

Schur Positivity and Kirillov–Reshetikhin Modules[★]

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Abstract. In this note, inspired by the proof of the Kirillov–Reshetikhin conjecture, we consider tensor products of Kirillov–Reshetikhin modules of a fixed node and various level. We fix a positive integer and attach to each of its partitions such a tensor product. We show that there exists an embedding of the tensor products, with respect to the classical structure, along with the reverse dominance relation on the set of partitions.

Key words: Kirillov–Reshetikhin modules; Q -systems; Schur positivity

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1 Introduction

This note is inspired by two results on certain modules of simple, finite-dimensional complex Lie algebras. The first one is an immediate consequence of the famous Clebsch–Gordan formula on decompositions of tensor product of simple $\mathfrak{sl}_2(\mathbb{C})$ -modules. Namely let $V(m) = \text{Sym}^m \mathbb{C}^2$, be the n -th symmetric power of the natural representation, then

$$V(n) \otimes V(m) \cong_{\mathfrak{sl}_2} V(n+m) \oplus V(n+m-2) \oplus \cdots \oplus V(n+m-2\min\{n,m\}).$$

Which implies that for $m_1, m_2 \leq m$ there exists a surjective map of \mathfrak{sl}_2 -modules

$$V(m_2) \otimes V(m-m_2) \twoheadrightarrow V(m_1) \otimes V(m-m_1)$$

if and only if $\min\{m_1, m-m_1\} \leq \min\{m_2, m-m_2\}$. Using this inequality, we obtain an order \preceq on partitions of n of length 2, $\mathcal{P}(m, 2)$. By taking the point of view from symmetric functions, s_m being the character of $V(m)$, we have

$$(m_1, m-m_1) \preceq (m_2, m-m_2) \Leftrightarrow s_{m_2} s_{m-m_2} - s_{m_1} s_{m-m_1} \in \sum_{k \geq 0} \mathbb{Z}_{\geq 0} s_k.$$

As the characters are also known as Schur functions, this property of the left hand side is also known as *Schur positivity*. A generalization of this order to $\mathfrak{sl}_n(\mathbb{C})$ and further to a simple finite-dimensional Lie algebra \mathfrak{g} of arbitrary type was investigated in [7] (resp. [3, 10]).

The other inspiration comes from certain character identities for classical limits of Kirillov–Reshetikhin modules for $U_q(\hat{\mathfrak{g}})$, the untwisted quantum affine algebra associated to \mathfrak{g} , namely the Q -systems. Kirillov–Reshetikhin modules $W_{m,a}^{(i)}$ are indexed by a node of the Dynkin diagram,

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say $i \in I$, a positive level m and a parameter $a \in \mathbb{C}^*$. For more details on Kirillov–Reshetikhin modules and their importance in the category of finite-dimensional $U_q(\hat{\mathfrak{g}})$ -modules we refer to [4].

We denote by $\text{KR}(m\omega_i)$ the \mathfrak{g} -module obtained through the limit $q \rightarrow 1$ of $W_{m,a}^{(i)}$, note that the classical structure is independent of a . Further denote $\text{char}_{\mathfrak{g}} \text{KR}(m\omega_i)$ the classical character of $\text{KR}(m\omega_i)$, then the Q -system is the following character identity [14, 18]

$$\text{char}_{\mathfrak{g}} \text{KR}(m\omega_i) \text{char}_{\mathfrak{g}} \text{KR}(m\omega_i) = \text{char}_{\mathfrak{g}} \text{KR}((m+1)\omega_i) \text{char}_{\mathfrak{g}} \text{KR}((m-1)\omega_i) + \text{char}_{\mathfrak{g}} S_m^{(i)},$$

where $S_m^{(i)}$ denotes the classical limit of a tensor product of certain Kirillov–Reshetikhin modules (depending on i and m).

From the Q -system one has the immediate consequence, that there exists a surjective map of \mathfrak{g} -modules

$$\text{KR}(m\omega_i) \otimes \text{KR}(m\omega_i) \twoheadrightarrow \text{KR}((m+1)\omega_i) \otimes \text{KR}((m-1)\omega_i).$$

Considering the partial order on partitions, we have $(m-1, m+1) \prec (m, m) \in \mathcal{P}(2m, 2)$. So for the maximal element and its predecessor this might be seen as a generalization of Schur positivity (to Lie algebras of arbitrary type). Note that in [6] it is proved that there exists a surjective map of modules for the current algebra of \mathfrak{g} , namely of fusion products of modules of Kirillov–Reshetikhin modules for the partitions $(m-1, m+1)$, (m, m) . Their work is also motivated by the Q -system.

