

Flowing in Group Field Theory Space: a Review^{*}

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Abstract. We provide a non-technical overview of recent extensions of renormalization methods and techniques to Group Field Theories (GFTs), a class of combinatorially non-local quantum field theories which generalize matrix models to dimension $d \geq 3$. More precisely, we focus on GFTs with so-called closure constraint, which are closely related to lattice gauge theories and quantum gravity spin foam models. With the help of recent tensor model tools, a rich landscape of renormalizable theories has been unravelled. We review our current understanding of their renormalization group flows, at both perturbative and non-perturbative levels.

Key words: group field theory; quantum gravity; quantum field theory; renormalization

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1 Introduction

From a mathematical perspective, a GFT [9, 43, 57, 70, 71] is a quantum field theory defined on d copies of a compact Lie group G , in which point-like interactions are replaced by non-trivial combinatorial objects. At the level of the field theory action, this translates into peculiar non-localities of the interactions, which are given by pairwise identifications and integrations of individual group variables of the elementary fields. One could thus say that a given GFT interaction is local in each pair of copies of the group G thus identified, but non-local from the point of view of the full configuration space G^d . This leads to subtleties in the construction and analysis of such models, which for a long time precluded the definition of renormalizable theories.

Alternatively, GFTs can be seen as generalized matrix and tensor models, with group-valued rather than discrete indices. Progress in tensor models (see, e.g., [25, 26, 53, 55] and references therein) can therefore be directly imported into the GFT formalism. The continuous structure of the group allows to considerably enrich this purely combinatorial background and define more general classes of tensorial theories, which go under the name of Tensorial Group Field Theories (TGFTs) or simply Tensorial Field Theories (TFTs). The introduction of derivative couplings, and in particular of non-trivial propagators, leads to a first class of such models [10, 11, 12, 15, 16, 18, 20, 21, 82, 83]. They are proper field theories whose (perturbative) definitions already require a non-trivial renormalizability analysis. Since the group plays a limited role and enjoys no particular physical interpretation in such models, they are preferably referred to as TFTs. Second, the group structure can be used to impose particular constraints on the elementary fields, which endow Feynman amplitudes with the structure of generalized lattice gauge theories. This latter class of models is usually referred to as Tensorial Group Field Theories (TGFTs) to emphasize the central importance of the group¹ (both technically and for

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¹Note however that the distinction between TFTs and TGFTs is somehow loose, as it relies more on questions of intent and interpretation than on a mathematically precise definition. These terms are therefore sometimes used interchangeably in the literature. In the present review, we will also refer to TFTs as *TGFTs without gauge invariance*.

the physical interpretation of the amplitudes), or simply group field theories whenever more general interactions than tensor invariants are allowed (see Section 2 below).

GFTs were originally introduced in the context of Loop Quantum Gravity (LQG) [37, 75] for the purpose of resumming Spin Foam amplitudes² [1, 74] and hence completing the definition of the dynamics of LQG [80, 85]. While LQG aims at solving the quantum gravity conundrum through a mere quantization of General Relativity (GR), it naturally leads to quantum state spaces of geometry in which discrete structures such as graphs [2] or triangulations [39] are center stage. Such structures percolate to the dynamical level and lead to interesting quantizations of discretized GR [8, 40, 41, 45], but the question of whether or not a smooth space-time structure can be recovered in some limit remains a great challenge in this approach. In order to address this question, it is in particular crucial to develop new renormalization tools, which should allow to efficiently explore the phase space of spin foam models. Two renormalization programmes have recently emerged to meet this challenge. One is based on an interpretation of spin foam models as direct space regularizations of quantum gravity [4, 38], and therefore explores generalizations of lattice renormalization techniques. The other one interprets spin foam amplitudes as Feynman contributions in the perturbative expansion of a specific GFT [44, 69, 78], and therefore requires generalizations of local field theory renormalization techniques to non-local field theories such as (T)GFTs. We here review this second research programme, in the particular context of TGFTs with *gauge invariance condition* (or equivalently *closure constraint*), which have been extensively studied in the literature [17, 22, 30, 31, 32, 35, 36, 60, 61, 62, 63, 64, 81, 84]. Though not as sophisticated as tentative GFT models for 4d quantum gravity [7, 8], such theories generate non-trivial spin foam amplitudes and require key generalizations of ordinary renormalization methods. They therefore provide a natural test bed for future and more challenging studies of quantum gravity models.

Our goal is to present the reader with a bird's eye view on the recent literature, and to clearly explain the motivations and the status of the subject. We will as much as possible refrain from delving into technical details. Given the rapid development of the subject, we will also not claim to be exhaustive. The choice of topics to be developed in the main text was at least to some extent a matter of personal taste.

The plan of this review is as follows. In Section 2 we motivate further the GFT renormalization programme, as well as the specific class of models which has been investigated up to now. Perturbative renormalizability is reviewed in detail in Section 3. A full classification of renormalizable model based on rigorous power-counting arguments is in particular provided. We then go on to investigations of the properties of the renormalization group flows of these models in Section 4. Emphasis is put on functional renormalization methods, which are of great practical interest even though they have so far only been applied to GFTs in the crudest truncations.

2 From simplicial to tensorial GFTs

2.1 Renormalization of GFTs: motivations and basic ingredients

We begin with the introduction of basic GFT structures, which may not be general enough to encompass all models relevant to full 4d quantum gravity, but will be sufficient in the present context. We define a GFT as a quantum field theory for a single complex scalar field φ leaving on d copies of a fixed compact Lie group G . Unless specified otherwise, we will use a vector notation for configuration space variables and its Haar measure

$$\mathbf{g} = (g_1, \dots, g_d) \in G^d, \quad d\mathbf{g} = dg_1 \cdots dg_d.$$

²Spin foam amplitudes fall in the class of generalized lattice gauge theories which may be generated by a GFT.

We will also use the short-hand notation:

$$\bar{\varphi}_1 \cdot \varphi_2 = \int d\mathbf{g} \bar{\varphi}_1(\mathbf{g}) \varphi_2(\mathbf{g}),$$

for any two square-integrable functions φ_1 and φ_2 on G^d . The dynamics of the GFT field is specified by a probability measure

$$d\mu_{C_\Lambda}(\varphi, \bar{\varphi}) \exp(-S_\Lambda[\varphi, \bar{\varphi}]) \quad (2.1)$$

or equivalently a generating functional

$$\mathcal{Z}_\Lambda[J, \bar{J}] = \int d\mu_{C_\Lambda}(\varphi, \bar{\varphi}) \exp(-S_\Lambda[\varphi, \bar{\varphi}] + \bar{J} \cdot \varphi + \bar{\varphi} \cdot J). \quad (2.2)$$

The measure $d\mu_{C_\Lambda}$ is the Gaussian measure associated to the covariance C_Λ , which is a positive operator with kernel:

$$C_\Lambda(\mathbf{g}; \mathbf{g}') = \int d\mu_{C_\Lambda}(\varphi, \bar{\varphi}) \bar{\varphi}(\mathbf{g}) \varphi(\mathbf{g}').$$

In other words, C_Λ is the propagator of the GFT, or equivalently its free 2-point function. We explicitly introduced an extra regularization or scale parameter $\Lambda > 0$, as the covariance is in general plagued with divergences. The role of this parameter is central in our renormalization programme; we will describe it in greater details once a specific choice of propagator is made. Perturbations around the Gaussian theory are introduced through the (interaction part of the) GFT action $S_\Lambda[\varphi, \bar{\varphi}]$, which we will parametrize as

$$S_\Lambda[\varphi, \bar{\varphi}] = \sum_{b \in \mathcal{B}} t_b(\Lambda) I_b[\varphi, \bar{\varphi}],$$

where \mathcal{B} is a set of elementary interactions, I_b is the specific monomial in the fields associated to the interaction b , and $t_b(\Lambda)$ is the running coupling constant associated to b at scale Λ . One main objective of renormalization is to determine how these coupling constants should be adjusted to compensate for a change in the cut-off Λ . More precisely, as in ordinary field theory, we will require that the infrared content of the connected $2k$ -point functions (or Schwinger functions)

$$S_\Lambda^{(2k)}(\mathbf{g}_1, \dots, \mathbf{g}_k, \mathbf{g}'_1, \dots, \mathbf{g}'_k) = \left(\prod_{i=1}^k \frac{\delta}{\delta \bar{J}(\mathbf{g}_i)} \frac{\delta}{\delta J(\mathbf{g}'_i)} \right) \ln \mathcal{Z}_\Lambda[J, \bar{J}] \Big|_{J=\bar{J}=0},$$

which also characterizes the random measure (2.1), is invariant under a change of Λ . Unlike ordinary field theories, we do not have any prior notion of energy scale which we can rely on to determine what 'infrared' means in the GFT context. We will therefore need to adopt a more abstract notion of scale, *a priori* unrelated to familiar space-time concepts.

For specific choices of group, propagator and basic interactions, the formal Feynman expansion of (for instance) the partition function

$$\mathcal{Z}_\Lambda := \mathcal{Z}_\Lambda[0, 0] = \sum_{\mathcal{G}} \prod_{b \in \mathcal{B}} (-t_b(\Lambda))^{n_b(\mathcal{G})} \mathcal{A}_{\mathcal{G}}(\Lambda), \quad (2.3)$$

generates Feynman diagrams \mathcal{G} which are in one-to-one correspondence with specific (closed) 2-complexes and are weighted by spin foam amplitudes $\mathcal{A}_{\mathcal{G}}$ ³. We remind the reader that, more

³Note that symmetry factors may have to be included in formula (2.3), depending on the detailed definition of the model and of the Feynman amplitudes. They are not relevant to the present discussion.

generally, spin foams are combinatorial objects interpolating between spin-network boundary states, whose amplitudes are constructed as discrete gravity path-integrals taking the form of generalized lattice gauge theory amplitudes. They are therefore interpreted as quantum space-time processes encoding the dynamics of loop quantum gravity spin-network functionals. In this review, we ignore the extra combinatorial structure provided by properly closed boundary spin-network states, since they do not play an essential role in what we want to discuss: the renormalization of such quantum gravity amplitudes is a particular case of that of general n -point functions (associated to n open boundary spin-network vertices), we can therefore focus entirely on the latter. In view of equation (2.3), the GFT formalism provides natural prescriptions for resumming infinite classes of spin foam amplitudes, with weights parametrized by the GFT coupling constants. This is the sense in which GFTs allow to complete the definition of spin foam models, which cannot be claimed to fully specify a dynamics for quantum gravity unless an extra organization principle for its amplitudes is clearly spelled out. The summing prescription implemented through GFT is one such possible organization principle, which effectively removes a large class of discretization ambiguities entering the definition of spin foam models (but not all). What remains to be checked is: 1) whether this formal procedure is mathematically consistent; and 2) whether it is physically relevant, in the sense that general relativity can be recovered in some limit. Renormalization will presumably play an important role in order to meet both of these challenges.

The formal relation between GFTs and spin foam models being an intrinsically perturbative statement, checking its validity and consistency is essentially equivalent to proving renormalizability of the GFT. At the very least one needs to check that – again formally – the set of GFT interactions I_b is stable under a shift of the cut-off Λ . This translates into a formal stability of the set of spin foams summed over on the right-hand side of equation (2.3). But turning this rather vague statement into a sensible perturbative definition requires that the same stability holds with only finitely many GFT interactions turned on, and that is equivalent to the perturbative renormalizability of the GFT. The relevant GFT interactions will then uniquely determine which (finitely many) elementary spin foam interaction vertices dominate in this perturbative phase. This is a more concrete and more rigorous illustration of how the GFT formalism may be powerfully used to remove spin foam discretization ambiguities and make predictions.

