

Quantum Dimension and Quantum Projective Spaces

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Abstract. We show that the family of spectral triples for quantum projective spaces introduced by D’Andrea and Dąbrowski, which have spectral dimension equal to zero, can be reconsidered as modular spectral triples by taking into account the action of the element $K_{2\rho}$ or its inverse. The spectral dimension computed in this sense coincides with the dimension of the classical projective spaces. The connection with the well known notion of quantum dimension of quantum group theory is pointed out.

Key words: quantum projective spaces; quantum dimension; modular spectral triples

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1 Introduction

Quantum homogeneous spaces provide an excellent testing ground to study how quantum groups fit into the framework of non-commutative geometry developed by Connes [5]. An important result in this respect is given in [17], where a Dirac operator D is defined on quantized irreducible generalized flag manifolds, which yields a Hilbert space realization of the covariant first-order differential calculus constructed in [13]. This in particular means that the commutator of D with an element of the coordinate algebra is a bounded operator, which is one of the defining properties of a spectral triple. On the other hand the compactness of the resolvent of D , which is another essential requirement of the theory, has not been proven yet, even though it is expected to hold. In particular it can be checked for the simplest case of this construction, the Podleś sphere, in which case D coincides with the Dirac operator introduced in [8], which has compact resolvent. In this respect we should also mention the work [19], where Dolbeault–Dirac operators are constructed for Hermitian symmetric spaces, which can be seen as a first step towards checking the compactness condition.

Among the class of q -deformed irreducible flag manifolds are the quantum projective spaces $\mathbb{C}P_q^\ell$, the simplest example of which is again the Podleś sphere, which is obtained for $\ell = 1$. The case of $\mathbb{C}P_q^2$ has been studied in [7] and then generalized in [6] to $\mathbb{C}P_q^\ell$ with $\ell \geq 2$. The starting point is the introduction of the q -analogue of the module of antiholomorphic differential k -forms Ω^k . More generally the modules Ω_N^k are considered, with $N \in \mathbb{Z}$, corresponding essentially to $\Omega^k = \Omega_0^k$ twisted by certain line bundles. The Hilbert space completion of these is denoted by H_N . For each of these an unbounded self-adjoint operator D_N is introduced, which has bounded commutators with the coordinate algebra $\mathcal{A}(\mathbb{C}P_q^\ell)$. The main result is that $(\mathcal{A}(\mathbb{C}P_q^\ell), H_N, D_N)$ is a family of equivariant spectral triples.

It turns out that these spectral triples are 0^+ -summable, in the sense that the operator $(D_N^2 + 1)^{-\epsilon/2}$ is trace-class for every $\epsilon > 0$. The detailed computation of the spectrum clearly reveals why this is the case: the eigenvalues of this operator grow like a q -number, so exponentially, while their multiplicities grow like a polynomial. We recall that in the classical case it is the balance between the growth of the eigenvalues and their multiplicities that allows to recover the

dimension of the manifold in consideration. In this case the eigenvalues grow much faster than their multiplicities, which explains the 0^+ -summability.

In this paper we consider a simple modification of the above construction, which fits into the framework of modular spectral triples. The idea is to consider the action of the element $K_{2\rho}$, which implements the modular group of the Haar state of $\mathcal{A}(\mathbb{C}\mathbb{P}_q^\ell)$. In particular we compute the spectral dimension associated to D with respect to the weight $\text{Tr}(K_{2\rho}\cdot)$, with the result that it coincides with the classical dimension. This computation is linked with an important concept in the theory of quantum groups, which is the notion of quantum dimension. We also point out that, as a consequence of a property of the quantum dimension, the same result for the spectral dimension is obtained by considering $K_{2\rho}^{-1}$. This in turn is connected with results from twisted Hochschild (co)homology, as we will discuss in the last part.

One of the motivations for this paper comes from the notion of integration which is defined in the context of spectral triples. Let us briefly review how this works in the classical case, by considering the canonical spectral triple $(C^\infty(M), L^2(M, S), D)$ associated to a compact spin manifold M . First of all, the dimension of such a manifold can be extracted from the spectrum of D . Indeed the operator $(D^2 + 1)^{-s/2}$, with $s \in \mathbb{R}$, turns out to be trace-class for all $s > n$, where n is the dimension of M . The operator D also allows to define a notion of integration. One possible formulation is via the linear functional $\psi : C^\infty(M) \rightarrow \mathbb{C}$ given by

$$\psi(f) = \lim_{s \rightarrow n^+} (s - n) \text{Tr}(f(D^2 + 1)^{-s/2}),$$

where f is considered to be acting via left multiplication. This functional turns out to coincide with the usual integral of f , up to a constant. Therefore this procedure allows to recover the dimension and the integration of functions for the manifold in consideration.

As we mentioned above, in the case of quantum projective spaces the analogue of this procedure gives a spectral dimension equal to zero. But, more importantly, it does not allow to recover the natural notion of integration that is available on these spaces, which is given by the faithful Haar state. The reason for this failure is quite clear: the analogue of the functional ψ is by construction a trace, as it follows from the defining properties of a spectral triple, while on the other hand the Haar state is not. A way out of this problem is to replace the trace in the definition of ψ by a weight, which immediately brings us into the realm of modular spectral triples. Similar ideas were discussed in [21] for the case of $\text{SU}_q(2)$.

The plan of the paper is as follows. In Section 2 we recall some basic notions on quantum projective spaces and their family of spectral triples. In Section 3 we recall the notion of modular spectral triple and discuss its role in the context of this paper. In Section 4 we show how the computation of the spectral dimension is connected to the notion of quantum dimension, and compute it in the cases of interest to us. Finally in Section 5 we prove that the spectral dimension coincides with the dimension of the classical projective spaces. We also discuss the connection with Hochschild twisted (co)homology.