Combining the partial order on $\mathcal{P}(m, 2)$ and this consequence of the Q -system relation was the starting point of this paper. We have generalized the arguments in the proof of the Kirillov–Reshetikhin conjecture, e.g. the character of the Kirillov–Reshetikhin modules satisfy the Q -system. Using this, we have proved that for all \mathfrak{g} of arbitrary type and $(m_1, m-m_1) \preceq (m_2, m-m_2) \in \mathcal{P}(m, 2)$ there exists a surjective map of \mathfrak{g} -modules

$$\text{KR}(m_2\omega_i) \otimes \text{KR}((m-m_2)\omega_i) \twoheadrightarrow \text{KR}(m_1\omega_i) \otimes \text{KR}((m-m_1)\omega_i).$$

This might be seen as a generalization of the \mathfrak{sl}_2 -case as well as of the consequence of the Q -system property.

More generally, we associate to each partition $(m_1 \geq m_2 \geq \dots \geq m_k > 0)$ of m a tensor product of Kirillov–Reshetikhin modules

$$\text{KR}(m_1\omega_i) \otimes \dots \otimes \text{KR}(m_k\omega_i).$$

By considering the reverse dominance relation on partitions of m , $\mathcal{P}(m)$, we can show further (using that the cover relation is induced by the cover relations on the set of partitions of length 2) that if $(m_1 \geq \dots \geq m_{k_1} > 0) \preceq (n_1 \geq \dots \geq n_{k_2} > 0)$, then there exists a surjective map of \mathfrak{g} -modules

$$\text{KR}(n_1\omega_i) \otimes \dots \otimes \text{KR}(n_{k_2}\omega_i) \twoheadrightarrow \text{KR}(m_1\omega_i) \otimes \dots \otimes \text{KR}(m_{k_1}\omega_i).$$

In the \mathfrak{sl}_n -case, the last statement can be deduced from the results on row shuffles in [16]. Even more general, in the case where ω_i is minuscule, e.g. $\text{KR}(m\omega_i)$ is a simple \mathfrak{g} -module, this last statement has been proved in [3]. The authors were constructing an explicit bijection of the highest weight vectors in terms of LS-paths. Our approach avoids these combinatorics.

Since $\text{KR}(m\omega_i)$ can be also constructed as a module for $\mathfrak{g} \otimes \mathbb{C}[t]$ (by using some “evaluation parameter” $a \in \mathbb{C}$), see Section 2.5, one might ask if there is a surjection also as $\mathfrak{g} \otimes \mathbb{C}[t]$ -modules. The natural object to be considered here is the fusion product introduced in [8]. This is the associated graded module (with respect to the degree filtration on $U(\mathfrak{g} \otimes \mathbb{C}[t])$) of the tensor product of the Kirillov–Reshetikhin modules with pairwise distinct evaluations. Can the

surjection in Theorem 4.1 be actually obtained from a surjective map of the corresponding fusion products?

We should remark here that a similar result on Schur positivity on tensor products of simple \mathfrak{g} -modules of arbitrary highest weight λ was conjectured in [7] and [3] (see Section 3.5 for more details). Our result suggests, that this generalized Schur positivity may hold along the partial order on tensor products of $q \mapsto 1$ limits of minimal affinizations of $V(\lambda)$ (the “minimal” module of the quantum affine algebra having a simple quotient whose limit is isomorphic to $V(\lambda)$, see [5] for more details).

In the \mathfrak{sl}_n -case, the restriction to \mathfrak{sl}_n of the limit of such a minimal affinization is nothing but the simple \mathfrak{sl}_n -module $V(\lambda)$, so this is the conjecture of Schur positivity by Lam, Postnikov and Pylyavskyy, cited in [7]. For other types, the limit of a minimal affinization is not a simple \mathfrak{g} -module in general, for example Kirillov–Reshetikhin modules are minimal affinizations of $V(m\omega_i)$. It might be interesting to investigate on minimal affinizations of other than rectangular weights.