Furthermore, assuming that general relativity may only be recovered in a phase in which macroscopically large spin-network boundary states acquire large amplitudes, addressing the second open problem will presumably require to go beyond perturbation theory. This suggests that the GFT phase space should be more systematically explored away from its perturbative regime, and the existence of non-trivial fixed points of the renormalization group investigated. It is important to realize that such non-trivial fixed points would correspond to non-perturbative resummations of spin-foam amplitudes and would therefore be very hard to grasp without recourse to GFT. They will generate new vacua, supporting new and possibly inequivalent representations of the GFT, and therefore leading to new GFT phases and phase transitions. The mechanism of Bose–Einstein condensation has in particular been investigated in this context, leading to interesting reconstructions of smooth homogeneous and spherically symmetric space-time geometries from the GFT formalism, with fascinating applications to cosmology [47, 48, 73] and black holes [72]. If such a scenario based on a collective reorganization of the spin foam amplitudes is correct, the field theory language provided by GFT and the powerful effective methods it entails seems hardly avoidable.

We now briefly outline some basic features of GFT model-building, the interested reader is referred to reviews on the subject [9, 43, 57, 70, 71] and references therein for further details. The specific GFTs which generate quantum gravity spin foam amplitudes (for instance [8, 13, 58]) require G to be a local symmetry group of space-time (or space), e.g., the Lorentz group

$\text{SO}(1, d-1)$ or its universal covering. In this review, we will only consider Euclidean groups – in particular $\text{SU}(2)$ in dimension $d = 3$ – and ignore complications arising from the Lorentzian signature⁴. Another important ingredient is the so-called *gauge invariance condition*, defined as a global symmetry of the GFT field under simultaneous translation of its group variables:

$$\forall h \in G, \quad \varphi(g_1 h, \dots, g_d h) = \varphi(g_1, \dots, g_d). \quad (2.4)$$

This condition is common to all known proposals of GFT models for quantum gravity, and is responsible for the generalized lattice gauge theory form of the amplitudes appearing on the right-hand side of equation (2.3). Within the general GFT formalism spelled out at the beginning of this section, we implement this symmetry by requiring that the covariance C_Λ is of the form

$$C_\Lambda = \mathcal{P} \tilde{C}_\Lambda \mathcal{P},$$

where \tilde{C}_Λ is again a positive operator, and \mathcal{P} is the projector on translation invariant GFT fields with kernel

$$\mathcal{P}(\mathbf{g}, \mathbf{g}') = \int_G dh \prod_{\ell=1}^d \delta(g_\ell h g_\ell'^{-1}).$$

Hence C_Λ is degenerate and its image lies within the space of fields verifying (2.4). The gauge invariance condition, also called *closure constraint*, is the main dynamical ingredient of GFT models for quantum BF theory in arbitrary dimension. In dimension 3, it turns out that $\text{SU}(2)$ BF theory can be interpreted as a theory of Euclidean gravity, and therefore $\text{SU}(2)$ GFT with closure constraint provides a natural arena in which to formulate $3d$ Euclidean quantum gravity models. A typical example is the Boulatov model [29] (which generates Ponzano–Regge spin foam amplitudes), a more recent version of which will be introduced below. In higher dimensions, further conditions on the GFT fields – which are known as *simplicity constraints* – need to be implemented, possibly leading to further degeneracies of the covariance. In what follows, we will however ignore such constraints and focus on examples in which \tilde{C}_Λ is non-degenerate. Finally, in most of the quantum-gravity literature, the 2-complexes supporting spin foam amplitudes are assumed to be dual to simplicial decompositions of manifolds. This is a simplification entering the construction of discrete gravity path-integrals which, though very natural, is as far as we can tell not very well motivated. At the GFT level, this corresponds to a choice of action S_Λ comprising a unique type of monomials I_b , which are of order $(d+1)$ and contract the field variables pairwise following the pattern of a d -simplex. Such models have very rigid combinatorial structure, and therefore their renormalization programme is more difficult to apprehend. Furthermore, radiative corrections which are not of the simplicial type are in general generated by the renormalization group flow, and therefore need to be added to S_Λ from the outset. The importance of this simple realization should not be underestimated: in ordinary quantum field theory, a similar argument implies that one should in principle allow any number of local interactions in the action; it is then left to the renormalization group to identify a finite relevant subset of local operators among all possible interactions. Likewise, a renormalization programme for GFTs requires the prior identification of an infinite reservoir of allowed interactions, providing a suitable generalization of ordinary locality.

The purpose of the next three subsections is to explain how recent developments in tensor models made such a generalized notion of locality available and allowed to launch a GFT renormalization programme. We will more specifically focus on GFTs with gauge invariance

⁴The renormalization of GFTs with Lorentzian signature remains largely open and will likely become an active field of research in the close future. See however [76] for a first attempt in this direction.

condition, and explain (including a new heuristic argument presented for the first time in this review) in which sense so-called *colored GFTs* [49] – which are more recent and better behaved versions of the simplicial GFTs briefly mentioned before – can be embedded in a larger and flexible enough class of models. These theories are known as *tensorial GFTs* and are the main focus of the rest of the review.

2.2 Simplicial GFT models and tensorial theory space

Colored tensor models [55] and GFTs were introduced in 2009 by Gurau [49] and have since then overcome two important caveats of older simplicial constructions [29, 68]: 1) in dimension $d \geq 3$, the combinatorial data of Feynman diagrams failed to unambiguously encode the combinatorial structure and topology of the simplicial complexes generated in perturbative expansion; 2) even though power counting results could be derived [14, 44, 66], no consistent organization of the amplitudes – such as the celebrated $1/N$ expansion of matrix models – could be proposed to make sense of the formal perturbative expansion. We therefore decided to gloss over the original simplicial models and work instead with colored structures from the outset.

Within the general GFT set-up we have described, the definition of a colored GFT model in d dimensions requires the introduction of d auxiliary complex GFT fields $\{\varphi_c | c = 1, \dots, d\}$, with covariance \bar{C}_Λ not necessarily identical to C_Λ . The label c is called *color*, and we conventionally associate the color 0 to the original field $\varphi \equiv \varphi_0$. The action S_Λ is then implicitly defined by

$$\exp(-S_\Lambda[\varphi, \bar{\varphi}]) = \int \left[\prod_{c=1}^d d\mu_{\bar{C}_\Lambda}(\varphi_c, \bar{\varphi}_c) \right] \exp(-S_\Lambda^{\text{col}}[\varphi, \bar{\varphi}; \varphi_c, \bar{\varphi}_c]), \quad (2.5)$$

where the colored GFT action is

$$\begin{aligned} S_\Lambda^{\text{col}}[\varphi_0, \bar{\varphi}_0; \varphi_c, \bar{\varphi}_c] &= \lambda(\Lambda) \int \left[\prod_{0 \leq \ell < \ell' \leq d} dg_{\ell\ell'} \right] \prod_{\ell=0}^d \varphi_\ell(\mathbf{g}_\ell) \\ &+ \bar{\lambda}(\Lambda) \int \left[\prod_{0 \leq \ell < \ell' \leq d} dg_{\ell\ell'} \right] \prod_{\ell=0}^d \bar{\varphi}_\ell(\mathbf{g}_\ell) \end{aligned} \quad (2.6)$$

and we have used the convention that $g_{\ell\ell'} = g_{\ell'\ell}$ together with the notation

$$\mathbf{g}_\ell = (g_{\ell\ell-1}, \dots, g_{\ell 0}, g_{\ell d}, \dots, g_{\ell\ell+1}).$$

The two interactions are interpreted as pairwise gluings of $(d+1)$ $(d-1)$ -simplices along $(d-1)$ -subsimplices, following to the shape of a d -simplex. This pattern of contractions can be pictorially represented by white (resp. black) nodes where $(d+1)$ colored half-edges meet as in Fig. 1. An half-edge of color ℓ is associated to a GFT field φ_ℓ or $\bar{\varphi}_\ell$, and is dual to a $(d-1)$ -simplex of color ℓ . A pair of edges of colors ℓ and ℓ' in turn encodes the integral over the variable $g_{\ell\ell'}$ in formula (2.6); it is interpreted as a pairwise gluing of two dual $(d-1)$ -simplices along a $(d-2)$ -subsimplex. In Fig. 2, we provide an equivalent stranded representation of the pattern of contractions associated to the $3d$ vertices, and illustrate how the dual tetrahedra can be reconstructed from the colored vertices: half-lines are dual to triangles, which are glued pairwise along their boundary edges.

Note that half-lines associated to fields φ and $\bar{\varphi}$ (with color 0) are dashed. This is to emphasize that the latter are the true dynamical variables of the theory; for instance, in equation (2.2) we remark that only them have been coupled to external sources. The colored fields φ_c for $c = 1, \dots, d$ can therefore be (formally) integrated out. This yields effective interactions I_b parametrized by d -colored graphs b involving colors $c = 1$ to d . These graphs are also called

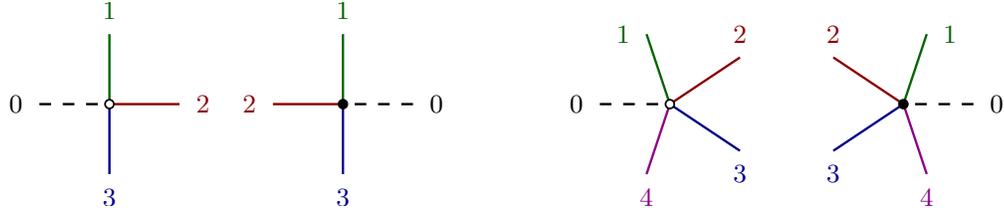


Figure 1. Colored interaction vertices in dimension $d = 3$ (left) and $d = 4$ (right).

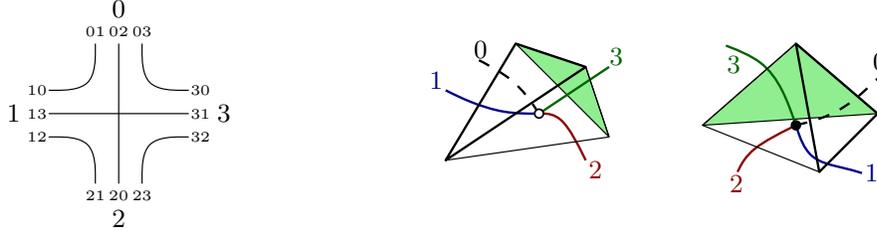


Figure 2. Stranded representation of the interactions in dimension $d = 3$ (left) and dual simplicial picture (right). Colors of boundary triangles are left implicit, except for green ones.

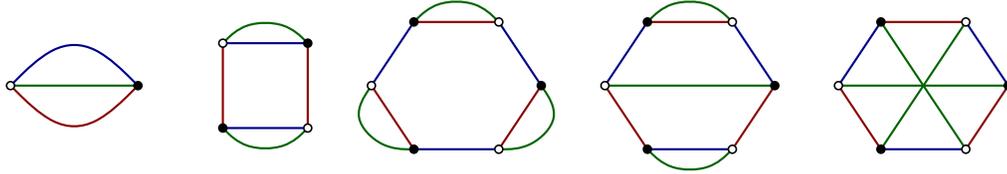


Figure 3. Bubble interactions up to order 6 in $d = 3$.

bubbles in the literature (e.g., in [25, 50]; examples in $d = 3$ are provided in Fig. 3. The coupling constant t_b is moreover proportional to⁵ $(\lambda\bar{\lambda})^{N_b/2}$, with N_b the number of nodes in the colored graph b .

