called fusion coefficients. They are the structure constants of the fusion ring (or Verlinde algebra) with generators e_a and product $e_a e_b = \sum_c N_{ab}^c e_c$. Then

$$\text{ch}_{L_\mu} = \sum_\lambda n_{\lambda,\mu} t^\lambda,$$

where $n_{\lambda,\mu}(a) = N_{a,\mu}^{a+\lambda}$.

The commutativity of the characters is the associativity of the Verlinde algebra.

A R -matrix at special values

The R -matrix (4.1) has a pole at $z = 1 \pmod{\Lambda}$. Because of the inversion relation we see that it is regular and invertible except for $z \in \pm 1 + \Lambda$. For $z = -1$ we have

$$\check{R}(-1, a) = - \sum_{i=1}^n E_{ii} \otimes E_{ii} + \sum_{i \neq j} \frac{[a_i - a_j + 1][1]}{[a_i - a_j][2]} (E_{ij} \otimes E_{ji} + E_{jj} \otimes E_{ii}).$$

Let $\check{R}_{\text{reg}}(1, a) = \text{res}_{z=1} \check{R}(z, a)$. Then

$$\check{R}_{\text{reg}}(1, a) = \sum_{i \neq j} \frac{[a_i - a_j + 1][1]}{[a_i - a_j]} (E_{ij} \otimes E_{ji} - E_{ii} \otimes E_{jj}).$$

Lemma A.1. *Let $e_i(a)$ denote the standard basis vector $e_i \in \mathbb{C}^n$ in $V_{(a, \epsilon_i)}$.*

- (i) *The image of $\check{R}(-1, a)$ coincides with the kernel of $\check{R}_{\text{reg}}(1, a)$ and is spanned by the linearly independent vectors*

$$e_i(a) \otimes e_j(a + \epsilon_i) + e_j(a) \otimes e_i(a + \epsilon_j),$$

where a, i, j are such that $i \leq j$ and $a, a + \epsilon_i, a + \epsilon_j, a + \epsilon_i + \epsilon_j \in P_{++}^r$.

- (ii) *The image of $\check{R}_{\text{reg}}(1, a)$ coincides with the kernel of $\check{R}(-1, a)$ and is spanned by the following linear independent vectors:*

$$e_i(a) \otimes e_j(a + \epsilon_i) - e_j(a) \otimes e_i(a + \epsilon_j),$$

where a, i, j are such that $i < j$ and $a, a + \epsilon_i, a + \epsilon_j, a + \epsilon_i + \epsilon_j \in P_{++}^r$;

$$e_{i-1}(a) \otimes e_i(a + \epsilon_{i-1}),$$

where $i = 2, \dots, n$ and $a_{i-1} = a_i + 1$;

$$e_n(a) \otimes e_1(a + \epsilon_n),$$

where $a_n = a_1 - r + 1$.

Proof. We consider these operators on the weight spaces $(V \otimes V)_{(a, \epsilon_i + \epsilon_j)}$, which are non-vanishing for $a, a + \epsilon_i + \epsilon_j \in P_{++}^r$. Recall that $a \in P_{++}^r$ means $a_1 > \dots > a_n > a_1 - r$. Thus the only case where the numerator $[a_i - a_j + 1]$ vanishes is when $j = i - 1$ and $a_{i-1} = a_i + 1$ for $i = 1, \dots, n$, where we set $a_0 = a_n + r$. In this case $a + \epsilon_i \notin P_{++}^r$. There are three cases to consider (we may assume that $i \geq j$):

