

Initial Value Problems for Integrable Systems on a Semi-Strip

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Abstract. Two important cases, where boundary conditions and solutions of the well-known integrable equations on a semi-strip are uniquely determined by the initial conditions, are rigorously studied in detail. First, the case of rectangular matrix solutions of the defocusing nonlinear Schrödinger equation with quasi-analytic boundary conditions is dealt with. (The result is new even for a scalar nonlinear Schrödinger equation.) Next, a special case of the nonlinear optics (N -wave) equation is considered.

Key words: Weyl–Titchmarsh function; initial condition; quasi-analytic functions; system on a semi-strip; nonlinear Schrödinger equation; nonlinear optics equation

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1 Introduction

Cauchy problems for wave equations were successfully investigated using the classical inverse scattering transform, Laplace transforms and also some other methods. The theory of initial-boundary value problems (and problems in a quarter-plane or semi-strip) is somewhat more complicated even for the case of linear wave equations. Some results, discussions and references on this topic are given in [3, 16, 19, 22, 36, 45]. The mentioned above results and discussions are also related to the integrable nonlinear equations. Although it is impossible to refer here to the whole variety of important publications on the initial-boundary value problems for integrable wave equations, we would like to list just some: [9, 10, 12, 16, 18, 19, 27, 30, 31, 32, 33, 37, 39, 40, 43, 48, 50, 55]. Usually (excluding, e.g., sine-Gordon case [32, 37, 42, 48]), initial-boundary value problems are overdetermined (see [11, 18, 48]) for such cases of integrable nonlinear equations, where exact procedures to recover solutions from initial and boundary values exist. Thus, the reduction of the initial-boundary conditions, which are necessary to recover solutions, becomes crucial for solving initial-boundary value problems.

In this paper we extend and develop further the work (which was started in [47]) on the reduction of the necessary initial-boundary conditions. Namely, a case, where boundary conditions provide direct information about an initial condition, was investigated in [47]. Here we study the cases, where an initial condition provides direct information about boundary conditions, and the solutions of the well-known nonlinear integrable equations in a semi-strip are uniquely defined by the initial conditions. (One could speak, perhaps, about additional symmetries of the solutions.)

We consider such situations using the inverse spectral transform approach [6, 7, 28]. More precisely, we follow the scheme introduced in [49, 51], see also [52, Chapter 12] and references therein. That is, we describe the evolution of Weyl–Titchmarsh (Weyl) function in terms of the linear-fractional transformations. The scheme is applicable to various integrable equations and several interesting uniqueness and existence theorems were proved in this way (see [48,

Chapter 6] and references therein for more details). Most of the mentioned above uniqueness and existence theorems were obtained for the equations with scalar solutions. Here we consider the matrix defocusing nonlinear Schrödinger (defocusing NLS or dNLS) equation

$$2v_t = i(v_{xx} - 2vv^*v), \quad v_t := \frac{\partial}{\partial t}v, \quad (1.1)$$

which is equivalent [59, 60] to the compatibility condition

$$G_t - F_x + [G, F] = 0 \quad [G, F] := GF - FG \quad (1.2)$$

of the auxiliary linear systems

$$y_x = Gy, \quad y_t = Fy, \quad (1.3)$$

where

$$G = i(zj + jV), \quad F = -i(z^2j + zjV - (iV_x - jV^2)/2), \quad (1.4)$$

$$j = \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix}, \quad V = \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}, \quad (1.5)$$

I_{m_1} is the $m_1 \times m_1$ identity matrix and v is an $m_1 \times m_2$ matrix function. We will consider dNLS equation on the semi-strip

$$\mathcal{D} = \{(x, t) : 0 \leq x < \infty, 0 \leq t < a\}, \quad (1.6)$$

and we note that the auxiliary system

$$y_x = Gy = i(zj + jV)y \quad (1.7)$$

is (for each fixed t) a well-known self-adjoint Dirac system, also called AKNS or Zakharov–Shabat system. Without changes in notation, we speak about usual derivatives inside domains and about left or right (which should be clear from the context) derivatives on the boundaries, and boundaries of \mathcal{D} in particular. We should mention that, in the usual PDE setting, solutions are often considered in open domains but, in view of certain regularity of the solutions treated in our paper, the boundaries are included in \mathcal{D} in our case.

Since v in dNLS (1.1) is an $m_1 \times m_2$ matrix function, interesting matrix, vector and multi-component dNLS equations from [2, Chapter 4] are included in the considered class.

Another equation that we study in this paper is the nonlinear optics (or N -wave) equation:

$$\left[D, \frac{\partial \varrho}{\partial t} \right] - \left[\widehat{D}, \frac{\partial \varrho}{\partial x} \right] = [D, \varrho][\widehat{D}, \varrho] - [\widehat{D}, \varrho][D, \varrho], \quad \varrho = \varrho^*, \quad (1.8)$$

$$D = \text{diag}\{d_1, d_2, \dots, d_m\}, \quad d_1 > d_2 > \dots > d_m > 0, \quad [D, \varrho] := D\varrho - \varrho D, \quad (1.9)$$

where $\varrho(x, t)$ is an $m \times m$ matrix function, $\text{diag}\{d_1, d_2, \dots\}$ stands for a diagonal matrix with the entries d_1, d_2, \dots on the main diagonal, and $\widehat{D} > 0$ is another diagonal matrix.

First, we obtain the evolution of the Weyl function for the equation (1.8). Next, we consider an interesting special case, where (similar to the inequalities for the entries of D) we have

$$\widehat{D} = \text{diag}\{\widehat{d}_1, \widehat{d}_2, \dots, \widehat{d}_m\}, \quad \widehat{d}_1 > \widehat{d}_2 > \dots > \widehat{d}_m > 0. \quad (1.10)$$

In Section 2 we formulate some necessary results on Weyl functions (and their evolution for the dNLS case), in Section 3 we recover the boundary conditions for dNLS from an initial

condition on a semi-axis (and in this way we solve the initial value problem for dNLS in a semi-strip), and Section 4 is dedicated to the evolution of the Weyl function and initial value problem in a semi-strip for the N -wave equation.

We note that the theory of Weyl functions (Weyl–Titchmarsh theory) is actively developing in recent years (see, e.g., [14, 15, 21, 23, 24, 35, 48, 52, 54, 56] and references therein) and its applications to initial-boundary value problems are of growing interest.

As usual, \mathbb{R} stands for the real axis, $\mathbb{R}_+ = (0, \infty)$, \mathbb{C} stands for the complex plane, and \mathbb{C}_+ for the open semi-plane $\{z: \Im(z) > 0\}$. We say that $v(x)$ is locally summable if its entries are summable on all finite intervals of $[0, \infty)$. We say that v is continuously differentiable if v is differentiable and its first derivatives are continuous. The notation $\|\cdot\|$ stands for the l^2 vector norm or the induced matrix norm. The partial derivative f_{xt} stands for $\partial f_x / \partial t$ and, correspondingly, $f_{tx} = \partial f_t / \partial x$.

2 Preliminaries

2.1 Dirac system and dNLS

We denote by u the fundamental solution of system (1.7) normalized by the condition

$$u(0, z) = I_m, \quad m = m_1 + m_2.$$

Definition 2.1. Let Dirac system (1.7) on $[0, \infty)$ be given and assume that V is locally summable. Then the Weyl function φ is an $m_2 \times m_1$ holomorphic matrix function, which satisfies the inequality

$$\int_0^\infty [I_{m_1} \quad \varphi(z)^*] u(x, z)^* u(x, z) \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix} dx < \infty. \quad (2.1)$$

The following proposition is proved in [20] (see also [48, Section 2.2]).

