

Twistor Topology of the Fermat Cubic^{*}

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Abstract. We describe topologically the discriminant locus of a smooth cubic surface in the complex projective space $\mathbb{C}\mathbb{P}^3$ that contains 5 fibres of the projection $\mathbb{C}\mathbb{P}^3 \rightarrow S^4$.

Key words: discriminant locus; Fermat cubic; twistor fibration

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1 Introduction

Given an algebraic surface Σ of degree d in $\mathbb{C}\mathbb{P}^3$, one can ask how it is configured relative to the twistor fibration $\pi : \mathbb{C}\mathbb{P}^3 \rightarrow S^4$. In particular, what can be said about the topology of the restricted map $\pi|_{\Sigma}$?

The motivation for studying this map is two fold.

1. By the twistor construction, complex hypersurfaces in $\mathbb{C}\mathbb{P}^3$ transverse to the fibres of π correspond to orthogonal complex structures on $S^4 = \mathbb{R}^4 \cup \{\infty\}$. Thus, by studying the interaction of Σ and π , we hope to better understand orthogonal complex structures on \mathbb{R}^4 .
2. The orientation-preserving conformal symmetries of S^4 induce fibre preserving biholomorphisms of $\mathbb{C}\mathbb{P}^3$. Thus one can attempt to classify algebraic surfaces in $\mathbb{C}\mathbb{P}^3$ up to equivalence under the conformal group of S^4 rather than the full group of biholomorphisms of $\mathbb{C}\mathbb{P}^3$. Topological invariants of $\pi|_{\Sigma}$ will automatically give invariants of Σ up to conformal transformation.

The first observation that one makes about $\pi|_{\Sigma}$ is that it gives a d -fold branched cover of S^4 . To see this note that each twistor fibre above a point x is a projective line in $\mathbb{C}\mathbb{P}^3$. When the defining equation for Σ is restricted to such a line and written out in homogenous coordinates it either vanishes or is a homogenous polynomial \tilde{p}_x in two variables of degree d . Equivalently we can write it in inhomogeneous coordinates to get a polynomial p_x of degree $\leq d$. Away from points where \tilde{p}_x vanishes, we have a d -branched cover by the fundamental theorem of algebra. This motivates the following definitions:

Definition 1.1. A *twistor line* of Σ is a fibre of π which lies entirely within the surface Σ .

Definition 1.2. The *discriminant locus* of Σ is the branch locus of the map $\pi|_{\Sigma}$. It is given by the points $x \in S^4$ where either the discriminant of p_x vanishes or the degree of p_x is less than $d - 1$.

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Thus, given a surface Σ one can ask how many twistor lines it has, and what is the topology of its discriminant locus. These questions were considered in [6] in the case of smooth degree 2 surfaces. It was shown that:

- The number of twistor lines on a smooth degree 2 surface is either 0, 1, 2 or ∞ .
- The topology of the discriminant locus for a degree 2 surface is determined by the number of twistor lines. It is either a torus (0 twistor lines), a torus pinched at one point (1 twistor line), a torus pinched at two points (2 twistor lines) or a circle (∞ twistor lines).

Furthermore, this rough classification of degree 2 surfaces by number of twistor lines was elaborated into a complete classification of degree 2 surfaces up to conformal equivalence extending work of Pontecorvo [4].

In this paper we shall consider the case of cubic surfaces and shall focus in particular upon the Fermat cubic

$$z_1^3 + z_2^3 + z_3^3 + z_4^3 = 0.$$

Our aim is to compute the topology of the discriminant locus in this particular case.

Note that, as it stands, this problem is not well defined. The discriminant locus does not depend solely upon the intrinsic geometry of the cubic surface, but also upon the embedding of the surface in $\mathbb{C}\mathbb{P}^3$. Equivalently, it depends upon the choice of twistor fibration. Fixing the twistor fibration, we will choose an embedding for which the twistor geometry is, in some sense, as simple as possible. We choose to measure the “simplicity” of an embedding of a surface Σ first by the number of twistor lines on Σ and then by the size of the group of symmetries preserving Σ and the twistor fibration. In other words, we wish to find the embedding of the Fermat cubic with the most twistor lines and the largest symmetry group, and then compute the topology of the discriminant locus in this case.

The remainder of this paper is structured as follows. In Section 2 we review the twistor fibration and the calculation of the discriminant locus. In Section 3 we will identify the “simplest” embedding of the Fermat cubic relative to the twistor fibration. We call the resulting embedded cubic surface the “transformed Fermat cubic”. Section 4 contains the main result of the paper: a computation of the topology of the discriminant locus of the transformed Fermat cubic. This gives the first known computation of the discriminant locus in degree $d > 2$.

This paper is to some extent a sequel to [1] and answers a question raised in [5]. For background on the twistor fibration we refer readers to [2].

2 Computing the discriminant locus

Let us describe in more detail how one can compute the discriminant locus. We begin by reviewing an explicit formulation for the twistor projection $\pi : \mathbb{C}\mathbb{P}^4 \rightarrow S^4$.