- (a) $i = j$. In this case the weight space is spanned by $e_i(a) \otimes e_i(a + \epsilon_i)$, $\check{R}(-1)$ acts by multiplication by $-[2]/[1] \neq 0$ and $\check{R}_{\text{reg}}(1)$ vanishes. Thus

$$\text{Im } \check{R}(-1) = \text{Ker } \check{R}_{\text{reg}}(1) = \text{span}(e_i(a) \otimes e_i(a + \epsilon_i)),$$

$$\text{Im } \check{R}_{\text{reg}}(1) = \text{Ker } \check{R}(-1) = 0.$$

- (b) $i > j$ and $a + \epsilon_i, a + \epsilon_j \in P_{++}^r$. A basis of the weight space consists of $e_j(a) \otimes e_i(a + \epsilon_j)$ and $e_i(a) \otimes e_j(a + \epsilon_i)$. The matrices of $\check{R}(-1)$ and $\check{R}_{\text{reg}}(1)$ in this basis are (up to factors of $[1]$ and $[2]$)

$$\begin{pmatrix} \frac{[a_i - a_j + 1]}{[a_i - a_j]} & \frac{[a_i - a_j - 1]}{[a_i - a_j]} \\ \frac{[a_i - a_j + 1]}{[a_i - a_j]} & \frac{[a_i - a_j - 1]}{[a_i - a_j]} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{[a_i - a_j - 1]}{[a_i - a_j]} & -\frac{[a_i - a_j + 1]}{[a_i - a_j]} \\ -\frac{[a_i - a_j - 1]}{[a_i - a_j]} & \frac{[a_i - a_j + 1]}{[a_i - a_j]} \end{pmatrix},$$

respectively. The matrix entries are all non-zero and we see that

$$\begin{aligned}\mathrm{Im} \check{R}(-1) &= \mathrm{Ker} \check{R}_{\mathrm{reg}}(1) = \mathrm{span}(e_j(a) \otimes e_i(a + \epsilon_j) + e_i(a) \otimes e_j(a + \epsilon_i)), \\ \mathrm{Im} \check{R}_{\mathrm{reg}}(1) &= \mathrm{Ker} \check{R}(-1) = \mathrm{span}(e_j(a) \otimes e_i(a + \epsilon_j) - e_i(a) \otimes e_j(a + \epsilon_i)).\end{aligned}$$

- (c) $i > j$ and one of $a + \epsilon_i, a + \epsilon_j \notin P_{++}^r$. This happens only if $j = i - 1$ and $a_{i-1} = a_i + 1$ (with $a_0 := a_n + r$). Then the weight space is spanned by $e_{i-1}(a) \otimes e_i(a + \epsilon_{i-1})$; $\check{R}(-1)$ acts by zero and $\check{R}_{\mathrm{reg}}(1)$ acts by multiplication by $[2]/[1] \neq 0$. Thus

$$\begin{aligned}\mathrm{Im} \check{R}(-1) &= \mathrm{Ker} \check{R}_{\mathrm{reg}}(1) = 0, \\ \mathrm{Im} \check{R}_{\mathrm{reg}}(1) &= \mathrm{Ker} \check{R}_{\mathrm{reg}}(-1) = \mathrm{span}(e_{i-1}(a) \otimes e_i(a + \epsilon_{i-1})), \quad i = 1, \dots, n,\end{aligned}$$