Proposition 2.2. *The Weyl function always exists and it is unique.*

In order to construct the Weyl function, we introduce a class of $m \times m_1$ matrix functions $\mathcal{P}(z)$, which are an immediate analog of the classical pairs of parameter matrix functions. Namely, the matrix functions $\mathcal{P}(z)$ are meromorphic in \mathbb{C}_+ and satisfy (excluding, possibly, a discrete set of points) the following relations

$$\mathcal{P}(z)^* \mathcal{P}(z) > 0, \quad \mathcal{P}(z)^* j \mathcal{P}(z) \geq 0, \quad z \in \mathbb{C}_+. \quad (2.2)$$

It is said that $\mathcal{P}(z)$ are nonsingular (i.e., the first inequality in (2.2) holds) and with property- j (i.e., the second inequality in (2.2) is valid). Relations (2.2) imply (see, e.g., [20]) that

$$\det \left([I_{m_1} \quad 0] u(x, z)^{-1} \mathcal{P}(z) \right) \neq 0.$$

Definition 2.3. The set $\mathcal{N}(x, z)$ of Möbius transformations is the set of values (at the fixed $x \in [0, \infty)$, $z \in \mathbb{C}_+$) of matrix functions

$$\varphi(x, z, \mathcal{P}) = [0 \quad I_{m_2}] u(x, z)^{-1} \mathcal{P}(z) \left([I_{m_1} \quad 0] u(x, z)^{-1} \mathcal{P}(z) \right)^{-1},$$

where $\mathcal{P}(z)$ are nonsingular matrix functions with property- j .

Remark 2.4. It was shown in [20] that a family $\mathcal{N}(x, z)$, where x increases to infinity and z is fixed ($z \in \mathbb{C}_+$), is a family of embedded matrix balls such that the right semi-radii are uniformly bounded and the left semi-radii tend to zero. (Recall that the $m_2 \times m_1$ matrix ball, or Weyl matrix ball, with the center \mathcal{M} , the left semi-radius \mathcal{R}_l and the right semi-radius \mathcal{R}_r is the set of $m_2 \times m_1$ matrices ω which may be presented in the form $\omega = \mathcal{M} + \mathcal{R}_l \mathcal{U} \mathcal{R}_r$, where \mathcal{U} are contractive $m_2 \times m_1$ matrices.)

Proposition 2.5 ([20]). *Let Dirac system (1.7) on $[0, \infty)$ be given and assume that V is locally summable. Then, the sets $\mathcal{N}(x, z)$ are well-defined. There is a unique matrix function $\varphi(z)$ defined in \mathbb{C}_+ and such that*

$$\{\varphi(z)\} = \bigcap_{x < \infty} \mathcal{N}(x, z). \quad (2.3)$$

This function is analytic and non-expansive (i.e., contractive). Furthermore, this function coincides with the Weyl function of system (1.7).

Formula (2.3) is supplemented by the asymptotic relation

$$\varphi(z) = \lim_{b \rightarrow \infty} \varphi_b(z), \quad (2.4)$$

which is valid for any set of functions $\varphi_b(z) \in \mathcal{N}(b, z)$. Relation (2.4) follows from (2.3) and Remark 2.4 (see also [48, Remark 2.24]).

Next, we consider the famous compatibility condition (zero curvature equation) (1.2).

Proposition 2.6 ([45]). *Let some $m \times m$ matrix functions G and F and their derivatives G_t and F_x exist on the semi-strip \mathcal{D} , let G , G_t and F be continuous with respect to x and t on \mathcal{D} , and let (1.2) hold. Then, we have the equality*

$$u(x, t, z)R(t, z) = R(x, t, z)u(x, 0, z), \quad R(t, z) := R(0, t, z), \quad (2.5)$$

where $u(x, t, z)$ and $R(x, t, z)$ are normalized fundamental solutions given, respectively, by

$$u_x = Gu, \quad u(0, t, z) = I_m; \quad R_t = FR, \quad R(x, 0, z) = I_m. \quad (2.6)$$

The equality (2.5) means that the matrix function

$$y(x, t, z) = u(x, t, z)R(t, z) = R(x, t, z)u(x, 0, z)$$

satisfies on \mathcal{D} both systems (1.3). Moreover, the fundamental solution u admits the factorization

$$u(x, t, z) = R(x, t, z)u(x, 0, z)R(t, z)^{-1}. \quad (2.7)$$

Proposition 2.6 and formula (2.4) yield [47] the following evolution theorem.

Theorem 2.7. *Let an $m_1 \times m_2$ matrix function $v(x, t)$ be continuously differentiable on \mathcal{D} and let v_{xx} exist. Assume that v satisfies the dNLS equation (1.1) as well as the following inequalities (for all $0 \leq t < a$ and some values $M(t) \in \mathbb{R}_+$):*

$$\sup_{x \in \mathbb{R}_+, 0 \leq s \leq t} \|v(x, s)\| \leq M(t). \quad (2.8)$$

Then, the evolution $\varphi(t, z)$ of the Weyl functions of Dirac systems (1.7) is given (for $\Im(z) > 0$) by the equality

$$\varphi(t, z) = (R_{21}(t, z) + R_{22}(t, z)\varphi(0, z))(R_{11}(t, z) + R_{12}(t, z)\varphi(0, z))^{-1}.$$

Remark 2.8. According to [46], the Dirac system $y_x = Gy$ (where G is given by (1.4) and (1.5) and v is locally square summable) is uniquely recovered from the Weyl function φ . In other words, v is uniquely recovered from φ , see the procedure in [46, Theorem 4.4]. The case of a more smooth (i.e., locally bounded) v was dealt with in [48], see also references therein.

2.2 Auxiliary linear systems for nonlinear optics equation

The nonlinear optics (N -wave) equation (1.8) is the compatibility condition of the systems (1.3), where

$$\begin{aligned} G(x, t, z) &= izD - \zeta(x, t), & F(x, t, z) &= iz\widehat{D} - \widehat{\zeta}(x, t); \\ \zeta &= [D, \varrho], & \widehat{\zeta} &= [\widehat{D}, \varrho]; \end{aligned} \quad (2.9)$$

see [58] for the case $N = 3$ and [1] for $N > 3$. We shall need some preliminary results on the Weyl theory of the auxiliary system $y_x = Gy$ from [48, Chapter 4] (see also [44]). The normalized fundamental solution w of such a system is defined by the formula

$$w_x(x, z) = (izD - \zeta(x))w(x, z), \quad w(0, z) = I_m, \quad \zeta = -\zeta^*. \quad (2.10)$$

Here and later we assume that D is a fixed matrix satisfying (1.9). We consider system (2.10) with locally bounded potentials ζ , that is, potentials satisfying (for each $l < \infty$) the inequality

$$\sup_{0 < x < l} \|\zeta(x)\| < \infty. \quad (2.11)$$

Definition 2.9. A generalized Weyl function (GW-function) of system (2.10), where ζ is locally bounded, is an $m \times m$ matrix function φ such that for some $M > 0$ it is analytic in the domain $\mathbb{C}_M^- = \{z: \Im(z) < -M\}$ and the inequality

$$\sup_{x \leq l, \Im(z) < -M} \|w(x, z)\varphi(z) \exp\{-izxD\}\| < \infty \quad (2.12)$$

holds for each $l < \infty$.

