

On the Integrability of Supersymmetric Versions of the Structural Equations for Conformally Parametrized Surfaces^{*}

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Abstract. The paper presents the bosonic and fermionic supersymmetric extensions of the structural equations describing conformally parametrized surfaces immersed in a Grassmann superspace, based on the authors' earlier results. A detailed analysis of the symmetry properties of both the classical and supersymmetric versions of the Gauss–Weingarten equations is performed. A supersymmetric generalization of the conjecture establishing the necessary conditions for a system to be integrable in the sense of soliton theory is formulated and illustrated by the examples of supersymmetric versions of the sine-Gordon equation and the Gauss–Codazzi equations.

Key words: supersymmetric models; Lie superalgebras; symmetry reduction; conformally parametrized surfaces; integrability

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1 Introduction

Over the last decades, the concept of supersymmetry has been used extensively in particle physics and string theory [4, 10, 17, 19, 30, 46, 49] as well as in hydrodynamic-type models [8, 16, 20, 26, 31, 33, 36, 37]. Systems involving even and odd Grassmann variables are interesting because even Grassmann variables have properties similar to those of bosonic particles and odd Grassmann variables have properties similar to those of fermionic particles. These particles appear in the standard model, bosons as interaction particles and fermions as matter particles. Supersymmetric (SUSY) extensions have been constructed, for example, for the Korteweg–de Vries equation [33, 37], the Chaplygin gas equation in $(1+1)$ - and $(2+1)$ -dimensions (using parametrizations of the action for a superstring and a Nambu–Goto supermembrane, respectively) [31], the scalar Born–Infeld equation [29], and the sine-Gordon equation [11, 14, 25, 27, 41, 42, 51]. Supersoliton solutions were obtained for a number of SUSY theories through a connection between the super-Bäcklund and super-Darboux transformations

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[1, 11, 24, 27, 35, 42, 47]. A Crum-type transformation was used to determine a number of supersoliton and multisupersoliton solutions, and the existence of infinitely many local conserved quantities was determined [25, 38, 41]. In many cases, the integrability of supersymmetric systems has been demonstrated by finding Lax pairs and conservation laws [33, 37].

Superpositions of solutions of nonlinear SUSY systems are not as well understood as superpositions of solutions of nonlinear classical systems. As a result, SUSY differential equations do not have as extensive a theoretical foundation as classical differential equations. However, the method of prolongation of infinitesimal vector fields for Lie symmetries, the methods for the classification of subalgebras and the symmetry reduction method can, to some extent, be adapted to the case of Grassmann-valued systems of differential equations (see, e.g., [2, 27]).

A supersymmetric generalization of the structural equations (the Gauss, Codazzi and Ricci equations), constructed through the use of the exterior geometry formalism, was proposed in [3, 4, 5, 43]. This generalization was used to study superstrings and different super- p -branes. It was shown [4], using the superembedding approach, that these structural equations have their doubly supersymmetric counterparts.

The subject of our investigation is conformally parametrized surfaces immersed in a Grassmann superspace. This study is based on the methodology for the construction of SUSY extensions of the Gauss–Weingarten (GW) and Gauss–Codazzi (GC) equations developed in the authors’ previous work [9]. It involves the use of a moving frame formalism, leading to an explicit formulation of the structural equations for surfaces immersed in a Grassmann superspace. These equations constitute the SUSY extensions of the GW and GC equations. In [9] we constructed two distinct extensions (one in terms of a bosonic superfield and the other in terms of a fermionic superfield) for each of these systems. For both SUSY extensions of the GC equations, Lie symmetry superalgebras were determined and the one-dimensional subalgebras of these superalgebras were classified into conjugacy classes under the action of their respective supergroups.

The main task undertaken in this paper is an analysis of the conditions for the existence of soliton and multisoliton solutions of the supersymmetric versions of differential equations. For this purpose we adapt the symmetry group approach to the problem of integrability in the sense of soliton theory to the SUSY case. This approach proved to be effective in the classical case when it was first proposed in the form of a conjecture for point symmetries of the GW and GC equations by D. Levi et al. [34] and next developed by J. Cieřliński [12, 13]. It establishes a spectral technique which enables us to explicitly construct one-parameter families of surfaces associated with a given integrable system.

To formulate an analogue of the classical conjecture for the SUSY case we had to determine the symmetries of the GW equations for the classical case as well as for the bosonic and fermionic SUSY extensions and to compare them to the symmetries of the associated GC equations. The conjecture states that, if the set of symmetries of the GC equations is larger than the set of symmetries of the GW equations, then we can introduce a spectral parameter into the GW equations and obtain a Lax pair associated with the GC equations, provided that the spectral parameter cannot be eliminated through a gauge transformation. This introduction can be done through the use of vector fields that are symmetries of the original system, but not symmetries of the associated linear system. We provide an algorithmic procedure for this analysis, facilitating the determination of the integrability of a system under consideration. We illustrate these results with the examples of the SUSY versions of the sine-Gordon equation and the GC equations.

The paper is organized as follows. In Section 2, we discuss the symmetry properties of the classical GW and GC equations, identify the Lie point symmetry algebras. Section 3 is devoted to a brief outline of the properties of Grassmann variables and Grassmann algebras. In Section 4, we analyze bosonic and fermionic SUSY extensions of the GW and GC equations. In Section 5, we adapt the classical conjecture distinguishing integrable systems to the SUSY extensions of the GW and GC equations. Finally, in Section 6, we present possibilities for future research.

2 Symmetries of the structural equations of conformally parametrized surfaces

Consider a moving frame Ω on a smooth orientable conformally parametrized surface in 3-dimensional Euclidean space \mathbb{R}^3 which satisfies the GW equations

$$\begin{aligned} \partial \begin{pmatrix} \partial F \\ \bar{\partial} F \\ N \end{pmatrix} &= \begin{pmatrix} \partial u & 0 & Q \\ 0 & 0 & \frac{1}{2} H e^u \\ -H & -2e^{-u} Q & 0 \end{pmatrix} \begin{pmatrix} \partial F \\ \bar{\partial} F \\ N \end{pmatrix}, & \partial \Omega &= V_1 \Omega, \\ \bar{\partial} \begin{pmatrix} \partial F \\ \bar{\partial} F \\ N \end{pmatrix} &= \begin{pmatrix} 0 & 0 & \frac{1}{2} H e^u \\ 0 & \bar{\partial} u & \bar{Q} \\ -2e^{-u} \bar{Q} & -H & 0 \end{pmatrix} \begin{pmatrix} \partial F \\ \bar{\partial} F \\ N \end{pmatrix}, & \bar{\partial} \Omega &= V_2 \Omega, \end{aligned} \quad (2.1)$$

where we define the space $\mathcal{X} = (z, \bar{z})$ of independent variables, where $z = x + iy$ and $\bar{z} = x - iy$ are complex variables, and the space $\mathcal{U} = (H, Q, \bar{Q}, u)$ of unknown functions. Here $\Omega = (\partial F, \bar{\partial} F, N)^T$ is a moving frame of a conformally parametrized surface with the vector-valued function $F = (F_1, F_2, F_3): \mathcal{R} \rightarrow \mathbb{R}^3$ (where \mathcal{R} is a Riemann surface) satisfying the following normalization for the tangent vectors ∂F and $\bar{\partial} F$ and the unit normal N

$$\langle \partial F, \partial F \rangle = \langle \bar{\partial} F, \bar{\partial} F \rangle = 0, \quad \langle \partial F, \bar{\partial} F \rangle = \frac{1}{2} e^u, \quad \langle \partial F, N \rangle = \langle \bar{\partial} F, N \rangle = 0, \quad \langle N, N \rangle = 1,$$

where the induced metric of the surface satisfies $I = e^u dz d\bar{z}$ with local z and \bar{z} coordinates on \mathcal{R} . We have used the abbreviated notation

$$\partial \equiv \partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \bar{\partial} \equiv \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y),$$

for the partial derivatives with respect to the complex variables z and \bar{z} , respectively. The bracket $\langle \cdot, \cdot \rangle$ denotes the scalar product in 3-dimensional Euclidean space

$$\langle a, b \rangle = \sum_{i=1}^3 a_i b_i, \quad a, b \in \mathbb{C}^3. \quad (2.2)$$