Define two equivalence relations on $(\mathbb{H} \times \mathbb{H}) \setminus \{0\}$, denoted $\sim_{\mathbb{C}}$ and $\sim_{\mathbb{H}}$, by

$$\begin{aligned} (q_1, q_2) &\sim_{\mathbb{H}} (\lambda q_1, \lambda q_2), & \lambda &\in \mathbb{H} \setminus \{0\}, \\ (q_1, q_2) &\sim_{\mathbb{C}} (\lambda q_1, \lambda q_2), & \lambda &\in \mathbb{C} \setminus \{0\}. \end{aligned}$$

We have

$$((\mathbb{H} \times \mathbb{H}) \setminus \{0\}) / \sim_{\mathbb{C}} \cong \mathbb{C}\mathbb{P}^3$$

and

$$((\mathbb{H} \times \mathbb{H}) \setminus \{0\}) / \sim_{\mathbb{H}} \cong \mathbb{H}P^1 \cong \mathbb{R}^4 \cup \{\infty\} \cong S^4.$$

An explicit map from $\mathbb{C}\mathbb{P}^3$ to $(\mathbb{H} \times \mathbb{H}) \setminus \{0\} / \sim_{\mathbb{C}}$ is given by

$$[z_1, z_2, z_3, z_4] \rightarrow [z_1 + z_2j, z_3 + z_4j]_{\sim_{\mathbb{C}}}.$$

The map $\pi : [q_1, q_2]_{\sim_{\mathbb{C}}} \rightarrow [q_1, q_2]_{\sim_{\mathbb{H}}}$ is the twistor fibration.

Left multiplication by the quaternion j induces an antiholomorphic map $\tau : \mathbb{C}\mathbb{P}^3 \rightarrow \mathbb{C}\mathbb{P}^3$. Writing this explicitly we have $\tau[q_1, q_2]_{\sim_{\mathbb{C}}} = [jq_1, jq_2]_{\sim_{\mathbb{C}}}$. The map τ acts on each fibre of π as the antipodal map. τ is a real structure, in other words it is antiholomorphic and satisfies $\tau^2 = -1$. Since there is a unique line through any two points of $\mathbb{C}\mathbb{P}^3$ one can recover the fibration from τ alone: it consists of all projective lines joining a point x to the point τx .

Let us now work out how to compute an explicit expression for the discriminant locus.

Let Σ be a complex surface in $\mathbb{C}\mathbb{P}^3$ defined by a homogeneous polynomial $f_{\Sigma}(z_1, z_2, z_3, z_4)$ of degree d . Define $p_1 : \mathbb{R}^4 \rightarrow \mathbb{H} \times \mathbb{H}$ by $p_1(x_1, x_2, x_3, x_4) = [x_1 + x_2i + x_3j + x_4k, 1]$. Define $p_2 = \tau \circ p_1$. Given a point $x \in \mathbb{R}^4$, define $\theta_x : \mathbb{C} \rightarrow \mathbb{H} \times \mathbb{H} \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}$ by $\theta_x(\lambda) = \lambda p_1(x) + p_2(x)$. So $\lambda \rightarrow [\theta_x(\lambda)]_{\sim_{\mathbb{C}}}$ gives inhomogenous coordinates for the fibre above x . We see that $f_{\Sigma}(\theta_x(\lambda))$ is a polynomial f_x in λ with coefficients given by polynomials in the x_i . The discriminant locus is given by the set of points where the degree of f_x is less than $d - 1$ or the discriminant of f_x is equal to 0.

Computing this discriminant explicitly, one obtains a complex-valued polynomial in the coordinates (x_1, x_2, x_3, x_4) . Since the algebraic expression for the discriminant of a degree d polynomial always vanishes when evaluated on a polynomial of degree $\leq d - 2$, the discriminant locus is given by the zero set of this polynomial. Taking real and imaginary parts, we obtain an explicit expression for the discriminant locus as the zero set of two real valued polynomials.

If one runs through this procedure for a generic cubic surface, the polynomials defining the discriminant locus will be degree 12 polynomials in 4 variables. Since the general degree 12 polynomial in 4 variables has 495 coefficients one begins to appreciate the difficulty in analysing the discriminant locus of a general cubic surface. In the next section we will identify the “simplest” possible non-singular cubic surface in $\mathbb{C}\mathbb{P}^3$ relative to the twistor fibration.

3 Twistor lines on cubic surfaces

A beautiful result of classical algebraic geometry due to Salmon states that any smooth cubic surface contains exactly 27 projective lines [3]. When studying the twistor geometry of cubic surfaces it is natural to ask: how many of these lines might be twistor lines?

The answer to this question can be found by studying the intersection properties of the lines. The starting point is the classical result, due to Schläfli [7] which states that the intersection properties of the 27 lines on a cubic are always the same. He showed that one can always label six of the lines a_i (with $1 \leq i \leq 6$), another six b_i (with $1 \leq i \leq 6$) and the remainder c_{ij} (with $1 \leq i < j \leq 6$) in such a way that:

- the 6 a_i lines are all disjoint,
- the 6 b_i lines are all disjoint,
- a_i intersects b_j if and only if $i \neq j$,
- a_i intersects c_{jk} if and only if $i \in \{j, k\}$,
- b_i intersects c_{jk} if and only if $i \in \{j, k\}$,
- c_{ij} intersects c_{kl} if and only if $\{i, j\} \cup \{j, k\} = \emptyset$.

This labelling of the 27 lines on a given cubic is far from unique.

It follows immediately that the maximum number of disjoint lines on a cubic surface is 6. Since twistor lines are necessarily disjoint fibres of π one sees immediately that there can be

no more than 6 twistor lines on a cubic surface. A more careful analysis allows one to improve upon this.

