

# Prequantization of the Moduli Space of Flat $\mathrm{PU}(p)$ -Bundles with Prescribed Boundary Holonomies<sup>\*</sup>

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**Abstract.** Using the framework of quasi-Hamiltonian actions, we compute the obstruction to prequantization for the moduli space of flat  $\mathrm{PU}(p)$ -bundles over a compact orientable surface with prescribed holonomies around boundary components, where  $p > 2$  is prime.

*Key words:* quantization; moduli space of flat connections; parabolic bundles

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## 1 Introduction

Let  $G$  be a compact connected simple Lie group and  $\Sigma$  a compact oriented surface with  $s$  boundary components. Given conjugacy classes  $\mathcal{C}_1, \dots, \mathcal{C}_s$ , let  $\mathfrak{M} = M_G(\Sigma; \mathcal{C}_1, \dots, \mathcal{C}_s)$  denote the moduli space of flat  $G$ -bundles on  $\Sigma$  with prescribed boundary holonomies in the conjugacy classes  $\mathcal{C}_j$ . Alternatively,  $\mathfrak{M}$  may be described as the character variety of the fundamental group of  $\Sigma$ ,

$$\mathfrak{M} = \mathrm{Hom}_{\mathcal{C}_1, \dots, \mathcal{C}_s}(\pi_1(\Sigma), G)/G.$$

Here,  $\mathrm{Hom}_{\mathcal{C}_1, \dots, \mathcal{C}_s}(\pi_1(\Sigma), G)$  consists of homomorphisms  $\rho : \pi_1(\Sigma) \rightarrow G$  whose restriction to (the homotopy class of) the  $j$ -th boundary circle of  $\Sigma$  lies in  $\mathcal{C}_j$ , and  $G$  acts by conjugation. Recall that  $\mathfrak{M}$  is a (possibly singular) symplectic space, where the symplectic form is defined by a choice of invariant inner product on the Lie algebra  $\mathfrak{g}$  of  $G$  [5, 12]. This paper considers the obstruction to the existence of a prequantization of  $\mathfrak{M}$  – that is, a prequantum (orbifold) line bundle  $L \rightarrow \mathfrak{M}$  (see Section 5.2 for details) – by expressing the corresponding integrality condition on the symplectic form in terms of the choice of inner product on the simple Lie algebra  $\mathfrak{g}$ , which is hence a certain multiple  $k$  of the basic inner product.

If the underlying structure group  $G$  is simply connected, the moduli space  $\mathfrak{M}$  is connected and the obstruction to prequantization is well known – a prequantization exists if and only if  $k \in \mathbb{N}$  and each conjugacy class  $\mathcal{C}_j$  corresponds to a level  $k$  weight (e.g., see [4, 6, 18]). If  $G$  is not simply connected,  $\mathfrak{M}$  may have multiple components. Moreover integrality of  $k$  is not sufficient to guarantee a prequantization even in the absence of markings/prescribed boundary holonomies: if  $\Sigma$  is closed and has genus at least 1, then  $k$  must be a multiple of an integer  $l_0(G)$  (computed in [15] for each  $G$ ). If  $\Sigma$  has boundary with prescribed holonomies, only the case  $G = \mathrm{SO}(3) \cong \mathrm{PU}(2)$  has been fully resolved [17].

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In this paper, we describe the connected components of  $\mathfrak{M}$  for non-simply connected structure groups  $G/Z$  in Corollary 4.2 and Proposition 4.3 (where  $G$  is simply connected and  $Z$  is a subgroup of the centre of  $G$ ). The decomposition into components makes use of an action of the centre  $Z(G)$  on a fundamental Weyl alcove  $\Delta$  in  $\mathfrak{t}$ , the Lie algebra of a maximal torus. The action is described concretely in [23] for classical groups and Appendix A records the action for the two remaining exceptional cases.

Finally, we compute the obstruction to prequantization in Theorem 5.8 in the case  $G = \mathrm{PU}(p)$  ( $p > 2$ , prime) for any number of boundary components  $s$ . We work within the theory of quasi-Hamiltonian group actions with group-valued moment map [2], where the moduli space  $\mathfrak{M}$  is a central example. In quasi-Hamiltonian geometry, quantization is defined as a certain element of the twisted  $K$ -theory of  $G$  [19], analogous to  $\mathrm{Spin}^c$  quantization for Hamiltonian group actions on symplectic manifolds. In this context, the obstruction to the existence of a prequantization is a cohomological obstruction (see Definition 5.1). The obstruction for other cases of non-simply connected structure group does not follow from the approach here (see Remark 5.7) and will be considered elsewhere.

## 2 Preliminaries

**Notation.** Unless otherwise indicated,  $G$  denotes a compact, simply connected, simple Lie group with Lie algebra  $\mathfrak{g}$ . We fix a maximal torus  $T \subset G$  and use the following notation:

$\mathfrak{t}$	– Lie algebra of $T$ ;
$\mathfrak{t}^*$	– dual of the Lie algebra of $T$ ;
$W = N(T)/T$	– Weyl group;
$I = \ker \exp_T$	– integer lattice;
$P = I^* \subset \mathfrak{t}^*$	– (real) weight lattice;
$Q \subset \mathfrak{t}^*$	– root lattice;
$Q^\vee \subset \mathfrak{t}$	– coroot lattice;
$P^\vee \subset \mathfrak{t}$	– coweight lattice.

Recall that since  $G$  is simply connected,  $I = Q^\vee$ . Moreover, the coroot lattice and weight lattice are dual to each other, as are the root lattice and coweight lattice. A choice of simple roots  $\alpha_1, \dots, \alpha_l$  (with  $l = \mathrm{rank}(G)$ ) spanning  $Q$ , determines the fundamental coweights  $\lambda_1^\vee, \dots, \lambda_l^\vee$  spanning  $P^\vee$ , defined by  $\langle \alpha_i, \lambda_j^\vee \rangle = \delta_{i,j}$ .

We let  $\langle -, - \rangle$  denote the basic inner product, the invariant inner product on  $\mathfrak{g}$  normalized to make short coroots have length  $\sqrt{2}$ . With this inner product, we will often identify  $\mathfrak{t} \cong \mathfrak{t}^*$ .

Given a subgroup  $Z$  of the centre  $Z(G)$  of  $G$ , we shall abuse notation and denote by  $q : G \rightarrow G/Z$  the resulting covering(s).

Finally, let  $\{e_1, \dots, e_n\}$  denote the standard basis for  $\mathbb{R}^n$ , equipped with the standard inner product that will also be denoted with angled brackets  $\langle -, - \rangle$ .

**Quasi-Hamiltonian group actions.** We recall some basic definitions and facts from [2]. (For the remainder of this section, we may take  $G$  to be any compact Lie group with invariant inner product  $\langle -, - \rangle$  on  $\mathfrak{g}$ .) Let  $\theta^L, \theta^R$  denote the left-invariant, right-invariant Maurer–Cartan forms on  $G$ , and let  $\eta = \frac{1}{12} \langle \theta^L, [\theta^L, \theta^L] \rangle$  denote the Cartan 3-form on  $G$ . For a  $G$ -manifold  $M$ , and  $\xi \in \mathfrak{g}$ , let  $\xi^\#$  denote the generating vector field of the action. The Lie group  $G$  is itself viewed as a  $G$ -manifold for the conjugation action.