In Section 2 we briefly recall the reverse dominance relation on partitions and some basics on Kirillov–Reshetikhin modules. In Section 3 we state the main theorem, while the proof follows in Section 4.

2 Preliminaries

Let \mathfrak{g} be a finite-dimensional simple, complex Lie algebra of rank n and Cartan matrix C . Let $\mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be a triangular decomposition. We denote the set of (positive) roots R (R^+ resp), the (dominant) integral weights P (resp. P^+). We denote the simple roots $\{\alpha_1, \dots, \alpha_n\}$, the fundamental weight $\{\omega_1, \dots, \omega_n\}$, $I = \{1, \dots, n\}$. For every $\alpha \in R^+$ we fix a \mathfrak{sl}_2 -triple $\{e_\alpha, h_\alpha, f_\alpha\}$.

2.1 Partial order

We recall the reverse dominance order on partitions. For this, let $m \geq 1$ be positive integer and denote by $\mathcal{P}(m)$ the set of partitions of m :

$$\mathcal{P}(m) = \{(m_1 \geq \dots \geq m_k > 0) \mid m_j \in \mathbb{Z} \text{ and } m_1 + \dots + m_k = m\}.$$

This is a finite set and the reverse dominance relation on $\mathcal{P}(m)$ is defined as follows: Let $\lambda = (m_1 \geq \dots \geq m_{k_1} > 0)$, $\mu = (n_1 \geq \dots \geq n_{k_2} > 0) \in \mathcal{P}(m)$. Then

$$\lambda \preceq \mu \quad :\Leftrightarrow \quad \forall j = 1, \dots, \min\{k_1, k_2\} : m_1 + \dots + m_j \geq n_1 + \dots + n_j.$$

Obviously, this gives a partial order on $\mathcal{P}(m)$ with a smallest element ($m > 0$) and largest element ($1 \geq 1 \geq \dots \geq 1 > 0$). Moreover, if we consider partitions of a fixed length k only, $\mathcal{P}(m, k)$, then there is also a unique maximal element. Namely if $m = \ell k + p$, where $0 \leq p < k$, then

$$\lambda = (\ell + 1 \geq \dots \geq \ell + 1 > \ell \geq \dots \geq \ell)$$

is the unique maximal element in $\mathcal{P}(m, k)$.

2.2 Cover relation

We recall the notion of the cover relation induced by \preceq , e.g. we say μ covers λ if

- 1) $\lambda \preceq \mu$ and
- 2) $\lambda \preceq \nu \preceq \mu$ implies $\nu = \lambda$ or $\nu = \mu$.

Since $\mathcal{P}(m)$ is a finite set, we can find for each pair $\lambda \preceq \mu$ partitions ν_0, \dots, ν_ℓ such that

$$\lambda = \nu_0 \preceq \nu_1 \preceq \dots \preceq \nu_\ell = \mu$$

and ν_i covers ν_{i-1} for all i . To understand the partial order on $\mathcal{P}(m)$ it is therefore sufficient to understand the cover relation on $\mathcal{P}(m)$. The following proposition was proved in [3, Proposition 3.5] (for simplicity of notation we assume that λ, μ have the same length by adding 0 parts to at most one of the both).

Proposition 2.1. *Let $\lambda = (m_1 \geq \dots \geq m_k \geq 0)$, $\mu = (n_1 \geq \dots \geq n_k \geq 0) \in \mathcal{P}(m)$ and n_k or $m_k \neq 0$. Suppose μ covers λ . Then there exists $i < j$ such that*

$$n_\ell = \begin{cases} m_\ell & \text{if } \ell \neq i, j, \\ m_\ell - 1 & \text{if } \ell = i, \\ m_\ell + 1 & \text{if } \ell = j. \end{cases}$$

The cover relation on partitions $\mathcal{P}(m)$ is completely determined by the cover relation on partitions of length 2.

2.3 Quantum affine algebras and their representations

We give a brief reminder on quantum affine algebras and their finite-dimensional representations. For more details we refer to [4, 14].