Theorem 3.1 ([1]). *A smooth cubic surface contains at most 5 twistor lines. If a smooth cubic surface contains 5 twistor lines, then the image of these lines under π must all lie on either a round 2-sphere or a plane in S^4 . Note that it follows from Theorem 3.2 below that such cubic surfaces do indeed exist.*

Proof. Suppose that Σ is a cubic surface containing 4 twistor lines. Since the lines are skew, it is not difficult to show that one can choose a labelling of the 27 lines such that these lines are a_1, a_2, a_3, a_4 .

The line b_5 intersects a_1, a_2, a_3 , and a_4 . Since the twistor lines are invariant under the antiholomorphic involution τ , the line $\tau(b_5)$ also intersects a_1, a_2, a_3 and a_4 . Thus $\tau(b_5)$ is a projective line that intersects Σ in 4 points. Thus $\tau(b_5)$ is one of the 27 lines.

Suppose for a contradiction that b_5 intersects $\tau(b_5)$ at a point p . Then b_5 also intersects $\tau(b_5)$ at $\tau(p) \neq p$. There is a unique line through each point so $b_5 = \tau(b_5)$. Hence b_5 is a fibre of π and so disjoint from a_1 . This is the desired contradiction.

We deduce that $\tau(b_5)$ intersects a_1, a_2, a_3 and a_4 and is disjoint from b_5 . By the intersection properties of the cubic, $\tau(b_5) = b_6$.

Suppose that we have a fifth twistor line L on Σ . We see that $L \in \{a_5, a_6, c_{56}\}$. So L intersects either b_5 or b_6 . So $L = \tau(L)$ also intersects $\tau(b_5) = b_6$ or $\tau(b_6) = b_5$. Thus L intersects b_5 and b_6 . Thus $L = c_{56}$.

Since there is only one possibility for the fifth twistor line, it is impossible to find a cubic with six twistor lines.

The lines b_5 and $b_6 = \tau(b_5)$ pass through all five twistor lines. So π maps the twistor lines onto the image $\pi(b_5) = \pi(b_6)$. The image of a projective line under π is always either a 2-sphere, two plane or point in S^4 . ■

Schläfli used his result on the intersection properties of the 27 lines as the basis of a classification of cubic surfaces up to projective transformation. In particular he was able to show that given two skew lines b_5 and b_6 , a generic list of five points $\{p_i\}$ on b_5 and a generic list of five points $\{q_i\}$ on b_6 , one can find a non-singular cubic surface containing b_5 and b_6 and the five lines connecting p_i to q_i . Moreover this cubic is unique up to projective transformation.

So given two lines with $b_5 = \tau(b_6)$ and 5 points p_i on b_5 , one expects to find a unique cubic surface up to projective transformation containing the lines b_5, b_6 and the lines joining p_i to $\tau(p_i)$. The only issue is that it is not clear that the points p_i and $\tau(p_i)$ will be sufficiently generic. Resolving this question requires a more detailed study of Schläfli's results, as was done in [1]. We summarize the main result.

Theorem 3.2. *Let b_5 be a line in $\mathbb{C}\mathbb{P}^3$ which is not a twistor fibre. Let $\{p_1, p_2, p_3, p_4, p_5\}$ be five points on b_5 . We can choose coordinates on b_5 to identify it with the Riemann sphere.*

There exists a smooth cubic surface containing $b_5, \tau(b_6)$ and the lines joining p_i and $\tau(p_i)$ if and only if no four of the points p_i lie on a circle under this identification.

The data $(b_5, p_1, p_2, p_3, p_4, p_5)$ in the above theorem uniquely determines the cubic surface up to projective transformation but not up to conformal transformation. It is shown in [1] that there is a one parameter family of conformally inequivalent cubic surfaces containing the given lines. This gives a classification of cubic surfaces with five twistor lines.

So we can now return to the question of determining the “simplest” non-singular cubic surface from the point of view of twistor geometry. According to the criteria for simplicity given in the introduction, we require the surface to have five twistor lines and to have as many conformal symmetries as possible.

Thus we will want to choose five points on the line $b_5 \cong \mathbb{C} \cup \{\infty\}$ to have as many conformal symmetries as possible while insisting that no 4 points lie on a circle in the Riemann sphere. Choosing inhomogeneous coordinates for b_5 , one sees that the most symmetrical such arrangement is given by the points $0, 1, \omega, \omega^2, \infty$ where ω is a complex cube root of unity. All equally symmetrical arrangements of five points on the sphere are conformally equivalent. The conformal symmetries of these 5 points are given by the symmetric group S_3 : each element of S_3 permutes the cube roots of unity; odd permutations swap the points 0 and ∞ .

