

Scalar Flat Kähler Metrics on Affine Bundles over $\mathbb{C}P^{1\star}$

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Abstract. We show that the total space of any affine \mathbb{C} -bundle over $\mathbb{C}P^1$ with negative degree admits an ALE scalar-flat Kähler metric. Here the degree of an affine bundle means the negative of the self-intersection number of the section at infinity in a natural compactification of the bundle, and so for line bundles it agrees with the usual notion of the degree.

Key words: scalar-flat Kähler metric; affine bundle; twistor space

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1 Introduction

If Γ is a finite subgroup of the unitary group $U(2)$ which acts freely on the unit sphere around the origin in \mathbb{C}^2 , it is natural to ask existence of a Kähler metric defined on the minimal resolution of the quotient space \mathbb{C}^2/Γ , which has ‘small curvature’ and which is asymptotically locally Euclidean (ALE) at infinity. If Γ is a finite subgroup of $SU(2)$, Kronheimer [6] constructed ALE Ricci-flat Kähler metrics on the minimal resolution by means of so called the hyperKähler quotient, and further showed [7] that the metrics are determined by the cohomology classes of a collection of Kähler forms associated to the hyper-Kähler structure. When Γ is a finite cyclic subgroup of $U(2)$ generated by scalar matrices, Γ is not included in $SU(2)$ unless $|\Gamma| = 2$, and the minimal resolution of \mathbb{C}^2/Γ is simply the total space of the line bundle $\mathcal{O}(-n)$, where $n = |\Gamma|$, and the unique negative section is the exceptional locus of the resolution. Because the section intersects positively with the canonical class if $n > 2$, there exists no Ricci-flat Kähler metric on $\mathcal{O}(-n)$ if $n > 2$. LeBrun [8] constructed on this complex surface a scalar-flat Kähler (SFK) metric which is also ALE. The metric is invariant under a natural $U(2)$ -action, and may be considered to be the natural Kähler metric on $\mathcal{O}(-n)$. Later, Calderbank–Singer [2] pointed out that, for any cyclic subgroup $\Gamma \subset U(2)$, the minimal resolution of \mathbb{C}^2/Γ admits an ALE SFK metric. All these metrics are anti-self-dual (ASD) with respect to the complex orientation.

ALE spaces can be compactified to be an orbifold by adding a point at infinity, and after an appropriate conformal change the metric can be extended to the compactification as an ASD metric on the orbifold. Small deformations of ASD conformal structures are governed by a deformation complex, and if the space is compact, the index of the complex is expressed in terms of topological invariants of the space. Viaclovsky [14] computed the index of the deformation complex for various compact orbifolds in explicit form, and show in particular that if the space is the compactification $\widehat{\mathcal{O}(-n)}$ of $\mathcal{O}(-n)$, the index is $12 - 4n$. As the obstruction for the deformation complex vanishes, this means that LeBrun’s metric has a non-trivial deformation as an ALE ASD metric (if $n > 3$). In [3] we computed the $U(2)$ -action on the relevant cohomology group in

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concrete form, and computed the dimension of the moduli space of ALE ASD metrics near the LeBrun metric. Also we found that there exists a real 1-parameter family of deformation of the LeBrun metric which preserves not only ASD-ALE property but also a Kähler representative.

In this article we investigate all small deformations of the LeBrun metric on $\mathcal{O}(-n)$ as an ALE SFK metric. If the complex structure is fixed, the following rigidity is shown:

Proposition 1.1 (= Proposition 3.6). *When we fix the complex structure on $\mathcal{O}(-n)$, LeBrun's metric on $\mathcal{O}(-n)$ cannot be deformed as an ALE SFK metric by small deformations.*

In order to explain what happens when we allow the complex structure on $\mathcal{O}(-n)$ to vary in deformations, we recall that the line bundle $\mathcal{O}(-n)$ is included as a special member of an $(n-1)$ -parameters family of affine \mathbb{C} -bundles over $\mathbb{C}\mathbb{P}^1$, whose transition law for fiber coordinates ζ_0 over $U_0 = \mathbb{C}(u) \subset \mathbb{C}\mathbb{P}^1$ and ζ_1 over $U_1 = \mathbb{C}(1/u) \subset \mathbb{C}\mathbb{P}^1$ is concretely given by

$$\zeta_0 = \frac{1}{u^n} \zeta_1 + \sum_{l=1}^{n-1} \frac{t_l}{u^l}, \quad (t_1, \dots, t_{n-1}) \in \mathbb{C}^{n-1}, \quad (1.1)$$

where t_l -s are parameters. If $(t_1, \dots, t_{n-1}) = (0, \dots, 0)$, this gives the line bundle $\mathcal{O}(-n)$, but if $(t_1, \dots, t_{n-1}) \neq (0, \dots, 0)$, the affine bundle (1.1) has no global section and it is just an affine bundle. For $t = (t_1, \dots, t_{n-1}) \in \mathbb{C}^{n-1}$ we denote by A_t for the affine \mathbb{C} -bundle over $\mathbb{C}\mathbb{P}^1$ defined by (1.1). Then we prove the following

Theorem 1.2. *There exists a neighborhood $B \subset \mathbb{C}^{n-1}$ of the origin for which LeBrun's ALE SFK metric on $\mathcal{O}(-n)$ extends naturally to A_t if $t \in B$, as ALE SFK metrics.*

This will be shown as Theorem 3.4, and from the proof, this family of metrics can be regarded as the versal family for the LeBrun metric on $\mathcal{O}(-n)$ as ALE SFK metrics. The 1-parameter family of ALE SFK metrics on the 4-manifold $\mathcal{O}(-n)$ obtained in [3] is exactly the restriction of the family of ALE SFK metrics in Theorem 1.2 to the first (or the last) coordinate axis.