2 Quantum projective spaces

In this section we provide some background on quantum projective spaces, which we denote by $\mathbb{C}\mathbb{P}_q^\ell$ for $\ell \in \mathbb{N}$ and $\ell \geq 2$. These are q -deformations of complex projective spaces of real dimension 2ℓ . The case $\ell = 1$ of this construction coincides with the standard Podleś sphere and is well known in the literature. We take our definitions and notations from [6].

To define quantum projective spaces we first define the Hopf $*$ -algebra $U_q(\mathfrak{su}(\ell + 1))$, which is a deformation of the universal enveloping algebra $U(\mathfrak{su}(\ell + 1))$, and its dual $\mathcal{A}(\text{SU}_q(\ell + 1))$, which can be considered as the algebra of representative functions on the quantum $\text{SU}(\ell + 1)$ group. Our reference for this material is the book [16], but one must keep in mind that what we

denote by $U_q(\mathfrak{g})$ corresponds to $\check{U}_q(\mathfrak{g})$ there. The coordinate algebra $\mathcal{A}(\mathbb{C}P_q^\ell)$ of the quantum projective space $\mathbb{C}P_q^\ell$ can then be defined as the fixed point subalgebra of $\mathcal{A}(\mathrm{SU}_q(\ell+1))$ for the action of a suitable Hopf subalgebra of $U_q(\mathfrak{su}(\ell+1))$. We now review these notions.

For $0 < q < 1$ we denote by $U_q(\mathfrak{su}(\ell+1))$ the $*$ -algebra generated by $K_i = K_i^*$, K_i^{-1} , E_i and $F_i = E_i^*$, with $i = 1, \dots, \ell$, and with relations

$$\begin{aligned} [K_i, K_j] &= 0, & K_i E_i K_i^{-1} &= q E_i, \\ K_i E_j K_i^{-1} &= q^{-1/2} E_j & \text{if } |i-j| &= 1, \\ K_i E_j K_i^{-1} &= E_j & \text{if } |i-j| &> 1, \\ [E_i, F_j] &= \delta_{ij} \frac{K_i^2 - K_i^{-2}}{q - q^{-1}}, \\ E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 &= 0 & \text{if } |i-j| &= 1, \\ [E_i, E_j] &= 0 & \text{if } |i-j| &> 1. \end{aligned}$$

We call $U_q(\mathfrak{su}(\ell))$ the Hopf $*$ -subalgebra generated by the elements $K_i = K_i^*$, K_i^{-1} , E_i and $F_i = E_i^*$ with $i = 1, \dots, \ell - 1$. Its commutant is the Hopf $*$ -subalgebra $U_q(\mathfrak{u}(1))$ generated by the element $K_1 K_2^2 \cdots K_\ell^\ell$ and its inverse. This is a positive operator in all the representations we consider, so we can define its root of order $2/(\ell+1)$ by

$$\hat{K} = (K_1 K_2^2 \cdots K_\ell^\ell)^{2/(\ell+1)}.$$

The following element will play a central role in this paper:

$$K_{2\rho} = (K_1^\ell K_2^{2(\ell-1)} \cdots K_j^{j(\ell-j+1)} \cdots K_\ell^\ell)^2.$$

Here the symbol ρ denotes the Weyl vector of the Lie algebra $\mathfrak{su}(\ell+1)$, see for example [16] for its role in q -deformations of semisimple Lie algebras. One important property of this element is that it implements the square of the antipode, in the sense that $S^2(h) = K_{2\rho} h K_{2\rho}^{-1}$ for any $h \in U_q(\mathfrak{su}(\ell+1))$. More importantly for us, it also implements the modular group of the Haar state of $\mathcal{A}(\mathrm{SU}_q(\ell+1))$, as we will see in a moment.

We are interested in representations in which K_j is represented by a positive operator. Such irreducible finite-dimensional $*$ -representations of $U_q(\mathfrak{su}(\ell+1))$ are labeled by ℓ non-negative integers. Writing $n = (n_1, \dots, n_\ell)$, we denote by V_n the vector space carrying the representation ρ_n with highest weight n . This means that there exists a vector v which is annihilated by all the E_j 's and satisfies $\rho_n(K_i)v = q^{n_i/2}v$.

We now introduce the coordinate algebra $\mathcal{A}(\mathrm{SU}_q(\ell+1))$. It is the Hopf $*$ -algebra generated by the elements u_j^i , with $i, j = 1, \dots, \ell+1$, and with relations

$$\begin{aligned} u_k^i u_k^j &= q u_k^j u_k^i, & u_i^k u_j^k &= q u_j^k u_i^k, & \text{for } i < j, \\ [u_i^i, u_k^j] &= 0, & [u_k^i, u_l^j] &= (q - q^{-1}) u_l^i u_k^j, & \text{for } i < j, \quad k < l. \end{aligned}$$

and with the determinant relation

$$\sum_{p \in S_{\ell+1}} (-q)^{\|p\|} u_{p(1)}^1 \cdots u_{p(\ell+1)}^{\ell+1} = 1,$$

where the sum is over all permutations p of the set $\{1, \dots, \ell+1\}$ and $\|p\|$ is the number of inversions in p . The $*$ -structure is defined as in [6].

There is a non-degenerate pairing $\langle \cdot, \cdot \rangle$ between $U_q(\mathfrak{su}(\ell+1))$ and $\mathcal{A}(\mathrm{SU}_q(\ell+1))$, which is used to define the canonical left and right actions as $h \triangleright a = a_{(1)} \langle h, a_{(2)} \rangle$ and $a \triangleleft h = \langle h, a_{(1)} \rangle a_{(2)}$,

where we use Sweedler's notation for the coproduct. In this way $\mathcal{A}(\mathrm{SU}_q(\ell + 1))$ can be seen as the algebra generated by the matrix coefficients of the finite-dimensional representations of $U_q(\mathfrak{su}(\ell + 1))$, similarly to the classical case. The pairing can be extended in a natural way to include also the action of the element \hat{K} and its inverse.