**Definition 2.1** ([2]). A quasi-Hamiltonian  $G$ -space is a triple  $(M, \omega, \Phi)$  consisting of a  $G$ -manifold  $M$ , a  $G$ -invariant 2-form  $\omega$  on  $M$ , and an equivariant map  $\Phi : M \rightarrow G$ , called the moment map, satisfying:

i)  $d\omega + \Phi^* \eta = 0$ ,

- ii)  $\iota_{\xi^\sharp}\omega + \frac{1}{2}\Phi^*((\theta^L + \theta^R) \cdot \xi) = 0$  for all  $\xi \in \mathfrak{g}$ ,
- iii) at every point  $x \in M$ ,  $\ker \omega_x \cap \ker d\Phi_x = \{0\}$ .

We will often denote a quasi-Hamiltonian  $G$ -space  $(M, \omega, \Phi)$  simply by the underlying space  $M$  when  $\omega$  and  $\Phi$  are understood from the context.

The *fusion product* of two quasi-Hamiltonian  $G$ -spaces  $M_j$  with moment maps  $\Phi_j : M_j \rightarrow G$  ( $j = 1, 2$ ) is the product  $M_1 \times M_2$ , with the diagonal  $G$ -action and moment map  $\Phi : M_1 \times M_2 \rightarrow G$  given by composing  $\Phi_1 \times \Phi_2$  with multiplication in  $G$ .

The *symplectic quotient* of a quasi-Hamiltonian  $G$ -space is the symplectic space  $M//G = \Phi^{-1}(1)/G$ , which is a symplectic orbifold whenever the group unit  $1 \in G$  is a regular value. If 1 is a singular value, then the symplectic quotient is a singular symplectic space as defined in [20].

The conjugacy classes  $\mathcal{C} \subset G$ , with moment map the inclusion into  $G$ , are basic examples of quasi-Hamiltonian  $G$ -spaces. Another important example is the *double*  $D(G) = G \times G$ , equipped with diagonal  $G$ -action and moment map  $\Phi(g, h) = ghg^{-1}h^{-1}$ , the group commutator. These two families of examples form the building blocks of the moduli space of flat  $G$ -bundles over a surface  $\Sigma$  with prescribed boundary holonomies. (See Section 4 for a sketch of this construction.)

### 3 Conjugacy classes invariant under translation by central elements

This section describes the set of conjugacy classes  $\mathcal{D} \subset G$  that are invariant under translation by a subgroup  $Z_{\mathcal{D}}$  of the centre  $Z(G)$  of  $G$ . We begin with the following Lemma, which identifies such a subgroup  $Z_{\mathcal{D}}$  with the fundamental group of a conjugacy class in  $G/Z$ , where  $Z \subset Z(G)$ .

**Lemma 3.1.** *Let  $Z$  be a subgroup of the centre  $Z(G)$  of  $G$  and let  $\mathcal{C} \subset G/Z$  be a conjugacy class. For any conjugacy class  $\mathcal{D} \subset G$  covering  $\mathcal{C}$ , the restriction  $q|_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{C}$  is the universal covering projection and hence the fundamental group  $\pi_1(\mathcal{C}) \cong Z_{\mathcal{D}} = \{z \in Z : z\mathcal{D} = \mathcal{D}\}$ .*

**Proof.** The inverse image  $q^{-1}(\mathcal{C})$  is a disjoint union of conjugacy classes in  $G$  that cover  $\mathcal{C}$ . Since conjugacy classes in a compact simply connected Lie group are simply connected and  $Z_{\mathcal{D}}$  acts freely on  $\mathcal{D}$ , the lemma follows. ■

Recall that every element in  $G$  is conjugate to a unique element  $\exp \xi$  in  $T$ , where  $\xi$  lies in a fixed (closed) alcove  $\Delta \subset \mathfrak{t}$  of a Weyl chamber. Therefore, the set of conjugacy classes in  $G$  is parametrized by  $\Delta$ . Since the  $Z(G)$ -action commutes with the conjugation action, we obtain an action  $Z(G) \times \Delta \rightarrow \Delta$ . Next we identify this description of the action of  $Z(G)$  on an alcove  $\Delta$  with a more concrete description of a  $Z(G)$ -action on  $\Delta$  given in [23, Section 4.1]. (See also [7, Section 3.1] for a similar treatment.)

Let  $\{\alpha_1, \dots, \alpha_l\}$  be a basis of simple roots for  $\mathfrak{t}^*$ , with highest root  $\tilde{\alpha} =: -\alpha_0$ . Let  $\Delta \subset \mathfrak{t}$  be the alcove

$$\Delta = \{\xi \in \mathfrak{t} : \langle \xi, \alpha_j \rangle \geq 0, \langle \xi, \tilde{\alpha} \rangle \leq 1\}.$$

Recall that the exponential map induces an isomorphism  $Z(G) \cong P^\vee/Q^\vee$ , and that the non-zero elements of the centre have representatives  $\lambda_i^\vee \in P^\vee$  given by minimal dominant coweights. By [23, Lemma 2.3] the non-zero minimal dominant coweights  $\lambda_i^\vee$  are dual to the special roots  $\alpha_i$ , which are those roots with coefficient 1 in the expression  $\tilde{\alpha} = \sum m_i \alpha_i$ . In Proposition 4.1.4 of [23], Toledo-Laredo provides a  $Z(G)$ -action on  $\Delta$  defined by

$$z \cdot \xi = w_i \xi + \lambda_i^\vee,$$

where  $z = \exp \lambda_i^\vee$ , and  $w_i \in W$  is a certain element of the Weyl group. The element  $w_i \in W$  is the unique element that leaves  $\Delta \cup \{\alpha_0\}$  invariant (i.e., induces an automorphism of the extended Dynkin diagram) and satisfies  $w_i(\alpha_0) = \alpha_i$  (see [23, Proposition 4.1.2]). The following proposition shows these actions coincide.

**Proposition 3.2.** *The translation action of  $Z(G)$  on  $G$  induces an action  $Z(G) \times \Delta \rightarrow \Delta$  and is given by the formula  $z \cdot \xi = w_i \xi + \lambda_i^\vee$ , where  $z = \exp \lambda_i^\vee$  and  $w_i$  is the unique element in  $W$  that leaves  $\Delta \cup \{\alpha_0\}$  invariant and satisfies  $w_i(\alpha_0) = \alpha_i$ .*

**Proof.** Observe that for any element  $w$  in  $W$ ,  $w\lambda_i^\vee - \lambda_i^\vee \in I = Q^\vee$  since  $w \exp \lambda_i^\vee = \exp \lambda_i^\vee$ . Therefore,  $w_i \xi + \lambda_i^\vee = w_i(\xi + \lambda_i^\vee + (w_i^{-1} \lambda_i^\vee - \lambda_i^\vee))$ . In other words,  $w_i \xi + \lambda_i^\vee = \hat{w}(\xi + \lambda_i^\vee)$  for some  $\hat{w}$  in the affine Weyl group. Letting  $z = \exp \lambda_i^\vee$ , this shows that  $z \exp \xi = \exp(\xi + \lambda_i^\vee)$  is conjugate to  $\exp(w_i \xi + \lambda_i^\vee)$ , which proves the proposition. ■

In fact, as the next proposition shows, the automorphism of the Dynkin diagram induced by  $w_i$  encodes the resulting permutation of the vertices of the alcove  $\Delta$ .