The precise form of the effective interaction I_b highly depends on the auxiliary covariance \bar{C}_Λ and is in general quite involved. Let us discuss the special and simple situation in which

$$\bar{C}_\Lambda(\mathbf{g}; \mathbf{g}') = \prod_{\ell=1}^d \delta_\Lambda(g_\ell g_\ell'^{-1}), \quad (2.7)$$

where δ_Λ is a regularized version of the delta function on G . More precisely, we assume that Λ is a sharp cut-off in the Fourier expansion of δ ⁶. Under this condition, it can be shown that the effective I_b are nothing but *tensor invariants* (up to constant factors and powers of the cut-off Λ that we ignore for the moment). We refer the reader to [50], in which tensor invariants were first introduced and where their derivation is described in greater details. Following the literature and by analogy with matrix models, we will use in this case a trace notation $I_b \equiv \text{Tr}_b$. A monomial $\text{Tr}_b(\varphi, \bar{\varphi})$ is uniquely determined by its d -colored bubble b , under the following rules:

⁵The proportionality factor depends on the combinatorial structure of the bubbles alone, not on the other ingredients of the model.

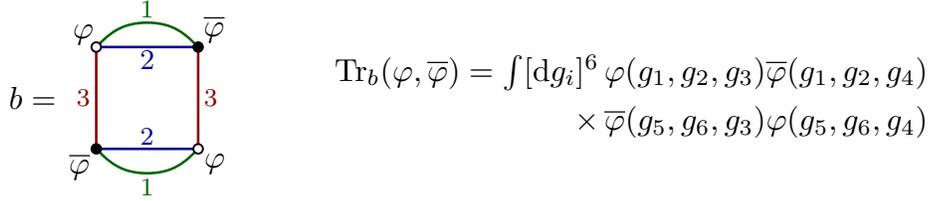
⁶For instance, when $G = \text{SU}(2)$, one may define

$$\delta_\Lambda(g) := \sum_{j \in \frac{\mathbb{N}}{2} | j(j+1) \leq \Lambda^2} (2j+1) \chi_j(g),$$

where χ_j are the characters of $\text{SU}(2)$.

- a white (resp. black) node of b is associated to a field φ (resp. $\bar{\varphi}$);
- an edge of color ℓ represents a convolution of two field variables, both appearing in the ℓ^{th} copy of the group G .

An example is provided in Fig. 4. In this simplified context, tensor invariant interactions generate an infinite-dimensional GFT theory space in which colored simplicial models are embedded as a one-parameter family of models⁷. Bubble interactions therefore provide a generalized notion of locality of the type we have been arguing for. Topologically, they represent elementary but non-simplicial cells with triangulated boundaries. A suggestive 3d example is given in Fig. 5⁸. Note however that bubbles may also be dual to topologically singular⁹ elementary cells, such as, e.g., a topological cone over a non-spherical $(d - 1)$ -dimensional manifold. This is precisely the case for the rightmost bubble of Fig. 3, which is dual to a topological cone over the 2-torus.



$$\text{Tr}_b(\varphi, \bar{\varphi}) = \int [dg_i]^6 \varphi(g_1, g_2, g_3) \bar{\varphi}(g_1, g_2, g_4) \times \bar{\varphi}(g_5, g_6, g_3) \varphi(g_5, g_6, g_4)$$

Figure 4. A four-valent 3d bubble and its corresponding tensor invariant.

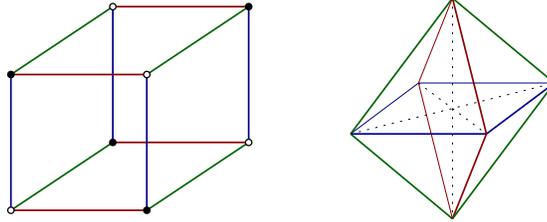


Figure 5. A 3d bubble of valency 8 dual to a double pyramid, which can equivalently be viewed as a gluing of 8 colored tetrahedra.

The relevance of the tensorial theory space for GFT renormalization has been first pointed out in a seminal paper of Ben Geloun and Rivasseau [18], who proved renormalizability of a tensorial GFT without gauge invariance condition. This is a context in which the argument we have just presented is applicable, and the relation between tensor invariant models and colored simplicial ones is therefore clear. The situation is more ambiguous as soon as one introduces gauge invariance or other quantum gravity ingredients. In this case one can make two *a priori* inequivalent choices.

1. The first possibility is to choose the covariances C_Λ and \bar{C}_Λ equal. Both are in particular degenerate, and the effective interactions I_b become quite complicated and hard to manipulate in concrete calculations. From the point of view of known spin foam models, which

⁷Colored simplicial models with covariance (2.7) generate coupling constants t_b which are functions of $\lambda\bar{\lambda}$, hence the one-dimensional character of this subspace of theories.

⁸Note that the boundary edges of the double pyramid on the right can be canonically colored. This illustrates one of the main advantages of the colored structure: it allows to canonically identify all subsimplices in the complex, and therefore faithfully encode its topology. See, e.g., [42, 55].

⁹A topological singularity is defined as a point whose neighbourhood is not homomorphic to a ball. This notion should not be confused with that of a metric singularity.

are derived from simplicial discretizations of formal quantum gravity path-integrals, this is however the most natural construction.

2. The second possibility is to assume, as we have done before, that the auxiliary colored fields φ_c have trivial covariance \bar{C}_Λ . One may argue in this case that imposing suitable spin foam constraints on the remaining dynamical field φ will again lead to legitimate discrete gravity path-integrals, however based on non-simplicial cellular complexes.

Which of these two alternatives is the most appropriate remains an open question, and we will not attempt to resolve it in the present article. We will stick to the second alternative, as GFT renormalization has only been substantially explored in this framework. But before that, we outline an additional heuristic calculation which provides a better grasp of the relation between the two approaches, at least in the context of $3d$ Euclidean quantum gravity.

2.3 Large N expansion and extended tensorial theory space: heuristic derivation

The colored Boulatov model (studied in, e.g., [5, 6, 33, 49, 51]) is a GFT for Euclidean quantum gravity in space-time dimension $d = 3$. The group G is therefore taken as the (universal covering) of the local symmetry group of Euclidean space: $G = \text{SU}(2)$. The propagator with cut-off may be defined as

$$C_N(g_1, g_2, g_3; g'_1, g'_2, g'_3) = \int_{\text{SU}(2)} dh \prod_{\ell=1}^3 K_{1/N^2}(g_\ell h g_\ell'^{-1}) \xrightarrow{N \rightarrow +\infty} \mathcal{P}(g_1, g_2, g_3; g'_1, g'_2, g'_3),$$

where K_α is the heat-kernel on $\text{SU}(2)$ at time α ¹⁰, and we denote the cut-off by N instead of Λ in reference to the original literature [51, 52, 54]. The auxiliary covariances appearing in formula (2.5) are furthermore taken equal to the propagator: $\bar{C}_N = C_N$. In [51] Gurau showed that, under the assumption that the coupling constant asymptotically behaves like

$$\lambda(N) \underset{N \rightarrow +\infty}{\sim} \frac{\lambda_0}{N^{3/2}}$$

for some fixed λ_0 , the colored Boulatov model admits a $1/N$ -expansion. In particular, the partition function can be expanded as

$$\mathcal{Z}_N = N^6 \mathcal{Z}_0(\lambda_0 \bar{\lambda}_0) + N^3 \mathcal{Z}_1(\lambda_0 \bar{\lambda}_0) + O(1), \quad (2.8)$$

where \mathcal{Z}_0 and \mathcal{Z}_1 sum over specific infinite families of Feynman diagrams representing spherical manifolds¹¹. What is of crucial interest for us is that singular topologies, and hence singular effective interactions, are all convergent. This clear separation between the first leading contributions in N and the first topologically singular spin foam structures, already established in [51], was more systematically investigated in [33, 34] by means of different methods¹². Sin-

¹⁰This defines a regularization of the delta function in which high spin representations are smoothly cut-off:

$$K_\alpha(g) = \sum_{j \in \frac{\mathbb{N}}{2}} e^{-\alpha j(j+1)} (2j+1) \chi_j(g).$$

¹¹The family summed over by \mathcal{Z}_0 – the melonic graphs – has been extensively studied in tensor models (see, e.g., [24, 55, 56] as well as in the present context [5]). The partition function \mathcal{Z}_1 has as far as we know not been studied in great details, but it is nonetheless known that it sums spherical manifolds [52].

¹²In particular, tighter bounds were derived, showing that singular topologies are suppressed in at least $N^{3(1-S)}$, where S is the number of singular bubbles. A similar result was shown to hold also in the case of the $4d$ colored Ooguri model [34].

gular topologies having no natural space-time interpretation at this point¹³, this is a welcomed property of the $1/N$ expansion.

Now, this means that we can truncate the effective action defined in equation (2.5) to non-singular bubbles without affecting \mathcal{Z}_0 and \mathcal{Z}_1 . By definition, such non-singular bubbles have moreover spherical boundaries, which implies that they lead to bulk amplitudes which are peaked around trivial holonomies. Let us give an illustration of what this means by focusing on the simplest possible bubble: the one with 2 nodes shown on the leftmost side of Fig. 3. It can be shown that it generates a term in the action S_N of the form

$$I_2(\varphi, \bar{\varphi}) = \int d\mathbf{g}d\mathbf{g}' \bar{\varphi}(\mathbf{g})\varphi(\mathbf{g}') \int [dh_i]^3 K_{1/N^2}(h_1 h_2^{-1}) K_{1/N^2}(h_1 h_3^{-1}) K_{1/N^2}(h_2 h_3^{-1}) \\ \times \prod_{i=1}^3 K_{1/N^2}(g_i h_i g_i'^{-1}).$$

Using the gauge invariant condition (2.4), one is free to translate the g_i variables by (say) h_3^{-1} . This reduces the h_i dependence of the last line to a dependence in $h_3^{-1}h_1$ and $h_3^{-1}h_2$. Performing the change of variables $h_1 \rightarrow h_3 h_1$ and $h_2 \rightarrow h_3 h_2$, we hence obtain an integral which is completely independent of h_3 . By normalization of the Haar measure, the new expression of I_2 is thus

$$I_2(\varphi, \bar{\varphi}) = \int d\mathbf{g}d\mathbf{g}' \bar{\varphi}(\mathbf{g})\varphi(\mathbf{g}') \int dh_1 dh_2 K_{1/N^2}(h_1 h_2^{-1}) K_{1/N^2}(h_1) K_{1/N^2}(h_2) \\ \times K_{1/N^2}(g_1 h_1 g_1'^{-1}) K_{1/N^2}(g_2 h_2 g_2'^{-1}) K_{1/N^2}(g_3 g_3'^{-1}). \quad (2.9)$$

This procedure is nothing else than a gauge fixing and is quite general: there is a gauge freedom associated to each node in the bubble, which allows to trivialize the holonomies along a tree of colored edges [46]. Now, in the large N limit, one realizes that: the heat-kernels appearing in the first line of equation (2.9) render the integrand sharply peaked around $h_1 = h_2 = \mathbf{1}$; together with the second line, this implies that the integrand is also sharply peaked around $g_i = g_i'$. This simple fact entitles us to expand $\varphi(\mathbf{g}')$ in Taylor expansion around \mathbf{g} :

$$\varphi(\mathbf{g}') = \varphi(\mathbf{g}) + \frac{d}{dt} \Big|_{t=0} \varphi(\mathbf{g}(t)) + \frac{1}{2} \frac{d^2}{dt^2} \Big|_{t=0} \varphi(\mathbf{g}(t)) + \dots,$$

where $\mathbf{g}'(t)$ is an affinely parametrized geodesic from \mathbf{g} to \mathbf{g}' in $SU(2)^3$. This reduces I_2 to an infinite sum over tensor invariant contractions of the fields *and their derivatives*. More precisely, this procedure can only generate $SU(2)$ -invariant differential operators acting on each copy of $SU(2)$ and one therefore obtains:

$$I_2(\varphi, \bar{\varphi}) = a(\Lambda) \int d\mathbf{g} \bar{\varphi}(\mathbf{g})\varphi(\mathbf{g}) + b(\Lambda) \int d\mathbf{g} \bar{\varphi}(\mathbf{g}) \left(- \sum_{\ell=1}^3 \Delta_\ell \right) \varphi(\mathbf{g}) + \dots, \quad (2.10)$$

where Δ_ℓ is the Laplace operator acting on the ℓ^{th} copy of $SU(2)$, and $a(\Lambda)$, $b(\Lambda)$ are computable functions. Higher order terms will involve invariant differential operators of arbitrary order.