The quantities Q , \bar{Q} and H in equations (2.1) involve the second derivatives of the immersion function F and are defined as follows

$$Q = \langle \partial^2 F, N \rangle \in \mathbb{C}, \quad H = 2e^{-u} \langle \partial \bar{\partial} F, N \rangle \in \mathbb{R},$$

where the differentials $Q dz^2$ and $\bar{Q} d\bar{z}^2$ defined on the Riemann sphere \mathcal{R} are called Hopf differentials while H is the mean curvature function of the surface.

The Gauss–Codazzi equations, which are the zero curvature condition (ZCC) for the potential matrices V_1 and V_2 taking values in a Lie algebra, are

$$\bar{\partial} V_1 - \partial V_2 + [V_1, V_2] = 0,$$

which reduce to the following three linearly independent equations

$$\begin{aligned} \partial \bar{\partial} u + \frac{1}{2} H^2 e^u - 2Q \bar{Q} e^{-u} &= 0, & \text{(the Gauss equation)} \\ \partial \bar{Q} = \frac{1}{2} e^u \bar{\partial} H, \quad \bar{\partial} Q = \frac{1}{2} e^u \partial H & & \text{(the Codazzi equations)} \end{aligned} \quad (2.3)$$

These equations guarantee the existence of conformally parametrized surfaces in \mathbb{R}^3 . A description of all infinitesimal symmetries of the GC equations was investigated [9] for conformally parametrized surfaces and the results can be summarized as follows.

In the case where the system of the GC equations has maximal rank over $M \subset \mathcal{X} \times \mathcal{U}$, it was found [9] that the set of all infinitesimal Lie point symmetries of the system forms an infinite-dimensional Lie algebra \mathcal{L}_1 spanned by the vector fields

$$\begin{aligned} X(\eta) &= \eta(z)\partial_z + \eta'(z)(-2Q\partial_Q - U\partial_U), & Y(\zeta) &= \zeta(\bar{z})\partial_{\bar{z}} + \zeta'(\bar{z})(-2\bar{Q}\partial_{\bar{Q}} - U\partial_U), \\ e_0 &= -H\partial_H + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + 2U\partial_U, \end{aligned}$$

where η and ζ are arbitrary functions of z and \bar{z} respectively, while η' and ζ' are the derivatives of η and ζ with respect to their arguments. Here and subsequently, we use the notation $U = e^u$. The generators $X(\eta)$ and $Y(\zeta)$ are two infinite-dimensional families of conformal transformations, while e_0 is a dilation in the dependent variables which constitutes the center of the algebra. The maximal finite-dimensional subalgebra L_1 of the algebra \mathcal{L}_1 was obtained by expanding η and ζ as power series with respect to their arguments. This algebra L_1 is spanned by the seven generators

$$\begin{aligned} e_0 &= -H\partial_H + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + 2U\partial_U, \\ e_1 &= \partial_z, & e_3 &= z\partial_z - 2Q\partial_Q - U\partial_U, & e_5 &= z^2\partial_z - 4zQ\partial_Q - 2zU\partial_U, \\ e_2 &= \partial_{\bar{z}}, & e_4 &= \bar{z}\partial_{\bar{z}} - 2\bar{Q}\partial_{\bar{Q}} - U\partial_U, & e_6 &= \bar{z}^2\partial_{\bar{z}} - 4\bar{z}\bar{Q}\partial_{\bar{Q}} - 2\bar{z}U\partial_U. \end{aligned}$$

Let us now perform an analysis of the infinitesimal symmetries of the GW equations (2.1). In the case where the system of GW equations has maximal rank over $M \subset \mathcal{X} \times \mathcal{U}$, the set of all infinitesimal symmetries of the system forms an infinite-dimensional Lie algebra \mathcal{L}_2 spanned by the vector fields

$$\begin{aligned} X(\eta) &= \eta(z)\partial_z - \eta'(z)(U\partial_U + 2Q\partial_Q), & Y(\zeta) &= \zeta(\bar{z})\partial_{\bar{z}} - \zeta'(\bar{z})(U\partial_U + 2\bar{Q}\partial_{\bar{Q}}), \\ \hat{e}_0 &= -H\partial_H + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + 2U\partial_U + F_i\partial_{F_i}, \\ T_i &= \partial_{F_i}, & \mathcal{D}_i &= F_i\partial_{F_i} + N_i\partial_{N_i}, & i &= 1, 2, 3, \\ R_{ij} &= (F_i\partial_{F_j} - F_j\partial_{F_i}) + (N_i\partial_{N_j} - N_j\partial_{N_i}), \\ S_{ij} &= (F_i\partial_{F_j} + F_j\partial_{F_i}) + (N_i\partial_{N_j} + N_j\partial_{N_i}), & i &< j = 2, 3. \end{aligned} \tag{2.4}$$

Here, we have used the notation $\eta'(z) = d\eta/dz$ and $\zeta'(\bar{z}) = d\zeta/d\bar{z}$, where η and ζ are arbitrary functions of z and \bar{z} respectively. The generators in (2.4) can be identified as follows: the T_i generate translations in the F_i directions respectively, R_{ij} represent rotations in the direction of the F_i and N_i variables, S_{ij} are local boost transformations and the vector fields e_0 , \mathcal{D}_1 and \mathcal{D}_2 correspond to scaling transformations. In addition, we obtain two infinite-dimensional families of infinitesimal transformations generated by $X(\eta)$ and $Y(\zeta)$. The non-zero commutation relations between the generators (2.4) are

$$\begin{aligned} [X(\eta_1), X(\eta_2)] &= (\eta_1\eta_2' - \eta_1'\eta_2)\partial_z + (\eta_1''\eta_2 - \eta_1\eta_2'')(U\partial_U + 2Q\partial_Q), \\ [Y(\zeta_1), Y(\zeta_2)] &= (\zeta_1\zeta_2' - \zeta_1'\zeta_2)\partial_{\bar{z}} + (\zeta_1''\zeta_2 - \zeta_1\zeta_2'')(U\partial_U + 2\bar{Q}\partial_{\bar{Q}}), \\ [\hat{e}_0, T_i] &= -T_i, & [T_i, \mathcal{D}_j] &= \delta_{ij}T_i, & [T_i, R_{jk}] &= \delta_{ij}T_k - \delta_{ik}T_j, \\ [T_i, S_{jk}] &= \delta_{ij}T_k + \delta_{ik}T_j, & [\mathcal{D}_i, R_{jk}] &= \delta_{ij}S_{ik} - \delta_{ik}S_{ij}, & [\mathcal{D}_i, S_{jk}] &= \delta_{ij}R_{ik} - \delta_{ik}R_{ji}, \\ [R_{ij}, S_{kl}] &= \delta_{jk}S_{il} + \delta_{jl}S_{ik} - \delta_{ik}S_{jl} - \delta_{il}S_{jk}, \end{aligned}$$

where δ_{jk} is the Kronecker delta function. The Lie algebra \mathcal{L}_2 can be decomposed into the direct sum