There is a faithful state on $\mathcal{A}(\mathrm{SU}_q(\ell + 1))$, called the Haar state and denoted by φ , which generalizes the properties of the Haar integral in the classical case. However, differently from the classical case, the Haar state is not a trace on $\mathcal{A}(\mathrm{SU}_q(\ell + 1))$. In particular its modular group is implemented by the element $K_{2\rho}$, in the sense that

$$\varphi(ab) = \varphi(bK_{2\rho} \triangleright a \triangleleft K_{2\rho}). \quad (1)$$

Consider now the left action of $U_q(\mathfrak{su}(\ell + 1))$ on $\mathcal{A}(\mathrm{SU}_q(\ell + 1))$ defined by

$$\mathcal{L}_h a = a \triangleleft S^{-1}(h).$$

It can be used to define the coordinate algebra $\mathcal{A}(S_q^{2\ell+1})$ of the quantum sphere $S_q^{2\ell+1}$ as

$$\mathcal{A}(S_q^{2\ell+1}) = \{a \in \mathcal{A}(\mathrm{SU}_q(\ell + 1)) : \mathcal{L}_h a = \varepsilon(h)a, \forall h \in U_q(\mathfrak{su}(\ell))\}.$$

Finally, using the generator of $U_q(\mathfrak{u}(1))$, which we denoted by \hat{K} , we define the coordinate algebra $\mathcal{A}(\mathbb{C}\mathbb{P}_q^\ell)$ of the quantum projective space $\mathbb{C}\mathbb{P}_q^\ell$ as

$$\mathcal{A}(\mathbb{C}\mathbb{P}_q^\ell) = \{a \in \mathcal{A}(S_q^{2\ell+1}) : \mathcal{L}_{\hat{K}} a = a\}.$$

Having defined the coordinate algebra $\mathcal{A}(\mathbb{C}\mathbb{P}_q^\ell)$, the next step in order to build a spectral triple is to introduce a Hilbert space, on which elements of this algebra act as bounded operators. Recall that the projective spaces $\mathbb{C}\mathbb{P}^\ell$ are only spin^c manifolds when ℓ is even. Then one possibility is to complete the space of antiholomorphic forms, with the idea of defining a Dolbeault–Dirac operator acting on it. This is the strategy followed in [7] for the case $\ell = 2$, where a q -analogue of the space of antiholomorphic forms is introduced.

This strategy is generalized in [6] for all quantum projective spaces. We denote by Ω^k their q -analogue of the space of antiholomorphic k -form. More generally, they also consider the possibility of twisting this module of k -forms by a line bundle Γ_N , with the resulting space being denoted by Ω_N^k , and with the space of forms corresponding to the case $N = 0$.

The space $\bigoplus_{k=0}^\ell \Omega_N^k$ carries a left action of $U_q(\mathfrak{su}(\ell + 1))$ and can be decomposed into irreducible representations. The resulting decomposition takes the following form:

$$\begin{aligned} \Omega_N^0 &\simeq \bigoplus_{m \in \mathbb{N}} V_{(m+c_1, 0, \dots, 0, m+c_2)}, \\ \Omega_N^k &\simeq \bigoplus_{m \in \mathbb{N}} V_{(m+c_3, 0, \dots, 0, m+c_4)+e_k} \oplus V_{(m+c_5, 0, \dots, 0, m+c_6)+e_{k+1}} \quad \text{for } 1 \leq k \leq \ell - 1, \\ \Omega_N^\ell &\simeq \bigoplus_{m \in \mathbb{N}} V_{(m+c_7, 0, \dots, 0, m+c_8)}. \end{aligned} \quad (2)$$

Here c_1, \dots, c_8 are integers depending on k and N , but independent of m . The Hilbert spaces obtained as a completions of $\bigoplus_{k=0}^\ell \Omega_N^k$ are denoted by H_N .

It is possible to introduce the q -analogue of the Dolbeault operator, which we denote by $\bar{\partial}$, which maps Ω_N^k into Ω_N^{k+1} and satisfies $\bar{\partial}^2 = 0$. Similarly the adjoint $\bar{\partial}^\dagger$ maps Ω_N^{k+1} into Ω_N^k and satisfies $(\bar{\partial}^\dagger)^2 = 0$. A family of Dolbeault–Dirac operators, denoted by D_N for $N \in \mathbb{Z}$, is defined by taking suitable linear combinations of $\bar{\partial}$ and $\bar{\partial}^\dagger$ on each Ω_N^k . The operator D_0 is the q -analogue of the Dolbeault–Dirac operator on $\mathbb{C}\mathbb{P}^\ell$, while D_N is the twist of D_0 with the

Grassmannian connection of a certain line bundle. In particular, if ℓ is odd and $N = (\ell + 1)/2$, then D_N is the q -analogue of the Dirac operator for the Fubini–Study metric.

In this paper we do not need the precise form of D_N , but only an asymptotic form of its eigenvalues. In particular, for our purposes, this turns out to be independent on the value of N . Using the decomposition (2), it is possible to compute the eigenvalues of $|D_N|$ when restricted to the space Ω_N^k . The information that we need is that these eigenvalues grow like q^{-m} with $m \in \mathbb{N}$, see the discussion at the end of [6].