Next for explaining an immediate consequence of Theorem 1.2, we recall that any affine \mathbb{C} -bundle over $\mathbb{C}\mathbb{P}^1$ can be naturally compactified to a Hirzebruch surface by attaching a section at infinity. We call the negative of the last self-intersection number as *degree* of the affine bundle. Then for any $t \in \mathbb{C}^{n-1}$ the affine bundle defined by the transition law (1.1) is of degree n . Conversely, if $n > 1$, any affine \mathbb{C} -bundle over $\mathbb{C}\mathbb{P}^1$ of degree n is of the form A_t for some $t \in \mathbb{C}^{n-1}$. Now because the equation (1.1) is linear in the variables $\zeta_0, \zeta_1, t_1, t_2, \dots, t_{n-1}$, the affine bundle A_t and A_{ct} is isomorphic for any $c \in \mathbb{C}^*$. Therefore, Theorem 1.2 implies the following

Corollary 1.3 (= Corollary 3.5). *Any affine \mathbb{C} -bundle over $\mathbb{C}\mathbb{P}^1$ of negative degree (in the above sense) admits an ALE SFK metric.*

Finally we explain some property of the family of ALE SFK metrics obtained in Theorem 1.2. In contrast with the LeBrun metric for which the rigidity holds as in Proposition 1.1, for the deformed metrics, we have the following

Proposition 1.4. *Let $B \subset \mathbb{C}^{n-1}$ be as in Theorem 1.2, and for $t \in B$ let g_t be the ALE SFK metric on A_t . Then if $t \neq 0$ and t is sufficiently close to the origin, there exists a smooth arc $\gamma_t \subset B$ passing through the point t which satisfies the following:*

- (i) *the complex structure of A_t is constant along the arc γ_t ,*
- (ii) *the conformal class of the ALE SFK metric g_t varies when t moves along γ_t .*

2 Preliminary computations for Hirzebruch surfaces

2.1 Notation and convention

For an integer $n \geq 0$, \mathbb{F}_n denotes the Hirzebruch surface of degree n ; namely $\mathbb{F}_n = \mathbb{P}(\mathcal{O}(-n) \oplus \mathcal{O})$ over $\mathbb{C}\mathbb{P}^1$. We write $\pi : \mathbb{F}_n \rightarrow \mathbb{C}\mathbb{P}^1$ for the projection, and f for a fiber (class) of π . We denote Γ_0 for $(-n)$ -section of π , which is unique when $n > 0$. We have $H^2(\mathbb{F}_n, \mathbb{Z}) \simeq \text{Pic}\mathbb{F}_n \simeq \mathbb{Z}[\Gamma_0] \oplus \mathbb{Z}[f]$, and $-K_{\mathbb{F}_n} \simeq \mathcal{O}(2\Gamma_0 + (n+2)f)$ for the anticanonical class. $\text{Aut}_0\mathbb{F}_n$ denotes the identity component of holomorphic transformation group of \mathbb{F}_n , and for a section L of $\mathbb{F}_n \rightarrow \mathbb{C}\mathbb{P}^1$, $\text{Aut}_0(\mathbb{F}_n, L)$ denotes the subgroup of $\text{Aut}_0\mathbb{F}_n$ consisting of transformations which keep L invariant. If $n > 0$ and $L = \Gamma_0$, we have $\text{Aut}_0\mathbb{F}_n = \text{Aut}_0(\mathbb{F}_n, L)$. Two pairs (\mathbb{F}_n, L) and (\mathbb{F}_n, L') are called isomorphic as a pair if there is a biholomorphic map $\phi : \mathbb{F}_n \rightarrow \mathbb{F}_n$ which satisfies $\phi(L) = L'$. Γ_∞ means a section whose self-intersection number is $(+n)$. $\text{Aut}_0\mathbb{F}_n$ acts transitively on the space of $(+n)$ -sections, and Γ_∞ may be identified with the section $\mathbb{P}(\mathcal{O}(-n))$. Thus the complement $\mathbb{F}_n \setminus \Gamma_\infty$ can be identified with the total space of the line bundle $\mathcal{O}(-n)$.

We write the linear system to which a section of $\mathbb{F}_n \rightarrow \mathbb{C}\mathbb{P}^1$ belongs in the form $|\Gamma_0 + kf|$, $k \geq 0$. It is well-known that this system has an irreducible member only when $k = 0$ or $k \geq n$. So if L is a section with positive self-intersection number, we have $L \in |\Gamma_0 + (n+l)f|$ for some $l \geq 0$. The letter l is always used in this meaning throughout the article. We have $\Gamma_\infty \in |\Gamma_0 + nf|$. The system $|\Gamma_0 + (n+l)f|$ is very ample if and only if $l > 0$. Moreover we have $h^0(\mathcal{O}_{\mathbb{F}_n}(\Gamma_0 + (n+l)f)) = n + 2l + 2$, where h^0 means $\dim H^0$. Therefore the complement of any member of these systems is realized in an affine space \mathbb{C}^{n+2l+1} .

We will also use the following result regarding the dimension of the cohomology group $H^i(\Theta_{\mathbb{F}_n})$ of the tangent sheaf. Namely if $n > 0$ we have [10]

$$h^0(\Theta_{\mathbb{F}_n}) = n + 5, \quad h^1(\Theta_{\mathbb{F}_n}) = n - 1, \quad h^2(\Theta_{\mathbb{F}_n}) = 0. \quad (2.1)$$

This will also be shown in the proof of Proposition 2.2.

2.2 Affine \mathbb{C} -bundles over $\mathbb{C}\mathbb{P}^1$

Let $\text{Af}(\mathbb{C})$ be the group of complex affine transformations of \mathbb{C} ; namely those of the form $\zeta \mapsto a\zeta + b$ for $\zeta \in \mathbb{C}$, where $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$. Let X be a projective algebraic manifold. By an affine \mathbb{C} -bundle over X , we mean (as usual) a \mathbb{C} -bundle $A \rightarrow X$ whose structure group is $\text{Af}(\mathbb{C})$. In this subsection, according to [1], we briefly explain a classification of affine \mathbb{C} -bundles over X , and then apply it to a concrete description of affine \mathbb{C} -bundles over $\mathbb{C}\mathbb{P}^1$.