Let $q \in \mathbb{C}^*$ which is not a root of unity. Let $U_q(\hat{\mathfrak{g}})$ be the untwisted quantum affine algebra associated to \mathfrak{g} . The simple objects of the category \mathcal{C} of (type 1) finite-dimensional representations of $U_q(\hat{\mathfrak{g}})$ are parametrized by dominant monomials of the ring $\mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{1 \leq i \leq n, a \in \mathbb{C}^*}$, that is for each such monomial $m = \prod_{i \in I, a \in \mathbb{C}^*} Y_{i,a}^{u_{i,a}}$ which is dominant (the $u_{i,a} \geq 0$), there is

a corresponding simple object $L(m)$ in \mathcal{C} . For example, for $i \in I$, $m \geq 0$, $a \in \mathbb{C}^*$, we have the Kirillov–Reshetikhin module

$$W_{m,a}^{(i)} = L(Y_{i,a} Y_{i,aq^2} \cdots Y_{i,aq_i^{2(m-1)}}).$$

Here $q_i = q^{r_i}$ where r_i is the length of simple root α_i .

The q -character morphism [12] is an injective ring morphism defined on the Grothendieck ring $\text{Rep}(U_q(\hat{\mathfrak{g}}))$ of the tensor category \mathcal{C} :

$$\chi_q : \text{Rep}(U_q(\hat{\mathfrak{g}})) \rightarrow \mathcal{Y}.$$

Theorem 2.1 ([11, 12]). *For m a dominant monomial, we have*

$$\chi_q(L(m)) \in m \mathbb{Z}_{\geq 0} [A_{i,a}^{-1}]_{1 \leq i \leq n, a \in \mathbb{C}^*},$$

where

$$A_{i,a} = Y_{i,aq_i^{-1}} Y_{i,aq_i} \prod_{\{j | C_{j,i} = -1\}} Y_{j,a}^{-1} \prod_{\{j | C_{j,i} = -2\}} Y_{j,aq^{-1}}^{-1} Y_{j,aq}^{-1} \prod_{\{j | C_{j,i} = -3\}} Y_{j,aq^2}^{-1} Y_{j,a}^{-1} Y_{j,aq^{-2}}^{-1}.$$

An element in $\text{Im}(\chi_q)$ is characterized by the multiplicity of its dominant monomials.

Note that we have a partial ordering on the monomials of \mathcal{Y} : $m \preceq m'$ if $m'm^{-1}$ is a product of monomials $A_{i,a}$. The first statement in the theorem can be reformulated by saying that all monomials in $\chi_q(L(m))$ are lower than m for this ordering, that is m is the highest monomial.

As consequence of the second statement, if we know that the q -character of a simple module has a unique dominant monomial, its q -character can be reconstructed (this is the Frenkel–Mukhin algorithm [11]). This property has been proved in the important case of Kirillov–Reshetikhin modules, which led to the proof of the Kirillov–Reshetikhin conjecture. It was first proved by Nakajima [18] for ADE -types and in [14] with a different proof which can be extended to the general case.

Theorem 2.2 ([14, 18]). *The q -character of Kirillov–Reshetikhin module has a unique dominant monomial. This implies the T -system in the Grothendieck ring $\text{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$:*

$$[W_{m,a}^{(i)} \otimes W_{m,aq_i^2}^{(i)}] = [W_{m+1,a}^{(i)} \otimes W_{m-1,aq_i^2}^{(i)}] + [S_{m,a}^{(i)}],$$

where $S_{m,a}^{(i)}$ is a tensor product of Kirillov–Reshetikhin modules. The representations $S_{m,a}^{(i)}$ and $W_{m+1,a}^{(i)} \otimes W_{m-1,aq_i^2}^{(i)}$ are simple.

2.4 The classical limit

Let $\text{KR}(m\omega_i)$ be the restriction of $W_{m,a}^{(i)}$ as a $U_q(\mathfrak{g})$ -module (it is well known it does not depend on a , see references in [14]). We denote by the same symbol its limit at $q = 1$ (that is we consider the corresponding \mathfrak{g} -module).

$\text{KR}(m\omega_i)$ decomposes into a direct sum of finitely many simple \mathfrak{g} -modules. This decomposition is computed for \mathfrak{g} of classical type, namely type A, B, C, D as well as for certain nodes for exceptional types (see [2, Theorem 1] and [2, Chapter 3]). For more on decompositions of Kirillov–Reshetikhin modules see [13].