**Proposition 3.3.** *Let  $v_0, \dots, v_l$  denote the vertices of  $\Delta$  with  $v_j$  opposite the facet parallel to  $\ker \alpha_j$ . Then  $\exp \lambda_i^\vee \cdot v_j = v_k$  whenever  $w_i \alpha_j = \alpha_k$ , where  $w_i$  is as in Proposition 3.2.*

**Proof.** Let  $v_0, \dots, v_l$  denote the vertices of  $\Delta$ , where the vertex  $v_j$  is opposite the facet (codimension 1 face) parallel to  $\ker \alpha_j$ . That is,  $v_0 = 0$  and for  $j \neq 0$ ,  $v_j$  satisfies:

$$\langle \alpha_0, v_j \rangle = -1 \quad \text{and} \quad \langle \alpha_r, v_j \rangle = 0 \quad \text{if and only if} \quad 0 \neq r \neq j.$$

(Hence, for  $j \neq 0$  we have  $\langle \alpha_j, v_j \rangle = \frac{1}{m_j}$ , where  $m_j$  is the coefficient of  $\alpha_j$  in the expression  $\tilde{\alpha} = \sum m_i \alpha_i$ .)

Suppose that  $w_i \alpha_0 = \alpha_i$  and let  $w_i \alpha_j = \alpha_k$  (where  $k$  depends on  $j$ ).

Consider  $\exp \lambda_i^\vee \cdot v_0$ . Since  $\langle \alpha_0, w_i v_0 + \lambda_i^\vee \rangle = \langle \alpha_0, \lambda_i^\vee \rangle = -1$ , and (for  $r \neq 0$ )  $\langle \alpha_r, w_i v_0 + \lambda_i^\vee \rangle = \langle \alpha_r, \lambda_i^\vee \rangle = \delta_{r,k}$ , we have  $\exp \lambda_i^\vee \cdot v_0 = v_i$ .

Next, consider  $\exp \lambda_i^\vee \cdot v_j = w_i v_j + \lambda_i^\vee$ , where  $j \neq 0$ . If  $k = 0$  so that  $w_i \alpha_j = \alpha_0$  then  $\alpha_j = w_i^{-1} \alpha_0$  is a special root (i.e.  $m_j = 1$ ) since  $w_i^{-1} = w_j$ . Therefore,  $\langle \alpha_0, w_i v_j + \lambda_i^\vee \rangle = \langle w_i^{-1} \alpha_0, v_j \rangle - 1 = \langle \alpha_j, v_j \rangle - 1 = 0$ . And if  $r \neq 0$ ,

$$\langle \alpha_r, w_i v_j + \lambda_i^\vee \rangle = \langle w_i^{-1} \alpha_r, v_j \rangle + \langle \alpha_r, \lambda_i^\vee \rangle. \quad (3.1)$$

If  $r \neq i$ ,  $w_i^{-1} \alpha_r$  is a simple root other than  $\alpha_j$ ; therefore, each term above is 0. Moreover, if  $r = i$ , then the above expression becomes  $\langle \alpha_0, v_j \rangle + \langle \alpha_i, \lambda_i^\vee \rangle = -1 + 1 = 0$ . Hence we have  $\exp \lambda_i^\vee \cdot v_j = v_0$  whenever  $w_i \alpha_j = \alpha_0$ .

On the other hand, if  $k \neq 0$  so that  $w_i \alpha_j = \alpha_k$  is a simple root, then  $\langle \alpha_0, w_i v_j + \lambda_i^\vee \rangle = \langle w_i^{-1} \alpha_0, v_j \rangle + \langle \alpha_0, \lambda_i^\vee \rangle = 0 - 1 = -1$  since the simple root  $w_i^{-1} \alpha_0 \neq \alpha_j$ . And if  $r \neq 0$ , we consider again the expression (3.1) and find (for the same reason as above) that (3.1) is trivial whenever  $r \neq k$ . If  $r = k$ , (3.1) becomes  $\langle \alpha_k, w_i v_j + \lambda_i^\vee \rangle = \langle w_i^{-1} \alpha_k, v_j \rangle + \langle \alpha_k, \lambda_i^\vee \rangle = \langle \alpha_j, v_j \rangle \neq 0$ . Hence we have that  $\exp \lambda_i^\vee \cdot v_j = v_k$ , as required. ■

The  $Z(G)$ -action on  $\Delta$  is explicitly described in [23] for all classical groups. (In Appendix A, we record the action of the centre on the alcove for the exceptional groups  $E_6$  and  $E_7$ , the remaining compact simple Lie groups with non-trivial centre.)

**Conjugacy classes in  $\text{SU}(n)$ .** We now specialize to the case  $G = \text{SU}(n)$  and consider the action of the centre on the alcove. Identify  $\mathfrak{t} \cong \mathfrak{t}^* \subset \mathbb{R}^n$  as the subspace  $\{x = \sum x_j e_j : \sum x_j = 0\}$  and recall that the basic inner product coincides with (the restriction of) the standard inner product on  $\mathbb{R}^n$ . The roots are the vectors  $e_i - e_j$  with  $i \neq j$ . Taking the simple roots to be  $\alpha_i = e_i - e_{i+1}$  ( $i = 1, \dots, n-1$ ) and the resulting highest root  $\tilde{\alpha} = e_1 - e_n$  gives the alcove

$$\Delta = \{x \in \mathfrak{t} : x_1 \geq x_2 \geq \dots \geq x_n, x_1 - x_n \leq 1\}.$$

Its vertices are

$$v_0 = 0 \quad \text{and} \quad v_j = \sum_{i=1}^j e_i - \frac{j}{n} \sum_{i=1}^n e_i, \quad j = 1, \dots, n.$$

The centre  $Z(\mathrm{SU}(n)) \cong \mathbb{Z}/n\mathbb{Z}$  is generated by (exp of) the minimal dominant coweight  $\lambda_1^\vee = e_1 - \frac{1}{n} \sum_{i=1}^n e_i$  corresponding to the special root  $\alpha_1 = e_1 - e_2$ . Since the element  $w_1$  inducing an automorphism of the extended Dynkin diagram for  $\mathrm{SU}(n)$  satisfies  $w_1 \alpha_0 = \alpha_1$ , by Proposition 3.3 the permutation of the vertices of  $\Delta$  induced by the action of  $\exp \lambda_1^\vee$  is the  $n$ -cycle  $(v_0 v_1 \cdots v_{n-1})$  (since  $v_j$  is the vertex opposite the facet parallel to  $\ker \alpha_j$ ).

It follows that the only point in  $\Delta$  fixed by the action of  $Z(G)$  is the barycenter

$$\zeta_* = \frac{1}{n} \sum_{j=0}^{n-1} v_j = \frac{n-1}{2n} e_1 + \frac{n-3}{2n} e_2 + \cdots + \frac{1-n}{2n} e_n.$$

Hence there is a unique conjugacy class in  $\mathrm{SU}(n)$  that is invariant under translation by the centre – namely, matrices in  $\mathrm{SU}(n)$  with eigenvalues  $z_1, \dots, z_n$ , the distinct  $n$ -th roots of  $(-1)^{n+1}$ . As the next proposition shows, however, restricting the action to a proper subgroup  $Z \cong \mathbb{Z}/\nu\mathbb{Z}$  ( $\nu|n$ ) of the centre results in larger  $Z$ -fixed point sets in  $\Delta$ .