$$\mathcal{L}_2 = \{X(\eta)\} \oplus \{Y(\zeta)\} \oplus \{\hat{e}_0, T_i, \mathcal{D}_i, R_{ij}, S_{ij}\},$$

which consists of two copies of the Virasoro algebra together with the 13-dimensional algebra generated by \hat{e}_0 , T_i , \mathcal{D}_i , R_{ij} and S_{ij} . If the functions η and ζ are analytic, they can be expanded as power series with respect to z and \bar{z} respectively. The maximal finite-dimensional subalgebra L_2 of \mathcal{L}_2 is spanned by the 19 generators

$$\begin{aligned}\hat{e}_0 &= -H\partial_H + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + 2U\partial_U + F_i\partial_{F_i}, \\ e_1 &= \partial_z, & e_3 &= z\partial_z - 2Q\partial_Q - U\partial_U, & e_5 &= z^2\partial_z - 4zQ\partial_Q - 2zU\partial_U, \\ e_2 &= \partial_{\bar{z}}, & e_4 &= \bar{z}\partial_{\bar{z}} - 2\bar{Q}\partial_{\bar{Q}} - U\partial_U, & e_6 &= \bar{z}^2\partial_{\bar{z}} - 4\bar{z}\bar{Q}\partial_{\bar{Q}} - 2\bar{z}U\partial_U, \\ T_i &= \partial_{F_i}, & \mathcal{D}_i &= F_i\partial_{F(i)} + N_i\partial_{N(i)}, & i &= 1, 2, 3, \\ R_{ij} &= (F_i\partial_{F_j} - F_j\partial_{F_i}) + (N_i\partial_{N_j} - N_j\partial_{N_i}), \\ S_{ij} &= (F_i\partial_{F_j} + F_j\partial_{F_i}) + (N_i\partial_{N_j} + N_j\partial_{N_i}), & i < j &= 2, 3,\end{aligned}$$

which have the non-zero commutation relations

$$\begin{aligned}[e_1, e_3] &= e_1, & [e_1, e_5] &= 2e_3, & [e_3, e_5] &= e_5, \\ [e_2, e_4] &= e_2, & [e_2, e_6] &= 2e_4, & [e_4, e_6] &= e_6, \\ [\hat{e}_0, T_i] &= -T_i, & [T_i, \mathcal{D}_j] &= \delta_{ij}T_i, & [T_i, R_{jk}] &= \delta_{ij}T_k - \delta_{ik}T_j, \\ [T_i, S_{jk}] &= \delta_{ij}T_k + \delta_{ik}T_j, & [\mathcal{D}_i, S_{jk}] &= \delta_{ij}R_{ik} - \delta_{ik}R_{ji}, \\ [\mathcal{D}_i, R_{jk}] &= \delta_{ij}S_{ik} - \delta_{ik}S_{ij}, & [R_{ij}, S_{kl}] &= \delta_{jk}S_{il} + \delta_{jl}S_{ik} - \delta_{ik}S_{jl} - \delta_{il}S_{jk}.\end{aligned}$$

The algebra L_2 can be decomposed as follows

$$L_2 = \{e_1, e_3, e_5\} \oplus \{e_2, e_4, e_6\} \oplus \{T_i, \mathcal{D}_i, R_{ij}, S_i\} \ni \{\hat{e}_0\}.$$

In the theory of solitons, there exists a conjecture [12, 13, 34] to isolate integrable systems which states that this characterization can be performed by comparing the sets of symmetries of the original system and of its associated linear system. In the case where the sets of symmetries of both the original system and the non-parametric linear system (the GW system) are finite-dimensional, we can compare the symmetries of the two systems by defining the differential projection operator π as the following operator

$$\pi(L_2) = L_2\omega, \quad \text{where } \omega = z\partial + \bar{z}\bar{\partial} + H\partial_H + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + U\partial_U,$$

which involves all independent and dependent variables. Here, ω is not necessarily an element of L_1 or L_2 . The projection operator π has the property that $\pi^n(L_2) = \pi(L_2)$ for any positive integer n and every element of the algebra L_2 . In fact, we have

$$\pi^2(L_2) = \pi(L_2\omega) = L_2\omega^2 = L_2\omega = \pi(L_2).$$

Under the above assumptions, the conjecture concerning integrable systems proposed in [12, 13, 34] can be formulated as follows.

Conjecture 2.1.

1. *In the case where $L_1 = \pi(L_2)$, the original system is non-integrable in the sense of soliton theory. In the case where there exist reductions of the original system (whose set of symmetries is L'_1) and the non-parametric linear system (whose set of symmetries is L'_2) such that $L'_1 \neq \pi(L'_2)$, the reduced subsystem of the original system can be integrable.*
2. *In the case where $L_1 \subset \pi(L_2)$, the system is a candidate to be integrable (in the sense of soliton theory) if it is possible to introduce a spectral parameter into the linear GW system, which represents a Lax pair, provided that the spectral parameter cannot be eliminated through a gauge transformation.*

It should be noted that, under the above conjecture, the GC equations (2.3) do not form an integrable system since $L_1 = \pi(L_2)$.

3 Certain aspects of Grassmann algebras

We present a brief overview of the concepts related to Grassmann variables and Grassmann algebras. The formalism is based on the theory of supermanifolds as described, e.g., in [6, 7, 10, 15, 18, 23, 32, 39, 40, 48]. We consider a complex Grassmann algebra Γ involving an arbitrary large (but finite) number \mathfrak{k} of Grassmann generators $(\xi_1, \xi_2, \dots, \xi_{\mathfrak{k}})$. The exact number of generators is not essential as long as there is a sufficient number of them to make all considered formulas meaningful. The Grassmann algebra Λ can be decomposed into its even (bosonic) and odd (fermionic) parts

$$\Lambda = \Lambda_{\text{even}} + \Lambda_{\text{odd}},$$

where Λ_{even} contains all terms involving a product of an even number of generators ξ_k , i.e., $1, \xi_1\xi_2, \xi_1\xi_3, \dots$, while Λ_{odd} contains all terms involving a product of an odd number of generators ξ_k , i.e., $\xi_1, \xi_2, \xi_3, \dots, \xi_1\xi_2\xi_3, \dots$. The space Λ and/or Λ_{even} replaces the field of complex numbers in the context of supersymmetry. The elements of Λ_{even} and Λ_{odd} are called even and odd supernumbers, respectively. An alternative decomposition for the Grassmann algebra Λ is

$$\Lambda = \Lambda_{\text{body}} + \Lambda_{\text{soul}},$$

where

$$\Lambda_{\text{body}} = \Lambda^0[\xi_1, \xi_2, \dots, \xi_{\mathfrak{k}}] \simeq \mathbb{C}, \quad \Lambda_{\text{soul}} = \sum_{k \geq 1} \Lambda^k[\xi_1, \xi_2, \dots, \xi_{\mathfrak{k}}].$$

Here $\Lambda^0[\xi_1, \xi_2, \dots, \xi_{\mathfrak{k}}]$ is used to refer to all elements of Λ that do not involve any of the generators ξ_i , while $\Lambda^k[\xi_1, \xi_2, \dots, \xi_{\mathfrak{k}}]$ refers to all elements of Λ that contain a product of k generators (for instance, if we have 5 generators $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5$ then $\Lambda^2[\xi_1, \xi_2, \xi_3, \xi_4, \xi_5]$ refers to all terms involving $\xi_1\xi_2, \xi_1\xi_3, \xi_1\xi_4, \xi_1\xi_5, \xi_2\xi_3, \xi_2\xi_4, \xi_2\xi_5, \xi_3\xi_4, \xi_3\xi_5$ and $\xi_4\xi_5$). Because of the \mathbb{Z}_0^+ -grading of the Grassmann algebra Λ , the bodiless elements in Λ_{soul} are non-invertible. Since the number \mathfrak{k} of Grassmann generators is finite, it follows that the bodiless elements are nilpotent of degree at most \mathfrak{k} .