Remark 2.10. We note that the Weyl function of the system (2.10) is defined (for ζ bounded on $[0, \infty)$) by the analog (2.17) of the inequality (2.1) and by normalization conditions (2.18). If ζ is bounded on $[0, \infty)$, this Weyl function coincides with the normalized GW-function. (See the discussion after formula (2.18) and Definition 2.9 of the GW-function above.) The fact that the Weyl function satisfies (2.12) explains the term “generalized Weyl function” (or “GW-function”).

The inverse spectral problem (ISpP) for system (2.10) is the problem of recovering (from a GW-function φ) a potential $\zeta(x) = -\zeta^*(x)$ such that (2.12) is valid and the diagonal entries ζ_{kk} of ζ equal zero (i.e., $\zeta_{kk} \equiv 0$).

Notation 2.11. The notation \mathfrak{M} stands for an operator mapping the pair D and φ into the corresponding potential ζ (i.e., $\mathfrak{M}(D, \varphi) = \zeta$). In other words, $\mathfrak{M}(D, \varphi)$ stands for a solution of the ISpP.

The following theorem (i.e., [48, Theorem 4.8]) is valid.

Theorem 2.12. *For any matrix function $\varphi(z)$ which is analytic and bounded in \mathbb{C}_M^- and has the property*

$$\int_{-\infty}^{\infty} (\varphi(z) - I_m)^*(\varphi(z) - I_m) d\xi < \infty, \quad z = \xi + i\eta, \quad \eta < -M, \quad (2.13)$$

there is at most one solution of the ISpP.

Our next corollary for systems on $(0, l)$ is immediate from the proof (see [48, pp. 108–109]) of Theorem 2.12.

Corollary 2.13. *Assume that (2.11) is valid and let relations (2.12) and (2.13) hold for an analytic and bounded matrix function φ . Then, $\zeta(x)$ is uniquely defined on $(0, l)$.*

Remark 2.14. Under somewhat stronger (than in (2.13)) restrictions, the solution of the ISpP always exists. Namely, if for some matrix ϕ_0 and some $M > 0$ we have

$$\sup_{\Im(z) < -M} \|z(\varphi(z) - I_m)\| < \infty, \quad \det \varphi(z) \neq 0 \quad \text{for } \Im(z) < -M, \quad (2.14)$$

$$(\xi + i\eta)(\varphi(\xi + i\eta) - I_m - \phi_0/(\xi + i\eta)) \in L_{m \times m}^2(-\infty, \infty) \quad \text{for all fixed } \eta < -M, \quad (2.15)$$

then $\mathfrak{M}(D, \varphi)$ is constructed in [48, Theorem 4.10].

The situation becomes simpler when $\zeta(x)$ is uniformly bounded on $[0, \infty)$, that is,

$$\sup_{0 < x < \infty} \|\zeta(x)\| \leq M_0. \quad (2.16)$$

We recall that a Weyl function of system (2.10) is introduced in another way than a GW-function. Namely, a Weyl function is an analytic $m \times m$ matrix function $\varphi(z)$, satisfying for certain $M > 0$ and $r > 0$ and for all z from the domain $\mathbb{C}_M^- = \{z: \Im(z) < -M\}$ the inequality

$$\int_0^\infty \exp\{i\bar{z}x D\} \varphi(z)^* w(x, z)^* w(x, z) \varphi(z) \exp\{-(izD + rI_m)x\} dx < \infty, \quad (2.17)$$

and the normalization conditions on the entries $\varphi_{ij}(z)$:

$$\varphi_{ij}(z) \equiv 1 \quad \text{for } i = j, \quad \varphi_{ij}(z) \equiv 0 \quad \text{for } i > j. \quad (2.18)$$

When (2.16) holds, a Weyl function of system (2.10) exists and is unique. Moreover, for that case φ is the unique GW-function (of system (2.10) with the given ζ) satisfying normalization conditions (2.18). In order to construct this Weyl (and simultaneously GW-) function we use matrices J of the form (1.5) for each $1 \leq m_1 < m$, that is, we set

$$J_k := \begin{bmatrix} I_k & 0 \\ 0 & -I_{m-k} \end{bmatrix}, \quad 1 \leq k < m.$$

Now, the Weyl function is constructed [48, pp. 103–106] in the following way.

First, for each k , we introduce a class of $m \times (m - k)$ matrix functions \mathcal{Q}_k , which are meromorphic in some semi-plane $\mathbb{C}_{M_k}^-$ and satisfy the inequalities

$$\mathcal{Q}_k(z)^* \mathcal{Q}_k(z) > 0, \quad \mathcal{Q}_k(z)^* J_k \mathcal{Q}_k(z) \leq 0, \quad (2.19)$$

excluding, possibly, isolated points. These \mathcal{Q}_k are called nonsingular with property- J_k . Assuming $M_k > 2M_0/(d_k - d_{k+1})$ and using (2.19), one can show that the matrix function

$$\psi_k(x, z) = \begin{bmatrix} I_k & 0 \end{bmatrix} w(x, z)^{-1} \mathcal{Q}_k(z) \left(\begin{bmatrix} 0 & I_{m-k} \end{bmatrix} w(x, z)^{-1} \mathcal{Q}_k(z) \right)^{-1} \quad (2.20)$$

is well-defined for $x \geq 0$, $z \in \mathbb{C}_{M_k}^-$, and satisfies the inequality

$$\begin{bmatrix} \psi_k(z)^* & I_{m-k} \end{bmatrix} J_k \begin{bmatrix} \psi_k(z) \\ I_{m-k} \end{bmatrix} \leq 0, \quad \text{i.e.,} \quad \psi_k(z)^* \psi_k(z) \leq I_{m-k}. \quad (2.21)$$

The set of matrices $\psi_k(x, z)$ given by (2.20), where x and z are fixed and matrix functions $\mathcal{Q}_k(z)$ are nonsingular with property- J_k , is denoted by $\mathcal{N}_k(x, z)$. These sets are embedded and have a point limit, that is, similar to (2.3) and (2.4) we have

$$\check{\psi}_k(z) = \bigcap_{x < \infty} \mathcal{N}_k(x, z), \quad \check{\psi}_k(z) = \lim_{x \rightarrow \infty} \psi_k(x, z), \quad z \in \mathbb{C}_{M_k}^-. \quad (2.22)$$

In this way we recover the $(k + 1)$ th column of the Weyl function φ . More precisely, we have

$$\{\varphi_{i,k+1}(z)\}_{i=1}^k = \check{\psi}_k(z) \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \end{bmatrix}, \quad \Im(z) < -M \quad (2.23)$$

for any $M > \max_{1 \leq k < m} (2M_0/(d_k - d_{k+1}))$. There is also an inverse transformation [48, Remark 4.6], which expresses $\check{\psi}_k$ via φ :

$$\check{\psi}_k(z) = [I_k \ 0] \varphi(z) \begin{bmatrix} 0 \\ I_{m-k} \end{bmatrix} \left([0 \ I_{m-k}] \varphi(z) \begin{bmatrix} 0 \\ I_{m-k} \end{bmatrix} \right)^{-1}. \quad (2.24)$$