Up to conformal transformation all choices for the line b_5 are equivalent. Thus we have narrowed down the search for the non-singular cubic surface with the simplest geometry to a search among a 1-parameter family of conformally inequivalent surfaces all of which have a symmetry group of order at least 6. One further symmetry which preserves the twistor fibration is possible: one may have a symmetry which swaps b_5 to $\tau(b_6)$ but which leaves the twistor lines fixed. This allows one to identify the most symmetrical smooth cubic surface with five twistor lines. It is unique up to conformal transformation. An explicit formula for this cubic was found in [1]. It is

$$z_1 z_4^2 + z_4 z_1^2 + z_2 z_3^2 + z_3 z_2^2 = 0. \quad (1)$$

The simplicity of this formula may lead one to suspect that this is projectively equivalent to a more familiar cubic surface. In fact it is projectively equivalent to the Fermat cubic. The equivalence is given by the following change of coordinates:

$$z'_1 = ax_2 + bx_3, \quad z'_2 = cx_2 - bx_3, \quad z'_3 = ax_1 + bx_4, \quad z'_4 = cx_2 - bx_3,$$

where the constants a, b and c are given by $a = \frac{1}{2} + \frac{\sqrt{3}}{6}i$, $b = \bar{a}$ and $c = \frac{i}{\sqrt{3}}$.

We call the embedded cubic surface defined by equation (1) the *transformed Fermat cubic* since it is equivalent to the Fermat cubic via a projective transformation. Note that this projective transformation is not a conformal equivalence. Since the transformed Fermat cubic has 5 twistor lines and the Fermat cubic only has 3 these two cubic surfaces are not conformally equivalent.

4 The discriminant locus of the transformed Fermat cubic

Now that we have identified an appropriately simple cubic surface, we will now compute the topology of its discriminant locus.

The discriminant locus describes a singular surface in $S^4 \cong \mathbb{R}^4 \cup \{\infty\}$. Our strategy to determine its topology is to view it as a curve in \mathbb{R}^3 that changes over time. We can plot an animation of this curve in Mathematica. By studying this animation, we hope to deduce the homeomorphism class. Carefully choosing coordinates for the time axis and for viewing \mathbb{R}^3 is essential if one wants to generate an animation that is easily understood.

Let G be the group of conformal symmetries of the transformed Fermat cubic. Viewed in terms of its action on $S_4 \cong \mathbb{R}^4 \cup \{\infty\}$, G is the group of order 12 generated by the transformations:

- 1) the rotation θ given by

$$(x_1, x_2, x_3, x_4) \mapsto \left(\cos\left(\frac{2\pi}{3}\right)x_1 + \sin\left(\frac{2\pi}{3}\right)x_2, -\sin\left(\frac{2\pi}{3}\right)x_1 + \cos\left(\frac{2\pi}{3}\right)x_2, x_3, x_4 \right),$$

- 2) the map σ given by $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, -x_3, -x_4)$,

- 3) the inversion ι given by $(x_1, x_2, x_3, x_4) \mapsto \frac{(x_1, -x_2, -x_3, -x_4)}{x_1^2 + x_2^2 + x_3^2 + x_4^2}$.

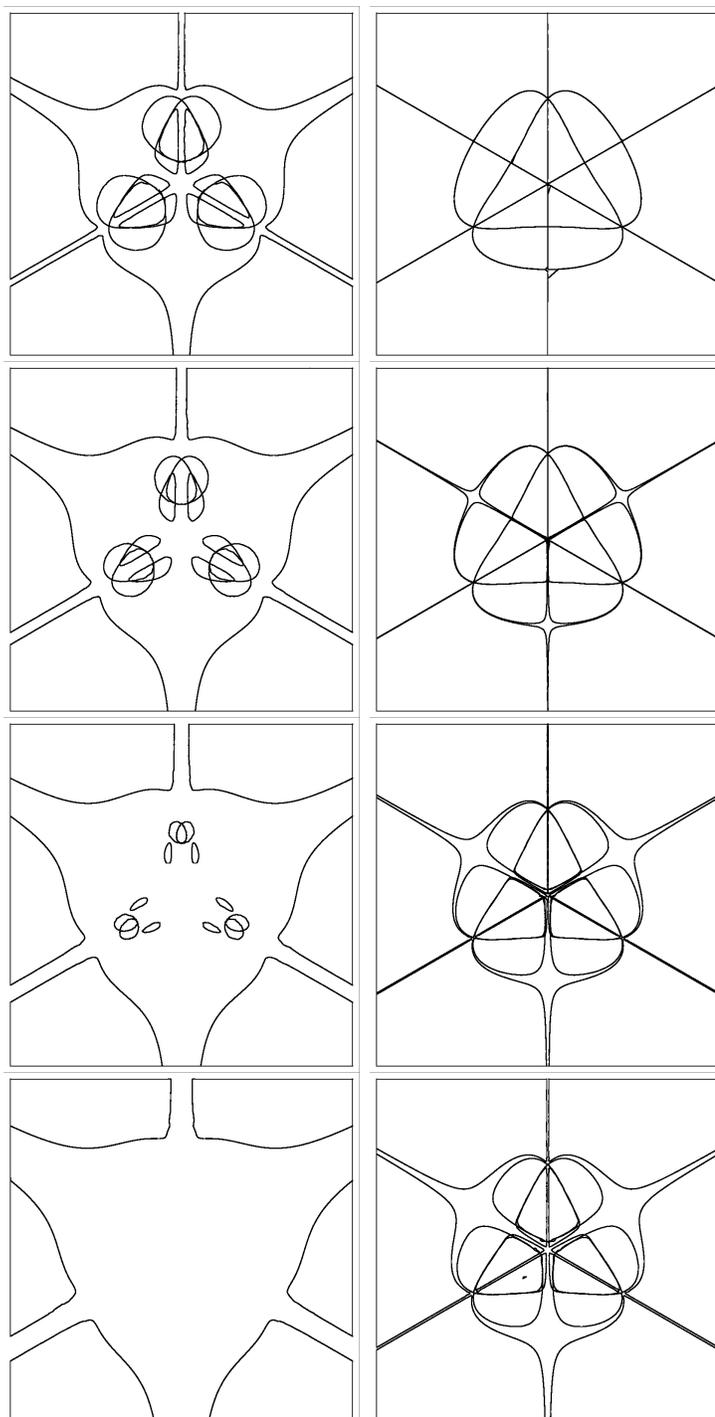


Figure 1. Animation of the discriminant locus from time $x_4 = 0$ (top left) to $x_4 = 0.1$ (bottom right).