In this paper, we use a \mathbb{Z}_2 -graded complex vector space V with even basis elements u_i , $i = 1, 2, \dots, N$ and odd basis elements v_μ , $\mu = 1, 2, \dots, M$ and consider $W = \Lambda \otimes_{\mathbb{C}} V$. The even part of W

$$W_{\text{even}} = \left\{ \sum_i a_i u_i + \sum_\mu \alpha_\mu v_\mu \mid a_i \in \Lambda_{\text{even}}, \alpha_\mu \in \Lambda_{\text{odd}} \right\},$$

is a Λ_{even} module which can be identified with $\Lambda_{\text{even}}^{\times N} \times \Lambda_{\text{odd}}^{\times M}$ (which consists of N copies of Λ_{even} and M copies of Λ_{odd}). To the original basis, consisting of the u_i and v_μ (although $v_\mu \notin W_{\text{even}}$), we associate the corresponding functionals

$$E_j: W_{\text{even}} \rightarrow \Lambda_{\text{even}}: E_j \left(\sum_i a_i u_i + \sum_\mu \alpha_\mu v_\mu \right) = a_j,$$

$$\Upsilon_\nu: W_{\text{even}} \rightarrow \Lambda_{\text{odd}}: \Upsilon_\nu \left(\sum_i a_i u_i + \sum_\mu \alpha_\mu v_\mu \right) = \alpha_\nu,$$

and view them as the coordinates (even and odd respectively) on W_{even} . Any topological manifold locally diffeomorphic to a suitable W_{even} is called a supermanifold [39]. Super-Minkowski space $\mathbb{R}^{(1,1|2)}$ is an example of such a supermanifold, being globally diffeomorphic to $\Lambda_{\text{even}}^{\times 2} \times \Lambda_{\text{odd}}^{\times 2}$,

with bosonic light-cone coordinates x_+ and x_- , and fermionic coordinates θ^+ and θ^- . Therefore, x_+ and x_- are linear combinations of terms containing an even number of generators: $1, \xi_1\xi_2, \xi_1\xi_3, \xi_1\xi_4, \dots, \xi_2\xi_3, \xi_2\xi_4, \dots, \xi_1\xi_2\xi_3\xi_4, \dots$. In contrast θ^+ and θ^- are linear combinations of terms containing an odd number of generators: $\xi_1, \xi_2, \xi_3, \xi_4, \dots, \xi_1\xi_2\xi_3, \xi_1\xi_2\xi_4, \xi_1\xi_3\xi_4, \xi_2\xi_3\xi_4, \dots$. Any fermionic (odd) variables θ^+ and θ^- satisfy the relation

$$(\theta^+)^2 = (\theta^-)^2 = \theta^+\theta^- + \theta^-\theta^+ = 0. \quad (3.1)$$

The supersymmetry transformations (4.3) presented in the next section can be understood as changes in the coordinates of $\mathbb{R}^{(1,1|2)}$ which transform solutions of the SUSY GW equations and the SUSY GC equations, respectively, into solutions of the same equations in new coordinates for both the bosonic and fermionic SUSY extensions. A bosonic or fermionic smooth superfield is a supersmooth G^∞ function from $\mathbb{R}^{(n_b|n_f)}$ to Λ (the values n_b and n_f of the superspace $\mathbb{R}^{(n_b|n_f)}$ stand for the number of bosonic and fermionic Grassmann coordinates respectively).

In this paper we use the convention that partial derivatives involving odd variables obey the following Leibniz rule for the product of two Grassmann-valued functions h and g

$$\partial_{\theta^\pm}(hg) = (\partial_{\theta^\pm}h)g + (-1)^{\deg(h)}h(\partial_{\theta^\pm}g),$$

where the degree of a homogeneous supernumber is given by

$$\deg(h) = \begin{cases} 0 & \text{if } h \text{ is even,} \\ 1 & \text{if } h \text{ is odd.} \end{cases}$$

We use the following ordering notation for partial derivatives $f_{\theta^+\theta^-} = \partial_{\theta^-}\partial_{\theta^+}f$. The partial derivatives with respect to the fermionic coordinates satisfy $\partial_{\theta^i}\theta^j = \delta_i^j$, where δ_i^j is the Kronecker delta function and the indices i and j each stand for $+$ or $-$. The fermionic operators ∂_{θ^\pm} , J_\pm and D_\pm in equations (4.1) and (4.4) alter the parity of a bosonic function to a fermionic function and vice versa. For instance, if ϕ is a bosonic function, then $\partial_{\theta^+}\phi$ is an odd superfield, while $\partial_{\theta^+}\partial_{\theta^-}\phi$ is an even superfield. For a Grassmann-valued composite function $f(g(x_+))$, the chain rule is ordered as follows

$$\frac{\partial f}{\partial x_+} = \frac{\partial g}{\partial x_+} \frac{\partial f}{\partial g}.$$

The interchange of mixed derivatives (with proper respect to the ordering of odd variables) is assumed throughout this paper. Additional details can be found in the books by Cornwell [15], DeWitt [18], Freed [23], Kac [32], Varadarajan [48] and references therein.

4 Supersymmetric versions of the Gauss–Weingarten and Gauss–Codazzi equations

In a previous paper [9], we constructed supersymmetric versions of the differential equations which define surfaces in super-Minkowski space. These versions consisted of supersymmetric extensions of the Gauss–Weingarten and Gauss–Codazzi equations using bosonic and fermionic superfields. The purpose of constructing such extensions was to construct surfaces immersed in a superspace ($\mathbb{R}^{(2,1|2)}$ for the bosonic extension and $\mathbb{R}^{(1,1|3)}$ for the fermionic extension). We use the variables $x_\pm = \frac{1}{2}(t \pm x)$ which are the bosonic light-cone coordinates, and θ^\pm which are fermionic (anticommuting) variables satisfying (3.1). Below, we present the outline of our procedure and its main results on which we base our further considerations.

Let \mathcal{S} be a smooth orientable conformally parametrized surface immersed in the superspace given by a vector-valued superfield $F(x_+, x_-, \theta^+, \theta^-)$ which, in view of (3.1), can be decomposed as

$$F = F_m(x_+, x_-) + \theta^+ \varphi_m(x_+, x_-) + \theta^- \psi_m(x_+, x_-) + \theta^+ \theta^- G_m(x_+, x_-), \quad m = 1, 2, 3.$$

In the bosonic case, the functions F_m and G_m are bosonic-valued, while the functions φ_m and ψ_m are fermionic-valued. Conversely, in the fermionic case, the functions F_m and G_m are fermionic-valued, while the functions φ_m and ψ_m are bosonic-valued. In both cases, we define the covariant superspace derivatives to be

$$D_{\pm} = \partial_{\theta^{\pm}} - i\theta^{\pm} \partial_{x_{\pm}}. \quad (4.1)$$