Finally, we will need a representation of $w(x, z)$ on intervals $0 \leq x \leq l$, $l < \infty$, [48, equation (4.37)] :

$$w(x, z) = \exp\{izxD\} + \int_{d_mx}^{d_1x} \exp\{izt\} N(x, t) dt, \quad \sup_{x \leq l} \|N(x, t)\| < \infty. \quad (2.25)$$

3 NLS with quasi-analytic boundary conditions

First publications on initial-boundary value problems for integrable systems (see, e.g., [28, 32]) appeared only several years after the great breakthrough for Cauchy problems for such systems. Interesting numerical [13], uniqueness [10, 57] and local existence [27, 37] results followed. Special *linearizable* cases of boundary conditions were found using symmetrical reduction [55] or BT (Bäcklund transformation) method [9, 25]. Global existence results for Dirichlet and Neumann initial-boundary value problems (for cubic NLS equations) were obtained using PDE methods in [12] and [29], respectively. Interesting approaches were developed by D.J. Kaup and H. Steudel [33], by P. Sabatier (elbow scattering) [39, 40] and by A.S. Fokas (global relation method) [18], see also some discussions on the corresponding difficulties and open problems in [3, 11]. Since many publications were dedicated to the initial-boundary value problems for NLS equations, it is of special interest that a wide class of solutions of NLS in a semi-strip is uniquely determined by the initial condition.

We note that the case of quasi-analytic boundary (or initial) conditions is important also because related suggestions that initial and boundary conditions (or even solutions) belong to the so-called Schwartz class of functions are often used for simplicity (see, e.g., [17]). Our result shows that one should be rather careful with such suggestions, so that they agree with the established interrelations between initial and boundary conditions.

Recall that the domain \mathcal{D} is defined in (1.6).

Notation 3.1. We consider $m_1 \times m_2$ matrix functions $v(x, t)$, which are continuously differentiable and are such that v_{xx} exists on the semi-strip \mathcal{D} . Moreover, we require that for each k there is a value $\varepsilon_k = \varepsilon_k(v) > 0$ such that v is k times continuously differentiable with respect to x in the square

$$\mathcal{D}(\varepsilon_k) = \{(x, t) : 0 \leq x \leq \varepsilon_k, 0 \leq t \leq \varepsilon_k\}, \quad \mathcal{D}(\varepsilon_k) \subset \mathcal{D}. \quad (3.1)$$

The class of such functions $v(x, t)$ is denoted by $C_\varepsilon(\mathcal{D})$.

Without loss of generality, we assume that the values ε_k in (3.1) monotonically decrease.

Proposition 3.2. *Assume that $v \in C_\varepsilon(\mathcal{D})$ satisfies the dNLS equation (1.1) on \mathcal{D} . Then, for each integer $r \geq 0$ and values $0 \leq k \leq r$, the functions $\left(\frac{\partial^k}{\partial t^k} v\right)(x, 0)$ and $\left(\frac{\partial^k}{\partial t^k} v_x\right)(x, 0)$ may be uniquely recovered (on the interval $0 \leq x \leq \varepsilon_{4r}$) from the initial condition*

$$v(x, 0) = \mathcal{V}(x). \quad (3.2)$$

Moreover, on the domain $\mathcal{D}(\varepsilon_{4(r+1)})$ the functions $\left(\frac{\partial^{k+1}}{\partial t^{k+1}} v\right)$, $\left(\frac{\partial^{k+1}}{\partial t^{k+1}} v\right)_x$ and $\left(\frac{\partial^{k+1}}{\partial t^{k+1}} v\right)_{xx}$ exist and are continuous, and the equalities

$$\left(\frac{\partial^k}{\partial t^k} v\right)_{xt} = \left(\frac{\partial^k}{\partial t^k} v\right)_{tx}, \quad \left(\frac{\partial^k}{\partial t^k} v\right)_{xxt} = \left(\frac{\partial^k}{\partial t^k} v\right)_{txx} \quad (3.3)$$

hold, whereas both sides of these equalities are again continuous. For $0 \leq k \leq r$ and $0 \leq \ell \leq s$ the functions $\left(\frac{\partial^\ell}{\partial x^\ell} \frac{\partial^{k+1}}{\partial t^{k+1}} v\right)$ exist and are continuous in the domains $\mathcal{D}(\varepsilon_{s+4(r+1)})$.

In order to prove this proposition, we need a stronger version of the well-known Clairaut's (or Schwarz's) theorem on mixed derivatives. We need this version for the closed square $\mathcal{D}(\varepsilon)$ (as in Proposition 3.2 from [47]), which statement easily follows from the proofs of the mixed derivatives theorem for open domains (see, e.g., [53]).

Proposition 3.3 ([47]). *If the functions f , f_t and f_{tx} exist and are continuous on $\mathcal{D}(\varepsilon)$ and the derivative $f_x(x, 0)$ exists for $0 \leq x \leq \varepsilon$, then f_x and f_{xt} exist on $\mathcal{D}(\varepsilon)$ and $f_{xt} = f_{tx}$.*

Proof of Proposition 3.2. We prove Proposition 3.2 by induction. First, consider the case $r = 0$. Clearly, $v(x, 0)$ and $v_x(x, 0)$ are given by the initial condition (3.2). Since the right-hand side of (1.1) is two times continuously differentiable with respect to x in $\mathcal{D}(\varepsilon_4)$, we derive that v_t , v_{tx} and v_{txx} exist and are continuous in $\mathcal{D}(\varepsilon_4)$:

$$2v_t = i(v_{xx} - 2vv^*v), \quad 2v_{tx} = i\left(v_{xxx} - 2\frac{\partial}{\partial x}(vv^*v)\right),$$

$$2v_{txx} = i\left(v_{xxxx} - 2\frac{\partial^2}{\partial x^2}(vv^*v)\right).$$

Moreover, putting $f = v$ we see that conditions of Proposition 3.3 are fulfilled and the first equality in (3.3) holds for $k = 0$. Putting $f = v_x$ and taking into account that the first equality in (3.3) yields $v_{xt} = v_{tx}$ and $v_{xtx} = v_{txx}$, we see that conditions of Proposition 3.3 hold also for $f = v_x$. That is, v_{xxt} exists and equals $v_{xtx} = v_{txx}$. Thus, (3.3) is proved for $k = 0$ (i.e., for $r = 0$). In view of (1.1), it is immediate that the last statement of Proposition 3.2 is also valid for $r = 0$.

Next, assuming that the statements of Proposition 3.2 hold for all $0 \leq r \leq r_0$, let us prove them for $r = r_0 + 1$. Differentiating both sides of (1.1) r_0 times with respect to t and taking into account (for $r_0 > 0$ and $k \leq r_0 - 1$) the second equality in (3.3), we express $\left(\frac{\partial^{r_0+1}}{\partial t^{r_0+1}} v\right)(x, 0)$ via derivatives which we already know. Then, from the first equality in (3.3), we obtain the formula $\left(\frac{\partial^{r_0+1}}{\partial t^{r_0+1}} v_x\right)(x, 0) = \left(\frac{\partial^{r_0+1}}{\partial t^{r_0+1}} v\right)_x(x, 0)$ and an expression for $\left(\frac{\partial^{r_0+1}}{\partial t^{r_0+1}} v_x\right)(x, 0)$ follows.