As in the case of any fiber bundle with prescribed structure group, isomorphic classes of affine \mathbb{C} -bundles over X are naturally in 1-1 correspondence with the cohomology set $H^1(X, \mathcal{A}f)$, where $\mathcal{A}f$ means the sheaf of germs of holomorphic maps from open sets in X to the group $\text{Af}(\mathbb{C})$. The set $H^1(X, \mathcal{A}f)$ is of course the inductive limit of $H^1(\mathcal{U}, \mathcal{A}f)$ with respect to open covering \mathcal{U} -s of X . For each \mathcal{U} , there is a natural map

$$\rho_{\mathcal{U}} : H^1(\mathcal{U}, \mathcal{A}f) \longrightarrow H^1(\mathcal{U}, \mathcal{O}^*),$$

which is induced from the natural homomorphism $\text{Af}(\mathbb{C}) \rightarrow \mathbb{C}^*$ that takes the coefficient of the linear part. These naturally induce a map $\rho : H^1(X, \mathcal{A}f) \rightarrow H^1(X, \mathcal{O}^*)$. Therefore we have

$$H^1(X, \mathcal{A}f) \simeq \bigsqcup_{\xi \in H^1(X, \mathcal{O}^*)} \rho^{-1}(\xi). \quad (2.2)$$

Geometrically, for an (isomorphism class of) affine bundle $A \in H^1(X, \mathcal{A}f)$, the image $\rho(A) \in H^1(X, \mathcal{O}^*)$ is exactly (the isomorphism class of) the dual line bundle of the normal bundle of the

section $\bar{A} \setminus A$ in \bar{A} , where \bar{A} means the compactified $\mathbb{C}\mathbb{P}^1$ -bundle which is obtained from $A \rightarrow X$ by the standard inclusion $\text{Af}(\mathbb{C}) \subset \text{PGL}(2, \mathbb{C})$.

Returning to the Čech cohomology group, analogously to (2.2), we clearly have, for each open covering \mathcal{U} of X ,

$$H^1(\mathcal{U}, \mathcal{A}f) = \bigsqcup_{\xi \in H^1(\mathcal{U}, \mathcal{O}^*)} \rho_{\mathcal{U}}^{-1}(\xi).$$

In order to describe the set $\rho_{\mathcal{U}}^{-1}(\xi)$, we write $\mathcal{U} = \{U_i\}$, and let $\xi \in H^1(\mathcal{U}, \mathcal{O}^*)$ be represented by a 1-cocycle $\{a_{ij}\}$, so that $a_{ij} \in \mathcal{O}^*(U_{ij})$ where $U_{ij} = U_i \cap U_j$. We fix a collection $\{h_i\}$ of meromorphic functions, where h_i is defined on U_i , that satisfy $h_i = a_{ij}h_j$ on U_{ij} (this is possible from the projectivity assumption for X), and let D be the divisor defined by $\{h_i = 0\}$. Though this is not necessarily effective, it is ‘linear equivalent’ to the line bundle ξ . Under these fixing of $\{a_{ij}\}$ and $\{h_i\}$, let $\{(a'_{ij}, b'_{ij})\} \in H^1(\mathcal{U}, \mathcal{A}f)$, where $a'_{ij} \in \mathcal{O}^*(U_{ij})$ and $b'_{ij} \in \mathcal{O}(U_{ij})$, be a representative of an element of $\rho_{\mathcal{U}}^{-1}(\xi)$. If we choose any $\phi_i \in \mathcal{O}^*(U_i)$ and $\psi_i \in \mathcal{O}(U_i)$ for each i and apply a fiber coordinate change $\tilde{\zeta}_i = \phi_i \zeta_i + \psi_i$ on U_i , then the new 1-cocycle $\{(\tilde{a}_{ij}, \tilde{b}_{ij})\}$ associated to $\{\tilde{\zeta}_i\}$, which is another representative of the same element of $H^1(\mathcal{U}, \mathcal{A}f)$, is readily seen to be given by

$$\tilde{a}_{ij} = \frac{\phi_i}{\phi_j} a'_{ij}, \quad \tilde{b}_{ij} = \phi_i b'_{ij} + \psi_i - \frac{\phi_i}{\phi_j} \psi_j a'_{ij}. \quad (2.3)$$

The first equation of these shows that any element of $\rho_{\mathcal{U}}^{-1}(\xi)$ can be represented by a cocycle of the form $\{(a_{ij}, b'_{ij})\}$ (namely, by using the original representative $\{a_{ij}\}$ for $\xi \in H^1(\mathcal{U}, \mathcal{O}^*)$), and in the following, for any element of $\rho_{\mathcal{U}}^{-1}(\xi)$, we only consider such representatives. This means that we only consider fiber coordinate changes $\{(\phi_i, \psi_i)\}$ which satisfy $\phi_i = \phi_j$ on U_{ij} , and hence we can write $\phi_i = t$ for all i for some constant $t \in \mathbb{C}^*$. Then the second equation of (2.3) becomes (after replacing a'_{ij} by a_{ij})

$$\tilde{b}_{ij} = t b'_{ij} + \psi_i - \psi_j a_{ij}. \quad (2.4)$$

This is the transformation law for representatives, under coordinate changes that satisfy the above constraint.

We are still fixing $\mathcal{U} = \{U_i\}$, $\xi \in H^1(\mathcal{U}, \mathcal{O}^*)$, a representative $\{a_{ij}\}$ of ξ , and $\{h_i\}$ that satisfies $h_i = a_{ij}h_j$ on U_{ij} . If $\{(a_{ij}, b_{ij})\}$ is a 1-cocycle that represents an element of $\rho_{\mathcal{U}}^{-1}(\xi)$, we define