These facts suggest that we use the coordinate x_4 as our time coordinate so that the symmetry σ becomes a time and parity reversal symmetry. They also suggest a perspective from which to view the animation: we should look down the x_3 axis onto the (x_1, x_2) plane.

Fig. 1 shows frames of an animation of the discriminant locus from precisely this viewpoint. It shows the evolution from time $x_4 = 0$ over positive times. We have omitted the behaviour over negative times since it can be deduced from the symmetry σ .

Fig. 1 gives a rather stylised view of the discriminant locus. One needs to look at the animation from a number of angles to understand the situation completely. However, the main points are already reasonably clear. At time 0, the discriminant locus has a relatively simple shape consisting of a number of vertices and curved edges. At subsequent times, the discriminant locus consists of a collection of disjoint loops and open curves that leave the edge of our picture. The loops do not appear to be disjoint in Fig. 1 but that is simply because they overlap when viewed from this angle.

The loops all shrink to a point and then disappear (as shown in the fifth and sixth frames). It is less clear from the picture what will happen to the open curves. However, we know that the discriminant locus is invariant under the inversion symmetry. After applying this symmetry, the open curves correspond to disjoint loops.

A loop that shrinks to a point describes a surface isomorphic to \mathbb{R}^2 in spacetime. Using the inversion symmetry, we see that the open curves also have worldsheets isomorphic to \mathbb{R}^2 . Thus from a topological perspective, we can think of our animation as providing a triangulation of the discriminant locus. The vertices and edges of our triangulation are given by the picture at time 0. The faces are given by the worldsheets of the open curves and loops.

As we have already remarked, our picture is something of an oversimplification. Several curves are overlapping at time 0. We will need to view the discriminant locus from another angle to fully understand what it looks like at time 0. In Figs. 2 and 3 we have given two perspectives of the discriminant locus at time $x_4 = 0$: a view from “above” showing the (x_1, x_2) plane rotated so that the x_1 axis is vertical; a view from “the side” looking at the (x_2, x_3) plane along the x_1 axis. The pair of eyes in Fig. 2 indicates the perspective taken in Fig. 3.

One can identify how the vertices in edges in the view from above correspond to the vertices and edges in the view from the side. In Figs. 2 and 3 we have labelled all the vertices and edges to show the correspondence. There is an obvious action of the group S_3 on the view from above which we have incorporated into our labelling. First we have labelled each 60 degree sector in the view from above with a number from 0 to 5, we have then labelled each element of the view from above with a letter that indicates its orbit under the S_3 followed by a numeric subscript indicating which sector it lies in. Edges are labelled in lower case, vertices in upper case. An element such as the edge g_2 in the view from above corresponds to a number of elements in the view from the side at different heights. Corresponding to an element in the view from above, we label the elements in the view from the side with a number in brackets to indicate the height. To reduce clutter, in Fig. 3 we have only labelled edges in the upper half of the figure and vertices in the lower half of the figure: the labels in the other half can be deduced by reflecting the picture and changing the signs in brackets. For example, the edge g_2 in the view from above corresponds to six edges $g_2(3)$, $g_2(2)$, $g_2(1)$, $g_2(-1)$, $g_2(-2)$ and $g_2(-3)$ in the view from the side. We have also omitted the number in brackets for elements such as $e_2(1)$ where only the two numbers (1) and (-1) would be needed.

One vertex is missing from our picture, a vertex at infinity. We label this $Z(\infty)$. On a few occasions below we use the notation $\mathcal{E}_m(n)$ for a generic edge; in such a formula m and n are integers and \mathcal{E} is a symbol like a or b .

We have now identified the vertices and edges of our triangulation and we have identified the boundary of each edge. We now need to identify the faces and the boundary of each face. To do this we examine the view from the side at time $x_4 = 0.02$. This is shown in Fig. 4. The