3 Modular spectral triples

Consider now the restriction of the Haar state of $\mathcal{A}(\mathrm{SU}_q(\ell + 1))$ to $\mathcal{A}(\mathbb{C}\mathrm{P}_q^\ell)$, denoted again by the symbol φ . It follows, using the definitions given in the previous section, that any element $a \in \mathcal{A}(\mathbb{C}\mathrm{P}_q^\ell)$ is invariant under the right action of $K_{2\rho}$, that is $a \triangleleft K_{2\rho} = a$. Therefore the modular property of the Haar state of $\mathcal{A}(\mathrm{SU}_q(\ell + 1))$, given by (1), becomes

$$\varphi(ab) = \varphi(bK_{2\rho} \triangleright a), \quad a, b \in \mathcal{A}(\mathbb{C}\mathrm{P}_q^\ell).$$

As we have remarked in the introduction, the non-commutative integral, defined in the usual sense of spectral triples in terms of D_N , does not coincide with the Haar state. Indeed the former is a trace while the latter is not. This fact provides a motivation to introduce a twist in the definition of the non-commutative integral, as we now proceed to explain.

We denote by $K_{2\rho}$ the closure of the unbounded operator acting via the left action of $K_{2\rho}$ on $\mathcal{A}(\mathrm{SU}_q(\ell + 1))$, which is a positive and invertible operator. We assume for the moment that the operator $K_{2\rho}(D_N^2 + 1)^{-s/2}$ is trace-class for all $s > p$, for some $p \geq 0$. Let also assume that the following linear functional on $\mathcal{A}(\mathbb{C}\mathrm{P}_q^\ell)$ makes sense

$$\psi(a) = \lim_{s \rightarrow p^+} (s - p) \mathrm{Tr} (K_{2\rho} a (D_N^2 + 1)^{-s/2}).$$

Then it can be shown, as will be done in Appendix A, that we have

$$\begin{aligned} \psi(ab) &= \lim_{s \rightarrow p^+} (s - p) \mathrm{Tr} (K_{2\rho} ab (D_N^2 + 1)^{-s/2}) = \lim_{s \rightarrow p^+} (s - p) \mathrm{Tr} (bK_{2\rho} a (D_N^2 + 1)^{-s/2}) \\ &= \lim_{s \rightarrow p^+} (s - p) \mathrm{Tr} (K_{2\rho} K_{2\rho}^{-1} b K_{2\rho} a (D_N^2 + 1)^{-s/2}). \end{aligned} \quad (3)$$

Here the non-trivial equality is the second one. Then, since we have $K_{2\rho}^{-1} b K_{2\rho} = K_{2\rho}^{-1} \triangleright b$ for all $b \in \mathcal{A}(\mathbb{C}\mathrm{P}_q^\ell)$, we find the modular property

$$\psi(ab) = \psi(K_{2\rho}^{-1} \triangleright ba),$$

which is equivalent to that of equation (1). Therefore in this way we should obtain a linear functional on $\mathcal{A}(\mathbb{C}\mathrm{P}_q^\ell)$ which has the modular property of the Haar state.

Of course we should check that the assumptions made above are justified. This will be done in the next sections, where we will check them explicitly in the case $a = 1$. It turns out that this is enough to conclude in the general case, since [22, Lemma 2.1] guarantees that if $\psi(1)$ is well defined then $\psi(a)$ is proportional to the Haar state.

Before getting into that, we should mention that this kind of construction fits into the framework of *modular spectral triples*, which was introduced in [2]. The main motivation for this concept was to study algebras that do not admit non-trivial traces, but it was later realized that it can also be used to study algebras which have faithful states but not faithful traces, as in [4]. The latter is the relevant case for this paper. This notion has been formalized on the basis of examples where the modular group comes from a circle action, so that modifications might be needed to handle more complicated examples.

Definition 1. Let \mathcal{A} be a unital $*$ -subalgebra of N , where N is a semifinite von Neumann algebra acting on a Hilbert space H . Fix a faithful normal strictly semifinite weight ϕ with modular group σ^ϕ . We call the triple (\mathcal{A}, H, D) a modular spectral triple if

- 1) \mathcal{A} is invariant under σ^ϕ and consist of analytic vectors for σ^ϕ ,
- 2) D is a self-adjoint operator affiliated with the fixed point algebra N^{σ^ϕ} ,
- 3) $[D, a]$ extends to a bounded operator in N for all $a \in \mathcal{A}$,
- 4) $(D^2 + 1)^{-1/2}$ is compact with respect to the trace $\tau = \phi|_{N^{\sigma^\phi}}$.

We recall that a semifinite weight ϕ is *strictly* semifinite if its restriction to N^{σ^ϕ} is semifinite, and that the ideal of compact operators with respect to a semifinite weight is the norm-closed ideal generated by projections on which the weight takes finite values.

It is worth noting that if \mathcal{A} is pointwise invariant under the modular group σ^ϕ then we are essentially back to the semifinite case. This observation makes clear the fact that the fixed point algebra plays an important role in this definition. However, in examples it might well be that no element of \mathcal{A} is invariant under the modular group.

Regarding summability, the notion of spectral dimension can be adapted straightforwardly to this case by replacing the trace with the state or weight under consideration.

Definition 2. A modular spectral triple (\mathcal{A}, H, D) is called finitely summable if there exists some $s_0 > 0$ such that $\phi((D^2 + 1)^{-s_0/2}) < \infty$. In this case, we define the *spectral dimension* as

$$p = \inf \{s > 0 : \phi((D^2 + 1)^{-s/2}) < \infty\}.$$

A modification of this notion has appeared in [14], by replacing the condition of boundedness of the commutator with the analogue one for a twisted commutator. An interesting example that makes use of this condition is the one provided for $SU_q(2)$ in [15].

4 Quantum dimension

Motivated by the previous section, we now want to introduce the tools needed to compute the spectral dimension of D_N with respect to the weight defined by $\text{Tr}(K_{2\rho})$. This computation is strictly related to the notion of quantum dimension, that we now review.