The previous argument can be applied in full generality. Any effective vertex I_b associated to a non-singular bubble b can be expanded into tensor invariant contractions of the fields and their derivatives. Moreover, only invariant differential operators are allowed in this expansion. By picking up a basis of such operators, one can thus define a generalized space of bubbles $\bar{\mathcal{B}} \supset \mathcal{B}$ labelling generalized trace invariants $\text{Tr}_{\bar{b}}(\varphi, \bar{\varphi})$. The original bubble interactions $\text{Tr}_b(\varphi, \bar{\varphi})$ (with

¹³While metric singularities are generic in general relativity, topological singularities are completely absent of standard models of classical space-time.

$b \in \mathcal{B}$) therefore generate a small subset of generalized tensor invariants, those which do not involve any non-trivial differential operator. What our analysis proves is that, up to convergent and topologically singular contributions in the $1/N$ expansion (2.8), the colored Boulatov model generates a one-parameter family of effective actions in the space of generalized tensor invariants. They therefore provide a suitable GFT theory space for the implementation of the first strategy proposed at the end of the previous subsection, and also shows that the second approach is actually a truncation of the first.

Finally, we point out that the same argument can be implemented for the colored Ooguri model, and leads to the definition of generalized tensor invariant interactions for $\text{Spin}(4)$ in dimension 4. It remains however to understand how this heuristic calculation may be generalized to $4d$ models with simplicity constraints, which have not yet been shown to admit $1/N$ expansions.

3 Perturbatively renormalizable TGFTs with closure constraint

3.1 A general class of models: local 'potential' approximation

We are now ready to introduce the class of TGFTs with gauge invariant condition, as defined in the literature. We are still in the general set-up introduced at the beginning of the preceding section. Namely, we consider a complex GFT field φ over $d \geq 3$ copies of a compact Lie group G . Its free 2-point function is assumed to be of the form

$$C_\Lambda(\mathbf{g}; \mathbf{g}') = \int_{1/\Lambda^2}^{+\infty} d\alpha \int_G dh \prod_{\ell=1}^d K_\alpha(g_\ell h g_\ell'^{-1}), \quad (3.1)$$

which is a regulated version of the formal operator $C = \mathcal{P} \left(- \sum_{\ell=1}^d \Delta_\ell \right)^{-1} \mathcal{P}$, K_α and Δ_ℓ being respectively heat-kernels and Laplace operators on G . Equation (3.1) is known as the Schwinger representation of the propagator, and α is accordingly called a Schwinger parameter. As for the interaction action S_Λ , we assume that it is generated by tensor invariants¹⁴:

$$S_\Lambda[\varphi, \bar{\varphi}] = \sum_{b \in \mathcal{B}} t_b(\Lambda) \text{Tr}_b[\varphi, \bar{\varphi}]. \quad (3.2)$$

Furthermore, since color labels have been introduced as purely auxiliary objects, we will also assume that the action is invariant under permutations of the colors. This implies specific dependences between some of the coupling constants t_b appearing in equation (3.2).

From the point of view of the extended GFT space described at the end of the last section, this is the exact analogue of a local potential approximation in ordinary quantum field theory. Indeed, locality is here embodied by tensor invariance, which entitles us to call the action S_Λ a local potential¹⁵. The only non-local terms are restricted to the first non-trivial differential operators appearing in the Taylor expansion of the most general propagator (see equation (2.10)): the Laplace operators Δ_ℓ .

Note that in this review we focus more precisely on *connected* tensor invariants. Non-connected bubble contributions may also be included in the action (3.2), and sometimes cannot be dispensed with. For instance, the diagram shown in Fig. 6 is associated to the non-connected

¹⁴Note that in this review we will always include mass terms in the action rather than the covariance, but the opposite convention is sometimes found in the literature.

¹⁵Even though we cannot define a potential function, due to the combinatorially non-trivial nature of tensorial locality.

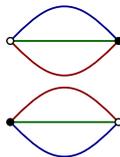


Figure 6. A bubble with two connected components.

tensor invariant:

$$\left(\int d\mathbf{g} \bar{\varphi}(\mathbf{g}) \varphi(\mathbf{g}) \right) \cdot \left(\int d\mathbf{g} \bar{\varphi}(\mathbf{g}) \varphi(\mathbf{g}) \right),$$

that is to the product of the tensor invariants encoded by its connected components. Such interactions have for instance appeared in [84], and have been more systematically studied in [60]. In the models we will more particularly discuss below, non-connected interactions are irrelevant (as implicitly shown in, e.g., [35], but more systematically derived in [60]), and for simplicity we have decided to ignore them altogether.

In order to legitimize the class of TGFTs thus defined as a perfectly honest arena for renormalization, we need to comment a bit more on the notion of scale in this context. For definiteness, let us specialize to $G = \text{SU}(2)$, which is one of the most relevant example as far as quantum gravity is concerned. The heat-kernel regularization we have introduced amounts to a smooth regularization of the quantity

$$p^2 = \sum_{\ell=1}^d j_{\ell}(j_{\ell} + 1),$$

where $\{j_{\ell}\}$ are the spin labels associated to the harmonic expansion of the fields. Modes associated to $p \geq \Lambda$ are exponentially suppressed, while the theory is essentially untouched at small p . By analogy with ordinary field theories, we may call p momentum. The fact that we need to regularize large momenta is dictated by the theory itself: this region of the GFT state space is where most of the degrees of freedom lie, and where they produce divergences. Therefore the renormalization group may only flow from large to small cut-off¹⁶. Because of that, and by analogy with high energy particle physics, it is conventional in the TGFT literature to dub large (resp. small) momenta 'ultraviolet' (resp. 'infrared'); we will stick to this nomenclature.

The purpose of our renormalization programme may now be explicitly stated: the goal is to develop the necessary tools for determining the functional dependence of the coupling constants $t_b(\Lambda)$, under the condition that the infrared sector of the GFT is kept fixed. We will in particular aim at a complete classification of perturbatively renormalizable models. This programme is interesting *per se*, in the sense that it proposes to extend the scope of renormalization theory to quantum field theories with exotic notions of locality. From a quantum gravity perspective its relevance is on the other hand conditioned by a key conjecture: that it is possible to assume that there is a large separation of scales between the cut-off and the support of interesting $3d$ Euclidean quantum gravity states. It is reassuring to see that this hypothesis is at least superficially consistent. Given that spins label the eigenvalues of the LQG length operator in $3d$ and that small spins are associated to small lengths, we may for example expect that smooth quantum gravity states can be approximated by (large superpositions of) spin-networks comprising a large number of nodes and edges (or, equivalently, dual triangles and dual edges), but only bounded spins. This also suggests that the study of these smooth quantum gravity states

¹⁶We remind the reader that the renormalization group is actually not a group: it has a fundamentally directed character since its whole purpose is to erase (irrelevant) physical information.

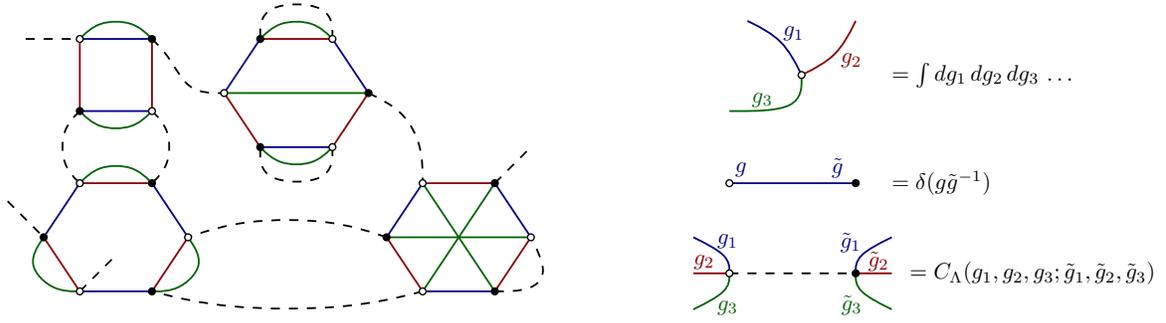


Figure 7. Example of TGFT Feynman graph with $N = 4$, $V = 4$ and $L = 9$ in $3d$ (left). The amplitude is reconstructed from the drawing by means of the Feynman rules given on the right-hand side.

will necessitate a non-perturbative treatment of the GFT renormalization group¹⁷. The present section is devoted entirely to perturbative questions, while some non-perturbative aspects will be discussed in the following one.

3.2 Power-counting theorem and classification of models

We now outline the power-counting arguments leading to the full classification of renormalizable TGFTs with closure constraint. A very nice feature of TGFTs is that they are amenable to general multiscale methods developed in the context of constructive field theory [77], allowing rigorous proofs of renormalizability at all orders.

The Feynman amplitudes of these models are labelled by $(d + 1)$ -colored graphs in which only dashed (or, equivalently, color-0) lines may be open. The d -colored connected components without dashed lines, in other words the bubbles, are the interaction vertices, while dashed lines are propagators. Given a Feynman graph \mathcal{G} , we will denote by $L(\mathcal{G})$, $V(\mathcal{G})$ and $N(\mathcal{G})$ its set of (internal) dashed lines, bubble vertices and external legs¹⁸. Accordingly, the amplitude associated to a graph \mathcal{G} is determined by the following Feynman rules: each node is associated to an integration over G^d ; each colored line internal to a bubble represents a delta function on G ; and finally, dashed lines must be replaced by kernels of C_Λ . We provide a $3d$ example in Fig. 7.

The multiscale analysis¹⁹ relies on a discrete slicing of the propagator:

$$C_\Lambda = C_{M^\rho} = \sum_{i \in \mathbb{N} | i \leq \rho} C_i, \quad (3.3)$$

where

$$C_0 := \int_1^{+\infty} d\alpha \int_G dh \prod_{\ell=1}^d K_\alpha(g_\ell h g_\ell'^{-1}),$$

$$\forall i \geq 1, \quad C_i := \int_{M^{-2i}}^{M^{2(i-1)}} d\alpha \int_G dh \prod_{\ell=1}^d K_\alpha(g_\ell h g_\ell'^{-1}).$$

$M > 1$ is a fixed but arbitrary slicing parameter and we have assumed that the UV cut-off is of the form $\Lambda = M^\rho$ with $\rho \in \mathbb{N}$. Each covariance C_i is then essentially responsible for the

¹⁷Another related observation is that quantum gravity may require the inclusion of differential operators of arbitrary orders, as found in the generalized tensor invariant space evoked previously.

¹⁸As is customary in the literature, we will allow ourselves to use the same notation for the cardinals of these sets.