2.5 Chari modules

For the readers convenience we should clarify the relation to *Kirillov–Reshetikhin modules for current algebras*, although it is not used in this note at all:

Denote $\mathfrak{g} \otimes \mathbb{C}[t]$ the current algebra of \mathfrak{g} . In [2], finite-dimensional modules for $\mathfrak{g} \otimes \mathbb{C}[t]$ were introduced. These are supposed to be classical analogs of Kirillov–Reshetikhin modules. Namely for $m \in \mathbb{Z}_{\geq 0}$, $a \in \mathbb{C}$, $i \in I$, $C(m\omega_i, a)$ is the $\mathfrak{g} \otimes \mathbb{C}[t]$ -module generated by a non-zero vector w subject to the relations

$$\begin{aligned} \mathfrak{n}^+ \otimes \mathbb{C}[t].w &= 0, & (h \otimes 1).w &= m\omega_i(h)w, & \mathfrak{h} \otimes (t - a)\mathbb{C}[t].w &= 0, \\ (f_\alpha)^{m\omega_i(h_\alpha)+1}.w &= 0, & (f_{\alpha_i} \otimes (t - a)).w &= 0. \end{aligned}$$

The following was proven for \mathfrak{g} of classical type, namely of type A, B, C, D in [2, Corollary 2.1]. The general case (e.g., an arbitrary simple, finite-dimensional complex Lie algebra) can be deduced as the special case of a single tensor factor of [15, Corollary 5.1], this has been proved by using a pentagram of identities see [15, Section 1.2].

Theorem 2.3. *For $i \in I$ and $m \geq 0$ we have*

$$C(m\omega_i, a) \cong_{\mathfrak{g}} \text{KR}(m\omega_i).$$

3 Main result

3.1. Main theorem. For $\lambda = (m_1 \geq \dots \geq m_k > 0) \in \mathcal{P}(m)$ and $i \in I$ we denote the tensor product of the associated KR modules as:

$$\text{KR}(\lambda, i) := \text{KR}(m_1\omega_i) \otimes \dots \otimes \text{KR}(m_k\omega_i).$$

Note that $\mathrm{KR}(\mathbf{0}, \omega_i) \cong \mathbb{C}$ is the trivial representation. We are mainly interested in the \mathfrak{g} -structure of these modules. Since this is the tensor product of finite-dimensional \mathfrak{g} -modules, it decomposes into the direct sum of simple modules. Due to the structure of the Kirillov–Reshetikhin modules, we see immediately that the maximal weight occurring is $m\omega_i$ and that the corresponding weight space has dimension 1.

The main purpose of this paper is the study of the \mathfrak{g} -structure of these modules along the partial order and the main theorem is the following

Theorem 3.1. *Let $m \geq 1$, $i \in I$ and $\lambda \preceq \mu \in \mathcal{P}(m)$, then*

$$\dim(\mathrm{Hom}_{\mathfrak{g}}(\mathrm{KR}(\lambda, i), V(\tau))) \leq \dim(\mathrm{Hom}_{\mathfrak{g}}(\mathrm{KR}(\mu, i), V(\tau)))$$

for all $\tau \in P^+$.

In other words, there exists a surjective map of \mathfrak{g} -modules

$$\mathrm{KR}(\mu, i) \longrightarrow \mathrm{KR}(\lambda, i).$$

Proof. Let $\lambda \preceq \mu \in \mathcal{P}(m)$. We denote $\mu = (n_1 \geq \dots \geq n_{k_2} \geq 0)$. It is sufficient to prove the statement for the case where μ covers λ . In this case, we know by Proposition 2.1 there exists $p < q$ such that $n_\ell = m_\ell$ for all $\ell \neq p, q$. This implies that

$$\mathrm{KR}(\mu, i) = \mathrm{KR}((m_p - 1)\omega_i) \otimes \mathrm{KR}((m_q + 1)\omega_i) \otimes \bigotimes_{\ell \neq p, q} \mathrm{KR}(m_\ell \omega_i).$$

So to prove the statement it is enough to give a proof for partitions of m of length 2. So let $m_1 \geq m_2 > 0$, we have to show that we have a surjective map of \mathfrak{g} -modules

$$\mathrm{KR}(m_1 \omega_i) \otimes \mathrm{KR}(m_2 \omega_i) \twoheadrightarrow \mathrm{KR}((m_1 + 1)\omega_i) \otimes \mathrm{KR}((m_2 - 1)\omega_i).$$