Differentiating both sides of (1.1) $r_0 + 1$ times and using (3.3) for $r = r_0$, we see also that the derivative $\frac{\partial^{r_0+2}}{\partial t^{r_0+2}} v$ exists and is continuous. Furthermore, differentiating (1.1) $r_0 + 1$ times with respect to t and once or twice with respect to x , from the last statement of our proposition (for the case $r = r_0$) we derive that the derivatives $\left(\frac{\partial^{r_0+2}}{\partial t^{r_0+2}} v\right)_x$ and $\left(\frac{\partial^{r_0+2}}{\partial t^{r_0+2}} v\right)_{xx}$ exist and are continuous in $\mathcal{D}(\varepsilon_{4(r_0+2)})$. Now, we see that the conditions of Proposition 3.3 are fulfilled for $f = \frac{\partial^{r_0+1}}{\partial t^{r_0+1}} v$, and so the first equality in (3.3) holds for $k \leq r_0 + 1$. Using this first equality

in (3.3), we derive that the conditions of Proposition 3.3 are fulfilled for $f = \left(\frac{\partial^{r_0+1}}{\partial t^{r_0+1}} v \right)_x$ and therefore the second equality in (3.3) holds for $k \leq r_0 + 1$. Differentiating again both sides of (1.1), we show that the last statement in Proposition 3.2 holds for $r = r_0 + 1$. ■

The class $C(\{\widetilde{M}_k\})$ consists of all infinitely differentiable on $[0, a)$ scalar functions f such that for some $c(f) \geq 0$ and for fixed constants $\widetilde{M}_k > 0$ ($k \geq 0$) we have

$$\left| \frac{d^k f}{dx^k}(x) \right| \leq c(f)^{k+1} \widetilde{M}_k \quad \text{for all } x \in [0, a).$$

Here, we use the notation \widetilde{M}_k (as well \widetilde{M} below) because the upper estimates M (without tilde) were already used in Section 2. Recall that $C(\{\widetilde{M}_k\})$ is called quasi-analytic if for the functions f from this class and for any $0 \leq x < a$ the equalities $\frac{d^k f}{dx^k}(x) = 0$ ($k \geq 0$) yield $f \equiv 0$. According to the famous Denjoy–Carleman theorem, the equality

$$\sum_{n=1}^{\infty} \frac{1}{L_n} = \infty, \quad L_n := \inf_{k \geq n} \widetilde{M}_k^{1/k}$$

implies that the class $C(\{\widetilde{M}_k\})$ is quasi-analytic.

Corollary 3.4. *If $v(x, t)$ satisfies conditions of Proposition 3.2 and the entries of $v(0, t)$ or $v_x(0, t)$ are quasi-analytic, then the matrix functions $v(0, t)$ or $v_x(0, t)$, respectively, are uniquely defined by the initial condition (3.2).*

Let us consider the case, where both matrix functions $v(0, t)$ and $v_x(0, t)$ are quasi-analytic. More precisely, we assume that the entries $v_{ij}(0, t)$ of $v(0, t)$ belong to some quasi-analytic classes $C(\{\widetilde{M}_k(i, j)\})$, the entries $(v_{ij})_x(0, t)$ of $v_x(0, t)$ belong to some quasi-analytic classes $C(\{\widetilde{M}_k^+(i, j)\})$, and, in this case, we say that $v(0, t) \in C([0, a); \widetilde{M})$ and $v_x(0, t) \in C([0, a); \widetilde{M}^+)$, where

$$\widetilde{M} = \{\widetilde{M}_k(i, j)\} \quad \text{and} \quad \widetilde{M}^+ = \{\widetilde{M}_k^+(i, j)\}.$$

Now, using Proposition 2.5, Theorem 2.7, Remark 2.8 and Corollary 3.4 we obtain the main theorem in this section.

Theorem 3.5. *Assume that $v \in C_\varepsilon(\mathcal{D})$ satisfies the dNLS equation (1.1) on \mathcal{D} , that (2.8) holds and that boundary values $v(0, t)$ and $v_x(0, t)$ belong to quasi-analytic classes $C([0, a); \widetilde{M})$ and $C([0, a); \widetilde{M}^+)$, respectively. Then, v is uniquely defined by the initial condition (3.2).*

Remark 3.6. We see that the scheme to recover v (in the semi-strip \mathcal{D}) from the initial condition follows from Proposition 2.5, Theorem 2.7 and the proof of Proposition 3.2. The only step that we did not describe in detail is the recovery of the functions $v(0, t)$ and $v_x(0, t)$ from their Taylor coefficients at $t = 0$. Although Taylor coefficients uniquely determine quasi-analytic functions $v(0, t)$ and $v_x(0, t)$, the recovery of these functions presents an interesting problem, which is not solved completely so far. See [4, 34] and [8, Section III.8] for some important results.

Another important case, where the boundary conditions of the nonlinear Schrödinger equation determine a quasi-analytic initial condition, is discussed in [47, Section 3]. We note that our solutions are not (in general) quasi-analytic.

Remark 3.7. An interesting class of such solutions of a scalar dNLS that the Weyl functions $\tilde{\varphi}(t, z)$ may be presented as the series $\tilde{\varphi}(t, z) = \sum_{k=0}^{\infty} \alpha_k(t)/z^k$ (for sufficiently large values of z) was treated in [50]. (We note that the Weyl functions $\tilde{\varphi}(z)$ from [50] are Herglotz functions and can be easily mapped into the Weyl functions $\varphi(z)$ considered here via a linear-fractional transformation with constant coefficients.) According to [50, Theorem 1], if $\tilde{\varphi}(0, z) = \sum_{k=0}^{\infty} \alpha_k(0)/z^k$, there is one and only one solution of dNLS from this class in some semi-strip.

Important results on the asymptotics of Weyl functions are given, for instance, in [14, 26]. However, it would be fruitful to know, under which conditions the asymptotic series (for Weyl functions) from [14, 26] converge or at least uniquely define the corresponding Weyl function.

4 Nonlinear optics equation on a semi-strip

In this section we consider the nonlinear optics equation (1.8), where D has the form (1.9). We consider equation (1.8) on the semi-strip \mathcal{D} , which is given by (1.6). First, using Weyl theoretic results from Section 2.2, we express Weyl function $\varphi(t, z)$ of the auxiliary system (2.10), where $\zeta(x) = \zeta(x, t) = [D, \varrho(x, t)]$, in terms of $\varphi(0, z)$ and the boundary condition $\varrho(0, t) = \hat{\rho}(t)$. In other words, we express in these terms the evolution of the Weyl function. For that purpose, following formulas (2.5) and (2.6) in Proposition 2.6, we introduce matrix function $R(t, z)$ by the relations

$$R_t(t, z) = (iz\hat{D} - [\hat{D}, \hat{\rho}(t)])R(t, z), \quad R(0, z) = I_m. \quad (4.1)$$

Recall that \hat{D} is a diagonal matrix and that $\hat{D} > 0$.