¹⁹In the present context, see for instance [30, 35, 36, 84]. For a more general and in-depth discussion in local field theory, [77] is recommended.

propagation of the modes with $M^{i-1} \lesssim p \lesssim M^i$. This procedure induces a decomposition of Feynman amplitudes

$$\mathcal{A}_{\mathcal{G}} = \sum_{\mu} \mathcal{A}_{\mathcal{G},\mu}, \quad (3.4)$$

according to scale attributions $\mu := \{i_l \in \mathbb{N}, l \in L(\mathcal{G})\}$. In this formula, $\mathcal{A}_{\mathcal{G},\mu}$ is an amplitude constructed from the sliced propagators $\{C_i\}$ rather than the full propagator C_{Λ} : that is, to each line l , we now associate the propagator C_{i_l} . The multiscale strategy then consists in looking for estimates of each of the amplitudes $\mathcal{A}_{\mathcal{G},\mu}$ separately, rather than of the full amplitude $\mathcal{A}_{\mathcal{G}}$. The scale attributions μ have the considerable advantage that they allow to optimize the naive bounds one would derive for $\mathcal{A}_{\mathcal{G}}$, and as a consequence to more precisely understand the origin of the divergences. For instance, given a graph \mathcal{G} with scale attribution μ , one can realize that divergences may only be generated by *high subgraphs*: these are defined as subgraphs $\mathcal{H} \subset \mathcal{G}$ which have internal scales higher than the scales of their external legs. In order to implement a renormalization procedure, one first and foremost needs to understand the structure of divergent high subgraphs, and study their behaviour in the limit in which the separation of scales between internal lines and external legs is large.

Without going too much into details, a general Abelian power-counting theorem [14, 35, 84] can be derived which, when G is commutative, provides a combinatorial characterization of the divergent subgraphs. When G is non-Abelian, further subtleties enter the picture because, although the Abelian power-counting still holds as a bound [27, 28], it is not necessarily optimal in this case. However, since it turns out *a posteriori* that all the divergent graphs encountered in TGFTs do saturate the Abelian bounds²⁰, we will simply ignore this subtlety, and the interested reader is referred to [30, 35] for more details.

Before introducing the notion of degree of divergence, we need to define the very central notion of *face*. In the present context, a face f of a graph \mathcal{G} is defined as a maximal bicolored path in \mathcal{G} , with the restriction that one of the two colors must be 0. For convenience, we will simply attribute the color ℓ to a face consisting of a path with colors 0 and ℓ . We will say that a line $l \in L(\mathcal{G})$ pertains to f ($l \in f$) if it coincides with one of the dashed lines of the path. Faces can furthermore be *open* (or, equivalently, *external*) or *closed* (or, equivalently, *internal*), depending on whether they are connected to external legs of \mathcal{G} or not. We will denote by $F(\mathcal{G})$ (resp. $F_{\text{ext}}(\mathcal{G})$) the set of closed (resp. open) faces of \mathcal{G} . Conventionally, we may also orient dashed lines positively from white to black nodes, and orient the faces accordingly. This allows to introduce an adjacency matrix ε_{lf} , of size $L \times F$ and with only 0 or 1 entries: $\varepsilon_{lf} = 1$ if $l \in f$, and $\varepsilon_{lf} = 0$ otherwise. Faces are particularly important because, given the form of the propagator and the Feynman rules, the integrand of an amplitude factorizes over its faces. More precisely, each closed face f yields a factor

$$K_{\sum_{l \in f} \alpha_l} \left(\prod_{l \in f}^{\rightarrow} h_l \right),$$

where α_l and h_l are respectively the Schwinger parameter and holonomy associated to the propagator line l , and the product over holonomies is taken accordingly to the orientation of f . See Fig. 8 for an example. An open face f yields on the other hand a factor

$$K_{\sum_{l \in f} \alpha_l} \left(g_{s(f)} \left[\prod_{l \in f}^{\rightarrow} h_l \right] g_{t(f)}^{-1} \right),$$

where $g_{s(f)}$ and $g_{t(f)}$ are boundary variables associated to the fields sitting at the source ($s(f)$) and target ($t(f)$) ends of f .

²⁰This is due to the fact that the associated 2-complexes are simply connected, see again [27, 28].

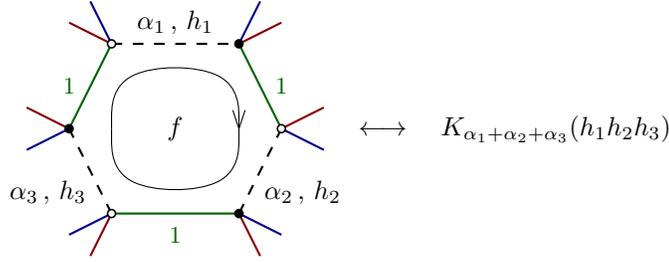


Figure 8. An internal face of color 1 and length 3, and its associated amplitude integrand.

Proposition 3.1. *The superficial degree of divergence of a (non-vacuum) graph \mathcal{G} is*

$$\omega(\mathcal{G}) := -2L(\mathcal{G}) + D(F(\mathcal{G}) - R(\mathcal{G})) \geq 0,$$

where $R(\mathcal{G})$ is the rank of the adjacency matrix ε_{lf} of \mathcal{G} , and D is the dimension of G .

The superficial degree of divergence (which we abbreviate to “degree of divergence” or simply “degree” in the sequel) captures the UV asymptotic behaviour of the amplitudes. For a single-slice amplitude at scale i (i.e., $\mu = \{i_\ell = i, \ell \in L(\mathcal{G})\}$), one can show that $\mathcal{A}_{\mathcal{G},\mu}$ has an exponential scaling of the form $M^{\omega(\mathcal{G})i}$ when $i \rightarrow +\infty$ ²¹. This corresponds to the situation in which \mathcal{G} contains a single high subgraph – itself. The fact that the divergences are in this case essentially controlled by the combinatorial quantity $(F - R)$ was already proven in [14], though in a slightly different context. The analysis of more general scale attributions is based on a step-by-step estimation of the contributions of high subgraphs, from higher to smaller scales, and was first detailed in [35, 84]. We will not need to go into such details here, which are only relevant for the full rigorous proof of renormalizability. We only point out that the concept of high subgraphs allows a very natural treatment of overlapping divergences, which otherwise lead to somewhat challenging recursive constructions (see [77] and references therein).

Once one understands how Feynman amplitudes diverge in the UV, one may try to devise simple criteria of renormalizability, for instance in terms of the dimensions d (space-time) and D (group). To this effect, it is important to find a more practical expression of the divergence degree, which puts combinatorial quantities such as the number of external legs N to the forefront. The presence of the rank R , which is a direct consequence of the gauge invariance condition we imposed on the fields, makes it more involved than in TGFTs without this ingredient. Invoking elementary combinatorial relations, one easily proves that²²

$$\omega = D(d-2) - \frac{D(d-2)-2}{2}N + \sum_{k \in \mathbb{N}^*} [(D(d-2)-2)k - D(d-2)]n_{2k} + D\rho, \quad (3.5)$$

where N is the number of external legs and n_{2k} the number of bubbles of valency k . The whole non-trivial dependence in the rank has been included in the combinatorial quantity

$$\rho := F - R - (d-2)(L - V + 1).$$

The key missing ingredient leading to a general classification of models is a bound on ρ . The following proposition, which was first derived in [35], serves this purpose.

Proposition 3.2. *Let \mathcal{G} be a non-vacuum graph. Then*

$$\rho(\mathcal{G}) \leq 0,$$

and $\rho(\mathcal{G}) = 0$ if and only if \mathcal{G} is a melonic graph.

²¹When $\omega(\mathcal{G}) = 0$ the divergences are logarithmic.

²²The variable ρ of this equation is a combinatorial quantity associated to a graph, and has nothing to do with the cut-off appearing in equation (3.3).

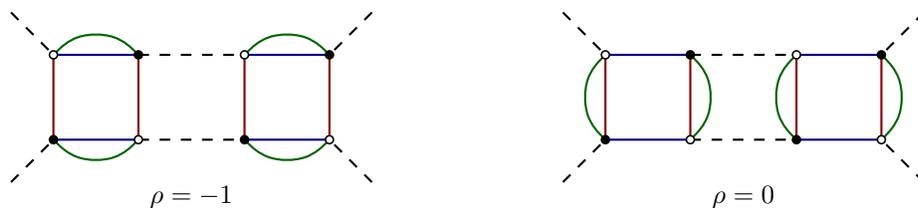


Figure 9. Melonic (right) and non-melonic (left) 4-point graphs in $d = 3$. $F = 2$ for the former and only 1 for the latter.

We refer the reader to, e.g., [19, 35] for a more precise combinatorial characterization of melonic graphs in this context. As illustrated in Fig. 9, melonic graphs tend to maximize the number of internal faces. Proposition 3.2 can alternatively be taken as a definition of melonic graphs. We only mention two important properties. First, melonic graphs are associated to and generated by a specific subset of bubbles, which are accordingly called *melonic bubbles*. In Fig. 3, all bubbles are melonic except for the rightmost one, showing that in dimension $d = 3$ the first non-trivial interactions are necessarily melonic. Second, melonic graphs and melonic bubbles have both trivial topology: in dimension d they represent d -balls, and are therefore topologically suitable building blocks of $(d + 1)$ -dimensional space-time.

Proposition 3.2 shows that melonic bubbles and melonic graphs lead to the most severe divergences. Following the literature, we now proceed with a classification of what we may call *melonic models*. We define them as TGFTs which: 1) include melonic interactions; and 2) are perturbatively consistent under renormalization. We emphasize that the first hypothesis is non-trivial, and is somewhat implicitly assumed in the literature. We will come back to this interesting aspect below.

Power-counting renormalizability requires the degree of divergence to be bounded from above. Moreover, in the presence of divergences, ω should decrease with the number of external legs, in such a way that only finitely many n -point functions need to be renormalized. From these conditions alone, one can derive a full classification of melonic models allowed by the power-counting analysis, in terms of the dimensions d and D , and the maximal valency v_{\max} of the renormalizable bubbles. The complete list, established in [35]²³, is reported in Table 1. Models of type A to E are candidate just-renormalizable GFTs, and are in principle the most interesting ones: they have infinitely many divergent Feynman amplitudes, which leads to universal properties of the flows. Models of type F and G are on the other hand super-renormalizable, which means that their divergences are generated by a finite family of single-vertex graphs (also known as tadpoles). Finally, models of type H are finite and are therefore not very interesting from the point of view of renormalization: the renormalization group is in this case unable to provide a physical hierarchy for the amplitudes and interactions, which may well all contribute with roughly the same intensity.

A striking feature of this classification is that the only combination of d and D which is compatible with a quantum space-time interpretation of the amplitudes is $d = D = 3$. Indeed, only in this case is the would-be space-time dimension d consistent with the dimension D of the local symmetry group G . This is quite remarkable: we have first motivated the general class of TGFTs with gauge invariance condition from Euclidean quantum gravity in three dimensions, and reciprocally, pure quantum field theory arguments allow us to in a sense *derive* dimension 3 as the only consistent one.

This prompted an in-depth study of the $d = 3$ model with $G = \text{SU}(2)$, which was proven renormalizable at all orders in [35]. Its flow equations were then studied in greater details

²³The classification of [35] includes only just-renormalizable models, we have added super-renormalizable and finite models for completeness.

Table 1. List of power-counting renormalizable melonic models.

Type	d	D	v_{\max}	ω	Explicit examples
A	3	3	6	$3 - N/2 - 2n_2 - n_4 + 3\rho$	$G = \text{SU}(2)$ [35, 31]
B	3	4	4	$4 - N - 2n_2 + 4\rho$	$G = \text{SU}(2) \times \text{U}(1)$ [32]
C	4	2	4	$4 - N - 2n_2 + 2\rho$	–
D	5	1	6	$3 - N/2 - 2n_2 - n_4 + \rho$	$G = \text{U}(1)$ [84, 81]
E	6	1	4	$4 - N - 2n_2 + \rho$	$G = \text{U}(1)$ [84, 81, 22]
F	3	2	arbitrary	$2 - 2V$	–
G	4	1	arbitrary	$2 - 2V$	$G = \text{U}(1)$ [36, 63]
H	3	1	arbitrary	$1 - L - V < 0$	$G = \text{U}(1)$ [62]

in [31]. Note that, because of the subtleties associated to non-Abelian amplitudes, this particular example required extra care, which we are glossing over in this review. Still in $d = 3$, a renormalizable model of type B based on the group $\text{SU}(2) \times \text{U}(1)$ has been considered in [32]. Examples of Abelian $\text{U}(1)$ models of type D and E were proposed in [84] and also proven renormalizable at all orders. Interestingly, and as is clearly allowed by the power-counting arguments we have reviewed, the φ^6 model of type D requires the inclusion of a non-melonic and non-connected interaction of the form $(\bar{\varphi} \cdot \varphi)^2$, sometimes called “anomalous”. The model of type E, which according to our power-counting arguments might also have necessitated the inclusion of non-melonic bubbles²⁴, remains consistent with only melonic bubbles included²⁵. The beta functions of these two models were then studied in [81], and the functional renormalization group of the model of type D was investigated in [22].