$$c_{ij} := \frac{b_{ij}}{h_i} \quad \text{on } U_{ij}.$$

Then from the cocycle condition for $\{(a_{ij}, b_{ij})\}$, it follows that $\{c_{ij}\}$ is a 1-cocycle whose value is in $\mathcal{O}(D)$, where $\mathcal{O}(D)$ is the sheaf of holomorphic functions f for which fh_i is holomorphic for any i . Moreover if we apply fiber coordinate changes of the form $\{(\phi_i, \psi_i)\} = \{(t, \psi_i)\}$, it follows readily from (2.4) that the new 1-cocycle $\{\tilde{c}_{ij} = \tilde{b}_{ij}/h_i\}$ is cohomologous to the 1-cocycle $\{tc_{ij}\}$. Thus the assignment $\{(a_{ij}, b_{ij})\} \mapsto \{c_{ij} = b_{ij}/h_i\}$ induces a map $\rho_{\mathcal{U}}^{-1}(\xi) \rightarrow H^1(\mathcal{U}, \mathcal{O}(D))/\mathbb{C}^*$, where \mathbb{C}^* acts on $H^1(\mathcal{U}, \mathcal{O}(D))$ as the scalar multiplication. Conversely the assignment $\{c_{ij}\} \mapsto \{(a_{ij}, h_i c_{ij})\}$ induces a map $H^1(\mathcal{U}, \mathcal{O}(D)) \rightarrow \rho_{\mathcal{U}}^{-1}(\xi)$, which descends (by (2.4)) to a map from $H^1(\mathcal{U}, \mathcal{O}(D))/\mathbb{C}^*$. The last map is clearly the inverse of the above map $\rho_{\mathcal{U}}^{-1}(\xi) \rightarrow H^1(\mathcal{U}, \mathcal{O}(D))$. Thus, under fixing $\{a_{ij}\}$ for $\xi \in H^1(\mathcal{U}, \mathcal{O}^*)$ and $\{h_i\}$ satisfying $h_i = a_{ij}h_j$, we obtained a bijection

$$\rho_{\mathcal{U}}^{-1}(\xi) \xrightarrow{\sim} H^1(\mathcal{U}, \mathcal{O}(D))/\mathbb{C}^*. \quad (2.5)$$

Here the quotient space $H^1(\mathcal{U}, \mathcal{O}(D))/\mathbb{C}^*$ is of course a single point if $H^1(\mathcal{U}, \mathcal{O}(D)) = 0$ and otherwise a single point plus a projective space. The single point corresponds to the line bundle ξ itself. In [1] it was proved that this map is independent of the choice of $\{a_{ij}\}$ and $\{h_i\}$. So by taking the inductive limit in (2.5) with respect to open coverings, we obtain a bijection

$$\rho^{-1}(\xi) \xrightarrow{\sim} H^1(X, \mathcal{O}(D))/\mathbb{C}^*.$$

Thus from (2.2) there is a natural 1-1 correspondence

$$H^1(X, \mathcal{A}f) \xrightarrow{\sim} \bigsqcup_{\xi \in H^1(X, \mathcal{O}^*)} H^1(X, \mathcal{O}(\xi))/\mathbb{C}^*. \quad (2.6)$$

When $X = \mathbb{C}\mathbb{P}^1$, by natural isomorphisms $H^1(X, \mathcal{O}^*) \simeq H^2(\mathbb{C}\mathbb{P}^1, \mathbb{Z}) \simeq \mathbb{Z}$, (2.6) can be rewritten as

$$H^1(\mathbb{C}\mathbb{P}^1, \mathcal{A}f) \xrightarrow{\sim} \bigsqcup_{n \in \mathbb{Z}} H^1(\mathbb{C}\mathbb{P}^1, \mathcal{O}(n))/\mathbb{C}^*. \quad (2.7)$$

Definition 2.1. The *degree* of an affine bundle $A \rightarrow \mathbb{C}\mathbb{P}^1$ is the image of $A \in H^1(\mathbb{C}\mathbb{P}^1, \mathcal{A}f)$ by the composition of the natural map and identifications

$$H^1(\mathbb{C}\mathbb{P}^1, \mathcal{A}f) \xrightarrow{\rho} H^1(\mathbb{C}\mathbb{P}^1, \mathcal{O}^*) \simeq H^2(\mathbb{C}\mathbb{P}^1, \mathbb{Z}) \simeq \mathbb{Z}.$$

Evidently, if an affine bundle is a line bundle, its degree coincides with the usual degree as a line bundle. Denoting \bar{A} for the $\mathbb{C}\mathbb{P}^1$ -bundle naturally associated to A as before, the degree of A is exactly the negative of the self-intersection number of $\bar{A} \setminus A$ in \bar{A} .

If $n \geq -1$ we have $H^1(\mathbb{C}\mathbb{P}^1, \mathcal{O}(n)) = 0$ and $\rho^{-1}(n)$ consists of a single point which is exactly the line bundle $\mathcal{O}(n)$. Hence if the degree of a line bundle is more than -2 , it cannot be deformed even as an affine bundle. So we are mainly interested in the case where the degree is less than -1 . We write such line bundles in the form $\mathcal{O}(-n)$, so that $n \geq 2$. Then as $h^1(\mathcal{O}(-n)) = n - 1 > 0$, $\rho^{-1}(-n)$ consists of the single point (which corresponds to the line bundle $\mathcal{O}(-n)$) and the projective space $\mathbb{C}\mathbb{P}^{n-2}$ (which is also a point if $n = 2$). For each $n \geq 2$ we now compute transition law for fiber coordinates on arbitrary affine bundles with degree $-n$ in a concrete form. For this we take the standard covering $\mathcal{U}_0 := \{U_0, U_1\}$ where $U_0 = \{(z : w) \in \mathbb{C}\mathbb{P}^1 \mid z \neq 0\}$ and $U_1 = \{(z : w) \in \mathbb{C}\mathbb{P}^1 \mid w \neq 0\}$, and put $u = w/z$, $v = 1/u$. Then we have $H^1(\mathcal{U}_0, \mathcal{O}^*) \simeq H^1(\mathbb{C}\mathbb{P}^1, \mathcal{O}^*)$, and so for $\xi = \mathcal{O}(-n)$, as $\{a_{ij}\}$ and $\{h_i\}$ we can take

$$a_{01} = \frac{1}{u^n}, \quad h_0 = 1 \quad (\text{so that } a_{10} = u^n, \quad h_1 = u^n).$$