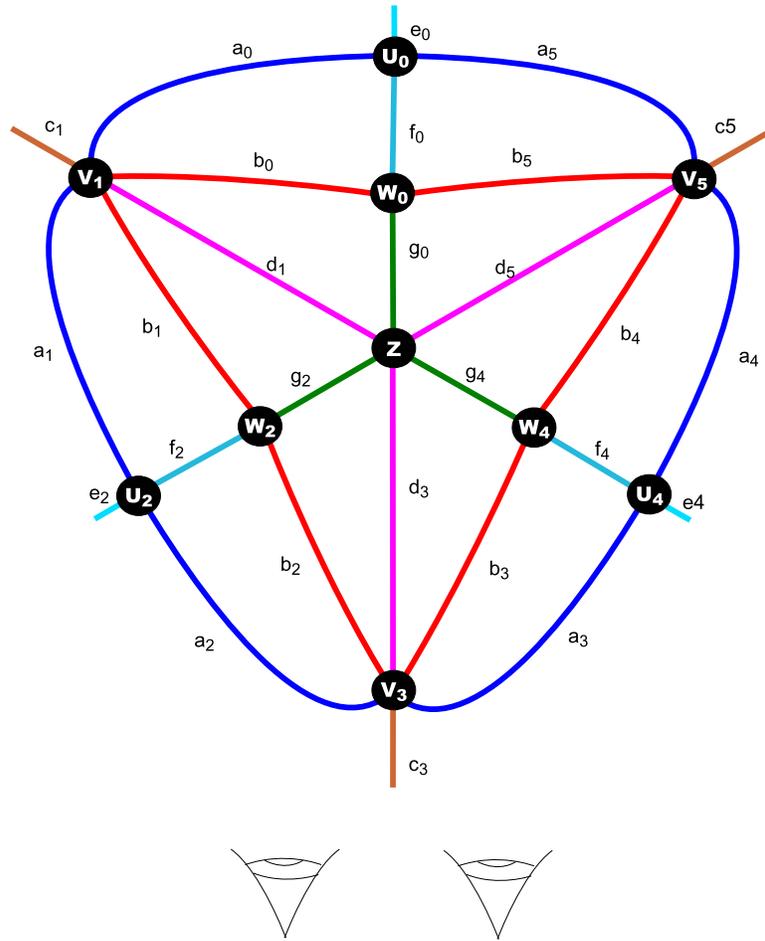


Figure 2. The discriminant locus seen from “above”.

value 0.02 has been chosen because at this time the curves which correspond to the faces in our triangulation are very near to the edges of our triangulation. This makes it easy to read off the boundary corresponding to each face.

Each curve in Fig. 4 corresponds to a face in our triangulation. We have labelled three of them as follows:

- R.* The red curve. The corresponding face in our triangulation has boundary $\{g_2(-1), b_1(-1), d_1(1)\}$.
- G.* The green curve. The corresponding face in our triangulation has boundary $\{e_2(1), a_1(1), c_1(-1)\}$.
- B.* The blue curve. The corresponding face in our triangulation has boundary $\{f_4(1), g_4(3), g_0(2), b_5(1), -a_4(1)\}$.

We can use the symmetries θ , σ and ι to write any face in our triangulation as $\theta^i \sigma^j \iota^k F$ where $F \in \{R, G, B\}$. We know that θ^i sends an edge or vertex $\mathcal{E}_m(n)$ to $\mathcal{E}_{(m+2i \bmod 6)}(n)$ and that σ sends $\mathcal{E}_m(n)$ to $\mathcal{E}_m(-n)$. Our notation does not give such a simple expression for the action of ι on edges and vertices. However, one can check that:

- ιR has boundary $\{e_4(1), a_4(1), c_5(-1)\}$,
- ιG has boundary $\{g_4(-1), b_4(-1), d_5(1)\}$,
- ιB has boundary $\{f_2(-1), g_2(-3), g_0(-2), b_0(-1), a_1(-1)\}$.