This will be proven in the next section, Theorem 4.1. ■

3.2. Before proving the last ingredient in the proof (Theorem 4.1), we shall make a couple of remarks. First of all, Theorem 3.1 was proved in [3, Theorem ii)] for the special case where ω_i is a minuscule weight. In this case, $\mathrm{KR}(m\omega_i) \cong V(m\omega_i)$ as a \mathfrak{g} -module (see, e.g., [2]). This case covers the \mathfrak{sl}_{n+1} case as well as certain special cases for other types. The proof given there uses the combinatorics of LS paths. Namely, an injection on the level of paths in the tensor product is given. This proof does not extend to other cases since the combinatorics of LS-paths are more complicated for non-minuscule weights.

3.3. It would be interesting to study the kernel of the map ($\lambda \preceq \mu \in \mathcal{P}(m)$)

$$\mathrm{KR}(\mu, i) \longrightarrow \mathrm{KR}(\lambda, i).$$

In the formerly studied case of Q -systems, this kernel turns out to be a tensor product of Kirillov–Reshetikhin modules again (Theorem 2.2). This is not true in general: for \mathfrak{sl}_n , $i = 2$ and the quotient

$$\mathrm{KR}(5, 2) \otimes \mathrm{KR}(1, 2) \longrightarrow \mathrm{KR}(6, 2),$$

the kernel is not a (non-trivial) tensor product at all. It is also not a minimal affinization as they are all simple \mathfrak{sl}_n -modules.

3.4. The partial order on $\mathcal{P}(m)$ is a special case of the more general poset $\mathcal{P}(\lambda)$ introduced in [3]. Here $\lambda \in P^+$ is a dominant weight and the elements in the set are partitions of λ , namely

$\lambda = (\lambda_1, \dots, \lambda_k \geq 0)$, with $\lambda_i \in P^+$ and $\lambda_1 + \dots + \lambda_k = \lambda$. Then the partial order on $\mathcal{P}(\lambda)$ is given by: $\lambda \preceq \mu$ if and only if for all $\alpha \in R^+$ and $\ell \geq 1$:

$$\min_{i_1 < \dots < i_\ell} \{(\lambda_{i_1} + \dots + \lambda_{i_\ell})(h_\alpha)\} \leq \min_{i_1 < \dots < i_\ell} \{(\mu_{i_1} + \dots + \mu_{i_\ell})(h_\alpha)\}.$$

Then [3, Theorem 1(i)] gives for $\lambda \preceq \mu$

$$\dim(V(\lambda_1) \otimes \dots \otimes V(\lambda_{k_1})) \leq \dim(V(\mu_1) \otimes \dots \otimes V(\mu_{k_2})).$$

3.5. It is conjectured [3, Conjecture 1] that if $(\lambda_1, \lambda_2) \preceq (\mu_1, \mu_2) \in \mathcal{P}(\lambda, 2)$ then

$$\dim(\text{Hom}_{\mathfrak{g}}(V(\lambda_1) \otimes V(\lambda_2), V(\tau))) \leq \dim(\text{Hom}_{\mathfrak{g}}(V(\mu_1) \otimes V(\mu_2), V(\tau)))$$

for all $\tau \in P^+$.

This conjecture was made before for \mathfrak{sl}_{n+1} by Lam, Postnikov, Pylyavskyy (cited in [7, Conjecture 1]). In this case, the conjecture is equivalent to the following statement on the level of characters:

$$s_{\mu_1} s_{\mu_2} - s_{\lambda_1} s_{\lambda_2} = \sum_{\tau \in P^+} c_\tau s_\tau, \quad \text{and} \quad c_\tau \geq 0 \quad \forall \tau \in P^+,$$

where s_λ is the Schur function corresponding to λ , e.g. the character of the simple \mathfrak{sl}_{n+1} -module $V(\lambda)$. In other words, it is conjectured that the difference of the products of the corresponding Schur functions is *Schur positive*.

A big step forward in proving this conjecture has been made with [7, Theorem 1]:

$$\dim(\text{Hom}_{\mathfrak{g}}(V(\lambda_1) \otimes V(\lambda_2), V(\tau))) \geq 1 \Rightarrow \dim(\text{Hom}_{\mathfrak{g}}(V(\mu_1) \otimes V(\mu_2), V(\tau))) \geq 1.$$

Further, in [3, Theorem 1(iii)] the conjecture was proved for \mathfrak{sl}_3 . Further evidence to this conjecture is due the link between fusion products and certain PBW graded modules as provided in [9].