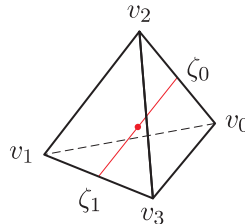
**Proposition 3.4.** *Let  $n = \nu m$  and consider the subgroup  $\mathbb{Z}/\nu\mathbb{Z} \subset \mathbb{Z}/n\mathbb{Z} \cong Z(\mathrm{SU}(n))$ . The  $\mathbb{Z}/\nu\mathbb{Z}$ -fixed points in the alcove  $\Delta$  for  $\mathrm{SU}(n)$  consist of the convex hull of the barycenters of the faces spanned by the orbits of the vertices  $v_0, \dots, v_{m-1}$  of  $\Delta$ .*

**Proof.** Write  $x = \sum t_i v_i$  in  $\Delta$  in barycentric coordinates (with  $t_i \geq 0$  and  $\sum t_i = 1$ ). Then a generator of  $\mathbb{Z}/\nu\mathbb{Z}$  sends  $x$  to  $\sum t'_i v_i$ , with  $t'_i = t_{i-m \bmod n}$ . Therefore  $x$  is fixed if and only if  $t_i = t_{i-m \bmod n}$ , and in this case we may write,

$$\begin{aligned} x &= t_0 \sum_{j=0}^{\nu-1} v_{jm} + t_1 \sum_{j=0}^{\nu-1} v_{1+jm} + \cdots + t_{m-1} \sum_{j=0}^{\nu-1} v_{m-1+jm} \\ &= \nu t_0 \frac{1}{\nu} \sum_{j=0}^{\nu-1} v_{jm} + \nu t_1 \frac{1}{\nu} \sum_{j=0}^{\nu-1} v_{1+jm} + \cdots + \nu t_{m-1} \frac{1}{\nu} \sum_{j=0}^{\nu-1} v_{m-1+jm}, \end{aligned}$$

which exhibits a fixed point in the desired form. ■

To illustrate, consider the subgroup  $Z \cong \mathbb{Z}/2\mathbb{Z}$  of the centre  $Z(\mathrm{SU}(4)) \cong \mathbb{Z}/4\mathbb{Z}$ , which acts by transposing the vertices  $v_0 \leftrightarrow v_2$  and  $v_1 \leftrightarrow v_3$ . The barycenters  $\zeta_0, \zeta_1$  of the edges  $\overline{v_0 v_2}$  and  $\overline{v_1 v_3}$ , respectively, are fixed and thus the  $Z$ -fixed points are those on the line segment joining  $\zeta_0$  and  $\zeta_1$  (see Fig. 1).



**Figure 1.** Alcove for  $\mathrm{SU}(4)$ . The indicated line segment through the barycenter parametrizes the set of conjugacy classes invariant under translation by  $\mathbb{Z}/2\mathbb{Z} \subset Z(\mathrm{SU}(4))$ .

## 4 Components of the moduli space with markings

In this section we recall the quasi-Hamiltonian description of the moduli space of flat bundles over a compact orientable surface with prescribed boundary holonomies. We refer to the original article [2] for the details regarding the construction sketched below.

Let  $\Sigma$  be a compact, oriented surface of genus  $h$  with  $s$  boundary components. For conjugacy classes  $\mathcal{C}_1, \dots, \mathcal{C}_s$  in  $G/Z$ , let  $M_{G/Z}(\Sigma; \mathcal{C}_1, \dots, \mathcal{C}_s)$  be the moduli space of flat  $G/Z$ -bundles over  $\Sigma$  with prescribed boundary holonomies lying in the conjugacy classes  $\mathcal{C}_j$  ( $j = 1, \dots, s$ ). Points in  $M_{G/Z}(\Sigma; \mathcal{C}_1, \dots, \mathcal{C}_s)$  are (gauge equivalence classes of) principal  $G/Z$ -bundles over  $\Sigma$  equipped with a flat connection whose holonomy around the  $j$ -th boundary component lies in the conjugacy class  $\mathcal{C}_j$ . This moduli space is an important example in the theory of quasi-Hamiltonian group actions, where it is cast as a symplectic quotient of a fusion product,

$$M_{G/Z}(\Sigma; \mathcal{C}_1, \dots, \mathcal{C}_s) = (D(G/Z)^h \times \mathcal{C}_1 \times \dots \times \mathcal{C}_s) // (G/Z), \quad (4.1)$$

which may have several connected components if  $Z$  is non-trivial. Extending the discussion in [17, Section 2.3], we describe the connected components of (4.1) as symplectic quotients of an auxiliary quasi-Hamiltonian  $G$ -space.

As in [17, Section 2.2], given a quasi-Hamiltonian  $G/Z$ -space  $N$  with group-valued moment map  $\Phi : N \rightarrow G/Z$ , let  $\check{N}$  be the fibre product defined by the Cartesian square,

$$\begin{array}{ccc} \check{N} & \xrightarrow{\check{\Phi}} & G \\ \downarrow & & \downarrow q \\ N & \xrightarrow{\Phi} & G/Z \end{array} \quad (4.2)$$

Then  $\check{N}$  is naturally a quasi-Hamiltonian  $G$ -space with moment map  $\check{\Phi}$ . The following proposition from [17] and its Corollary summarize some properties of this construction.

**Proposition 4.1** ([17, Proposition 2.2]). *Let  $\check{N}$  be the fibre product defined by (4.2), where  $\Phi : N \rightarrow G/Z$  is a group-valued moment map.*

- i) *We have a canonical identification of symplectic quotients  $\check{N} // G \cong N // (G/Z)$ .*
- ii) *For a fusion product  $N = N_1 \times \dots \times N_r$  of quasi-Hamiltonian  $G/Z$ -spaces, the space  $\check{N}$  is a quotient of  $\check{N}_1 \times \dots \times \check{N}_r$  by the group  $\{(c_1, \dots, c_r) \in Z^r \mid \prod_{j=1}^r c_j = e\}$ .*
- iii) *If  $\Phi : N \rightarrow G/Z$  lifts to a moment map  $\Phi' : N \rightarrow G$ , thus turning  $N$  into a quasi-Hamiltonian  $G$ -space then  $\check{N} = N \times Z$ .*

**Corollary 4.2.** *Let  $\check{N}$  be the fibre product defined by (4.2), where  $\Phi : N \rightarrow G/Z$  is a group-valued moment map, and write  $\check{N} = \bigsqcup X_j$  as a union of its connected components. Then the components of  $N // (G/Z)$  can be identified with the symplectic quotients  $X_j // G$ .*

**Proof.** The restrictions  $\check{\Phi}_j = \check{\Phi}|_{X_j}$  are  $G$ -valued moment maps whose fibres are connected by [2, Theorem 7.2]. Since  $\check{\Phi}^{-1}(e) = \bigsqcup \check{\Phi}_j^{-1}(e)$ , it follows that  $\check{N} // G = \check{\Phi}^{-1}(e) // G = \bigsqcup \check{\Phi}_j^{-1}(e) // G = \bigsqcup X_j // G$ . The result follows from Proposition 4.1(i).  $\blacksquare$

Hence to identify the components of (4.1), it suffices to identify the components of  $\check{N} // G$ , where  $N = D(G/Z)^h \times \mathcal{C}_1 \times \dots \times \mathcal{C}_s$  – namely,  $X_j // G$ , where  $X_j$  ranges over the components of  $\check{N}$ . In particular, we may view the moduli space (4.1) as a union of symplectic quotients of quasi-Hamiltonian  $G$ -spaces (as opposed to  $G/Z$ -spaces), which will be very important for the approach taken in Section 5.