The conformal parametrization of the surface \mathcal{S} gives the following normalization on the superfield

$$\langle D_i F, D_j F \rangle = f g_{ij}, \quad \langle D_i F, N \rangle = 0, \quad \langle N, N \rangle = 1, \quad i, j = 1, 2, \quad (4.2)$$

where $D_{\pm} F$ are the tangent vector superfields and N is a normal bosonic vector field which can be decomposed in the form

$$N = N_m(x_+, x_-) + \theta^+ \alpha_m(x_+, x_-) + \theta^- \beta_m(x_+, x_-) + \theta^+ \theta^- H_m(x_+, x_-), \quad m = 1, 2, 3,$$

where N_m and H_m are bosonic functions, while α_m and β_m are fermionic functions. In the bosonic case the function f which appears in (4.2) is a bodiless bosonic function (i.e., $f \in \Lambda_{\text{soul}}$) of x_+ and x_- which is a nilpotent function of some order k . In the fermionic case the bosonic function f may be bodiless or not. The values 1 and 2 of the indices i and j stand for + and - respectively. The bracket $\langle \cdot, \cdot \rangle$ denotes the scalar product (2.2) for 3-dimensional Euclidean space, where we use the property (3.1) for any fermionic variables. This scalar product takes its values in the Grassmann algebra Λ . The coefficients of the induced bosonic metric function g_{ij} on the surface \mathcal{S} are given by

$$g_{ii} = 0, \quad g_{12} = \frac{1}{2} e^{\phi}, \quad g_{21} = \frac{1}{2} \epsilon e^{\phi}, \quad i = 1, 2,$$

where $\epsilon = 1$ in the fermionic case and $\epsilon = -1$ in the bosonic case. It should be noted that the covariant metric tensor g_{ij} is anti-symmetric in the indices i and j in the bosonic case while it is symmetric in those indices in the fermionic case. Here, the superfield ϕ is assumed to be bosonic and can be expanded in terms of the fermionic variables θ^+ and θ^- :

$$\phi = \phi_0(x_+, x_-) + \theta^+ \phi_1(x_+, x_-) + \theta^- \phi_2(x_+, x_-) + \theta^+ \theta^- \phi_3(x_+, x_-),$$

where ϕ_0 and ϕ_3 are bosonic functions while ϕ_1 and ϕ_2 are fermionic functions. The exponential function can be expanded as follows in terms of θ^+ and θ^- :

$$e^{\pm\phi} = e^{\pm\phi_0} (1 \pm \theta^+ \phi_1 \pm \theta^- \phi_2 \pm \theta^+ \theta^- \phi_3 - \theta^+ \theta^- \phi_1 \phi_2).$$

The SUSY extensions of the GW and GC equations are constructed in such a way that they are invariant under the transformations

$$x_{\pm} \rightarrow x_{\pm} + i\eta_{\pm} \theta^{\pm}, \quad \theta^{\pm} \rightarrow \theta^{\pm} + i\eta_{\pm}, \quad (4.3)$$

which are generated by the differential SUSY operators

$$J_{\pm} = \partial_{\theta^{\pm}} + i\theta^{\pm} \partial_{x_{\pm}}, \quad (4.4)$$

respectively. Here η_{\pm} are fermionic-valued parameters. The SUSY operators J_{\pm} satisfy the following anticommutation relations:

$$\begin{aligned} \{J_n, J_m\} &= 2i\delta_{mn}\partial_{x_m}, & \{D_n, D_m\} &= -2i\delta_{mn}\partial_{x_m}, & \{J_m, D_n\} &= 0, & m, n &= 1, 2, \\ D_{\pm}^2 &= -i\partial_{\pm}, & J_{\pm}^2 &= i\partial_{\pm}, \end{aligned}$$

where δ_{mn} is the Kronecker delta function and the brace brackets denote anticommutation, unless otherwise specified. The values 1 and 2 of the indices m and n stand for $+$ and $-$ respectively. Here and subsequently, summation over repeated indices is understood.

In order to construct the SUSY version of the GW equations we assume that the second-order covariant derivatives of F and the first-order covariant derivatives of the normal unit vector N can be defined in terms of the moving frame $\Omega = (D_+F, D_-F, N)^T$ on a surface \mathcal{S} , i.e.,

$$D_j D_i F = \Gamma_{ij}{}^k D_k F + b_{ij} f N, \quad D_i N = b_i{}^k D_k F + \omega_i N, \quad i, j, k = 1, 2,$$

where the coefficients $\Gamma_{ij}{}^k$ and ω_i are fermionic functions. The functions b_{ij} and $b_i{}^k$ are bosonic-valued in the bosonic extension and are fermionic-valued in the fermionic extension. Here, the values 1 and 2 of the indices i, j and k stand for $+$ and $-$, respectively. We define the coefficients b_{ij} to be

$$b_{11} = Q^+, \quad b_{12} = -b_{21} = \frac{1}{2}e^{\phi}H, \quad b_{22} = Q^-. \quad (4.5)$$

In the bosonic extension, the moving frame Ω contains both bosonic and fermionic components. Under the above assumptions, we obtained the following results [9]

Proposition 4.1. *For any bosonic vector superfields $F(x_+, x_-, \theta^+, \theta^-)$ and $N(x_+, x_-, \theta^+, \theta^-)$ satisfying the normalization conditions (4.2) and (4.5), the moving frame $\Omega = (D_+F, D_-F, N)^T$ on a smooth conformally parametrized surface immersed in the superspace $\mathbb{R}^{(2,1|2)}$ satisfies the SUSY GW equations*

$$\begin{aligned} D_+\Omega &= A_+\Omega, & D_-\Omega &= A_-\Omega, \\ A_+ &= \begin{pmatrix} \Gamma_{11}{}^1 & \Gamma_{11}{}^2 & Q^+f \\ -\Gamma_{12}{}^1 & -\Gamma_{12}{}^2 & -\frac{1}{2}e^{\phi}Hf \\ H & 2e^{-\phi}Q^+ & 0 \end{pmatrix}, & A_- &= \begin{pmatrix} \Gamma_{12}{}^1 & \Gamma_{12}{}^2 & \frac{1}{2}e^{\phi}Hf \\ \Gamma_{22}{}^1 & \Gamma_{22}{}^2 & Q^-f \\ -2e^{-\phi}Q^- & H & 0 \end{pmatrix}. \end{aligned} \quad (4.6)$$

The zero curvature condition

$$D_+A_- + D_-A_+ - \{EA_+, EA_-\} = 0, \quad (4.7)$$

where

$$E = \pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

constitutes the GC equations and corresponds to the following six linearly independent equations

$$\begin{aligned} (i) \quad & D_-(\Gamma_{11}{}^1) + D_+(\Gamma_{22}{}^2) + D_+(\Gamma_{12}{}^1) - D_-(\Gamma_{12}{}^2) = 0, \\ (ii) \quad & D_-(\Gamma_{11}{}^1) - \Gamma_{11}{}^2\Gamma_{22}{}^1 + D_+(\Gamma_{12}{}^1) + \Gamma_{12}{}^2\Gamma_{12}{}^1 + \frac{1}{2}H^2e^{\phi}f - 2Q^+Q^-e^{-\phi}f = 0, \\ (iii) \quad & Q^+\Gamma_{22}{}^2 - \Gamma_{11}{}^2Q^- + D_-Q^+ - Q^+D_-\phi + \frac{1}{2}e^{\phi}D_+H = 0, \\ (iv) \quad & Q^-\Gamma_{11}{}^1 - \Gamma_{22}{}^1Q^+ + D_+Q^- - Q^-D_+\phi - \frac{1}{2}e^{\phi}D_-H = 0, \\ (v) \quad & D_-(\Gamma_{11}{}^2) - \Gamma_{12}{}^1\Gamma_{11}{}^2 - \Gamma_{11}{}^2\Gamma_{22}{}^2 - \Gamma_{11}{}^1\Gamma_{12}{}^2 + D_+(\Gamma_{12}{}^2) + 2Q^+Hf = 0, \\ (vi) \quad & D_+(\Gamma_{22}{}^1) + \Gamma_{12}{}^2\Gamma_{22}{}^1 - \Gamma_{22}{}^1\Gamma_{11}{}^1 + \Gamma_{22}{}^2\Gamma_{12}{}^1 - D_-(\Gamma_{12}{}^1) + 2Q^-Hf = 0. \end{aligned} \quad (4.8)$$

In the fermionic extension, the moving frame Ω contains only bosonic components. The fermionic counterpart of Proposition 4.1 can be summarized as follows.