3.6. Besides some numerical evidence and the partial cases stated above, the conjectures remain open. With the result of the current paper one may tempt to replace the simple modules $V(\lambda)$ in the tensor product by minimal affinizations of $V(\lambda)$ (see [5] for more details on these) and still conjecture a surjective map of \mathfrak{g} -modules. Note that for classical types, $\text{KR}(m\omega_i)$ is the minimal affinization of $V(m\omega_i)$ [1, 5]. Some evidence to this conjecture can be found in [17] where the Q -system relations are extended to minimal affinizations.

4 Proof of the main theorem

Theorem 4.1. *Let $i \in I$ and $m_1 > m_2 > 0$. Then there exists an embedding of \mathfrak{g} -modules*

$$\text{KR}(m_1\omega_i) \otimes \text{KR}(m_2\omega_i) \rightarrow \text{KR}((m_1 - 1)\omega_i) \otimes \text{KR}((m_2 + 1)\omega_i)$$

The proof uses arguments of the proofs of [14, Theorems 1.3 and 6.1]. We explain below the main differences between our present situation and the results proved in [14].

Proof. For $m_1 = m_2 + 1$ it is clear as the two \mathfrak{g} -modules are isomorphic. Let us suppose that $m_1 \geq m_2 + 2$. It follows directly from the following statement that we establish. The $U_q(\hat{\mathfrak{g}})$ -module

$$V = W_{m_1, q_i}^{(i)} \otimes_{-2(m_1 - m_2 - 1)} W_{m_2, 1}^{(i)}$$

is simple and occurs as a composition factor in $\text{Rep}(U_q(\hat{\mathfrak{g}}))$ of

$$V' = W_{m_1-1, q_i}^{(i)} \otimes W_{m_2+1, 1}^{(i)}.$$

For $m_1 = m_2 + 2$ this is a direct consequence of the T -system in Theorem 2.2 which gives the decomposition of $W_{m_1-1, q_i}^{(i)} \otimes W_{m_1-1, 1}^{(i)}$ in simple modules in $\text{Rep}(U_q(\hat{\mathfrak{g}}))$.

For a general $m_1 > m_2 + 2$, the two modules V and V' have the same highest monomial

$$M = Y_{i, q_i}^{2m_2} \left(Y_{i, q_i}^{2(m_2-1)} Y_{i, q_i}^{2(m_2-2)} \cdots Y_{i, 1} \right)^2 \left(Y_{i, q_i}^{-2} Y_{i, q_i}^{-4} \cdots Y_{i, q_i}^{-2(m_1-m_2-1)} \right).$$

Hence it suffices to prove that V is simple, that is $V \simeq L(M)$. To prove it, we write a proof as for [14, Theorem 6.1(2)]:

(1) We list the dominant monomials occurring in $\chi_q(V)$, and we get as in [14, Lemma 5.6(2)] the following set:

$$\{ M, M A_{i, q_i}^{-1}{}_{2m_2-1}, M A_{i, q_i}^{-1}{}_{2m_2-1} A_{i, q_i}^{-1}{}_{2m_2-3}, \dots, M A_{i, q_i}^{-1}{}_{2m_2-1} A_{i, q_i}^{-1}{}_{2m_2-3} \cdots A_{i, aq_i}^{-1} \}.$$

All of them occur with multiplicity 1.

(2) We prove with the same argument as in [14, Section 6.1] that these monomials do occur in $\chi_q(L(M))$: otherwise, we would have a monomial

$$M' = M A_{i, q_i}^{-1}{}_{2m_2-1} A_{i, q_i}^{-1}{}_{2m_2-3} \cdots A_{i, aq_i}^{-1}{}_{2m_2-1-2r} < M$$

in this list such that $L(M')$ is a simple constituent of V . But then all monomials of $\chi_q(L(M'))$ should occur in $\chi_q(L(M))$, in particular $M' A_{i, aq_i}^{-1}{}_{2m_2-1-2r}$, which is not as explained in [14, Section 6.1]. ■

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