Theorem 4.1. *Let $\varrho(x, t)$ satisfy the nonlinear optics equation (1.8) and the boundary condition $\varrho(0, t) = \hat{\rho}(t)$. Assume that ϱ is uniformly bounded and continuously differentiable on \mathcal{D} . Then, the matrix functions*

$$\check{\psi}_k(t, z) := [I_k \ 0] R(t, z) \varphi(0, z) \begin{bmatrix} 0 \\ I_{m-k} \end{bmatrix} \left([0 \ I_{m-k}] R(t, z) \varphi(0, z) \begin{bmatrix} 0 \\ I_{m-k} \end{bmatrix} \right)^{-1} \quad (4.2)$$

are well-defined for $1 \leq k < m$, and the evolution of the Weyl function is given by the formula

$$\{\varphi_{i,k+1}(t, z)\}_{i=1}^k = \check{\psi}_k(t, z) \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \end{bmatrix}, \quad \Im(z) < -M \quad (4.3)$$

and by the normalization conditions (2.18).

Proof. We set

$$M_0 = \sup \|\zeta(x, t)\|, \quad (x, t) \in \mathcal{D}, \quad M > \max_{1 \leq k < m} (2M_0/(d_k - d_{k+1})). \quad (4.4)$$

Recall that G and F for the case of the nonlinear optics equation are given by (2.2). Since ϱ is continuously differentiable, the conditions of Proposition 2.6 are fulfilled. Taking into account that the fundamental solution of (2.10) is denoted by w (instead of u in Proposition 2.6), we rewrite (2.7) in the form $w(x, t, z) = R(x, t, z)w(x, 0, z)R(t, z)^{-1}$ or, equivalently,

$$w(x, t, z)^{-1} = R(t, z)w(x, 0, z)^{-1}R(x, t, z)^{-1}. \quad (4.5)$$

Using (4.5), we express $w(x, t, z)^{-1}$ via $w(x, \tilde{t}, z)^{-1}$, $0 \leq t, \tilde{t} < a$:

$$\begin{aligned} w(x, t, z)^{-1} &= R(t, z)R(\tilde{t}, z)^{-1}R(\tilde{t}, z)w(x, 0, z)^{-1}R(x, \tilde{t}, z)^{-1}(R(x, t, z)R(x, \tilde{t}, z)^{-1})^{-1} \\ &= R(t, z)R(\tilde{t}, z)^{-1}w(x, \tilde{t}, z)^{-1}(R(x, t, z)R(x, \tilde{t}, z)^{-1})^{-1}. \end{aligned} \quad (4.6)$$

Recall that $R_t = FR$, where F is given in (2.2) and $\hat{D} > 0$. Hence, putting $\mathcal{R}(x, t, \tilde{t}, z) := (R(x, t, z)R(x, \tilde{t}, z)^{-1})^{-1}$ we derive

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{R}(x, t, \tilde{t}, z) &= -\mathcal{R}(x, t, \tilde{t}, z)F(x, t, z), \\ \frac{\partial}{\partial t} (\mathcal{R}(x, t, \tilde{t}, z)\mathcal{R}(x, t, \tilde{t}, z)^*) &< 0, \quad \Im(z) < 0, \quad \mathcal{R}(x, \tilde{t}, \tilde{t}, z) = I_m. \end{aligned}$$

From the relations above it is immediate that

$$\|\mathcal{R}(x, t, \tilde{t}, z)\| < 1 \quad \text{for } t > \tilde{t}, \quad \Im(z) < 0, \quad (4.7)$$

which allows us to estimate the difference

$$I_m - \mathcal{R}(x, t, \tilde{t}, z) = \int_{\tilde{t}}^t \mathcal{R}(x, s, \tilde{t}, z)F(x, s, z)ds. \quad (4.8)$$

According to (4.4), (4.7) and (4.8), for each $\delta > 0$ and $c > M$ there is $\varepsilon = \varepsilon(M_0) > 0$ such that

$$\begin{aligned} \|\mathcal{R}(x, t, \tilde{t}, z)\| &\leq \delta \\ \text{for all } x \in [0, \infty), \quad 0 \leq t - \tilde{t} \leq \varepsilon, \quad z \in \{z: |z| < c\} \cap \{z: \Im(z) < -M\}, \quad c > M. \end{aligned} \quad (4.9)$$

Modifying (2.20) (so that the functions ψ_k and w depend there also on an additional variable t), in view of (4.6), we derive

$$\begin{aligned} \psi_k(x, t, z) &= [I_k \quad 0] R(t, z)R(\tilde{t}, z)^{-1}w(x, \tilde{t}, z)^{-1}\mathcal{R}(x, t, \tilde{t}, z)\mathcal{Q}_k(z) \\ &\quad \times ([0 \quad I_{m-k}] R(t, z)R(\tilde{t}, z)^{-1}w(x, \tilde{t}, z)^{-1}\mathcal{R}(x, t, \tilde{t}, z)\mathcal{Q}_k(z))^{-1}. \end{aligned} \quad (4.10)$$

Moreover, putting

$$\tilde{\mathcal{Q}}_k(z) := \mathcal{R}(x, t, \tilde{t}, z)\mathcal{Q}_k(z), \quad \mathcal{Q}_k(z) := [0 \quad I_{m-k}] \quad (4.11)$$

we see that for sufficiently small δ the matrix function $\tilde{\mathcal{Q}}_k(z)$ satisfies (2.19) in the domain (for z) given in (4.9). Substituting $\tilde{\mathcal{Q}}_k$ (instead of \mathcal{Q}_k) into (2.20), we obtain

$$\begin{aligned} \begin{bmatrix} \psi_k(x, \tilde{t}, z) \\ I_{m-k} \end{bmatrix} &= w(x, \tilde{t}, z)^{-1}\tilde{\mathcal{Q}}_k(z) \left([0 \quad I_{m-k}] w(x, \tilde{t}, z)^{-1}\tilde{\mathcal{Q}}_k(z) \right)^{-1}, \quad \text{i.e.,} \\ w(x, \tilde{t}, z)^{-1}\tilde{\mathcal{Q}}_k(z) &= \begin{bmatrix} \psi_k(x, \tilde{t}, z) \\ I_{m-k} \end{bmatrix} \left([0 \quad I_{m-k}] w(x, \tilde{t}, z)^{-1}\tilde{\mathcal{Q}}_k(z) \right). \end{aligned} \quad (4.12)$$

Using the first equality in (4.11), we can substitute (4.12) into (4.10). Thus, we derive

$$\begin{bmatrix} \psi_k(x, t, z) \\ I_{m-k} \end{bmatrix} = R(t, z)R(\tilde{t}, z)^{-1} \begin{bmatrix} \psi_k(x, \tilde{t}, z) \\ I_{m-k} \end{bmatrix} \left([0 \quad I_{m-k}] R(t, z)R(\tilde{t}, z)^{-1} \begin{bmatrix} \psi_k(x, \tilde{t}, z) \\ I_{m-k} \end{bmatrix} \right)^{-1}$$

and in view of (2.22), passing to the limit $x \rightarrow \infty$, we have the following formula

$$\begin{aligned} \begin{bmatrix} \check{\psi}_k(t, z) \\ I_{m-k} \end{bmatrix} &= R(t, z)R(\tilde{t}, z)^{-1} \begin{bmatrix} \check{\psi}_k(\tilde{t}, z) \\ I_{m-k} \end{bmatrix} \\ &\quad \times \left([0 \quad I_{m-k}] R(t, z)R(\tilde{t}, z)^{-1} \begin{bmatrix} \check{\psi}_k(\tilde{t}, z) \\ I_{m-k} \end{bmatrix} \right)^{-1}. \end{aligned} \quad (4.13)$$