Abelian super-renormalizable models of type G actually provided the first examples of renormalizable TGFTs with gauge invariance [36]. Since only finitely many divergent graphs are generated in this case, renormalization could be implemented by means of a generalization of the standard Wick ordering prescription. Constructive aspects of a φ^4 model of this type, as well as of a finite model of type H, were more recently studied in [62, 63]. This led in both cases to a Borel resummation of the perturbative expansion, thus proving its analytical existence. It is a very interesting step towards a full non-perturbative definition of just-renormalizable TGFTs with closure constraint, including the more physically relevant $d = 3$ and $G = \text{SU}(2)$ situation.

Examples of renormalizable models of type C and F have not been explicitly exhibited in the literature. There is however no doubt that such example exists, for instance with the group $G = \text{U}(1)^2$. Indeed, the arguments and tools from [36] and [84] are directly applicable to this Abelian group. In particular, the analysis of non-melonic graphs proposed in [84] allows to demonstrate that melonic bubbles alone lead to a consistent model in situation C. We further conjecture that it is also possible to consistently include φ^4 necklace bubbles [23] in this context, which will generate divergent but non-melonic 2-point graphs (with $\rho = -1$). See Fig. 10.

3.3 Renormalization, subtraction schemes and contractible graphs

Once power-counting renormalizability has been checked, several standard quantum field theory techniques may be applied to prove full-fledged renormalizability. The main physical idea is to re-express the Feynman expansion in terms of new physically meaningful perturbative parameters. The bare coupling constants are indeed associated to processes occurring at arbitrarily large energies, and have therefore no empirical content. Instead, one should parametrize the theory

²⁴Graphs with $\rho = -1$ or -2 might in principle still lead to divergences.

²⁵Note that we are not claiming that non-melonic bubbles cannot be consistently included, only that the model is consistent without them. The construction of non-melonic phases remains a largely unexplored and interesting research direction.

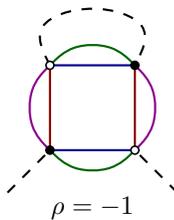


Figure 10. Non-melonic 2-point graph with $\rho = -1$ in $d = 4$. Its single vertex is a necklace bubble, which suggests that such an interaction may consistently be included in a TGFT model of type C. The graph has a degree $\omega = 0$ and is therefore logarithmically divergent in this case.

with the values of n -point functions at an arbitrarily chosen but physically accessible low energy scale. In this new expansion, Feynman amplitudes converge, and the divergences of the bare amplitudes are interpreted as spurious effects resulting from a misplaced parametrization of the field theory.

Different renormalization prescriptions may be used, leading to slightly different (but equivalent) definitions of renormalized quantities. One can for example rely on the celebrated Bogoliubov–Parasiuk–Hepp–Zimmermann (BPHZ) scheme, which amounts to using $p^2 = 0$ as a reference energy scale. This standard textbook procedure is rather simple at first loop orders, but may appear somewhat mysterious when it comes to overlapping divergences²⁶. The effective expansion, based on multiscale methods, is more in the spirit of Wilson’s brilliant reformulation of renormalization. It consists in a step-by-step recursive definition of effective coupling constants $t_{b,i}$, which measure the amplitudes of physical processes associated to the index scale i . The contributions of divergent graphs at a given scale are reabsorbed into the coupling constants at lower scales, resulting in a discrete renormalization group flow from higher to lower scales:

$$t_{b,i-1} - t_{b,i} = \beta_i(\{t_{b',j} \mid i \leq j \leq \rho\}).$$

The n -point functions may then be expressed as formal multi-series in the $t_{b,i}$ ’s, with convergent coefficients in the limit $\rho \rightarrow +\infty$. Moreover, the effective expansion provides a new perspective on the more standard renormalized BPHZ expansion: at fixed multiscale parameter μ (3.4), divergent graphs can simply not overlap, and this simple realization greatly clarifies the reason why the BPHZ procedure converges at all orders in perturbation theory. We refer the reader to [77] for a detailed discussion of both the BPHZ expansion and the effective expansion in the context of local scalar field theories.

Both effective and renormalized expansions have been successfully generalized and applied to TGFTs with gauge invariance condition [35, 36, 84]. Reviewing these constructions in detail would take us too far into technicalities, we therefore only expose the core argument explaining why these techniques can be applied at all. In ordinary quantum field theory, renormalization relies on the key realization that high energy processes look essentially local (as seen by an observer operating at much lower energy scales). This is true irrespectively of how complicated these high energy processes are, and is the main reason why the contributions of high divergent subgraphs at a given scale can always be absorbed into redefinitions of the coupling constants at lower momenta. Possibly severe complications arise in our TGFT context: first, the non-standard notion of locality encapsulated in tensor invariant interactions renders the analysis obviously more intricate; second, and more importantly, the main combinatorial building blocks of the amplitudes generated by TGFTs with closure constraint are the faces, which are intrinsically non-local objects.

²⁶We remind the reader that, given a graph \mathcal{G} , two divergent subgraphs $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{G}$ are said to overlap if neither of the three relations is verified: $\mathcal{H}_1 \cap \mathcal{H}_2 = \emptyset$, $\mathcal{H}_1 \subset \mathcal{H}_2$, $\mathcal{H}_2 \subset \mathcal{H}_1$.

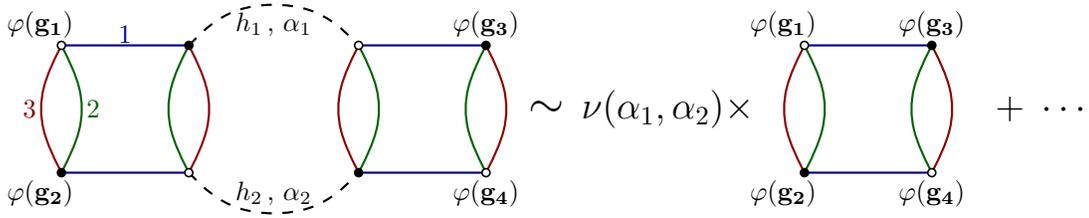


Figure 11. Approximation of a tracial graph as an effective trace invariant contribution plus corrections.

For concreteness, let us consider the example of Fig. 11. On the left one finds a typical effective contribution to the 4-point function generated by a melonic graph in $3d$. In terms of the Schwinger parameters α_1 and α_2 , the integrand of its amplitude is:

$$\int dh_1 dh_2 [K_{\alpha_1 + \alpha_2}(h_1 h_2)]^2 \int \left[\prod_{i < j} dg_{ij} \right] K_{\alpha_1}(g_{11} h_1 g_{31}^{-1}) K_{\alpha_2}(g_{21}^{-1} h_2 g_{41}) \\ \times \delta(g_{12} g_{22}^{-1}) \delta(g_{13} g_{22}^{-1}) \delta(g_{42} g_{32}^{-1}) \delta(g_{43} g_{33}^{-1}) \varphi(\mathbf{g}_1) \bar{\varphi}(\mathbf{g}_2) \bar{\varphi}(\mathbf{g}_3) \varphi(\mathbf{g}_4).$$

The question is whether this expression can be approximated by an elementary tensor invariant in the sector $\alpha_1, \alpha_2 \rightarrow 0$ ²⁷. Though it is not that obvious at first sight, the answer is yes. We can resort to a similar line of arguments as the one which led us to the expansion (2.10). The gauge symmetry associated to the amplitudes allows to gauge-fix one of the two holonomies and reduce this expression to:

$$\int dh [K_{\alpha_1 + \alpha_2}(h)]^2 \int \left[\prod_{i < j} dg_{ij} \right] K_{\alpha_1}(g_{11} h g_{31}^{-1}) K_{\alpha_2}(g_{21}^{-1} g_{41}) \\ \times \delta(g_{12} g_{22}^{-1}) \delta(g_{13} g_{22}^{-1}) \delta(g_{42} g_{32}^{-1}) \delta(g_{43} g_{33}^{-1}) \varphi(\mathbf{g}_1) \bar{\varphi}(\mathbf{g}_2) \bar{\varphi}(\mathbf{g}_3) \varphi(\mathbf{g}_4).$$

It is then obvious that, in the large scale limit, h is peaked around the identity and therefore g_{11} (resp. g_{21}) is identified to g_{31} (resp. g_{41}). A Taylor expansion of the external variables of color 1 along their external faces then allows to approximate the amplitude by the tensor invariant (and melonic) interaction shown on the right-hand side of Fig. 11, up to a scale-dependent constant $\nu(\alpha_1, \alpha_2)$.

We say that a graph is *contractible* if its bulk holonomies can be trivialized in the UV region, as illustrated in the example before²⁸. Only if all divergent graphs are contractible can we reabsorb UV divergences into tensor invariant effective interactions. It may however happen that disconnected effective bubbles are generated in this way. A contractible graph generating a connected tensor invariant interaction is furthermore called *tracial*. We now understand that the reasons why renormalization theory meaningfully applies to all the candidate models we have introduced in the previous subsection are that: 1) melonic graphs are tracial; and 2) all non-melonic divergent corrections are contractible. It is important to understand that these are highly non-trivial facts, which rely on intimate relations between the topology of colored graphs and the scaling of their amplitudes. On the one hand, trivial topology in the bulk of divergent graphs ensures that our tensor invariant truncation is stable under renormalization. But there is more as, in return, this consistent renormalization scheme guarantees that no topological singularities can be generated by radiative corrections. For instance, the rightmost interaction of Fig. 3 can be consistently set to 0 in the $d = 3$ model on $SU(2)$, even though it is allowed

²⁷We remind the reader that the Schwinger parameters should be thought of as inverse squared momenta p^{-2} .

²⁸This is equivalent to simple-connectedness of the 2-complex, i.e., flat connections are trivial up to gauge.

by our power-counting arguments [35]. Interestingly, divergent non-melonic graphs with non-melonic bubbles such as the one shown in Fig. 10 are also tracial, which suggests again that necklace terms may sometimes be consistently included into the picture.

4 Renormalization group and non-trivial fixed points

The purpose of this section is to illustrate some of the interesting properties of the renormalization group flows of TGFTs with gauge invariance condition. The goal of renormalization group investigations is two-fold: 1) understand the fate of the renormalized coupling constants in the deep UV, and hence determine whether the theory is consistent to arbitrary high scales or not; 2) systematically explore the theory space away from the perturbative fixed point, and investigate in particular the existence of non-trivial fixed points. The first objective can be first addressed within a perturbative scheme, and is embodied in this context by the question of *asymptotic freedom*. In the unfavourable case in which asymptotic freedom does not hold (which means that the renormalized coupling constants do not converge to 0 in the UV), there is still the possibility that the theory may be UV completed by means of a non-perturbative UV fixed point. The second objective is central to the whole GFT approach to quantum gravity and requires a non-perturbative treatment anyway. We therefore decide to focus on the Functional Renormalization Group (FRG), and more precisely on the Wetterich equation, which can conveniently be used to address both types of questions and will presumably play an important role in the future. The related Polchinski equation was on the other hand investigated in [59, 60, 61].