where for $i = 1$ the right-hand side is $e_n(a) \otimes e_1(a + \epsilon_n)$. ■

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References

- [1] Aganagic M., Okounkov A., Elliptic stable envelopes, *J. Amer. Math. Soc.* **34** (2021), 79–133, [arXiv:1604.00423](#).
- [2] Andersen H.H., Paradowski J., Fusion categories arising from semisimple Lie algebras, *Comm. Math. Phys.* **169** (1995), 563–588.
- [3] Andrews G.E., Baxter R.J., Forrester P.J., Eight-vertex SOS model and generalized Rogers–Ramanujan-type identities, *J. Statist. Phys.* **35** (1984), 193–266.
- [4] Baxter R.J., One-dimensional anisotropic Heisenberg chain, *Ann. Physics* **70** (1972), 323–337.
- [5] Baxter R.J., Exactly solved models in statistical mechanics, Academic Press, Inc., London, 1982.
- [6] Belavin A.A., Polyakov A.M., Zamolodchikov A.B., Infinite conformal symmetry in two-dimensional quantum field theory, *Nuclear Phys. B* **241** (1984), 333–380.
- [7] Bethe H., Zur Theorie der Metalle. I. Eigenwerte und Eigenfunktionen der linearen Atomkette, *Z. Phys.* **71** (1931), 205–226.
- [8] Chari V., Pressley A., A guide to quantum groups, Cambridge University Press, Cambridge, 1994.
- [9] Costello K., Witten E., Yamazaki M., Gauge theory and integrability, I, *ICCM Not.* **6** (2018), 46–119, [arXiv:1709.09993](#).
- [10] Costello K., Witten E., Yamazaki M., Gauge theory and integrability, II, *ICCM Not.* **6** (2018), 120–146, [arXiv:1802.01579](#).
- [11] Date E., Jimbo M., Miwa T., Okado M., Fusion of the eight vertex SOS model, *Lett. Math. Phys.* **12** (1986), 209–215.
- [12] Drinfeld V.G., Quantum groups, in Proceedings of the International Congress of Mathematicians, Vols. 1, 2 (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, 1987, 798–820.
- [13] Etingof P., Latour F., The dynamical Yang–Baxter equation, representation theory, and quantum integrable systems, *Oxford Lecture Series in Mathematics and its Applications*, Vol. 29, Oxford University Press, Oxford, 2005.
- [14] Etingof P., Schiffmann O., Lectures on the dynamical Yang–Baxter equations, in Quantum Groups and Lie Theory (Durham, 1999), *London Math. Soc. Lecture Note Ser.*, Vol. 290, Cambridge University Press, Cambridge, 2001, 89–129, [arXiv:math.QA/9908064](#).