Given a finite-dimensional irreducible representation T of a Drinfeld–Jimbo algebra $U_q(\mathfrak{g})$, its *quantum dimension* is defined as the number $\text{Tr}(T(K_{2\rho}))$, where the trace is taken over the vector space that carries the representation T , see for example [16]. In the classical case, that is for $q = 1$, the quantum dimension coincides with the dimension of the vector space. In the context of quantum groups the notion of quantum dimension appears, for example, in the q -analogue of the Schur orthogonality relations.

In the classical case, if we consider a finite-dimensional representation of a Lie algebra \mathfrak{g} with highest weight Λ , the dimension of the associated vector space V_Λ can be computed from the *Weyl dimension formula*, which reads as

$$\dim V_\Lambda = \prod_{\alpha > 0} \frac{(\Lambda + \rho, \alpha)}{(\rho, \alpha)},$$

where the product is over the positive roots and ρ is the Weyl vector, defined as the half-sum of the positive roots. There is also a q -analogue of this formula, see [9] and references within (notice that our q is $q^{1/2}$ in the notation of the book). It allows to compute the quantum dimension of a representation with highest weight Λ as

$$\dim_q V_\Lambda = \prod_{\alpha > 0} \frac{[(\Lambda + \rho, \tilde{\alpha})]}{[(\rho, \tilde{\alpha})]},$$

where we use the usual notion of q -number

$$[x] = \frac{q^{-x} - q^x}{q^{-1} - q}$$

and $\tilde{\alpha} = 2\alpha/(\theta, \theta)$ where θ is the highest root. Note that an explicit normalization is needed for the positive roots, differently from the classical case.

Our aim is now to compute the quantum dimension for any of the irreducible representations that appear in the decomposition (2). More precisely we are only interested in the asymptotics of this value when $m \rightarrow \infty$, since this is the only contribution that matters in the computation of the spectral dimension.

We need to review some facts about the root system of $\mathfrak{su}(\ell + 1)$, whose elements can be considered as vectors in $\mathbb{R}^{\ell+1}$. The simple roots are given by $\alpha_i = e_i - e_{i+1}$ with $1 \leq i \leq \ell$. The positive roots are given by $\alpha_{ij} = e_i - e_j$, with $1 \leq i < j \leq \ell + 1$, and we note that they can be written in terms of the simple roots as $\alpha_{ij} = \sum_{k=i}^{j-1} \alpha_k$. Their scalar product is $(\alpha_{ij}, \alpha_{ij}) = 2$. In particular $(\theta, \theta) = 2$, so that $\tilde{\alpha} = \alpha$ in the Weyl formula.

We also need the basis of the fundamental weights, which we denote by ω_i . They are connected to the simple roots via the Cartan matrix A as $\alpha_i = \sum_{j=1}^{\ell} A_{ij} \omega_j$. The fundamental weights are dual to the simple roots in the sense that

$$\frac{2(\alpha_i, \omega_j)}{(\alpha_i, \alpha_i)} = \delta_{ij}.$$

Since in our case $(\alpha_i, \alpha_i) = 2$ this relation becomes $(\alpha_i, \omega_j) = \delta_{ij}$. Finally we recall that the Weyl vector ρ , which is usually defined as the half-sum of the positive roots, can be written in the basis of the fundamental weights in the simple form $\rho = \sum_{j=1}^{\ell} \omega_j$.

In the following we will use the notation $f(m) \sim g(m)$ for $m \rightarrow \infty$ to mean that $\lim_{m \rightarrow \infty} f(m)/g(m) = C$, where C is non-zero. It is not difficult to determine C explicitly in the expressions we will present, but this is unnecessary for our purposes.

Proposition 1. *Let $\Lambda = n_1 \omega_1 + n_a \omega_a + n_\ell \omega_\ell$ be a dominant weight, where $n_1 = m + c_1$, $n_\ell = m + c_2$ with $m \in \mathbb{N}$, $c_1, c_2 \in \mathbb{N}$ and $n_a = 0, 1$ with $2 \leq a \leq \ell - 1$. Then for the corresponding quantum dimension we have $\dim_q(V_\Lambda) \sim q^{-2\ell m}$ for $m \rightarrow \infty$.*

Proof. First observe that $[x] = (q^x - q^{-x})/(q - q^{-1}) \sim q^{-x}$ for $x \rightarrow \infty$, since we are assuming that $0 < q < 1$. Then we introduce the notation

$$S_i = \prod_{j=i+1}^{\ell+1} \frac{[(\Lambda + \rho, \alpha_{ij})]}{[(\rho, \alpha_{ij})]},$$

in such a way that $\dim_q(V_\Lambda)$ is given by the product of the S_i , that is

$$\dim_q(V_\Lambda) = \prod_{i=1}^{\ell} S_i.$$

Let us consider first the case $i = 1$. Using the formulae $(\alpha_i, \omega_j) = \delta_{ij}$ and $\alpha_{ij} = \sum_{k=i}^{j-1} \alpha_k$ it is immediate to show that we have

$$(\Lambda, \alpha_{ij}) = \begin{cases} n_1, & j \leq a, \\ n_1 + n_a, & a < j < \ell + 1, \\ n_1 + n_a + n_\ell, & j = \ell + 1. \end{cases}$$

Then for $m \rightarrow \infty$ we obtain

$$S_1 = \prod_{j=2}^{\ell+1} \frac{[(\Lambda + \rho, \alpha_{ij})]}{[(\rho, \alpha_{ij})]} \sim q^{-(\ell-1)m} q^{-2m} = q^{-(\ell+1)m}.$$