With this in mind, choose conjugacy classes  $\mathcal{D}_j \subset G$  covering  $\mathcal{C}_j$  ( $j = 1, \dots, s$ ) and let

$$\tilde{N} = D(G)^h \times \mathcal{D}_1 \times \cdots \times \mathcal{D}_s.$$

Let

$$\Gamma = \left\{ (\gamma_1, \dots, \gamma_s) \in Z_{\mathcal{D}_1} \times \cdots \times Z_{\mathcal{D}_s} : \prod \gamma_j = 1 \right\} \subset Z^s \quad (4.3)$$

(cf. Lemma 3.1). We show next that the components of  $\tilde{N}$  are all homeomorphic to  $\tilde{N}/(Z^{2h} \times \Gamma)$  (generalizing the decomposition appearing in [17, Lemma 2.3] for  $G = \mathrm{SO}(3)$ ).

**Proposition 4.3.** *Let  $N = D(G/Z)^h \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_s$  for conjugacy classes  $\mathcal{C}_j \subset G/Z$  ( $j = 1, \dots, s$ ) and let  $\tilde{N}$  be the fibre product defined by (4.2). Then  $\tilde{N}$  may be written as a union of its connected components,*

$$\tilde{N} \cong \bigsqcup_{Z/(Z_{\mathcal{D}_1} \cdots Z_{\mathcal{D}_s})} D(G/Z)^h \times (\mathcal{D}_1 \times \cdots \times \mathcal{D}_s)/\Gamma,$$

where  $\mathcal{D}_j \subset G$  are conjugacy classes covering  $\mathcal{C}_j$  ( $j = 1, \dots, s$ ) and  $\Gamma$  is as in (4.3).

**Proof.** This is a straightforward application of the properties (ii) and (iii) listed in Proposition 4.1. By property (iii),  $\check{D}(G/Z)^h = D(G/Z)^h \times Z$ , and by Lemma 3.1,  $\check{\mathcal{C}}_j = \mathcal{D}_j \times Z/Z_{\mathcal{D}_j}$ . Therefore, by property (ii),

$$\check{N} \cong D(G/Z)^h \times (Z \times \mathcal{D}_1 \times Z/Z_{\mathcal{D}_1} \times \cdots \times \mathcal{D}_s \times Z/Z_{\mathcal{D}_s})/\Lambda,$$

where  $\Lambda = \{(c_0, \dots, c_s) \in Z^{s+1} : c_0 \cdots c_s = 1\}$ . Since

$$(Z \times \mathcal{D}_1 \times Z/Z_{\mathcal{D}_1} \times \cdots \times \mathcal{D}_s \times Z/Z_{\mathcal{D}_s})/\Lambda \cong (Z \times \mathcal{D}_1 \times \cdots \times \mathcal{D}_s)/\Gamma',$$

where  $\Gamma' = \{(\gamma_0, \dots, \gamma_s) \in Z \times Z_{\mathcal{D}_1} \times \cdots \times Z_{\mathcal{D}_s} : \prod \gamma_j = 1\}$ , we see that the components of  $\check{N}$  are in bijection with  $Z/(Z_{\mathcal{D}_1} \cdots Z_{\mathcal{D}_s})$ .

Consider the component corresponding to  $\bar{z} \in Z/(Z_{\mathcal{D}_1} \cdots Z_{\mathcal{D}_s})$  in which each point is of the form  $(\bar{g}, [(z, x_1, \dots, x_s)]_{\Gamma'})$ , where  $[\ ]_{\Gamma'}$  denotes a  $\Gamma'$ -orbit. (Note that there is always a representative of this form with  $z$  in the first coordinate.) This component is homeomorphic to  $D(G/Z)^h \times (\mathcal{D}_1 \times \cdots \times \mathcal{D}_s)/\Gamma$  by the map  $(\bar{g}, [(x_1, \dots, x_s)]_{\Gamma}) \mapsto (\bar{g}, [(z, x_1, \dots, x_s)]_{\Gamma'})$ . ■

**Remark 4.4.** The decomposition in Proposition 4.3 is consistent with Theorem 14 in Ho–Liu’s work [13] on the connected components of the moduli space for any compact connected Lie group  $G$ .

For the case  $G/Z = \mathrm{SU}(p)/(\mathbb{Z}/p\mathbb{Z}) = \mathrm{PU}(p)$ , where  $p$  is prime, the decomposition above simplifies. In particular, there is only one conjugacy class  $\mathcal{D}_* = \mathrm{SU}(p) \cdot \exp \zeta_*$ , corresponding to the barycenter  $\zeta_* \in \Delta$ , invariant under the action of the centre. Let  $\mathcal{C}_* = q(\mathcal{D}_*)$  be the corresponding conjugacy class in  $\mathrm{PU}(p)$ . Therefore, we obtain the following Corollary (cf. [17, Lemma 2.3]).

**Corollary 4.5.** *Let  $p$  be prime and let  $N = D(\mathrm{PU}(p))^h \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_s$  for conjugacy classes  $\mathcal{C}_j \subset \mathrm{PU}(p)$  ( $j = 1, \dots, s$ ) and let  $\tilde{N}$  be the fibre product defined by (4.2). Then,*

$$\tilde{N} \cong \begin{cases} D(\mathrm{PU}(p))^h \times (\mathcal{D}_1 \times \cdots \times \mathcal{D}_s)/\Gamma & \text{if } \exists j : \mathcal{C}_j = \mathcal{C}_*, \\ D(\mathrm{PU}(p))^h \times \mathcal{D}_1 \times \cdots \times \mathcal{D}_s \times Z & \text{otherwise,} \end{cases}$$

where  $\mathcal{D}_j \subset \mathrm{SU}(p)$  are conjugacy classes covering  $\mathcal{C}_j$  ( $j = 1, \dots, s$ ) and  $\Gamma$  is as in (4.3).

In particular, if (after re-labelling)  $\mathcal{C}_j = \mathcal{C}_*$  for all  $j \leq r$  ( $r > 0$ ), then we obtain

$$\tilde{N} \cong D(\mathrm{PU}(p))^h \times (\mathcal{D}_*)^r/\Gamma \times \mathcal{D}_{r+1} \times \cdots \times \mathcal{D}_s, \quad (4.4)$$

where, in this case,  $\Gamma = \{(\gamma_1, \dots, \gamma_r) \in Z^r : \prod \gamma_j = 1\}$ .

## 5 Obstruction to prequantization

### 5.1 Prequantization for quasi-Hamiltonian group actions

We recall some definitions and properties regarding prequantization of quasi-Hamiltonian group actions. Recall that the Cartan 3-form  $\eta \in \Omega^3(G)$  is integral – in fact,  $[\eta] \in H^3(G; \mathbb{R})$  is the image of a generator  $x \in H^3(G; \mathbb{Z}) \cong \mathbb{Z}$  under the coefficient homomorphism induced by  $\mathbb{Z} \rightarrow \mathbb{R}$ . Condition (i) in Definition 2.1 says that the pair  $(\omega, \eta)$  defines a relative cocycle in  $\Omega^3(\Phi)$ , the algebraic mapping cone of the pull-back map  $\Phi^*: \Omega^*(G) \rightarrow \Omega^*(M)$ , and hence a cohomology class  $[(\omega, \eta)] \in H^3(\Phi; \mathbb{R})$ . (See [8, Chapter I, Section 6] for the definition of relative cohomology.)