Here we used the fact that, according to (2.21) and (4.9),

$$\det \left(\begin{bmatrix} 0 & I_{m-k} \end{bmatrix} R(t, z) R(\tilde{t}, z)^{-1} \begin{bmatrix} \check{\psi}_k(\tilde{t}, z) \\ I_{m-k} \end{bmatrix} \right) \neq 0$$

for sufficiently small δ . Setting $\tilde{t} = k\varepsilon$ ($k = 0, 1, \dots$), recalling that $R(0, z) = I_m$ and applying each time (4.13), we easily prove (by induction) the equality

$$\begin{bmatrix} \check{\psi}_k(t, z) \\ I_{m-k} \end{bmatrix} = R(t, z) \begin{bmatrix} \check{\psi}_k(0, z) \\ I_{m-k} \end{bmatrix} \left(\begin{bmatrix} 0 & I_{m-k} \end{bmatrix} R(t, z) \begin{bmatrix} \check{\psi}_k(0, z) \\ I_{m-k} \end{bmatrix} \right)^{-1} \quad (4.14)$$

for t on all intervals $[0, (k+1)\varepsilon] \cap [0, a)$, that is, for t on $[0, a)$. Although (4.14) is proved for $z \in \{z: |z| < c\} \cap \{z: \Im(z) < -M\}$, the analyticity of both sides of (4.14) implies that the equality holds in the semi-plane $\Im(z) < -M$. Finally, we note that (2.24) at $t = 0$ yields

$$\begin{bmatrix} \check{\psi}_k(0, z) \\ I_{m-k} \end{bmatrix} = \varphi(0, z) \begin{bmatrix} 0 \\ I_{m-k} \end{bmatrix} \left(\begin{bmatrix} 0 & I_{m-k} \end{bmatrix} \varphi(0, z) \begin{bmatrix} 0 \\ I_{m-k} \end{bmatrix} \right)^{-1}. \quad (4.15)$$

Substituting (4.15) into (4.14), we obtain (4.2). The procedure to recover $\varphi(t, z)$ from $\{\check{\psi}_k(t, z)\}$ (for fixed values of t) is described in Section 2.2 (note that (4.3) coincides with (2.23)). ■

Now, let us prove a uniqueness result for the case of \widehat{D} of the form (1.10). Let initial condition be given by the equality

$$\varrho(x, 0) = \rho(x), \quad \sup_{x \in [0, \infty)} \|\rho(x)\| < \infty.$$

Denote the Weyl function of system

$$y_x(x, z) = (izD - \zeta(x))y(x, z), \quad x \geq 0, \quad \zeta = [D, \rho] \quad (4.16)$$

by $\varphi_0(z)$. (According to Section 2.2, this Weyl function exists and is unique.)

Theorem 4.2. *For the case where the entries of the matrix \widehat{D} in (1.8) are ordered as in (1.10), there is no more than one uniformly bounded and continuously differentiable on \mathcal{D} solution $\varrho = \varrho^*$ (of the nonlinear optics equation (1.8)), having the initial values $\varrho(x, 0)$ such that φ_0 is bounded and (2.13) holds. That is, there is no more than one solution of the corresponding initial value problem.*

Proof. Let $\varrho(x, t)$ satisfy conditions of the theorem. We fix M such (4.4) is valid, $\varphi_0(z)$ is bounded for $\Im(z) \leq -M$ and (2.13) holds for $\varphi_0(\xi + i\eta)$ when $\eta \leq -M$. We set also

$$\widehat{M} = \sup_{0 \leq t < a} \|\widehat{\zeta}(0, t)\|, \quad \widehat{\zeta}(0, t) = [\widehat{D}, \widehat{\rho}(t)], \quad (4.17)$$

where $\widehat{\rho}(t) = \varrho(0, t)$. First, we show that the inequality

$$\sup_{t \in [0, a), \Im(z) < -M} \|R(t, z)\varphi_0(z) \exp\{-izt\widehat{D}\}\| < \infty, \quad a < \infty \quad (4.18)$$

is valid. Indeed, according to (2.21) and (4.14) we have

$$\begin{bmatrix} \check{\psi}_k(0, z)^* & I_{m-k} \end{bmatrix} R(t, z)^* J_k R(t, z) \begin{bmatrix} \check{\psi}_k(0, z) \\ I_{m-k} \end{bmatrix} \leq 0. \quad (4.19)$$

Clearly, (4.19) yields the inequality

$$\begin{aligned} & -4\widehat{M} [\check{\psi}_k(0, z)^* \quad I_{m-k}] R(t, z)^* (I_m - J_k) R(t, z) \begin{bmatrix} \check{\psi}_k(0, z) \\ I_{m-k} \end{bmatrix} \\ & \leq -4\widehat{M} [\check{\psi}_k(0, z)^* \quad I_{m-k}] R(t, z)^* R(t, z) \begin{bmatrix} \check{\psi}_k(0, z) \\ I_{m-k} \end{bmatrix}. \end{aligned} \quad (4.20)$$

Taking into account that $R_t = FR$ and relations (1.10), (2.2), (4.17) and (4.20) hold, we derive

$$\begin{aligned} & \frac{d}{dt} \left(\exp\{i(\bar{z} - z)\widehat{d}_{k+1}t - 4\widehat{M}t\} [\check{\psi}_k(0, z)^* \quad I_{m-k}] \right. \\ & \quad \left. \times R(t, z)^* (I_m - J_k) R(t, z) \begin{bmatrix} \check{\psi}_k(0, z) \\ I_{m-k} \end{bmatrix} \right) \leq 0 \end{aligned} \quad (4.21)$$

for $\Im(z) < 0$. Formulas (4.19) and (4.21) imply that

$$\begin{aligned} & \exp\{i(\bar{z} - z)\widehat{d}_{k+1}t - 4\widehat{M}t\} [\check{\psi}_k(0, z)^* \quad I_{m-k}] R(t, z)^* R(t, z) \begin{bmatrix} \check{\psi}_k(0, z) \\ I_{m-k} \end{bmatrix} \\ & \leq [\check{\psi}_k(0, z)^* \quad I_{m-k}] (I_m - J_k) \begin{bmatrix} \check{\psi}_k(0, z) \\ I_{m-k} \end{bmatrix} = 2I_{m-k}, \quad 1 \leq k < m, \quad \Im(z) < -M. \end{aligned} \quad (4.22)$$

Recall that φ_0 is given by (2.23). Hence, (4.18) follows from (4.22).

Consider system (4.1). Since φ_0 is bounded and satisfies (2.13) and (4.18), according to Corollary 2.13, the matrix function $\widehat{\zeta}(0, t) = [\widehat{D}, \widehat{\rho}(t)]$ (and so R) is uniquely defined by φ_0 . Thus, from Theorem 4.1, we see that $\varphi(t, z)$ is uniquely defined by φ_0 .