Proposition 4.2. *For any fermionic vector superfield $F(x_+, x_-, \theta^+, \theta^-)$ and bosonic normal unit vector $N(x_+, x_-, \theta^+, \theta^-)$ satisfying the normalization conditions (4.2) and (4.5), the bosonic moving frame $\Omega = (D_+F, D_-F, N)^T$ on a smooth conformally parametrized surface immersed in the superspace $\mathbb{R}^{(1,1|3)}$ satisfies the SUSY GW equations*

$$\begin{aligned} D_+ \begin{pmatrix} D_+F \\ D_-F \\ N \end{pmatrix} &= \begin{pmatrix} \Gamma_{11}^1 & 0 & Q^+f \\ 0 & 0 & -\frac{1}{2}e^\phi Hf \\ H & -2e^{-\phi}Q^+ & 0 \end{pmatrix} \begin{pmatrix} D_+F \\ D_-F \\ N \end{pmatrix}, \\ D_- \begin{pmatrix} D_+F \\ D_-F \\ N \end{pmatrix} &= \begin{pmatrix} 0 & 0 & \frac{1}{2}e^\phi Hf \\ 0 & \Gamma_{22}^2 & Q^-f \\ -2e^{-\phi}Q^- & -H & 0 \end{pmatrix} \begin{pmatrix} D_+F \\ D_-F \\ N \end{pmatrix}. \end{aligned} \quad (4.9)$$

The GC equations, which are equivalent to the ZCC

$$D_+A_- + D_-A_+ - \{A_+, A_-\} = 0,$$

reduce to the following four linearly independent equations

$$\begin{aligned} (i) \quad & D_+(\Gamma_{22}^2) + D_-(\Gamma_{11}^1) = 0, \\ (ii) \quad & D_-(\Gamma_{11}^1) + 2e^{-\phi}Q^+Q^-f = 0, \\ (iii) \quad & D_+Q^- - \frac{1}{2}e^\phi D_-H + Q^-(D_+\phi - \Gamma_{11}^1) = 0, \\ (iv) \quad & D_-Q^+ + \frac{1}{2}e^\phi D_+H + Q^+(D_-\phi - \Gamma_{22}^2) = 0. \end{aligned} \quad (4.10)$$

5 Conjecture on supersymmetric integrable systems

In this section, we formulate a SUSY version of the Conjecture 2.1 on integrable systems described in Section 2. A symmetry supergroup G of a SUSY system of equations consists of a local supergroup of transformations acting on a Cartesian product of supermanifolds $\mathcal{X} \times \mathcal{U}$, where \mathcal{X} is the space of four independent variables $(x_+, x_-, \theta^+, \theta^-)$ and \mathcal{U} is the space of dependent superfields.

Let \mathcal{L}_1 be a maximal finite-dimensional superalgebra of Lie point symmetries associated with the system of nonlinear partial differential equations (NPDEs) under consideration. Let \mathcal{L}_2 be a maximal finite-dimensional superalgebra of Lie point symmetries of the linear system associated with the original system of NPDEs. Let π be a projection operator acting on the subalgebra \mathcal{L}_2 such that $\pi(\mathcal{L}_2) = \mathcal{L}_2\omega$, where ω is the differential operator

$$\omega = x_+\partial_{x_+} + x_-\partial_{x_-} + \theta^+\partial_{\theta^+} + \theta^-\partial_{\theta^-} + u^\alpha\partial_{u^\alpha} + \varphi^\beta\partial_{\varphi^\beta}$$

involving all independent bosonic and fermionic variables $(x_+, x_-, \theta^+, \theta^-)$ and all dependent bosonic and fermionic superfields, u^α and φ^β , respectively, appearing in the system of NPDEs. The common symmetries of the NPDEs and the linear spectral problem (LSP), associated with the original system of NPDEs, are the vector fields which span the set

$$\mathcal{L}_3 = \mathcal{L}_1 \cap \pi(\mathcal{L}_2) \neq \emptyset.$$

It should be noted that the set \mathcal{L}_3 is not necessarily an algebra. The prolongation of one of these vector fields acting on the LSP has to vanish for all wavefunctions of the LSP. In this

case, the integrated form of a two-dimensional surface in a Lie algebra is given by the Fokas–Gel’fand immersion formula [21, 22, 28], whenever the tangent vectors on the surface are linearly independent. Let us consider the set of vector fields defined by

$$\mathcal{L}_4 = \mathcal{L}_1 \setminus \{\mathcal{L}_1 \cap \pi(\mathcal{L}_2)\}.$$

Here, \mathcal{L}_4 consists of all symmetries of \mathcal{L}_1 that are not symmetries of \mathcal{L}_2 . Again, \mathcal{L}_4 is not necessarily an algebra. Under the above assumptions, an extension of the Conjecture 2.1 to SUSY integrable systems can be formulated as follows.

Conjecture 5.1.

1. If $\mathcal{L}_1 = \pi(\mathcal{L}_2)$ then the system of NPDEs is not integrable.
2. If the following conditions are satisfied
 - (a) $\pi(\mathcal{L}_2)$ is a proper subset of \mathcal{L}_1 , that is

$$\mathcal{L}_1 \supset \pi(\mathcal{L}_2).$$

A free parameter can be introduced into the linear system using a symmetry transformation generated by one of the vector fields appearing in \mathcal{L}_4 .

- (b) *The transformation given in (a) acts in a nontrivial way (i.e., cannot be eliminated through an \mathcal{L}_1 -valued gauge matrix function).*

Then the system of NPDEs is a candidate to be an integrable system.

The proposed conjecture is illustrated through the following examples.

Example 5.1. The bosonic extension of the GC equations (4.8) involves eleven unknown functions $\mathcal{U} = (\phi, H, Q^+, Q^-, R^+, R^-, S^+, S^-, T^+, T^-, f)$, where ϕ, H, Q^+, Q^-, f are bosonic functions while $R^+, R^-, S^+, S^-, T^+, T^-$ are fermionic functions. In what follows we use the notation

$$R^+ = \Gamma_{11}^1, \quad R^- = \Gamma_{11}^2, \quad S^+ = \Gamma_{12}^1, \quad S^- = \Gamma_{12}^2, \quad T^+ = \Gamma_{22}^1, \quad T^- = \Gamma_{22}^2.$$

The action of the supergroup G on the superfields \mathcal{U} of $(x_+, x_-, \theta^+, \theta^-)$ maps solutions of the bosonic version of the SUSY GC equations (4.8) to solutions of (4.8). The bodiless bosonic function f depends only on x_+ and x_- , in contrast with the other listed superfields in \mathcal{U} which can depend on $(x_+, x_-, \theta^+, \theta^-)$. Assuming that G is a Lie supergroup as described in [32, 50], we found that its associated Lie superalgebra \mathfrak{g}_1 , whose elements are infinitesimal symmetries of the bosonic SUSY GC equations (4.8), was generated by the following eight vector fields [9]

$$\begin{aligned} C_0 &= H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} - 2f\partial_f, \\ K_0 &= -H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} + 2\partial_\phi, \\ K_1^b &= -2x_+\partial_{x_+} - \theta^+\partial_{\theta^+} + R^+\partial_{R^+} + 2R^-\partial_{R^-} + S^-\partial_{S^-} - T^+\partial_{T^+} + 2Q^+\partial_{Q^+} + \partial_\phi, \\ K_2^b &= -2x_-\partial_{x_-} - \theta^-\partial_{\theta^-} - R^-\partial_{R^-} + S^+\partial_{S^+} + 2T^+\partial_{T^+} + T^-\partial_{T^-} + 2Q^-\partial_{Q^-} + \partial_\phi, \\ P_+ &= \partial_{x_+}, \quad P_- = \partial_{x_-}, \quad J_+ = \partial_{\theta^+} + i\theta^+\partial_{x_+}, \quad J_- = \partial_{\theta^-} + i\theta^-\partial_{x_-}. \end{aligned} \quad (5.1)$$