Moreover as a basis of $H^1(\mathcal{U}_0, \mathcal{O}(D)) = H^1(\mathcal{U}_0, \mathcal{O}(-n)) \simeq \mathbb{C}^{n-1}$ we can take $\{u^{-1}, u^{-2}, \dots, u^{1-n}\}$ where $u^{-l} \in H^0(U_{01}, \mathcal{O}(D))$. Thus for each element $b_{01} = t_1 u^{-1} + t_2 u^{-2} + \dots + t_{n-1} u^{1-n} \in H^1(\mathcal{U}_0, \mathcal{O}(D))$ we can associate an affine \mathbb{C} -bundle $A \rightarrow \mathbb{C}\mathbb{P}^1$ whose transition law for fiber coordinates is given by

$$\zeta_0 = \frac{1}{u^n} \zeta_1 + \sum_{l=1}^{n-1} \frac{t_l}{u^l} \quad \text{on } U_{01}, \quad (2.8)$$

where ζ_0 and ζ_1 are fiber coordinates over $U_0 = \mathbb{C}(u)$ and $U_1 = \mathbb{C}(v)$ respectively. This can be regarded as defining a holomorphic family of affine \mathbb{C} -bundles over $\mathbb{C}\mathbb{P}^1$ parametrized by \mathbb{C}^{n-1} , and over the origin we have the line bundle $\mathcal{O}(-n)$. We write the total space of this family by \mathcal{A}_n , thereby obtaining a holomorphic map

$$\mathcal{A}_n \rightarrow \mathbb{C}^{n-1}. \quad (2.9)$$

The equation (2.8) is linear in the variables $\zeta_0, \zeta_1, t_1, t_2, \dots, t_{n-1}$. Hence the total space of the family (2.9) admits a \mathbb{C}^* -action which is the multiplication to all these variables by weight 1. This \mathbb{C}^* -action clearly descends to the scalar multiplication on the base space \mathbb{C}^{n-1} , and fibers of the family (2.9) are mutually isomorphic along orbits of this \mathbb{C}^* -action on \mathbb{C}^{n-1} . This explains geometrically why two cocycles $\{(a_{ij}, b_{ij})\}$ and $\{(a_{ij}, tb_{ij})\}$ with values in $\mathcal{A}f$ determines mutually isomorphic affine bundles.

As above, for any $t \in \mathbb{C}^*$, the two cocycles $\{(a_{ij}, b_{ij})\}$ and $\{(a_{ij}, tb_{ij})\}$ determines the same (or isomorphic, more precisely) affine bundles. This can also be seen directly by noticing that the equation (2.8) is linear in the variables $\zeta_0, \zeta_1, t_1, t_2, \dots, t_{n-1}$. Hence by identifying fibers of (2.9) lying over the same linear 1-dimensional subspace, we have obtained a family of affine \mathbb{C} -bundles over $\mathbb{C}\mathbb{P}^1$ which is parametrized by $H^1(\mathbb{C}\mathbb{P}^1, \mathcal{O}(-n))/\mathbb{C}^*$. By varying n in \mathbb{Z} , this gives a concrete realization of the bijection (2.7).

Strictly speaking, in the argument of the last paragraph, we need to show that the natural map $H^1(\mathcal{U}_0, \mathcal{A}f) \rightarrow H^1(\mathbb{C}\mathbb{P}^1, \mathcal{A}f)$ is bijective; especially we need to show that any affine bundle over $\mathbb{C}\mathbb{P}^1$ can be trivialized over the open sets U_0 and U_1 respecting the structure of affine bundle. But this can be proved by standard adjusting argument using coboundaries, and we omit the detail.

2.3 Affine \mathbb{C} -bundles over $\mathbb{C}\mathbb{P}^1$ and Hirzebruch surfaces

We are concerned with ALE SFK metrics on the total spaces of affine \mathbb{C} -bundles over $\mathbb{C}\mathbb{P}^1$ whose degree is negative. We will investigate this through the natural compactification of the affine bundles to $\mathbb{C}\mathbb{P}^1$ bundles. The latter are of course Hirzebruch surfaces. In this subsection we will briefly explain relationship between affine \mathbb{C} -bundles over $\mathbb{C}\mathbb{P}^1$ and the Hirzebruch surfaces.

First let $A \rightarrow \mathbb{C}\mathbb{P}^1$ be an affine \mathbb{C} -bundle, and let $\bar{A} \rightarrow \mathbb{C}\mathbb{P}^1$ be the natural compactification to a $\mathbb{C}\mathbb{P}^1$ -bundle induced by the inclusion $\text{Af}(\mathbb{C}) \subset \text{PGL}(2, \mathbb{C})$ as before. We have $\bar{A} \simeq \mathbb{F}_n$ for some $n \geq 0$. We write $L := \bar{A} \setminus A$ for the added locus, which is of course a section of the projection $\bar{A} \rightarrow \mathbb{C}\mathbb{P}^1$. Then $-L^2$ is exactly the degree of $A \rightarrow \mathbb{C}\mathbb{P}^1$. Hence any affine \mathbb{C} -bundle over $\mathbb{C}\mathbb{P}^1$ of negative degree is naturally identified with the complement of a section of some \mathbb{F}_n whose self-intersection number is positive. We write by

$$\mathcal{F}_n \rightarrow \mathbb{C}^{n-1} \tag{2.10}$$

for the family of Hirzebruch surfaces that is obtained as the simultaneous compactification for members of the family $\mathcal{A}_n \rightarrow \mathbb{C}^{n-1}$ in (2.9). For this family, it is well-known that the Kodaira–Spencer map

$$T_0\mathbb{C}^{n-1} \rightarrow H^1(\mathbb{F}_n, \Theta) \tag{2.11}$$

at the origin is isomorphic (see [5, pp. 309–312]), and the family (2.10) gives the Kuranishi family of the Hirzebruch surface \mathbb{F}_n . Thus the parameter space \mathbb{C}^{n-1} of $\mathcal{A}_n \rightarrow \mathbb{C}^{n-1}$ and $\mathcal{F}_n \rightarrow \mathbb{C}^{n-1}$ may also be naturally identified with $H^1(\Theta_{\mathbb{F}_n})$.