Similarly for $2 \leq i \leq a$ we have

$$(\Lambda, \alpha_{ij}) = \begin{cases} 0, & j \leq a, \\ n_a, & a < j < \ell + 1, \\ n_a + n_\ell, & j = \ell + 1. \end{cases}$$

and for $m \rightarrow \infty$ we obtain

$$S_i = \prod_{j=i+1}^{\ell+1} \frac{[(\Lambda + \rho, \alpha_{ij})]}{[(\rho, \alpha_{ij})]} \sim q^{-m}.$$

Finally for $i \geq a + 1$ we have

$$(\Lambda, \alpha_{ij}) = \begin{cases} 0, & j < \ell + 1, \\ n_\ell, & j = \ell + 1, \end{cases}$$

and for $m \rightarrow \infty$ we obtain

$$S_i = \prod_{j=i+1}^{\ell+1} \frac{[(\Lambda + \rho, \alpha_{ij})]}{[(\rho, \alpha_{ij})]} \sim q^{-m}.$$

Putting all together we find

$$\begin{aligned} \dim_q(V_\Lambda) &= S_1 \left(\prod_{i=2}^a S_i \right) \left(\prod_{i=a+1}^{\ell} S_i \right) \sim q^{-(\ell+1)m} \left(\prod_{i=2}^a q^{-m} \right) \left(\prod_{i=a+1}^{\ell} q^{-m} \right) \\ &= q^{-(\ell+1)m} q^{-(a-1)m} q^{-(\ell-a)m} = q^{-2\ell m}. \end{aligned} \quad \blacksquare$$

5 Spectral dimension

Given the result of the previous section, it is now easy to prove the main result of the paper.

Theorem 1. *The operator $K_{2\rho}(D_N^2 + 1)^{-s/2}$ is trace-class for $s > 2\ell$, and the corresponding spectral dimension (in the sense of modular spectral triples) is 2ℓ . Moreover*

$$\lim_{s \rightarrow 2\ell^+} (s - 2\ell) \operatorname{Tr} (K_{2\rho}(D_N^2 + 1)^{-s/2})$$

exists and is non-zero.

Proof. The Hilbert space H_N is the completion of $\bigoplus_{k=0}^{\ell} \Omega_N^k$ and each Ω_N^k can be decomposed into irreducible representations of $U_q(\mathfrak{su}(\ell + 1))$ as in (2). As shown in [6], the operator D_N^2 restricted to the space Ω_N^k can be expressed in terms of the Casimir operator of $U_q(\mathfrak{su}(\ell + 1))$. Therefore it acts as a multiple of the identity in each irreducible representation.

The only representations which appear in the decomposition (2) are those of weight $(m + c_{1,k,N}, 0, \dots, 0, m + c_{2,k,N}) + e_k$, where $m \in \mathbb{N}$, $2 \leq k \leq \ell$ and $c_{1,k,N}$, $c_{2,k,N}$ are some positive integers depending on k and N . We denote the vector space that carries such a representation

by $V_{m,k,N}$ and the corresponding eigenvalue of D_N^2 by $\lambda_{m,k,N}^2$. Finally denoting by $\text{Tr}_{m,k,N}$ the trace on the vector space $V_{m,k,N}$ we have that

$$\text{Tr}_{m,k,N} (K_{2\rho}(D_N^2 + 1)^{-s/2}) = \dim_q(V_{m,k,N})(\lambda_{m,k,N}^2 + 1)^{-s/2}.$$

From Proposition 1 we know that $\dim_q(V_{m,k,N}) \sim q^{-2\ell m}$ for $m \rightarrow \infty$. Moreover we know from [6] that $\lambda_{m,k,N} \sim q^{-m}$. Then $\text{Tr}_{m,k,N}(K_{2\rho}(D_N^2 + 1)^{-s/2}) \sim q^{(s-2\ell)m}$.

Finally the trace can be written in the form

$$\text{Tr} (K_{2\rho}(D_N^2 + 1)^{-s/2}) = \sum_{k=0}^{\ell} \sum_{m=1}^{\infty} \text{Tr}_{m,k,N} (K_{2\rho}(D_N^2 + 1)^{-s/2}) \sim \sum_{m=1}^{\infty} q^{(s-2\ell)m}.$$

The series $\sum_{m=1}^{\infty} q^{(s-2\ell)m}$ is absolutely convergent for $s > 2\ell$, from which it follows that the spectral dimension is 2ℓ . Moreover we have $\sum_{m=1}^{\infty} q^{(s-2\ell)m} = q^{s-2\ell}/(1 - q^{s-2\ell})$, from which one easily finds that the limit exists and is non-zero. \blacksquare

It is also possible to prove that $\text{Tr}(K_{2\rho}(D_N^2 + 1)^{-s/2})$ extends to a meromorphic function on the complex plane, similarly to the classical case, but we will not show it here.

We now give a few comments on this result. As we mentioned in the introduction, in the classical case the computation of the spectral dimension hinges on the balance between the growth of the eigenvalues of D and the growth of their multiplicities. On the other hand, in the case of q -deformations the eigenvalues of D grow like q -numbers, therefore exponentially, while their multiplicities only grow polynomially. This has the consequence of giving a spectral dimension equal to zero for the spectral triples $(\mathcal{A}(\mathbb{C}\mathbb{P}_q^\ell), H_N, D_N)$. Roughly speaking, the effect of the weight $\text{Tr}(K_{2\rho})$ is to replace the multiplicities of the eigenvalues with their q -analogues, therefore restoring the balance in the computation. Indeed it can be argued that in this context the notion of quantum dimension is the most natural one, as seen from its role in the formulation of the quantum orthogonality relations.

The same result for the spectral dimension is obtained by considering $K_{2\rho}^{-1}$, as follows from a general property of the quantum dimension.