4.1 Effective average action and Wetterich equation

Irrespectively of one's preferred formulation of the renormalization group, consistent flow equations may only be formulated for *dimensionless* coupling constants $u_b(\Lambda)$, which are appropriate rescalings of $t_b(\Lambda)$ by powers of Λ . This important aspect may be formalized by the notion of *canonical dimension* d_b of tensor invariant bubbles, which has been discussed at length in [22, 31, 60]. For melonic models (with closure constraint), it is defined as

$$d_b := D(d-2) - (D(d-2) - 2)\frac{N_b}{2},$$

where N_b is the valency of the bubble b (i.e., its number of nodes). Hence the dimensionless coupling constants are defined as

$$u_b(\Lambda) := \frac{t_b(\Lambda)}{\Lambda^{d_b}},$$

and b is renormalizable (or, equivalently, perturbatively relevant) if and only if $d_b \geq 0$. In view of the expression (3.5) for the divergence degree ω , it appears that the most divergent graphs contributing to the running of t_b diverge like Λ^{d_b} , and yield corrections to the dimensionless couplings u_b of order 1 (as they should).

We insist on the fact that the notion of canonical dimension just defined is *well-suited to melonic models only*, as we have implicitly assumed that the most divergent contributions come from melonic graphs. A model which would for instance contain only necklace interactions would bring us out of the melonic world described in this review, and henceforth yield a different notion of canonical dimension²⁹.

²⁹This possibility is actively explored in the context of $4d$ models with Barrett–Crane simplicity constraints [Carrozza S., Lahoche V., Oriti D., work in progress].

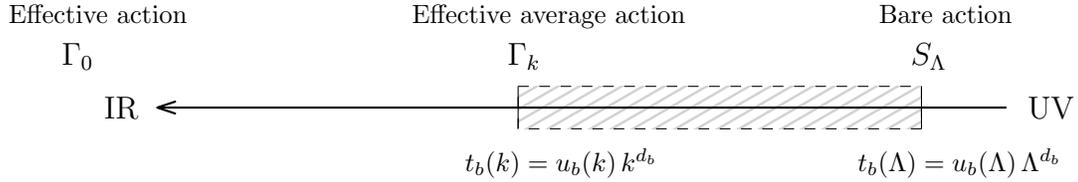


Figure 12. The effective average action interpolates between the bare action in the ultraviolet and the full effective action in the infrared.

In the Wetterich–Morris [67, 86] approach to the functional renormalization group, a one-parameter family of deformed generating functionals is introduced

$$\mathcal{Z}_{\Lambda,k}[J, \bar{J}] := \int d\mu_{C_\Lambda}(\varphi, \bar{\varphi}) \exp(-S_\Lambda[\varphi, \bar{\varphi}] - \bar{\varphi} \cdot R_k \cdot \varphi + \bar{J} \cdot \varphi + \bar{\varphi} \cdot J).$$

The new operator R_k has the function of regularizing the field modes below an infrared cut-off k while leaving the high energy sector unaffected. This allows the introduction of a new generating functional, the *effective average action* Γ_k , which is the Legendre transform of $W_k[J, \bar{J}] := \ln(\mathcal{Z}_{k,\Lambda}[J, \bar{J}])$, appropriately shifted by the 2-point counter-term $\bar{\varphi} \cdot R_k \cdot \varphi$ (see [22] for a detailed discussion). Interestingly, for suitable choices of R_k , Γ_k can be shown to interpolate between the bare action $S_\Lambda = \Gamma_\Lambda$ in the UV, and the full effective action Γ_0 (or in other words the generating functional of one-particle irreducible graphs) in the infrared, which we illustrate in Fig. 12.

Finally, as derived in detail in [22] for TGFTs with gauge invariant condition, the effective average action verifies a Wetterich equation

$$k\partial_k \Gamma_k[\varphi, \bar{\varphi}] = \int d\mathbf{g}_1 d\mathbf{g}_2 d\mathbf{g}_3 k\partial_k R_k(\mathbf{g}_1; \mathbf{g}_2) (\Gamma_k^{(2)} + R_k)^{-1}(\mathbf{g}_2, \mathbf{g}_3) \mathcal{P}(\mathbf{g}_3; \mathbf{g}_1), \quad (4.1)$$

where

$$\Gamma_k^{(2)}[\varphi, \bar{\varphi}](\mathbf{g}; \mathbf{g}') := \frac{\delta^2 \Gamma_k}{\delta\varphi(\mathbf{g})\delta\bar{\varphi}(\mathbf{g}')}[\varphi, \bar{\varphi}]$$

is the full interacting propagator at scale k and \mathcal{P} is as before the projector onto gauge invariant fields. Equation (4.1) defines a formal flow in an infinite-dimensional space of theories, a rigorous mathematical definition of which is for the time being out of reach. The standard procedure used in the literature to make sense of the Wetterich equation consists in choosing a finite-dimensional ansatz for the effective average action, and then systematically projecting the formal flow equation down to this finite-dimensional subspace of theories. The local potential approximation introduced in the previous section is one such possible ansatz, and yields a system of one-loop flow equations. Standard one-loop perturbative equations may be recovered with a truncation which only includes renormalizable interactions, while the computation of higher order loops requires the inclusion of non-renormalizable corrections to the potential. The same method can be used in the non-perturbative regime, with the important caveat that truncations are much harder to justify in this case. The functional renormalization group equation provides little analytical control over error terms, and one must therefore resort to more empirical justifications, based mainly on numerical tests of convergence of the truncation procedure. It turns out at the end of the day that the FRG provides reliable and effective methods for discovering and computing the properties of non-trivial fixed points in ordinary statistical field theories (see, e.g., [3]). This is what makes them particularly precious in GFTs, as they have the potential to unravel new and more physical phases. Applications of the FRG to TGFTs being rather recent, only the simplest truncations have been considered so far, and the question of their reliability remains to be further explored.

4.2 Example: perturbative treatment of rank-3 models

Let us start with a perturbative application of the formalism just introduced, which already suggests a variety of different properties for the renormalization group flows of TGFTs. We restrict our attention to $d = 3$ renormalizable models of the type A ($D = 3$) and B ($D = 4$). Interestingly, the situation is reminiscent of that of an ordinary local scalar field, which is renormalizable up to quartic interactions in space-time dimension 4, and up to order 6 interactions in dimension 3. The same statement holds for $d = 3$ melonic models once the space-time dimension is traded for the group dimension D . This remarkable fact allows a simple but informative comparison of the qualitative features of the flow equations of TGFTs against those of ordinary local field theories.

Let us start with a model of type B and choose for instance the group $SU(2) \times U(1)$ as in [32]. This theory is renormalizable up to order 4, therefore a natural ansatz for the effective average action is³⁰

$$\Gamma_k(\varphi, \bar{\varphi}) = -Z(k)\bar{\varphi} \cdot \Delta\varphi + Z(k)u_2(k)k^2 \text{ (bubble)} + Z(k)^2u_4(k) \text{ (square)},$$

where for convenience we directly represented tensor interactions by their associated colored graphs³¹, and we have introduced a wave-function parameter $Z(k)$. In this group dimension, the mass term has canonical dimension 2 and the marginal φ^4 interactions have as they should dimension 0. It was shown in [32] that the perturbative renormalization group flow reduces in this truncation to

$$\begin{aligned} k \frac{\partial u_2(k)}{\partial k} &= -2u_2(k) - 3\pi u_4(k), \\ k \frac{\partial u_4(k)}{\partial k} &= -2\pi u_4(k)^2. \end{aligned} \quad (4.2)$$

One notices a major difference with ordinary scalar field theories: the derivative of the φ^4 coupling is negative, which means that it decreases towards 0 in the ultraviolet. We therefore obtain an asymptotically free and UV complete perturbative definition of the theory! This is due to a quite general and remarkable property of TGFTs at large. As was first remarked in [10] in the context of TGFTs without gauge invariance condition, and later on generalized to models with closure constraint in [81], wave-function counter-terms generally dominate over vertex renormalization ones, and are ultimately responsible for changes in the signs of some of the coefficients of the flow equations. A beautiful explanation based on a symmetry argument has been furthermore proposed in [79], thereby proving that asymptotic freedom is a completely general feature of quartic renormalizable TGFTs. In particular, Abelian models of type E are asymptotically free, as argued for in [81].

Let us now move on to the $d = 3$ model with $G = SU(2)$, which is renormalizable up to order 6 interactions. Accordingly, one can choose the following ansatz for the effective average action:

$$\begin{aligned} \Gamma_k(\varphi, \bar{\varphi}) &= -Z(k)\bar{\varphi} \cdot \Delta\varphi + Z(k)u_2(k)k^2 \text{ (bubble)} + Z(k)^2u_4(k)k \text{ (square)} \\ &+ Z(k)^3u_{6,1}(k) \text{ (hexagon 1)} + Z(k)^3u_{6,2}(k) \text{ (hexagon 2)}, \end{aligned}$$

³⁰We use the short-hand notation $\Delta := \sum_{\ell=1}^3 \Delta_\ell$.

³¹Note that each colored graph in this equation is to be thought of as representing the equivalent class of bubbles with the same combinatorial structure up to permutation of the color labels. There is for instance a sum of three distinct melonic interactions at order 4, corresponding to three inequivalent permutations of the color labels.

and we note the change of dimensionality of the φ^4 interactions with respect to the previous situation. The one-loop perturbative flow can in this case be approximated by³²

$$\begin{aligned} k \frac{\partial u_2(k)}{\partial k} &\approx -2u_2(k) - au_4(k), \\ k \frac{\partial u_4(k)}{\partial k} &\approx -u_4(k) - b(u_{6,1}(k) + 2u_{6,2}(k)), \\ k \frac{\partial u_{6,1}(k)}{\partial k} &\approx -cu_4(k)u_{6,1}(k), \\ k \frac{\partial u_{6,2}(k)}{\partial k} &\approx -du_4(k)u_{6,2}(k), \end{aligned} \quad (4.3)$$

where a , b , c and d are strictly positive constants. This is a rather complicated system of equations, but we notice again the negative signs in the last two equations. It is therefore tempting to conjecture that, if one assumes that all the coupling constants are positive, then the marginal constants $u_{6,1}$ and $u_{6,2}$ both converge to 0 in the ultraviolet. As shown in [31]³³, this is actually misleading because, given the form of the flow equations, it can be proven that $u_4(k)$ necessarily reaches negative values for large k , which has the effect of making the φ^6 interactions grow again (see [31]). Hence, one is forced to conclude that this model *cannot* be asymptotically free if we assume that both positive marginal coupling constants are positive³⁴. This example shows that the question of asymptotic freedom can be tricky in φ^6 renormalizable models, because the φ^4 super-renormalizable coupling constants have a non-trivial influence on the marginal φ^6 interactions. In particular, the combinatorial TGFT of [10] and the type D model of [81], which have both been argued to be asymptotically free on the basis of an analysis which neglected the flow of super-renormalizable constants, may possibly suffer from a similar back-reaction effect.