Although the transition law for each member of the family (2.10) is concretely given as in (2.8), it is not easy to identify them with \mathbb{F}_m for a precise value of m . This was intensively studied in [13, p. 143, Theorem], where an explicit answer was given, but it is too complicated to write the result here. Some exceptions are identification for fibers on the coordinate axes of \mathbb{C}^{n-1} . Namely letting $\mathbb{C}(t_l)$ to be the l -th coordinate axis of \mathbb{C}^{n-1} , if we introduce new fiber coordinates $\tilde{\zeta}_0$ and $\tilde{\zeta}_1$ by

$$\tilde{\zeta}_0 = \frac{u^l \zeta_0 - t_l}{t_l \zeta_0} \quad \text{and} \quad \tilde{\zeta}_1 = \frac{\zeta_1}{t_l v^{n-l} \zeta_1 + t_l^2} \tag{2.12}$$

on the open sets $U_0 = \mathbb{C}(u)$ and $U_1 = \mathbb{C}(v)$ respectively, then with the aid of (2.8), we readily obtain the relation $\tilde{\zeta}_0 = v^{n-2l}\tilde{\zeta}_1$, which means that the ruled surface over the axis $\mathbb{C}(t_i)$ is isomorphic to \mathbb{F}_{n-2l} , except the central fiber. Here we are allowing the case $n - 2l < 0$ and in that case \mathbb{F}_{n-2l} means \mathbb{F}_{2l-n} . We also note that, as a natural extension of the \mathbb{C}^* -action on \mathcal{A}_n , the total space of $\mathcal{F}_n \rightarrow \mathbb{C}^{n-1}$ has a \mathbb{C}^* -action, and it also descends to the scalar multiplication on \mathbb{C}^{n-1} .

Conversely if L is a section of $\pi : \mathbb{F}_m \rightarrow \mathbb{C}\mathbb{P}^1$ for some $m \geq 0$, then the complement $\mathbb{F}_m \setminus L$ is biholomorphic to an affine \mathbb{C} -bundle over $\mathbb{C}\mathbb{P}^1$. This can be seen in the following way. Let Γ_0 and Γ_∞ be sections satisfying $\Gamma_0^2 = -m$ and $\Gamma_\infty^2 = m$. If $m = 0$, we assume $\Gamma_0 \neq \Gamma_\infty$. Let $\mathcal{U} = \{U_i\}$ be an open covering of $\mathbb{C}\mathbb{P}^1$ which satisfies for any i at least one of $L \cap \Gamma_0 \cap \pi^{-1}(U_i) = \emptyset$ or $L \cap \Gamma_\infty \cap \pi^{-1}(U_i) = \emptyset$ holds for any i . Let ζ_i be any fiber coordinate over U_i of the line bundle $\mathcal{O}(-m) \subset \mathbb{F}_m$ (so that $\Gamma_0 \cap \pi^{-1}(U_i)$ and $\Gamma_\infty \cap \pi^{-1}(U_i)$ are defined by $\zeta_i = 0$ and $\zeta_i = \infty$ respectively), and f_i be a meromorphic function on U_i such that $L \cap \pi^{-1}(U_i)$ is defined by $\zeta_i = f_i$. From the choice, f_i does not have both a zero and a pole. Then for any i such that f_i does not have a pole, we define a new fiber coordinate $\tilde{\zeta}_i$ over U_i as an affine bundle by setting

$$\tilde{\zeta}_i = \frac{1}{\zeta_i - f_i}. \quad (2.13)$$

Then from the choice of i , this may be used as a fiber coordinate on $\mathbb{F}_m \rightarrow \mathbb{C}\mathbb{P}^1$, and we have $L \cap \pi^{-1}(U_i) = \{\tilde{\zeta}_i = \infty\}$. For the remaining i -s, f_i does not have a zero. $L \cap \Gamma_0 \cap \pi^{-1}(U_i) = \emptyset$. For these i -s we put

$$\tilde{\zeta}_i = \frac{f_i \zeta_i}{f_i - \zeta_i}. \quad (2.14)$$

Then this can also be used as a fiber coordinate over U_i , and we again have $L \cap \pi^{-1}(U_i) = \{\tilde{\zeta}_i = \infty\}$. From (2.13) and (2.14) we readily see that the transition law for the new coordinate system $\{\tilde{\zeta}_i\}$ is included in the affine group $\text{Af}(\mathbb{C})$. Therefore $\mathbb{F}_m \setminus L$ is actually an affine bundle. However, even if the equation for a section L is given in a concrete form, it is not immediate again to trivialize the affine bundle $\mathbb{F}_m \setminus L \rightarrow \mathbb{C}\mathbb{P}^1$ over $U_0 = \mathbb{C}(u)$ and $U_1 = \mathbb{C}(v)$ and write down the transition function in the form (2.8).

2.4 Computations for Hirzebruch surfaces

As we mentioned we will investigate ALE SFK metrics on the affine \mathbb{C} -bundles over $\mathbb{C}\mathbb{P}^1$ through the compactification to Hirzebruch surfaces. More precisely the Hirzebruch surfaces are included in the twistor spaces of a conformal compactification of the ALE SFK metrics on the affine bundles, and the added section will be the twistor line over the added point at infinity, whose self-intersection number in the surface is positive. For this purpose, in this subsection, we make computations for pairs (\mathbb{F}_n, L) where L is a section satisfying $L^2 > 0$. (So $\mathbb{F}_n \setminus L$ is an affine \mathbb{C} -bundle over $\mathbb{C}\mathbb{P}^1$ of negative degree.) Especially we compute the dimension $h^i(\Theta_{\mathbb{F}_n, L})$ for arbitrary pairs, where $\Theta_{\mathbb{F}_n, L}$ is the sheaf of germs of holomorphic vector fields on \mathbb{F}_n which are tangent to L .

If L is a section of $\mathbb{F}_n \rightarrow \mathbb{C}\mathbb{P}^1$ which satisfies $L^2 > 0$, we have $L \in |\Gamma_0 + (n+l)f|$ for some $l \geq 0$ (see Section 2.1 for notation). Of course the value of $h^i(\Theta_{\mathbb{F}_n, L})$ depends on the number l . We begin with the case $l = 0$. In this case we have $n > 0$ as we are supposing $L^2 > 0$. Moreover if we identify \mathbb{F}_n with \bar{A} where $A = \mathcal{O}(-n)$, then L can be identified with $\bar{A} \setminus A$, the section at infinity. Hence regarding $\mathcal{O}(-n)$ (or \mathbb{F}_n) as the minimal resolution of $\mathbb{C}^2/\mathbb{Z}_n$ (or $\mathbb{C}\mathbb{P}^2/\mathbb{Z}_n$) where $\mathbb{Z}_n \subset \text{GL}(2, \mathbb{C})$ is a cyclic subgroup of scalar matrices of order n , the pair (\mathbb{F}_n, L) has an effective action of $\text{GL}(2, \mathbb{C})/\mathbb{Z}_n$. In particular we have $h^0(\Theta_{\mathbb{F}_n, L}) \geq 4$.