Corollary 1. *The results of Theorem 1 remain valid if $K_{2\rho}$ is replaced by $K_{2\rho}^{-1}$.*

Proof. This follows from the identity $\text{Tr}(K_{2\rho}^{-1}) = \text{Tr}(K_{2\rho})$, where the trace is taken on the vector space of an irreducible finite-dimensional representation, see [16, § 7.1.6]. We give an outline of the proof for the benefit of the reader. There is an algebra automorphism η of $U_q(\mathfrak{su}(\ell + 1))$ which is given on the generators as

$$E_i \mapsto F_{\ell+1-i}, \quad F_i \mapsto E_{\ell+1-i}, \quad K_i \mapsto K_{\ell+1-i}^{-1}, \quad 1 \leq i \leq \ell,$$

as can be checked directly from the defining relations of $U_q(\mathfrak{su}(\ell + 1))$. Since $\eta(K_{2\rho}) = K_{2\rho}^{-1}$, the trace of $K_{2\rho}$ in V_n is equal to the trace of $K_{2\rho}^{-1}$ in V_n twisted by η . But the latter module is isomorphic to V_n itself, since they are both simple finite-dimensional modules with the same highest weight. This proves the claimed identity. \blacksquare

This simple corollary is interesting in view of its possible applications to twisted Hochschild (co)homology, see [12, § 2.2] and references therein. It is known that for quantum groups there is a dimension drop in Hochschild homology: this means that, if G is a semisimple group and we denote by $\mathcal{A}(G_q)$ the associated quantized algebra of functions, then we have $H_n(\mathcal{A}(G_q)) = 0$, where n denotes the classical dimension of G . On the other hand, by using twisted Hochschild

homology, that is by twisting appropriately the notion of Hochschild homology, it is possible to avoid this dimension drop. This was observed first in [12] for $\mathrm{SL}_q(2)$ by direct computation and then generalized in [1] to the general case.

Similar results hold for quantum homogeneous spaces as the Podleś spheres, as shown by the computations in [11]. For results on a more general class of quantum homogeneous spaces see [18]. For the standard Podleś sphere the dimension drop is avoided by considering the twist ϑ_P^{-1} , where ϑ_P the modular group of the Haar state. Then the volume form, being a twisted cycle, will pair non-trivially with a twisted cocycle with twist ϑ_P^{-1} .

In view of the results mentioned above, we expect that they continue to hold also for the projective spaces $\mathcal{A}(\mathbb{C}\mathbb{P}_q^\ell)$. Therefore, if we denote by ϑ the modular group in this case, we expect to avoid the dimension drop in homology by twisting with ϑ^{-1} . Then, in view of our results, we can write a natural candidate for a twisted cocycle that has a chance of pairing non-trivially with the volume form.

Corollary 2. *The functional on $\mathcal{A}(\mathbb{C}\mathbb{P}_q^\ell)^{\otimes(2\ell+1)}$ defined by*

$$\tilde{\psi}(a_0, \dots, a_{2\ell}) = \lim_{s \rightarrow 2\ell^+} (s - 2\ell) \mathrm{Tr} (K_{2\rho}^{-1} a_0 [D_N, a_1] \cdots [D_N, a_{2\ell}] (D_N^2 + 1)^{-s/2})$$

is a twisted cocycle with twist ϑ^{-1} .

Proof. It follows from Corollary 1 that this functional is well-defined. That it is a twisted cocycle with twist ϑ^{-1} follows from the twisted trace property shown in equation (3), with $K_{2\rho}$ replaced by $K_{2\rho}^{-1}$, and from standard computations. ■

For the case of the Podleś sphere, it is shown in [20] that such a twisted cocycle is indeed non-trivial, when D_N is taken to be the Dirac operator introduced in [8].

A The twisted trace property

In this appendix we give a proof, under suitable assumptions, of the equality appearing in equation (3). The proof holds quite generally, so that we do not make any reference to quantum projective spaces. We consider a triple (\mathcal{A}, H, D) , where \mathcal{A} is a unital algebra acting as bounded operators on the Hilbert space H , that is $\mathcal{A} \subset B(H)$, and D is a self-adjoint unbounded operator. Moreover let Δ_ϕ be a positive invertible operator acting on H . The assumptions we make on this data are essentially those of a modular spectral triple:

- 1) $\Delta_\phi a \Delta_\phi^{-1} \in \mathcal{A}$ for any $a \in \mathcal{A}$,
- 2) Δ_ϕ and $(D^2 + 1)^{-1/2}$ commute,
- 3) $[D, a]$ extends to a bounded operator for any $a \in \mathcal{A}$.

We also make the following summability assumptions:

- 1) $\Delta_\phi (D^2 + 1)^{-s/2}$ is trace-class for all $s > p$, with $p \geq 0$,
- 2) $\lim_{s \rightarrow p^+} (s - p) \mathrm{Tr} (\Delta_\phi a (D^2 + 1)^{-s/2})$ exists for all $a \in \mathcal{A}$.

Here by $\lim_{s \rightarrow p^+}$ we mean the one-sided limit from the right. We note in passing that these two summability conditions can be related to the semifinite theory, see [3].