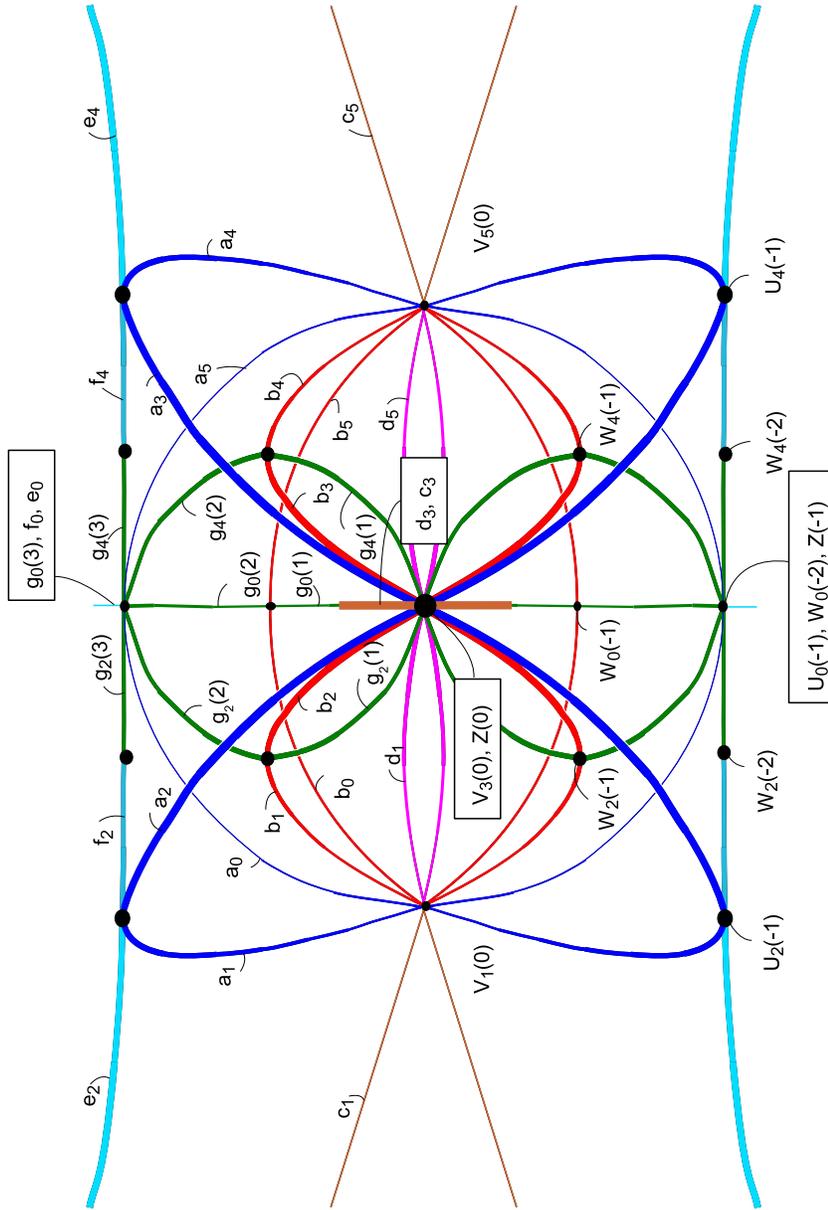


Figure 3. The discriminant locus seen from “the side”.

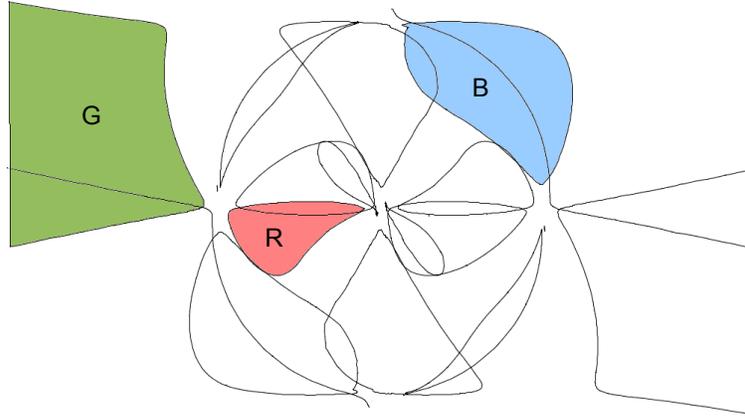


Figure 4. The view from the side at time $x_4 = 0.02$.

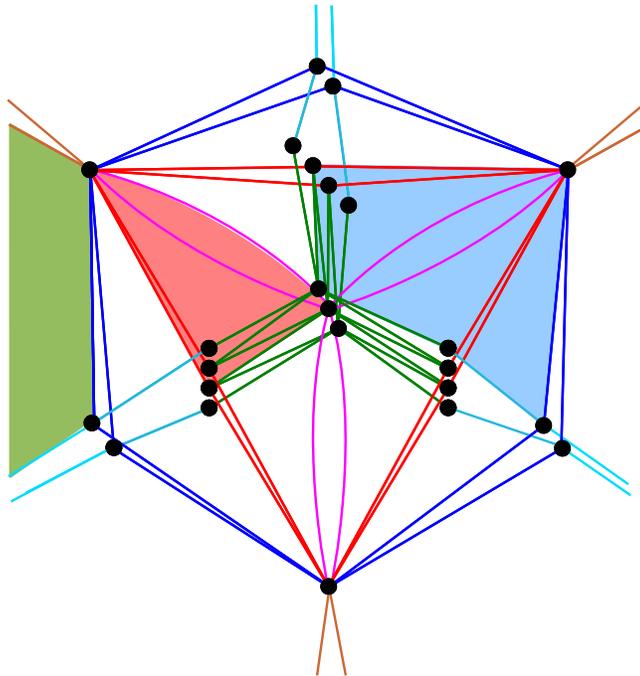


Figure 5. Schematic view of the triangulation of the discriminant locus.

So one can compute the boundary of the general face $\theta^i \sigma^j \iota^k$ from these formulae. Thus we have written down all the combinatorial information contained in our triangulation: the list of edges, vertices and faces and the boundary map ∂ . We have found 25 vertices, 66 edges and 36 faces.

The reader may well find this combinatorial data rather indigestible, but it is summarized schematically in Fig. 5. This shows the view from above perturbed slightly so that all vertices separate. One should think of this perturbation as corresponding to a movement along the x_3 axis out of the page. The shaded faces correspond to the R , G and B faces shown in the view from the side.

The identification of this triangulation depends upon a visual examination of the discriminant locus rather than a formal algebraic proof. If one is willing to accept the results of this visual examination we can now compute the topology of the discriminant locus.