The generators P_+ and P_- correspond to translations in the bosonic variables x_+ and x_- respectively. We also have four dilations, of which two, C_0 and K_0 , involve only the bosonic dependent variables, while the other two, K_1^b and K_2^b , involve both independent and dependent, and both bosonic and fermionic variables. Finally, we also recover the two supersymmetry generators J_+ and J_- which were identified previously in (4.4).

The algebra \mathfrak{g}'_1 of infinitesimal symmetries of the SUSY GW equations (4.6) are spanned by the following vector fields

$$\begin{aligned} P_{\pm} &= \partial_{x_{\pm}}, & J_{\pm} &= \partial_{\theta^{\pm}} + i\theta^{\pm}\partial_{x_{\pm}}, & \hat{C}_0 &= H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} - 2f\partial_f + N_i\partial_{N_i}, \\ \hat{K}_0 &= -H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} + 2\partial_{\phi} - N_i\partial_{N_i}, \\ K_1^b &= -2x_+\partial_{x_+} - \theta^+\partial_{\theta^+} + R^+\partial_{R^+} + 2R^-\partial_{R^-} + S^-\partial_{S^-} - T^+\partial_{T^+} + 2Q^+\partial_{Q^+} + \partial_{\phi}, \\ K_2^b &= -2x_-\partial_{x_-} - \theta^-\partial_{\theta^-} - R^-\partial_{R^-} + S^+\partial_{S^+} + 2T^+\partial_{T^+} + T^-\partial_{T^-} + 2Q^-\partial_{Q^-} + \partial_{\phi}, \\ G_i &= F_i\partial_{F_i} + N_i\partial_{N_i}, & B_i &= \partial_{F_i}, & \text{for } i &= 1, 2, 3, \\ R_{ij} &= F_i\partial_{F_j} - F_j\partial_{F_i} + N_i\partial_{N_j} - N_j\partial_{N_i}, & & & i < j &= 2, 3. \end{aligned}$$

Using the projection operator $\pi(\mathfrak{g}'_1) = \mathfrak{g}'_1\omega$ involving all dependent and independent variables of the SUSY GC equations (4.8), where

$$\begin{aligned} \omega &= x_+\partial_{x_+} + x_-\partial_{x_-} + \theta^+\partial_{\theta^+} + \theta^-\partial_{\theta^-} + \phi\partial_{\phi} + H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} + f\partial_f \\ &\quad + R^+\partial_{R^+} + R^-\partial_{R^-} + S^+\partial_{S^+} + S^-\partial_{S^-} + T^+\partial_{T^+} + T^-\partial_{T^-}, \end{aligned}$$

and comparing the resulting vector fields with the generators of \mathfrak{g}_1 , given by (5.1), we conclude that $\mathfrak{g}_1 = \pi(\mathfrak{g}'_1)$, which implies that the SUSY GC equations are non-integrable as in the classical case.

Example 5.2. As another example, we apply the conjecture to the SUSY sine-Gordon equation as formulated in [11]

$$D_+D_-\Phi = i \sin \Phi, \tag{5.2}$$

where Φ is a bosonic superfield. Its Lie symmetry superalgebra \mathfrak{g}_3 is spanned by the vector fields

$$P_{\pm} = \partial_{x_{\pm}}, \quad J_{\pm} = \partial_{\theta^{\pm}} + i\theta^{\pm}\partial_{x_{\pm}}, \quad \mathcal{K} = 2x_+\partial_{x_+} - 2x_-\partial_{x_-} + \theta^+\partial_{\theta^+} - \theta^-\partial_{\theta^-}. \tag{5.3}$$

The non-parametric linear problem (the GW equations) associated with the SUSY sine-Gordon equation (5.2) is given by

$$\begin{aligned} D_{\pm}\Psi &= B_{\pm}\Psi, & \text{where } \Psi &= \begin{pmatrix} \psi_{11} & \psi_{12} & f_{13} \\ \psi_{21} & \psi_{22} & f_{23} \\ f_{31} & f_{32} & \psi_{33} \end{pmatrix}, \\ B_+ &= \frac{1}{2} \begin{pmatrix} 0 & 0 & ie^{i\Phi} \\ 0 & 0 & -ie^{-i\Phi} \\ -e^{-i\Phi} & e^{i\Phi} & 0 \end{pmatrix}, & B_- &= \begin{pmatrix} iD_-\Phi & 0 & -i \\ 0 & -iD_-\Phi & i \\ -1 & 1 & 0 \end{pmatrix}. \end{aligned} \tag{5.4}$$

Here the ψ_{ij} are bosonic superfields and the f_{ij} are fermionic superfields, $i, j = 1, 2, 3$. The infinitesimal symmetry generators \mathfrak{g}'_3 of equations (5.4) are spanned by the vector fields

$$\begin{aligned} P_{\pm} &= \partial_{x_{\pm}}, & J_{\pm} &= \partial_{\theta^{\pm}} + i\theta^{\pm}\partial_{x_{\pm}}, & G_1 &= \psi_{11}\partial_{\psi_{11}} + \psi_{21}\partial_{\psi_{21}} + f_{31}\partial_{f_{31}}, \\ G_2 &= \psi_{12}\partial_{\psi_{12}} + \psi_{22}\partial_{\psi_{22}} + f_{32}\partial_{f_{32}}, & G_3 &= f_{13}\partial_{f_{13}} + f_{23}\partial_{f_{23}} + \psi_{33}\partial_{\psi_{33}}. \end{aligned}$$

Using the projection operator π defined as $\pi(\mathfrak{g}'_3) = \mathfrak{g}'_3\omega$, where

$$\omega = x_+\partial_{x_+} + x_-\partial_{x_-} + \theta^+\partial_{\theta^+} + \theta^-\partial_{\theta^-} + \Phi\partial_{\Phi},$$

we obtain the relation $\mathfrak{g}_3 \supset \pi(\mathfrak{g}'_3)$, which implies that the SUSY sine-Gordon equation may be integrable, as in the classical case. The fact that the generator \mathcal{K} in (5.3) does not appear in

the symmetries of the linear problem (5.4) of the SUSY sine-Gordon equations (5.2) allows us to introduce a bosonic spectral parameter λ through \mathcal{K} . This is accomplished by introducing a one-parameter group associated with the dilation \mathcal{K} through the transformation $\tilde{x}_+ = \lambda x_+$, $\tilde{x}_- = \lambda^{-1} x_-$, $\tilde{\theta}^+ = \lambda^{1/2} \theta^+$ and $\tilde{\theta}^- = \lambda^{-1/2} \theta^-$, $\lambda = \pm e^\mu$, where $\mu \in \Lambda_{\text{even}}$, into the linear system (5.4) which gives us

$$D_+ \Psi = B_+ \Psi, \quad D_- \Psi = B_- \Psi,$$

$$B_+ = \frac{1}{2\sqrt{\lambda}} \begin{pmatrix} 0 & 0 & ie^{i\Phi} \\ 0 & 0 & -ie^{-i\Phi} \\ -e^{-i\Phi} & e^{i\Phi} & 0 \end{pmatrix}, \quad B_- = \begin{pmatrix} iD_- \Phi & 0 & -i\sqrt{\lambda} \\ 0 & -iD_- \Phi & i\sqrt{\lambda} \\ -\sqrt{\lambda} & \sqrt{\lambda} & 0 \end{pmatrix}, \quad (5.5)$$

which coincide with the results found in [41]. The ZCC of equation (5.5) takes the form (4.7), where the matrices A_+ and A_- are replaced by the matrices B_+ and B_- , respectively. The connection between the super-Darboux transformations and the super-Bäcklund transformations for the sine-Gordon equation (5.2) allows the construction of explicit multi-super-soliton solutions [11, 25, 42].