Given these assumptions, we can define a linear functional on \mathcal{A} by

$$\psi(a) = \lim_{s \rightarrow p^+} (s - p) \mathrm{Tr} (\Delta_\phi a (D^2 + 1)^{-s/2}).$$

Proposition 2. *With the same assumptions as above, the linear functional $\psi : \mathcal{A} \rightarrow \mathbb{C}$ satisfies the twisted trace property $\psi(ab) = \psi(\Delta_\phi^{-1}b\Delta_\phi a)$ for all $a, b \in \mathcal{A}$.*

Proof. The crucial step of the proof is to show that

$$\psi(ab) = \lim_{s \rightarrow p^+} (s - p) \operatorname{Tr} (\Delta_\phi ab (D^2 + 1)^{-s/2}) = \lim_{s \rightarrow p^+} (s - p) \operatorname{Tr} (\Delta_\phi a (D^2 + 1)^{-s/2} b),$$

or equivalently that the following quantity vanishes

$$\lim_{s \rightarrow p^+} (s - p) \operatorname{Tr} (\Delta_\phi a [(D^2 + 1)^{-s/2}, b]) = 0. \quad (4)$$

It is enough to consider $a = 1$ in this last equation, since using Hölder's inequality

$$|\operatorname{Tr} (\Delta_\phi a [(D^2 + 1)^{-s/2}, b])| \leq \|\Delta_\phi a \Delta_\phi^{-1}\| \operatorname{Tr} (|\Delta_\phi [(D^2 + 1)^{-s/2}, b]|).$$

We proceed similarly to [10, Theorem 10.20], but taking care of the presence of the modular operator Δ_ϕ . First of all we write $p = k\bar{r}$, with fixed $k \in \mathbb{N}$ and $0 < \bar{r} < 1$ (notice that if p is an integer we can set $k = 2p$ and $\bar{r} = 1/2$). With this convention we can write any s in a sufficiently small neighbourhood of p as $s = kr$, for some $0 < r < 1$.

Then, using simple commutator identities, we obtain

$$\begin{aligned} [(D^2 + 1)^{-s/2}, b] &= \sum_{j=1}^k (D^2 + 1)^{-(j-1)r/2} [(D^2 + 1)^{-r/2}, b] (D^2 + 1)^{-(k-j)r/2} \\ &= - \sum_{j=1}^k (D^2 + 1)^{-jr/2} [(D^2 + 1)^{r/2}, b] (D^2 + 1)^{-(k-j+1)r/2}. \end{aligned}$$

We introduce the notation

$$R_j = (D^2 + 1)^{-jr/2} [(D^2 + 1)^{r/2}, b] (D^2 + 1)^{-(k-j+1)r/2}.$$

Let p_j and q_j be numbers such that $p_j^{-1} + q_j^{-1} = 1$. Then we have

$$\begin{aligned} \Delta_\phi R_j &= \Delta_\phi^{p_j^{-1}} \Delta_\phi^{q_j^{-1}} (D^2 + 1)^{-jr/2} \Delta_\phi^{-q_j^{-1}} \Delta_\phi^{q_j^{-1}} \\ &\quad \times [(D^2 + 1)^{r/2}, b] \Delta_\phi^{-q_j^{-1}} \Delta_\phi^{q_j^{-1}} (D^2 + 1)^{-(k-j+1)r/2}. \end{aligned}$$

Since we assumed that D and Δ_ϕ commute, this can be rewritten as

$$\Delta_\phi R_j = \Delta_\phi^{p_j^{-1}} (D^2 + 1)^{-jr/2} [(D^2 + 1)^{r/2}, \Delta_\phi^{q_j^{-1}} b \Delta_\phi^{-q_j^{-1}}] \Delta_\phi^{q_j^{-1}} (D^2 + 1)^{-(k-j+1)r/2}.$$

Now from Hölder's inequality it follows that

$$\operatorname{Tr} (|\Delta_\phi R_j|) \leq C_j \operatorname{Tr} (\Delta_\phi (D^2 + 1)^{-jp_j r/2})^{p_j^{-1}} \operatorname{Tr} (\Delta_\phi (D^2 + 1)^{-(k-j+1)q_j r/2})^{q_j^{-1}},$$

where $C_j = \|[(D^2 + 1)^{r/2}, \Delta_\phi^{q_j^{-1}} b \Delta_\phi^{-q_j^{-1}}]\|$. It follows from general arguments, which use the boundedness of $[D, a]$ for every $a \in \mathcal{A}$, that this quantity is finite, see [10, Lemma 10.17].

Now we want to choose p_j and q_j in such a way that the operators $\Delta_\phi (D^2 + 1)^{-jp_j r/2}$ and $\Delta_\phi (D^2 + 1)^{-(k-j+1)q_j r/2}$ are trace-class, which in turn would show that $\Delta_\phi R_j$ is trace-class. Since by assumption we have that $\Delta_\phi (D^2 + 1)^{-s/2}$ is trace-class for all $s > p$, this implies the inequalities $jp_j r > p$ and $(k - j + 1)q_j r > p$. Let us set

$$p_j = \frac{s}{r(j - 1/2)}, \quad q_j = \frac{s}{r(k - j + 1/2)},$$

and notice that they satisfy the equality $p_j^{-1} + q_j^{-1} = 1$, as they should. Then it is immediate to see that the inequalities $jp_jr > p$ and $(k - j + 1)q_jr > p$ are satisfied for $s \geq p$ (notice the equality sign). Therefore we have proven that $\Delta_\phi R_j$ is trace-class and, since

$$\Delta_\phi [(D^2 + 1)^{-s/2}, b] = - \sum_{j=1}^k \Delta_\phi R_j,$$

the same is true for this operator. This shows that the limit in (4) is equal to zero and proves the equality claimed at the beginning.

The rest of the proof is now trivial. Using the trace property we get

$$\begin{aligned} \psi(ab) &= \lim_{s \rightarrow p^+} (s - p) \operatorname{Tr} (\Delta_\phi a (D^2 + 1)^{-s/2} b) = \lim_{s \rightarrow p^+} (s - p) \operatorname{Tr} (b \Delta_\phi a (D^2 + 1)^{-s/2}) \\ &= \lim_{s \rightarrow p^+} (s - p) \operatorname{Tr} (\Delta_\phi \Delta_\phi^{-1} b \Delta_\phi a (D^2 + 1)^{-s/2}). \end{aligned}$$

But this shows that $\psi(ab) = \psi(\Delta_\phi^{-1} b \Delta_\phi a)$, which concludes the proof. ■

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