As mentioned already, if a model is not asymptotically free, one may still contemplate the idea of finding a non-perturbation UV completion of it. This is an interesting but notoriously hard question to investigate, since this requires to establish the existence of a non-perturbative fixed point of the renormalization group. An elementary standard method often invoked in statistical physics to test this assumption is the ε -expansion [87], which has the advantage of being essentially perturbative. In scalar field theories for instance, one can formally define statistical models in dimension $4 - \varepsilon$, which smoothly interpolate between dimension 4 and dimension 6. A TGFT generalization of this construction was proposed in [32]. The procedure consists in defining a $d = 3$ TGFT on the group $SU(2) \times U(1)^{D-3}$ for arbitrary $D \geq 3$, and then analytically continue the parameter $D := 4 - \varepsilon$ to the interval $3 \leq D \leq 4$. When ε is small enough, one may assume that the φ^4 truncation remains pertinent:

$$\Gamma_k(\varphi, \bar{\varphi}) = -Z(k)\bar{\varphi} \cdot \Delta\varphi + Z(k)u_2(k)k^2 \text{ (loop diagram)} + Z(k)^2u_4(k)k^\varepsilon \text{ (square diagram)}.$$

Note that u_4 acquired a small canonical dimension ε . This has the effect of slightly modifying the flow equations (4.2) to

$$k \frac{\partial u_2(k)}{\partial k} \approx -2u_2(k) - 3\pi u_4(k), \quad k \frac{\partial u_4(k)}{\partial k} \approx -\varepsilon u_4(k) - 2\pi u_4(k)^2. \quad (4.4)$$

³²These flow equations were not explicitly evaluated in [32], but they immediately follow from the computations reported there.

³³This paper relied on different methods, in the language of the multiscale expansion. It also went further in that 2-loop contributions were included to account for quadratic terms (in $u_{6,1}^2$, $u_{6,1}^2$ and $u_{6,1}u_{6,2}$) which have been neglected in (4.3). The main conclusions are however the same.

³⁴The situation is a more subtle when $u_{6,1}u_{6,2} < 0$, we do not exclude the possibility that one could define an asymptotically free theory with positive action in this particular case.

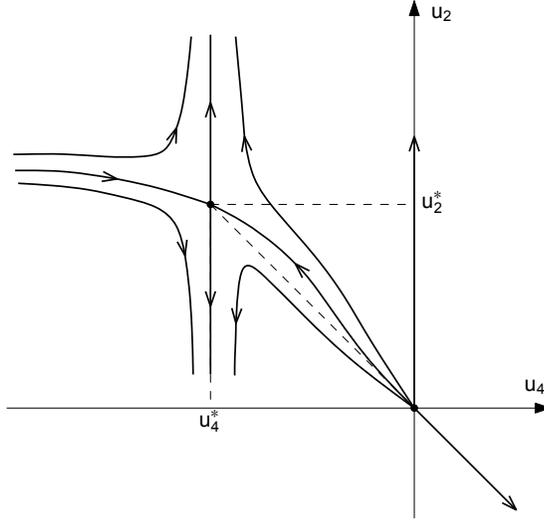


Figure 13. Qualitative phase portrait of the renormalization group flow equations (4.4), with arrows oriented from high to small scales.

Accordingly, one formally finds a new solution to the fixed point equation:

$$u_2^* \approx \frac{3}{4}\varepsilon + \mathcal{O}(\varepsilon^2), \quad u_4^* \approx -\frac{1}{2\pi}\varepsilon + \mathcal{O}(\varepsilon^2).$$

Qualitatively, the renormalization group flow (4.4) is therefore as represented in Fig. 13. Extrapolating to $\varepsilon = 1$, this suggests the existence of a TGFT analogue of the Wilson–Fisher fixed point of $3d$ local scalar field theory in the $SU(2)$ model of [35]. This hypothesis should of course be taken with a grain of salt, since the first terms in the ε -expansion may only give a crude idea of what is really going on at $\varepsilon = 1$. Note also that u_4^* has the ‘wrong’ sign (again because wave-function counter-terms dominate in TGFT), which might be taken as a sign that the formal fixed point we found is only a spurious effect. This anyway provides solid motivations for performing a non-perturbative study of the Wetterich equation directly in the case $\varepsilon = 1$ [Carrozza S., Lahoche V., work in progress].

4.3 Non-perturbative aspects and truncations

As far as TGFTs with gauge invariance are concerned, the non-perturbative aspects of the Wetterich equation have been studied in two complementary papers [17, 22].

In [22], the role of gauge invariance was carefully analyzed and the Wetterich equation (4.1) was formally derived. The $d = 6$ melonic model on $U(1)$ (type E) was studied in the φ^4 melonic truncation

$$\Gamma_k(\varphi, \bar{\varphi}) = -Z(k)\bar{\varphi} \cdot \Delta\varphi + Z(k)u_2(k)k^2 \text{ (melonic diagram)} + Z(k)^2u_4(k) \text{ (melonic diagram)}.$$

For such Abelian models, it is convenient to use a Litim cut-off operator R_k [65] which, in momentum space, is defined by the kernel

$$R_k(\mathbf{p}; \mathbf{p}) = Z(k)(k^2 - \mathbf{p}^2)\Theta(k^2 - \mathbf{p}^2) \prod_{\ell=1}^6 \delta_{p_\ell p'_\ell},$$

where $p_\ell \in \mathbb{Z}$ label $U(1)$ representations. The merit of this type of cut-off is that it greatly simplifies the structure of the truncated flow equations, leading to beta functions which are algebraic fractions in the coupling constants. In the perturbative regime one can check again that the model is asymptotically free, as first proven in [81] with different methods. An important aspect of TGFTs on compact Lie groups is that, because of finite size effects, the renormalization group flow is not autonomous. The beta functions of the dimensionless coupling constants explicitly depend on the infrared cut-off k , which complicates the search for non-trivial fixed points. The UV ($k \gg 1$) and IR ($k \rightarrow 0$) asymptotic regimes were analyzed separately in [22]. One finds in both a non-perturbative fixed point analogous to the Wilson–Fisher fixed point of statistical field theory. The quantitative values of the coupling constants are slightly different in the two regimes, but the qualitative structure of the phase portrait is the same. In particular, and unlike the formal fixed point found in the ε -expansion of $3d$ models, the coupling constant u_4 at the fixed point is positive (and u_2 is negative).

A non-compact version of this model, based on the Abelian group \mathbb{R} , was investigated in [17]. The renormalization group analysis requires in this case a further regularization of infinite volume divergences. This was implemented by a compactification of \mathbb{R} into $U(1)$, thus resulting in a regularized theory identical to [22]. The authors could then define an appropriate thermodynamic limit, capturing the properties of the model in the limit in which the volume of $U(1)$ is scaled to infinity. Once more, a fixed point of the Wilson–Fisher type is found in this truncation, consistently with the results of [22]. It remains to be seen whether this qualitative behaviour will survive closer scrutiny. One will in particular need to check its regularization independence and its stability under extensions of the truncation. This preliminary result may nonetheless be interpreted as a hint of a phase transition between a symmetric and a broken phase of a condensate type, in which the field φ acquires a vacuum expectation value. In view of recent applications of Bose–Einstein condensation, which is a phase transition with an order parameter of the same type, these are particularly interesting results. They may help justifying scenarios which have been recently proposed in the GFT literature, with important applications in cosmology [47, 48, 73] and black hole physics [72].

Non-Abelian models may also be explored in this formalism, and a particularly interesting one is again the $d = 3$ theory on $SU(2)$ [35]. Checking its perturbative behaviour requires to push the truncation to order 6 at least, and even to higher orders if one wants to also account for perturbative 2-loop contributions. Moreover, the Litim cut-off is not very convenient in this setting, because of the gauge invariant condition. There was no difficulty in the Abelian model mentioned before because the closure constraint translates into a simple momentum constraint $p_1 + \dots + p_6 = 0$, which can be explicitly dealt with. With $SU(2)$ the situation is not so simple: the gauge invariance condition encodes complicated recoupling relations among the $SU(2)$ harmonic modes, and results in quite challenging expressions. It is therefore better to work in direct space and to rely on a heat-kernel regularization, as was already done in the perturbative Section 3. Such a construction is under way [Carrozza S., Lahoche V., work in progress]: even if the flow equations obtained within this renormalization scheme are not algebraic fractions, they are nonetheless computable and can be integrated out numerically. We should therefore soon be able to directly compute the properties of the flow of model [35] in a φ^6 truncation, and compare them with the features we extrapolated from the formal ε -expansion.

5 Summary and outlook

We hope we have managed to convince the reader that GFT renormalization is an active field of research which has already born interesting fruits. For one thing, the fact that such non-local field theories can be defined and analyzed by means of renormalization methods may at first sound like a contradiction in the terms. Locality is indeed a primary concept in relativistic

quantum field theories, and is absolutely key to the formulation of renormalization theory. While at the fundamental level, GFT in a sense goes away with space-time altogether, it is remarkable that tensor invariance may successfully be used as a substitute for locality. On top of providing a (for a long time missing) structure encoding the topology of GFT interactions and Feynman diagrams, it introduces just enough of flexibility to allow for a GFT theory space which is stable under renormalization.

The inclusion of the GFT gauge invariance condition (2.4) into renormalizable tensor field theories was a necessary and by no means obvious step in the direction of quantum gravity applications. We have explained in some detail the remarkable interplay between topology and renormalization which makes it possible (Section 3.3): in perturbative expansion, the most divergent spin foam amplitudes turn out to be supported on simply connected 2-complexes, which allows to trivialize the bulk holonomies associated to ultraviolet scales, and therefore reabsorb the associated divergences into effective tensor invariant coupling constants.

This produced a rather large class of renormalizable TGFTs with closure constraint (Table 1), and consequently a natural test bed for GFT renormalization. Renormalization group studies have in particular shown that asymptotic freedom is a generic feature of φ^4 perturbative models, which may therefore be interpreted as ultraviolet complete theories. Moreover, the non-perturbative truncations investigated so far suggest that non-trivial fixed points with properties analogous to that of the Wilson–Fisher fixed point of local scalar field theories are generic. This opens the way to the study of GFT phases and phase transitions.

We conclude with a non-exhaustive list of open problems which we consider particularly interesting.

Inclusion of non-melonic interactions. The renormalizable models presented in this review are governed by melonic interactions and melonic radiative corrections. We have however pointed out on several occasions that non-melonic bubbles (such as the necklace bubble of Fig. 10) may also be included and potentially lead to the definition of yet other perturbative phases. This question deserves to be explored more systematically, as it should in particular help us understand to which extent the polymer phases typically generated by melonic families of graphs – which have a tree-like structure – may be escaped [56].

Local potential approximation and beyond. A heuristic derivation of an extended tensor theory space, which includes interactions with arbitrary G -invariant differential operators, has been proposed in Section 2.3. This suggests to interpret all the renormalizable models studied so far as local potential approximations within this extended tensor theory space, and therefore to explore the properties of more general truncations. Our heuristic argument also shows that in $3d$ and with $G = \text{SU}(2)$, the generalized tensor theory space contains in principle the colored Boulatov model. This might open the way to a proper quantum gravity interpretation of this three-dimensional theory.

Lorentzian signature. The whole literature on GFT and GFT renormalization is focused on models with compact Lie groups, which may at best result in consistent formulations of Euclidean quantum gravity. The GFT formulation of quantum gravity models with Lorentz signature necessitates to go beyond this framework. While a renormalization scheme taking the additional infrared divergences associated to non-compact linear groups is already available [16, 17], the physically relevant $\text{SL}(2, \mathbb{R})$ and $\text{SL}(2, \mathbb{C})$ are much more challenging. Indeed, the invariant Laplace operators on such groups are not positive, leading to complications associated to the definition of natural propagators. To a large extent, these theories resemble quantum fields on Minkowski space-time, therefore Euclidean multiscale methods are not easily applicable. Alternative techniques, based for instance on Epstein–Glaser renormalization (as it is applicable in any signature) or on a Wick rotation, will be necessary to explore this question in greater detail.

4d quantum gravity constraints. Eventually one will need to check whether the now well-understood closure constraint can be consistently complemented with spin foam simplicity con-

straints. If a renormalizable $4d$ model can be defined in this context, we will obtain a consistent perturbative sum over spin foam transition amplitudes, and therefore a tentative definition of the dynamics of LQG. All the renormalization group methods that have been developed for and tested on our simpler toy-models will then be of great practical use, in particular to determine whether a sector of the quantum dynamics reproduces general relativity in a suitably defined classical limit.

Acknowledgements

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