- [15] Etingof P., Varchenko A., Geometry and classification of solutions of the classical dynamical Yang–Baxter equation, *Comm. Math. Phys.* **192** (1998), 77–120, [arXiv:q-alg/9703040](#).
- [16] Etingof P., Varchenko A., Solutions of the quantum dynamical Yang–Baxter equation and dynamical quantum groups, *Comm. Math. Phys.* **196** (1998), 591–640, [arXiv:q-alg/9708015](#).
- [17] Etingof P., Varchenko A., Exchange dynamical quantum groups, *Comm. Math. Phys.* **205** (1999), 19–52, [arXiv:math.QA/9801135](#).
- [18] Faddeev L., Integrable models in $(1 + 1)$ -dimensional quantum field theory, in *Recent Advances in Field Theory and Statistical Mechanics* (Les Houches, 1982), North-Holland, Amsterdam, 1984, 561–608.
- [19] Felder G., Conformal field theory and integrable systems associated to elliptic curves, in *Proceedings of the International Congress of Mathematicians*, Vols. 1, 2 (Zürich, 1994), Birkhäuser, Basel, 1995, 1247–1255, [arXiv:hep-th/9407154](#).
- [20] Felder G., Elliptic quantum groups, in *XIth International Congress of Mathematical Physics* (Paris, 1994), Int. Press, Cambridge, MA, 1995, 211–218, [arXiv:hep-th/9412207](#).
- [21] Felder G., Tarasov V., Varchenko A., Solutions of the elliptic qKZB equations and Bethe ansatz. I, in *Topics in Singularity Theory*, *Amer. Math. Soc. Transl. Ser. 2*, Vol. 180, Amer. Math. Soc., Providence, RI, 1997, 45–75, [arXiv:q-alg/9606005](#).
- [22] Felder G., Varchenko A., Algebraic Bethe ansatz for the elliptic quantum group $E_{\tau,\eta}(\mathfrak{sl}_2)$, *Nuclear Phys. B* **480** (1996), 485–503, [arXiv:q-alg/9605024](#).
- [23] Felder G., Varchenko A., On representations of the elliptic quantum group $E_{\tau,\eta}(\mathfrak{sl}_2)$, *Comm. Math. Phys.* **181** (1996), 741–761, [arXiv:q-alg/9601003](#).
- [24] Felder G., Varchenko A., Resonance relations for solutions of the elliptic QKZB equations, fusion rules, and eigenvectors of transfer matrices of restricted interaction-round-a-face models, *Commun. Contemp. Math.* **1** (1999), 335–403, [arXiv:math.QA/9901111](#).
- [25] Gautam S., Toledano-Laredo V., Elliptic quantum groups and their finite-dimensional representations, [arXiv:1707.06469](#).
- [26] Gervais J.-L., Neveu A., Novel triangle relation and absence of tachyons in Liouville string field theory, *Nuclear Phys. B* **238** (1984), 125–141.
- [27] Huse D.A., Exact exponents for infinitely many new multicritical points, *Phys. Rev. B* **30** (1984), 3908–3915.
- [28] Inozemtsev V.I., On the connection between the one-dimensional $S = 1/2$ Heisenberg chain and Haldane–Shastry model, *J. Statist. Phys.* **59** (1990), 1143–1155.
- [29] Jimbo M., Konno H., Odake S., Shiraishi J., Quasi-Hopf twistors for elliptic quantum groups, *Transform. Groups* **4** (1999), 303–327, [arXiv:q-alg/9712029](#).
- [30] Jimbo M., Kuniba A., Miwa T., Okado M., The $A_n^{(1)}$ face models, *Comm. Math. Phys.* **119** (1988), 543–565.
- [31] Jimbo M., Miwa T., Okado M., Solvable lattice models related to the vector representation of classical simple Lie algebras, *Comm. Math. Phys.* **116** (1988), 507–525.
- [32] Kalmykov A., Safronov P., A categorical approach to dynamical quantum groups, [arXiv:2008.09081](#).
- [33] Kassel C., Quantum groups, *Graduate Texts in Mathematics*, Vol. 155, Springer-Verlag, New York, 1995.
- [34] Kirillov A.N., Reshetikhin N.Yu., Representations of the algebra $U_q(\mathfrak{sl}(2))$, q -orthogonal polynomials and invariants of links, in *Infinite-Dimensional Lie Algebras and Groups* (Luminy-Marseille, 1988), *Adv. Ser. Math. Phys.*, Vol. 7, World Sci. Publ., Teaneck, NJ, 1989, 285–339.
- [35] Kirillov A.N., Reshetikhin N.Yu., Formulas for multiplicities of occurrence of irreducible components in the tensor product of representations of simple Lie algebras, *J. Math. Sci.* **80** (1996), 1768–1772.
- [36] Konno H., Elliptic quantum groups: representations and related geometry, *SpringerBriefs in Mathematical Physics*, Vol. 37, Springer, Singapore, 2020.
- [37] Kulish P.P., Reshetikhin N.Yu., Sklyanin E.K., Yang–Baxter equations and representation theory. I, *Lett. Math. Phys.* **5** (1981), 393–403.
- [38] Lundström P., The category of groupoid graded modules, *Colloq. Math.* **100** (2004), 195–211.
- [39] Lusztig G., Introduction to quantum groups, *Progress in Mathematics*, Vol. 110, Birkhäuser Boston, Inc., Boston, MA, 1993.
- [40] McGuire J.B., Study of exactly soluble one-dimensional N -body problems, *J. Math. Phys.* **5** (1964), 622–636.
- [41] Sawin S.F., Quantum groups at roots of unity and modularity, *J. Knot Theory Ramifications* **15** (2006), 1245–1277, [arXiv:math.QA/0308281](#).

-
- [42] Shibukawa Y., Hopf algebroids and rigid tensor categories associated with dynamical Yang–Baxter maps, *J. Algebra* **449** (2016), 408–445.
 - [43] Stokman J., Reshetikhin N., N -point spherical functions and asymptotic boundary KZB equations, [arXiv:2002.02251](https://arxiv.org/abs/2002.02251).
 - [44] Turaev V., Quantum invariants of knots and 3-manifolds, *De Gruyter Studies in Mathematics*, Vol. 18, [Walter de Gruyter & Co.](https://www.walterdegruyter.com/), Berlin, 1994.
 - [45] Turaev V., Virelizier A., Monoidal categories and topological field theory, *Progress in Mathematics*, Vol. 322, [Birkhäuser/Springer](https://www.birkhauser.com/), Cham, 2017.
 - [46] Verlinde E., Fusion rules and modular transformations in 2D conformal field theory, *Nuclear Phys. B* **300** (1988), 360–376.
 - [47] Yang C.N., Some exact results for the many-body problem in one dimension with repulsive delta-function interaction, *Phys. Rev. Lett.* **19** (1967), 1312–1315.