Proposition 2.2. *Suppose $n > 0$ and let L be any $(+n)$ -section of $\mathbb{F}_n \rightarrow \mathbb{C}\mathbb{P}^1$. Then we have the following:*

- (i) $h^0(\Theta_{\mathbb{F}_n, L}) = 4$ and $H^2(\Theta_{\mathbb{F}_n, L}) = 0$.
- (ii) The natural homomorphism $H^1(\Theta_{\mathbb{F}_n, L}) \rightarrow H^1(\Theta_{\mathbb{F}_n})$ is isomorphic, and these are $(n-1)$ -dimensional vector spaces.
- (iii) The complex structure of the pair (\mathbb{F}_n, L) is independent of the choice of L .
- (iv) We have $\text{Aut}_0(\mathbb{F}_n, L) \simeq \text{GL}(2, \mathbb{C})/\mathbb{Z}_n$ (see above).

Proof. Let $\Theta_{\mathbb{F}_n/\mathbb{C}\mathbb{P}^1} \subset \Theta_{\mathbb{F}_n}$ be the subsheaf consisting of germs of holomorphic vector fields which are tangent to fibers of $\pi : \mathbb{F}_n \rightarrow \mathbb{C}\mathbb{P}^1$. Scalar matrices in $\text{GL}(2, \mathbb{C})$ induce a \mathbb{C}^* -action on \mathbb{F}_n which preserves each fiber of $\mathbb{F}_n \rightarrow \mathbb{C}\mathbb{P}^1$, and it defines a vector field which is tangent to each fiber of π . Hence we obtain a section of $\Theta_{\mathbb{F}_n/\mathbb{C}\mathbb{P}^1}$. Moreover, as the vector field has simple zeros on $\Gamma_0 \sqcup L$ and no other zeros, we obtain $\Theta_{\mathbb{F}_n/\mathbb{C}\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{F}_n}(\Gamma_0 + L)$. With the aid of this isomorphism, we have the following commutative diagram of exact sequences of sheaves on \mathbb{F}_n :

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{F}_n}(\Gamma_0) & \longrightarrow & \Theta_{\mathbb{F}_n/\mathbb{C}\mathbb{P}^1} & \longrightarrow & N_{L/\mathbb{F}_n} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & \Theta_{\mathbb{F}_n, L} & \longrightarrow & \Theta_{\mathbb{F}_n} & \longrightarrow & N_{L/\mathbb{F}_n} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \pi^*\Theta_{\mathbb{C}\mathbb{P}^1} & \xlongequal{\quad} & \pi^*\Theta_{\mathbb{C}\mathbb{P}^1} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

Since the group $\text{GL}(2, \mathbb{C})$ acts transitively on $\mathbb{C}\mathbb{P}^1$, the natural map $H^0(\Theta_{\mathbb{F}_n, L}) \rightarrow H^0(\pi^*\Theta_{\mathbb{C}\mathbb{P}^1})$ is surjective. Hence from the first column of the diagram, since $h^0(\mathcal{O}_{\mathbb{F}_n}(\Gamma_0)) = 1$ as $n > 0$ and $h^0(\Theta_{\mathbb{C}\mathbb{P}^1}) = 3$, we obtain $h^0(\Theta_{\mathbb{F}_n, L}) = 1 + 3 = 4$. Also from the same column, as we readily have $h^1(\mathcal{O}_{\mathbb{F}_n}(\Gamma_0)) = n-1$ and $h^2(\mathcal{O}_{\mathbb{F}_n}(\Gamma_0)) = 0$, we obtain $h^1(\Theta_{\mathbb{F}_n, L}) = n-1$ and $h^2(\Theta_{\mathbb{F}_n, L}) = 0$.

From the isomorphism $\Theta_{\mathbb{F}_n/\mathbb{C}\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{F}_n}(\Gamma_0 + L)$ we also obtain $h^0(\Theta_{\mathbb{F}_n/\mathbb{C}\mathbb{P}^1}) = h^0(\mathcal{O}_{\mathbb{F}_n}(\Gamma_0 + L)) = n + 2$. Hence as $h^0(N_{L/\mathbb{F}_n}) = h^0(\mathcal{O}_{\mathbb{C}\mathbb{P}^1}(n)) = n + 1$ and $h^0(\mathcal{O}_{\mathbb{F}_n}(\Gamma_0)) = 1$, from the first row, we obtain that the map $H^0(\Theta_{\mathbb{F}_n/\mathbb{C}\mathbb{P}^1}) \rightarrow H^0(N_{L/\mathbb{F}_n})$ is surjective. Hence from the commutative diagram the map $H^0(\Theta_{\mathbb{F}_n}) \rightarrow H^0(N_{L/\mathbb{F}_n})$ is also surjective. Therefore from the middle row, as $h^0(\Theta_{\mathbb{F}_n, L}) = 4$, we obtain $h^0(\Theta_{\mathbb{F}_n}) = 4 + (n + 1) = n + 5$. Moreover, as $h^i(N_{L/\mathbb{F}_n}) = h^i(\mathcal{O}(n)) = 0$ for $i \in \{1, 2\}$ and $h^2(\Theta_{\mathbb{F}_n, L}) = 0$, we obtain from the same row that the natural map $H^1(\Theta_{\mathbb{F}_n, L}) \rightarrow H^1(\Theta_{\mathbb{F}_n})$ is isomorphic and $h^2(\Theta_{\mathbb{F}_n}) = 0$.

The assertion (iii) is clear since $\mathbb{F}_n \setminus L$ is isomorphic to the line bundle $\mathcal{O}(-n)$ for any $(+n)$ -section L . (iv) follows from the remark preceding to Proposition 2.2 and the assertion (i) which is already shown. ■

Next we consider the case $l = 1$, which requires some care.

Proposition 2.3. *Suppose $n > 0$. If L is a section of $\pi : \mathbb{F}_n \rightarrow \mathbb{C}\mathbb{P}^1$ belonging to the system $|\Gamma_0 + (n+1)f|$, we have the following:*

- (i) $h^0(\Theta_{\mathbb{F}_n, L}) = 2$, $h^1(\Theta_{\mathbb{F}_n, L}) = n-1$, and $h^2(\Theta_{\mathbb{F}_n, L}) = 0$.