**Definition 5.1** ([15, 19]). Let  $k \in \mathbb{N}$ . A *level  $k$  prequantization* of a quasi-Hamiltonian  $G$ -space  $(M, \omega, \Phi)$  is an integral lift  $\alpha \in H^3(\Phi; \mathbb{Z})$  of the class  $k[(\omega, \eta)] \in H^3(\Phi; \mathbb{R})$ .

**Remark 5.2.** The definition of prequantization in Definition 5.1 uses the assumption in this paper that  $G$  is simply connected. The general definition of prequantization [19, Definition 3.2] (with  $G$  semi-simple and compact) requires an integral lift in  $H_G^3(\Phi; \mathbb{Z})$  of an equivariant extension of the class  $k[(\omega, \eta)]$ . When  $G$  is simply connected, [15, Proposition 3.5] shows that the definition above is equivalent. Our main goal is to apply the quasi-Hamiltonian viewpoint on prequantization to the moduli space of flat bundles with prescribed holonomies; therefore, by Corollary 4.2 it suffices to work on each component of the quasi-Hamiltonian  $G$ -space  $\check{N}$  in Proposition 4.3 using Definition 5.1.

We list some basic properties of level  $k$  prequantizations that we shall encounter.

- (a) If  $M_1$  and  $M_2$  are pre-quantized quasi-Hamiltonian  $G$ -spaces at level  $k$ , then their fusion product  $M_1 \times M_2$  inherits a prequantization at level  $k$ . Conversely, a prequantization of the product induces prequantizations of the factors. See [15, Proposition 3.8].
- (b) A level  $k$  prequantization of  $M$  induces a prequantization of the symplectic quotient  $M//G$ , equipped with the  $k$ -th multiple of the symplectic form.
- (c) The long exact sequence in relative cohomology gives a necessary condition  $k\Phi^*(x) = 0$  for the existence of a level  $k$ -prequantization. If  $H^2(M; \mathbb{R}) = 0$ ,  $k\Phi^*(x) = 0$  is also sufficient [15, Proposition 4.2] to conclude a level  $k$ -prequantization exists.

The following examples relate to the moduli space of flat bundles with prescribed boundary holonomies.

**Example 5.3.** The double  $D(G) = G \times G$  with moment map  $\Phi : D(G) \rightarrow G$  equal to the group commutator admits a prequantization at all levels  $k \in \mathbb{N}$ . For non-simply connected groups, the double  $D(G/Z)$  with moment map  $\Phi : D(G/Z) \rightarrow G$  the canonical lift of the group commutator admits a level  $k$ -prequantization if and only if  $k$  is a multiple of  $l_0 \in \mathbb{N}$ , where  $l_0$  is a positive integer depending on  $G/Z$  computed for all compact simple Lie groups in [15]. For  $G/Z = \text{PU}(n)$ ,  $l_0 = n$ .

**Example 5.4.** Conjugacy classes  $\mathcal{D} \subset G$  admitting a level  $k$ -prequantization are those  $\mathcal{D} = G \cdot \exp \xi$  ( $\xi \in \Delta$ ) with  $(k\xi)^{\flat} \in P$  [18], where  $(k\xi)^{\flat} = \langle k\xi, - \rangle$  (i.e., a *level  $k$  weight*). For simply laced groups (such as  $G = \text{SU}(n)$ ), under the identification  $\mathfrak{t} \cong \mathfrak{t}^*$ ,  $P^{\vee} \cong P$ . Therefore, in this case,  $\mathcal{D}$  admits a level  $k$ -prequantization if and only if  $k\xi \in P^{\vee}$ . Since  $\exp^{-1} Z(G) = P^{\vee}$ , we see that  $\mathcal{D}$  admits a level  $k$ -prequantization if and only if  $g^k \in Z(G)$  for all  $g \in \mathcal{D}$ . (So in particular if  $k$  is a multiple of the *order of  $\mathcal{D}$*  [6, Definition 5.76], then  $\mathcal{D}$  admits a level  $k$  prequantization.)



## 5.2 Quasi-Hamiltonian prequantization and symplectic quotients

To provide some context, we further elaborate on property (b) following Definition 5.1 since we view the moduli space of flat bundles as a symplectic quotient (4.1) in quasi-Hamiltonian geometry. By [15, Proposition 3.6], a level  $k$  prequantization of a quasi-Hamiltonian  $G$ -space  $(M, \omega, \Phi)$  gives an integral lift of the equivariant cohomology class  $[j^*\omega] \in H_G^2(\Phi^{-1}(1); \mathbb{R})$ , where  $j : \Phi^{-1}(1) \rightarrow M$  denotes inclusion. Hence, there is a  $G$ -equivariant line bundle  $L \rightarrow \Phi^{-1}(1)$  with connection of curvature  $j^*\omega$ . If 1 is a regular value, the symplectic quotient  $M//G = \Phi^{-1}(1)/G$  is a symplectic orbifold [2] and the  $G$ -equivariant line bundle over the level set descends to a prequantum orbifold line bundle  $L/G \rightarrow M//G$ . (See [1, Example 2.29] for a discussion of orbifold vector bundles in this context.)

**Remark 5.5.** The orbifold line bundle  $L/G \rightarrow M//G$  need not be an ordinary line bundle over the underlying topological space  $M//G$  (i.e., the coarse moduli space of the orbifold). (For orbifolds that arise as quotients  $X/G$  of a smooth, proper, locally free action of a Lie group  $G$  on a smooth manifold  $X$ , this distinction is apparent from the observation that  $H_G^2(X; \mathbb{Z})$  is not necessarily isomorphic to  $H^2(X/G; \mathbb{Z})$ .) Some works in the literature require a prequantization to be an ordinary line bundle, and hence obtain a further obstruction to the existence of a prequantization (e.g. [14, Theorem 4.2], [10, Theorems 4.12 and 6.1], [21, Lemme 3.2]). We gratefully acknowledge the referee's comments that led to this important clarification.

## 5.3 The obstruction to prequantization for the moduli space of $\mathrm{PU}(p)$ bundles, $p$ prime

Let  $p$  be an odd prime. In this section we obtain the obstruction to prequantization for the quasi-Hamiltonian  $\mathrm{SU}(p)$ -space  $\check{N}$ , where  $N = D(\mathrm{PU}(p))^h \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_s$  for conjugacy classes  $\mathcal{C}_j \subset \mathrm{PU}(p)$  ( $j = 1, \dots, s$ ). Let  $M \subseteq \check{N}$  be a connected component (by Corollary 4.5),

$$M = D(\mathrm{PU}(p))^h \times (\mathcal{D}_1 \times \cdots \times \mathcal{D}_s) / \Gamma,$$

where  $\Gamma$  is as in (4.3). As we shall see in the proof of Theorem 5.8, we will find property (a) in Section 5.1 very useful in order to proceed 'factor by factor', using the decomposition (4.4).

To begin, we establish the following proposition which allows us to use property (c) in Section 5.1 to compute the obstruction to prequantization for the factor  $(\mathcal{D}_*)^r / \Gamma$  in (4.4).