Example 5.3. The fermionic extension of the GC equations (4.10) involves seven unknown functions $\mathcal{U} = (\phi, H, Q^+, Q^-, R^+, T^-, f)$ where ϕ and f are bosonic functions, while $H, Q^+, Q^-, R^+ = \Gamma_{11}^1, T^- = \Gamma_{22}^2$ are fermionic functions. Proceeding in a similar manner as in the bosonic SUSY case, we obtain a Lie symmetry superalgebra \mathfrak{g}_2 consisting of the following six bosonic infinitesimal generators [9]

$$P_+ = \partial_{x_+}, \quad P_- = \partial_{x_-},$$

$$C_0 = H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} - 2f\partial_f,$$

$$K_0 = -H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} + 2\partial_\phi,$$

$$K_1^f = -2x_+\partial_{x_+} - \theta^+\partial_{\theta^+} + 2Q^+\partial_{Q^+} + R^+\partial_{R^+} + \partial_\phi,$$

$$K_2^f = -2x_-\partial_{x_-} - \theta^-\partial_{\theta^-} + 2Q^-\partial_{Q^-} + T^-\partial_{T^-} + \partial_\phi,$$

together with the three fermionic generators

$$J_+ = \partial_{\theta^+} + i\theta^+\partial_{x_+}, \quad J_- = \partial_{\theta^-} + i\theta^-\partial_{x_-}, \quad W = \partial_H.$$

The generators W and P_\pm correspond to translations in the fermionic variable H and the bosonic variables x_\pm respectively. We obtain two dilations C_0 and K_0 involving only dependent variables, together with two additional dilations K_1^f and K_2^f , which involve both dependent and independent variables. We also recover the two supersymmetry generators J_+ and J_- .

The Lie symmetry algebra \mathfrak{g}'_2 of the SUSY GW equations (4.9) is spanned by the vector fields

$$P_\pm = \partial_{x_\pm}, \quad J_\pm = \partial_{\theta^\pm} + i\theta^\pm\partial_{x_\pm},$$

$$\hat{C}_0 = H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} - 2f\partial_f + N_i\partial_{N_i},$$

$$\hat{K}_0 = -H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} + 2\partial_\phi - N_i\partial_{N_i},$$

$$K_1^f = -2x_+\partial_{x_+} - \theta^+\partial_{\theta^+} + 2Q^+\partial_{Q^+} + R^+\partial_{R^+} + \partial_\phi,$$

$$K_2^f = -2x_-\partial_{x_-} - \theta^-\partial_{\theta^-} + 2Q^-\partial_{Q^-} + T^-\partial_{T^-} + \partial_\phi,$$

$$G_i = F_i\partial_{F_i} + N_i\partial_{N_i}, \quad B_i = \partial_{F_i}, \quad \text{for } i, j = 1, 2, 3,$$

$$R_{ij} = F_i\partial_{F_j} - F_j\partial_{F_i} + N_i\partial_{N_j} - N_j\partial_{N_i}, \quad i < j = 2, 3.$$

By using the projector π , defined as $\pi(\mathfrak{g}'_2) = \mathfrak{g}_2\omega$, where

$$\omega = x_+\partial_{x_+} + x_-\partial_{x_-} + \theta^+\partial_{\theta^+} + \theta^-\partial_{\theta^-} + H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} \\ + \phi\partial_\phi + f\partial_f + R^+\partial_{R^+} + T^-\partial_{T^-},$$

we obtain that the set of symmetries of the SUSY GW equations (4.9) is a proper subset of the set of Lie symmetries of the SUSY GC equations (4.10). More specifically the translation in H generated by W is not a symmetry of the SUSY GW equations. Therefore we can introduce a fermionic parameter $\underline{\lambda}$ in the SUSY GW equations (4.9) with the potential matrices

$$A_+ = \begin{pmatrix} R^+ & 0 & Q^+ f \\ 0 & 0 & -\frac{1}{2}e^\phi(H + \underline{\lambda})f \\ H + \underline{\lambda} & -2e^{-\phi}Q^+ & 0 \end{pmatrix},$$

$$A_- = \begin{pmatrix} 0 & 0 & \frac{1}{2}e^\phi(H + \underline{\lambda})f \\ 0 & T^- & Q^- f \\ -2e^{-\phi}Q^- & -(H + \underline{\lambda}) & 0 \end{pmatrix}.$$

The parameter $\underline{\lambda}$ cannot be eliminated through a gauge transformation. This suggests that the fermionic version of the SUSY GC equations (4.10) may be integrable.

6 Concluding remarks and outlook

The objective of this paper was to compare the symmetries of the SUSY GW equations with those of the SUSY GC equations for both the bosonic and fermionic extensions. This comparison allowed us to formulate a generalization of the conjecture establishing the necessary conditions for a system to be integrable in the sense of soliton theory. The symmetry analysis developed in this paper could be extended in several directions. First, it should be noted that the list of symmetries for the bosonic and fermionic SUSY structural equations found in this paper is not necessarily exhaustive since the symmetry criterion has not been proven for equations involving Grassmann variables. A comprehensive list of all symmetries of the bosonic and fermionic SUSY Gauss–Weingarten and SUSY Gauss–Codazzi equations could be compiled. This would require the development of a computer Lie algebra symmetry package capable of handling both bosonic and fermionic symmetries. Another possibility would be to extend the procedure to hypersurfaces in higher dimensions. It could also be worth attempting to establish a SUSY version of Noether’s theorem in order to determine conserved quantities, and to derive a SUSY version of the Weierstrass–Enneper formula for the immersion of surfaces in a multidimensional superspace. One could also investigate how characteristics associated with integrable models such as Hamiltonian structures and conserved quantities manifest themselves in the SUSY case. Another worthwhile subject is a variational problem of geometric functionals (e.g., Willmore functionals), which can be interpreted as actions from which we can determine the Euler–Lagrange equations for a given surface immersed in a superspace. Recurrence operators of generalized symmetries of the SUSY GC equations could be used to obtain the recurrence relations for the surfaces. A complete invariant geometrical characterization of these surfaces in the superspace remains to be done. A singularity analysis of the SUSY system under consideration could be performed in connection with Lie groups in order to verify the Painlevé property. This would be motivated by the goal of obtaining explicit analytic solutions. Such analytic solutions can be useful for observing the qualitative behaviour of solutions which would otherwise be difficult to detect numerically. The existence of different types of soliton solutions constitutes such an example. An essential step in the further development of the theory of surfaces associated with SUSY integrable systems would be a generalization of the known formulas for constructing soliton surfaces immersed in Lie algebras, namely the Sym–Tafel [44, 45] and the Fokas–Gel’fand [22] formulas. Our procedure for introducing a spectral parameter in the GW equations could make this task feasible.

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