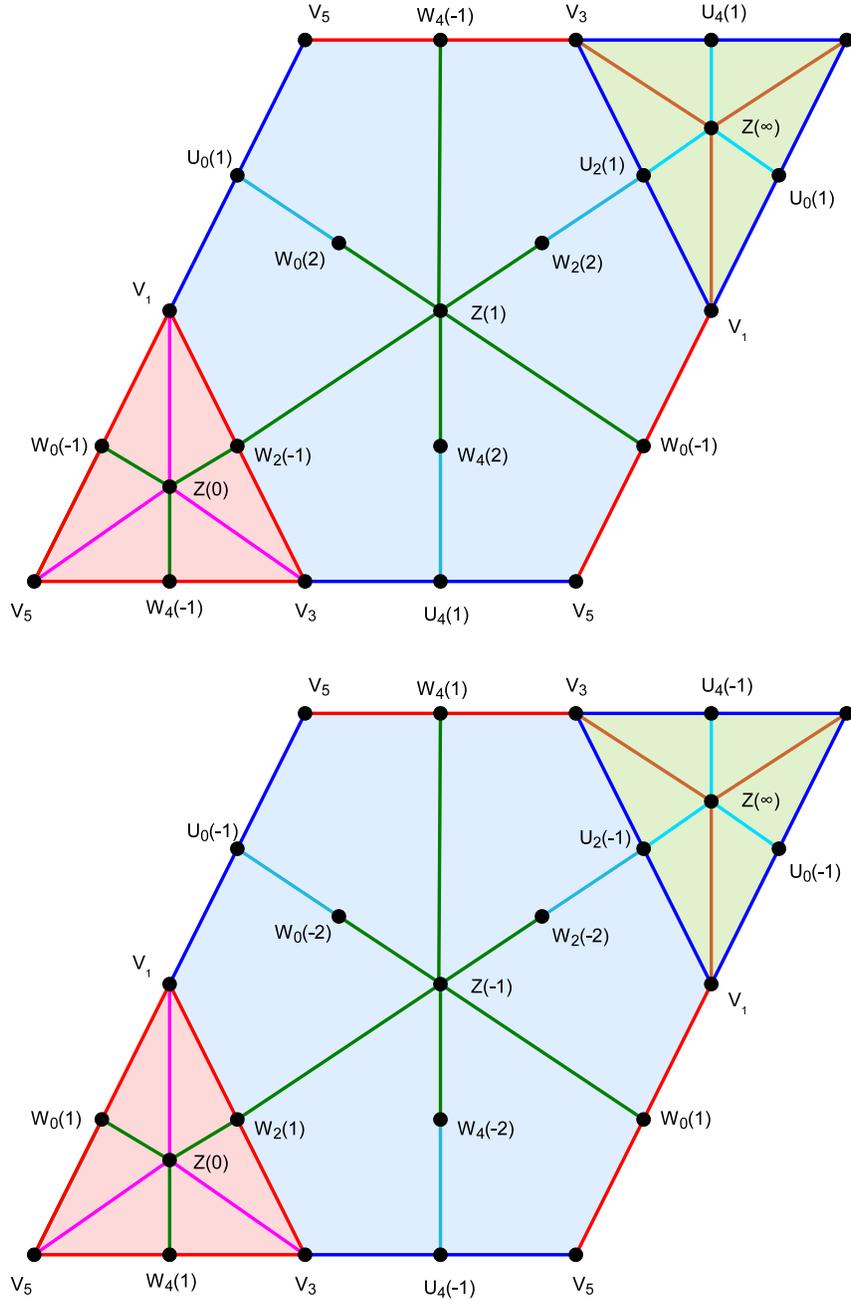


Figure 6. A triangulation of two tori with 5 pairs of points identified.

Theorem 4.1. *The discriminant locus of the transformed Fermat cubic $z_1z_4^2 + z_4z_1^2 + z_3z_2^2 + z_2z_3^2 = 0$ is homeomorphic to the space $(T_1 \sqcup T_2)/\sim$ where T_1 and T_2 are 2-dimensional tori and \sim is an equivalence relation identifying 5 distinct points on T_1 with another 5 distinct points on T_2 .*

Proof. Let v be an involution defined on the edges of our triangulation by:

$$v(\mathcal{E}_m(n)) = \begin{cases} \mathcal{E}_{(5-m \bmod 6)}(n), & \mathcal{E} \in \{a, b\}, \\ \mathcal{E}_{(-m \bmod 6)}(n), & \text{otherwise.} \end{cases}$$

In terms of Fig. 5 the map v corresponds to a reflection of the edges in the vertical, x_1 , axis.

Let ∂ denote the map sending an oriented surface to its boundary. Notice that $\partial(\iota R) = v\partial G$, $\partial(\iota G) = v\partial R$ and $\partial(\iota B) = \sigma v\partial B$. Thus the set of all boundaries of faces in our triangulation is generated by the action of θ , σ and v on the boundaries of R , G and B . This means that the triangulation of the discriminant locus is equivalent to another triangulation with edges also given by the schematic diagram 5 but with faces generated by the faces in 5 under the maps θ , σ and v . The point is that we have eliminated the rather awkward inversion symmetry ι and replaced it with the simple reflection v .

Using these symmetries, it is easy to check that our triangulation is isomorphic to that shown in Fig. 6 once opposite edges of the parallelograms have been identified to obtain tori and the vertices

$$Z(0), \quad Z(\infty), \quad V_1, \quad V_3, \quad V_5 \tag{2}$$

have been identified. The mappings of edges and faces under the isomorphism can be easily deduced.

Thus the discriminant locus is homeomorphic to two tori with five pairs of points identified. The five points (2) are the images of the five twistor lines in the cubic Σ . ■

The Mathematica code used to generate the figures in this article is available from [https://kclpure.kcl.ac.uk/portal/en/publications/twistor-topology-of-the-fermat-cubic--mathematica-code\(0f8ceb5f5-a9f1-41cb-8511-e79f54a1c819\).html](https://kclpure.kcl.ac.uk/portal/en/publications/twistor-topology-of-the-fermat-cubic--mathematica-code(0f8ceb5f5-a9f1-41cb-8511-e79f54a1c819).html).

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