**Proposition 5.6.** *Let  $\mathcal{D}_* \subset \mathrm{SU}(p)$  denote the conjugacy class of the barycenter  $\zeta_*$  of the alcove  $\Delta$  and let  $\Gamma = \{(\gamma_1, \dots, \gamma_r) \in Z^r : \prod \gamma_j = 1\}$  with  $r > 1$ . Then  $H^2((\mathcal{D}_*)^r / \Gamma; \mathbb{R}) = 0$ .*

**Proof.** Since  $(\mathcal{D}_*)^r \rightarrow (\mathcal{D}_*) / \Gamma$  is a covering projection,  $H^2((\mathcal{D}_*)^r / \Gamma; \mathbb{R}) \cong H^2((\mathcal{D}_*)^r; \mathbb{R})^\Gamma$ . By the Künneth Theorem,  $H^2((\mathcal{D}_*)^r; \mathbb{R}) \cong \bigoplus H^2(\mathcal{D}_*, \mathbb{R})$ . Since the  $\Gamma$ -action factors through  $Z^m$ ,  $H^2((\mathcal{D}_*)^r; \mathbb{R})^\Gamma = \bigoplus H^2(\mathcal{D}_*; \mathbb{R})^Z$ .

Recall that since  $\zeta_*$  lies in the interior of the alcove, the centralizer  $\mathrm{SU}(p)_{\exp \zeta_*} = T$  and hence  $\mathcal{D}_* \cong \mathrm{SU}(p)/T$ . Moreover, we have  $H^*(\mathcal{D}_*; \mathbb{R}) \cong \mathbb{R}[t_1, \dots, t_p] / (\sigma_1, \dots, \sigma_p)$ , where  $\sigma_i$ 's are the elementary symmetric polynomials. In particular, we may write

$$H^2(\mathcal{D}_*; \mathbb{R}) \cong (\mathbb{R}t_1 \oplus \cdots \oplus \mathbb{R}t_p) / (t_1 + \cdots + t_p = 0).$$

The  $Z$ -action on  $\mathcal{D}_*$  corresponds to an action on  $\mathrm{SU}(p)/T$  by a cyclic subgroup of the Weyl group (e.g., see the proof of Proposition 3.2). Since the Weyl group (i.e., symmetric group  $\Sigma_p$ ) acts by permuting the  $t_i$ ,  $Z$  acts by a  $p$ -cycle on the  $t_i$ . Therefore,  $H^2(\mathcal{D}_*; \mathbb{R})^Z = 0$ , which establishes the result.  $\blacksquare$

**Remark 5.7.** The analogue of Proposition 5.6 for the factors  $(\mathcal{D}_1 \times \cdots \times \mathcal{D}_s) / \Gamma$  that appear in the decomposition in Proposition 4.3 need not hold when considering other non-simply connected structure groups  $G/Z$ .

**Theorem 5.8.** *The quasi-Hamiltonian  $SU(p)$ -space  $M = D(\mathrm{PU}(p))^h \times (\mathcal{D}_1 \times \cdots \times \mathcal{D}_s)/\Gamma$  admits a level  $k$ -prequantization if and only if the following conditions are satisfied:*

- i) if  $h \geq 1$ , then  $k \in p\mathbb{N}$ ;*
- ii)  $g^k \in Z(\mathrm{SU}(p))$  for every  $g \in \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_s$ .*

Moreover, if in addition the identity matrix  $1 \in \mathrm{SU}(p)$  is a regular value of the restriction of the group-valued moment map  $\check{\Phi} : M \rightarrow \mathrm{SU}(p)$ , the prequantization descends to a prequantization of the corresponding component of the moduli space  $\mathfrak{M} = M_{\mathrm{PU}(p)}(\Sigma; \mathcal{C}_1, \dots, \mathcal{C}_s)$ , where  $\mathcal{C}_j = q(\mathcal{D}_j) \subset \mathrm{PU}(p)$ .

**Proof.** By property (a) in Section 5.1,  $M$  admits a level  $k$ -prequantization if and only if each factor does. Since  $D(\mathrm{PU}(p))$  admits a level  $k$ -prequantization if and only if condition (i) is satisfied (see Example 5.3), we may assume from now on  $h = 0$ .

We first verify the necessity of condition (ii). A prequantization of  $M = (\mathcal{D}_1 \times \cdots \times \mathcal{D}_s)/\Gamma$  induces a prequantization of its universal cover  $\tilde{M} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_s$ , and hence each  $\mathcal{D}_j$  must admit a prequantization, which is equivalent to condition (ii).

Next we verify that condition (ii) is sufficient for a level  $k$ -prequantization of  $M$  (with  $h = 0$ ). As in the decomposition (4.4), write (possibly after re-labelling)

$$M = \underbrace{(\mathcal{D}_* \times \cdots \times \mathcal{D}_*)}_{r \text{ factors}}/\Gamma \times \mathcal{D}_{r+1} \times \cdots \times \mathcal{D}_s$$

Using property (a) in Section 5.1 again, it suffices to consider the case  $1 < r = s$ . (Note that if  $s = r = 1$ ,  $\Gamma$  is trivial.) In this case, condition (ii) is simply that  $\mathcal{D}_*$  admit a level  $k$  prequantization. Since  $\mathcal{D}_*$  consists of matrices in  $\mathrm{SU}(p)$  conjugate to

$$\exp \zeta_* = \mathrm{diag}\left(\exp\left(\frac{p-1}{p}\pi\sqrt{-1}\right), \exp\left(\frac{p-3}{p}\pi\sqrt{-1}\right), \dots, \exp\left(\frac{1-p}{p}\pi\sqrt{-1}\right)\right),$$

$\mathcal{D}_*$  admits a level  $k$ -prequantization if and only if  $(\exp \zeta_*)^k$  is a scalar matrix; if and only if  $k$  is a multiple of  $p$ . By property (c) in Section 5.1 and Proposition 5.6, it suffices to show that  $p \cdot \check{\Phi}^*x = 0$ , where  $\check{\Phi} : M \rightarrow \mathrm{SU}(p)$  is the group-valued moment map.

By Corollary 7.6 in [3],  $\mathfrak{h}^\vee \check{\Phi}^*x = W_3(M)$ , the third integral Stiefel–Whitney class, where  $\mathfrak{h}^\vee$  denotes the dual Coxeter number. Recall that  $W_3(M) = \beta w_2(M)$ , where  $\beta : H^2(M; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^3(M; \mathbb{Z})$  is the (integral) Bockstein homomorphism and  $w_2(M)$  is the second Stiefel–Whitney class. Since  $\Gamma$  has odd order,  $H^2(M; \mathbb{Z}/2\mathbb{Z}) \cong H^2((\mathcal{D}_*)^r; \mathbb{Z}/2\mathbb{Z})^\Gamma$ , which is trivial (by an argument similar to the proof of Proposition 5.6). Since  $\mathfrak{h}^\vee = p$ , we are done.

As discussed in Section 5.2, a quasi-Hamiltonian prequantization descends to a prequantization of the symplectic quotient  $\mathfrak{M}$ . ■

## A The action of the centre on the alcove of exceptional Lie groups

Below we record the action of the centre  $Z(G)$  on an alcove for the exceptional Lie groups  $G = E_6$  and  $G = E_7$ . (The action for classical groups appears in [23].)

The vertices of the alcove were obtained using `polymake` [11], which outputs the vertices of a polytope presented as an intersection of half-spaces. The relevant Weyl group element from Proposition 3.2 – one which gives an automorphism of the extended Dynkin diagram – was found with the help of John Stembridge’s `coxeter-weyl` package for Maple [22]; a direct calculation then shows that this element has the desired properties in Proposition 3.2.

Let  $\{e_1, \dots, e_8\}$  denote the standard basis in  $\mathbb{R}^8$ , equipped with the usual inner product. Given a vector  $\alpha$  in  $\mathbb{R}^8$ ,  $s_\alpha : \mathbb{R}^8 \rightarrow \mathbb{R}^8$  denotes reflection in the subspace orthogonal to  $\alpha$ ,

$$s_\alpha(v) = v - \frac{2\langle \alpha, v \rangle}{\langle \alpha, \alpha \rangle} \alpha.$$

The notation used below is consistent with that found in [9, Planches V-VI].