In order to prove our theorem, it remains to show that $\varphi(t, z)$ satisfies conditions of Theorem 2.12 for each t . Indeed, in view of (4.1), we can rewrite for R the representation (2.25):

$$R(t, z) = \exp\{izt\widehat{D}\} + \int_{\widehat{d}_m t}^{\widehat{d}_1 t} \exp\{izs\} \widehat{N}(t, s) ds, \quad \sup_{t < a} \|\widehat{N}(t, s)\| < \infty. \quad (4.23)$$

By virtue of (4.18), the matrix function $R(t, z)\varphi_0(z)e^{-izt\widehat{D}} - I_m$ is bounded in the domain $\Im(z) \leq -M$. Since $R(t, z)$ satisfies (4.23) and M is chosen so that φ_0 satisfies (2.13) for $z = \xi - iM$, we see that $R(t, \xi - iM)\varphi_0(\xi - iM)e^{-i(\xi - iM)t\widehat{D}} - I_m \in L_{m \times m}^2(-\infty, \infty)$ for each $0 \leq t < a$, where $L_{m \times m}^2(0, \infty)$ is the class of $m \times m$ matrix functions, the entries of which belong to $L^2(0, \infty)$. Hence, the well-known Theorems V and VIII (Sections 4 and 5 in [38], respectively) on the Fourier transform in complex domains yield the representation (see [48, formula (E11)]):

$$R(t, z)\varphi_0(z)e^{-izt\widehat{D}} = I_m + \int_0^\infty e^{-izx} \mathcal{F}(x) dx, \quad e^{-xM} \mathcal{F}(x) \in L_{m \times m}^2(0, \infty). \quad (4.24)$$

It is immediate also that formula (4.2) can be modified slightly:

$$\begin{aligned} \check{\psi}_k(t, z) &= [I_k \quad 0] R(t, z)\varphi_0(z)e^{-izt\widehat{D}} \begin{bmatrix} 0 \\ I_{m-k} \end{bmatrix} \\ & \quad \times \left([0 \quad I_{m-k}] R(t, z)\varphi_0(z)e^{-izt\widehat{D}} \begin{bmatrix} 0 \\ I_{m-k} \end{bmatrix} \right)^{-1}. \end{aligned} \quad (4.25)$$

According to (4.24), the normalized GW-function $\varphi(t, z)$ constructed via equalities (4.3) and (4.25) satisfies conditions of Theorem 2.12. In other words, there is no more than one solution of ISpP for $\varphi(t, z)$, that is, $\varrho(x, t)$ is unique. \blacksquare

Remark 4.3. In the case of system (4.1), by virtue of (4.17) and (4.18), the requirements of Corollary 2.13 are fulfilled for φ_0 , and so φ_0 uniquely determines the boundary condition $\widehat{\rho}$. Moreover, if φ_0 satisfies (2.14) and (2.15) there is a rigorous procedure to recover $\widehat{\rho}$ from φ_0 (see Remark 2.14).

Although Theorem 4.2 was announced in [41], its proof is published for the first time. It is essential to know, for which initial conditions $\rho(x)$, the restrictions on φ_0 (from Theorem 4.2) are fulfilled. First, let us formulate a particular case of Theorem 6.1 from [5].

Proposition 4.4. *Suppose that the $m \times m$ matrix function $\rho(x)$ is absolutely continuous on \mathbb{R} and $\rho(x), \rho'(x) \in L^1_{m \times m}(-\infty, \infty)$. Then, for some $M > 0$, there is an analytic with respect to z fundamental (unnormalized) solution $\mathcal{M}(x, z)$ of the equation*

$$\mathcal{M}_x = iz[D, \mathcal{M}] - \zeta \mathcal{M}, \quad x \in \mathbb{R}, \quad \zeta = [D, \rho], \quad (4.26)$$

such that uniformly with respect to x we have

$$\mathcal{M}(x, z) = I_m + \frac{1}{z} \mathcal{M}_1(x) + o(|z|^{-1}), \quad |z| \rightarrow \infty, \quad \Im z < -M, \quad (4.27)$$

where $\mathcal{M}_1(x)$ is absolutely continuous.

We note that the fact that \mathcal{M}_1 is absolutely continuous is immediate from the proof of [5, Theorem 6.1] (more precisely, from formulas (6.6) and (6.8)). Now, since we can always extend ρ' on \mathbb{R} , we consider $\rho(x)$ on $[0, \infty)$ only and assume that ρ is absolutely continuous and $\rho, \rho' \in L^1_{m \times m}(\mathbb{R}_+)$. Without loss of generality, we assume that M is chosen so that $\mathcal{M}(0, z)$ is invertible for $\Im(z) < -M$. Then, according to (4.26), the matrix function

$$w(x, z) = \mathcal{M}(x, z) e^{izxD} \mathcal{M}(0, z)^{-1} \quad (4.28)$$

is the normalized (by $w(0, z) = I_m$) fundamental solution of the equation (4.16) (and we don't require so far that $\rho = \rho^*$). Moreover, (4.27) implies that (2.12) holds for $\varphi(z) = \mathcal{M}(0, z)$. That is, assuming $\rho = \rho^*$ and taking into account Definition 2.9, we see that $\mathcal{M}(0, z)$ is a GW-function of (4.16).

Recall that in Theorem 4.2 we speak about the Weyl function φ_0 or, equivalently for a bounded function $\rho = \rho^*$, about the normalized GW-function. Thus, we should normalize $\mathcal{M}(0, z)$. For that purpose we construct a lower triangular matrix function $\widehat{\mathcal{M}}(z)$ via the right lower $k \times k$ blocks $\mathcal{P}_k(z)$ of $\mathcal{M}(0, z)$. Namely, we construct $\widehat{\mathcal{M}}(z)$ columnwise via the equalities

$$\widehat{\mathcal{M}}(z) \{\delta_{i, m-k+1}\}_{i=1}^m := \begin{bmatrix} 0 \\ \mathcal{P}_k(z)^{-1} \{\delta_{i1}\}_{i=1}^k \end{bmatrix}, \quad 1 \leq k \leq m, \quad (4.29)$$

where $\{\delta_{i, m-k+1}\}_{i=1}^m$ and $\{\delta_{i1}\}_{i=1}^k$ are column vectors. It follows from (4.29) that the normalization conditions (2.18) hold for

$$\varphi_0(z) = \mathcal{M}(0, z) \widehat{\mathcal{M}}(z). \quad (4.30)$$

Since $\mathcal{M}(z)$ is lower triangular and D satisfies (1.9), we see that $e^{izxD} \widehat{\mathcal{M}}(z) e^{-izxD}$ is bounded for $\Im(z) < -M$. Hence, taking into account that $\mathcal{M}(0, z)$ is a GW-function, we derive that (2.12) is also valid for $\varphi(z) = \mathcal{M}(0, z) \widehat{\mathcal{M}}(z)$ (i.e., φ_0 given by (4.30) is the normalized GW-function). Finally, relations (4.27), (4.29) and (4.30) show that φ_0 is bounded and that (2.13) also holds for φ_0 . Thus, we proved the statement below.

Proposition 4.5. *Suppose that the initial condition $\rho(x) = \rho(x)^*$ is absolutely continuous on $[0, \infty)$ and $\rho(x), \rho'(x) \in L^1_{m \times m}(\mathbb{R}_+)$. Then, the Weyl function $\varphi_0(z)$ of the system (4.16), where $\zeta = \zeta_0 = [D, \rho]$, exists. Moreover, $\varphi_0(z)$ is analytic and bounded (in some semi-plane $\Im(z) < -M$, $M > 0$), and it satisfies (2.13).*

An existence result for a solution of an initial value problem (for the nonlinear optics equation) is given in [44, Remark 4.7].

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