**$G = E_6$ .** Let  $\mathfrak{t} \cong \mathfrak{t}^* \cong \{(x_1, \dots, x_8) \in \mathbb{R}^8 : x_6 = x_7 = -x_8\}$ . The simple roots  $\alpha_1, \dots, \alpha_6$  and highest root  $\tilde{\alpha}$  determine the half-spaces whose intersection is the alcove  $\Delta \subset \mathfrak{t}$ . The vertices of  $\Delta$  (opposite the facets parallel to the corresponding root hyperplanes) are given in Table 1.

**Table 1.** Alcove data for  $E_6$ .

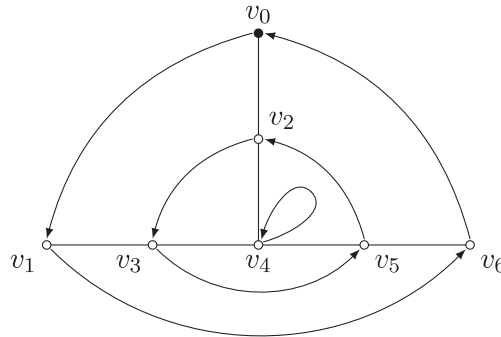
Simple or dominant root	Opposite vertex
$\alpha_1 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	$v_1 = (0, 0, 0, 0, 0, -\frac{2}{3}, -\frac{2}{3}, \frac{2}{3})$
$\alpha_2 = (1, 1, 0, 0, 0, 0, 0, 0)$	$v_2 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4})$
$\alpha_3 = (-1, 1, 0, 0, 0, 0, 0, 0)$	$v_3 = (-\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{5}{12}, -\frac{5}{12}, \frac{5}{12})$
$\alpha_4 = (0, -1, 1, 0, 0, 0, 0, 0)$	$v_4 = (0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})$
$\alpha_5 = (0, 0, -1, 1, 0, 0, 0, 0)$	$v_5 = (0, 0, 0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})$
$\alpha_6 = (0, 0, 0, -1, 1, 0, 0, 0)$	$v_6 = (0, 0, 0, 0, 1, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})$
$\tilde{\alpha} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	$v_0 = 0$

The non-zero elements of the centre  $Z(E_6) \cong \mathbb{Z}/3\mathbb{Z}$  are given by (exp of) the minimal dominant coweights  $\lambda_1^\vee = \frac{2}{3}(e_8 - e_7 - e_6)$  and  $\lambda_6^\vee = e_5 + \frac{1}{3}(e_8 - e_7 - e_6)$ . The corresponding elements  $w_1$  and  $w_6$  of the Weyl group (as in Proposition 3.2), inducing automorphisms of the extended Dynkin diagram are:

$$w_1 = s_{\alpha_1} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_5} s_{\alpha_4} s_{\alpha_3} s_{\alpha_1} s_{\alpha_6} s_{\alpha_5} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_6},$$

$$w_6 = s_{\alpha_6} s_{\alpha_5} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_1} s_{\alpha_4} s_{\alpha_3} s_{\alpha_5} s_{\alpha_4} s_{\alpha_2} s_{\alpha_6} s_{\alpha_5} s_{\alpha_4} s_{\alpha_3} s_{\alpha_1}.$$

The permutation of the vertices induced by the action of  $\exp(\lambda_1^\vee)$  (encoded by the automorphism  $w_1$  of the underlying extended Dynkin diagram) is shown schematically in Fig. 2.



**Figure 2.** Permutation induced by action of  $\exp \lambda_1^\vee$  on the vertices of the alcove for  $E_6$ .

**$G = E_7$ .** Let  $\mathfrak{t} \cong \mathfrak{t}^* \cong \{(x_1, \dots, x_8) \in \mathbb{R}^8 : x_7 = -x_8\}$ . The simple roots  $\alpha_1, \dots, \alpha_7$  and highest root  $\tilde{\alpha}$  determine the half-spaces whose intersection is the alcove  $\Delta \subset \mathfrak{t}$ . The vertices of  $\Delta$  (opposite the facets parallel to the corresponding root hyperplanes) are given in Table 2.

The non-zero element of the centre  $Z(E_7) \cong \mathbb{Z}/2\mathbb{Z}$  is given by (exp of) the minimal dominant coweight  $\lambda_7^\vee = e_6 + \frac{1}{2}(e_8 - e_7)$ . The corresponding element  $w_7$  of the Weyl group (as in

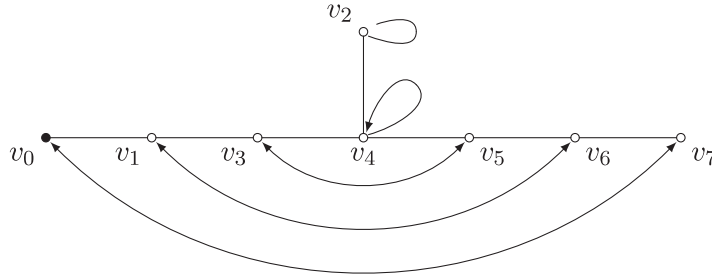
**Table 2.** Alcove data for  $E_7$ .

Simple or dominant root	Opposite vertex
$\alpha_1 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	$v_1 = (0, 0, 0, 0, 0, 0, -\frac{1}{2}, \frac{1}{2})$
$\alpha_2 = (1, 1, 0, 0, 0, 0, 0, 0)$	$v_2 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{2}, \frac{1}{2})$
$\alpha_3 = (-1, 1, 0, 0, 0, 0, 0, 0)$	$v_3 = (-\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{2}, \frac{1}{2})$
$\alpha_4 = (0, -1, 1, 0, 0, 0, 0, 0)$	$v_4 = (0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{2}, \frac{1}{2})$
$\alpha_5 = (0, 0, -1, 1, 0, 0, 0, 0)$	$v_5 = (0, 0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{2}, \frac{1}{2})$
$\alpha_6 = (0, 0, 0, -1, 1, 0, 0, 0)$	$v_6 = (0, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$
$\alpha_7 = (0, 0, 0, 0, -1, 1, 0, 0)$	$v_7 = (0, 0, 0, 0, 0, 1, -\frac{1}{2}, \frac{1}{2})$
$\tilde{\alpha} = (0, 0, 0, 0, 0, 0, -1, 1)$	$v_0 = 0$

Proposition 3.2), inducing an automorphism of the extended Dynkin diagram is

$$w_7 = s_{\alpha_7} s_{\alpha_6} s_{\alpha_5} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_1} s_{\alpha_4} s_{\alpha_3} s_{\alpha_5} s_{\alpha_4} s_{\alpha_2} s_{\alpha_6} s_{\alpha_5} s_{\alpha_4} s_{\alpha_3} s_{\alpha_1} \\ \times s_{\alpha_7} s_{\alpha_6} s_{\alpha_5} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_6} s_{\alpha_7}.$$

The permutation of the vertices induced by the action of  $\exp(\lambda_7^\vee)$  (encoded by the automorphism  $w_7$  of the underlying extended Dynkin diagram) is shown schematically in Fig. 3.

**Figure 3.** Permutation induced by action of  $\exp \lambda_7^\vee$  on the vertices of the alcove for  $E_7$ .

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