(ii) The natural map $H^1(\Theta_{\mathbb{F}_n, L}) \rightarrow H^1(\Theta_{\mathbb{F}_n})$ is isomorphic.

(iii) The complex structure of the pair (\mathbb{F}_n, L) is independent of the choice of the section L .

Proof. Let $L \in |\Gamma_0 + (n+1)f|$ be a section as in the proposition. Then we readily have

$$(L, \Gamma_0) = 1, \quad (L, \Gamma_0 + nf) = n + 1. \quad (2.15)$$

Note that we have not specified a $(+n)$ -section Γ_∞ yet. We write $p = L \cap \Gamma_0$ and let $q \in L$ be any point which is different from p . Then as $\dim |\Gamma_0 + nf| = n + 1$, by dimension counting, there exists a section $\Gamma_\infty \in |\Gamma_0 + nf|$ which touches L at the point q by multiplicity $(n+1)$. Let $T_{\mathbb{C}} \subset \text{Aut } \mathbb{F}_n$ be the maximal torus which is determined by the property that it preserves the two sections Γ_0, Γ_∞ and fixes the two points p, q . The complement $\mathbb{F}_n \setminus \Gamma_\infty$ may be identified with the line bundle $\mathcal{O}(-n)$. Let u be an affine coordinate on $\mathbb{C}\mathbb{P}^1 \setminus \pi(q)$ (where π is the projection $\mathbb{F}_n \rightarrow \mathbb{C}\mathbb{P}^1$ as before), and ζ a fiber coordinate of the line bundle $\mathcal{O}(-n)$ over $U_0 = \mathbb{C}(u)$, so that

$$p = (0, 0), \quad q = (\infty, \infty), \quad \Gamma_0 = \{\zeta = 0\}, \quad \text{and} \quad \Gamma_\infty = \{\zeta = \infty\}.$$

Then as L intersects Γ_0 transversally at p by (2.15), a defining equation for L has to be of the form, in the above coordinates,

$$\zeta = uh(u), \quad h(0) \neq 0,$$

where $h = h(u)$ is a holomorphic function on $U_0 = \mathbb{C}(u)$. In the coordinates $(v, \eta^{-1}) := (u^{-1}, u^{-n}\zeta^{-1})$ around the point $q = (\infty, \infty)$, this can be rewritten as $\eta^{-1} = v^{n+1}/h(v^{-1})$. Then since L touches Γ_∞ at the point q by multiplicity $(n+1)$, the function $h(v^{-1})$ cannot have a pole at $v = 0$. This means that $h(u)$ is a constant. Hence L is defined by the equation $\zeta = cu$ for some $c \in \mathbb{C}^*$. But we may assume $c = 1$ by changing the fiber coordinate ζ to $c^{-1}\zeta$. Thus L is defined by $\zeta = u$ in the coordinates (u, ζ) . In particular this proves the assertion (iii).

Next in order to determine $h^0(\Theta_{\mathbb{F}_n, L})$, we recall that, in terms of the coordinates (u, ζ) , any vector field $\theta \in H^0(\Theta_{\mathbb{F}_n})$ is concretely written as (see [10, pp. 43–44])

$$\theta = g(u) \frac{\partial}{\partial u} + (f(u)\zeta^2 + c\zeta u) \frac{\partial}{\partial \zeta}, \quad (2.16)$$

where $g(u) = a_1u^2 + a_2u + a_3$ ($a_i \in \mathbb{C}$), $f(u) = b_1u^n + b_2u^{n-1} + \dots + b_{n+1}$ ($b_i \in \mathbb{C}$) and $c \in \mathbb{C}$. (So we have $3 + (n+1) + 1 = n+5$ parameters in total, which agrees with $h^0(\Theta_{\mathbb{F}_n}) = n+5$.)

For later use, we let $l \geq 1$ and let the section L to be defined by $\zeta = u^l$, and define $F(u, \zeta) := \zeta - u^l$, so that F is a defining equation of L . Then $\theta \in H^0(\Theta_{\mathbb{F}_n, L})$ iff the derivation θF satisfies $\theta F|_L = 0$. By (2.16) we have

$$\theta F = f\zeta^2 + cu\zeta - lgu^{l-1}.$$

Hence by substituting $\zeta = u^l$, the restriction becomes

$$\begin{aligned} \theta F|_L &= (b_1u^n + b_2u^{n-1} + \dots + b_{n+1})u^{2l} + cu^{l+1} - l(a_1u^2 + a_2u + a_3)u^{l-1} \\ &= (b_1u^{2l+n} + b_2u^{2l+n-1} + \dots + b_{n+1}u^{2l}) + \{(c - la_1)u^{l+1} - la_2u^l - la_3u^{l-1}\}. \end{aligned} \quad (2.17)$$

When $l = 1$, we have $2l = l + 1$, and we obtain

$$\begin{aligned} \theta F|_L &= (b_1u^{n+2} + b_2u^{n+1} + \dots + b_{n+1}u^2) + \{(c - a_1)u^2 - a_2u - a_3\} \\ &= b_1u^{n+2} + b_2u^{n+1} + \dots + b_{n+1}u^3 + \{b_{n+1} + (c - a_1)\}u^2 - a_2u - a_3. \end{aligned}$$

Thus $\theta F|_L = 0$ iff

$$b_1 = b_2 = \cdots = b_n = 0, \quad b_{n+1} + c - a_1 = a_2 = a_3 = 0.$$

These imply $h^0(\Theta_{\mathbb{F}_n, L}) = 2$.

It remains to compute $h^i(\Theta_{\mathbb{F}_n, L})$ for $i \in \{1, 2\}$ and show the isomorphicity in (ii). But these follow readily from (2.1), the standard exact sequence $0 \rightarrow \Theta_{\mathbb{F}_n, L} \rightarrow \Theta_{\mathbb{F}_n} \rightarrow N_{L/\mathbb{F}_n} \rightarrow 0$ and $h^0(\Theta_{\mathbb{F}_n, L}) = 2$. The assertion (iii) is already shown. \blacksquare

By the proposition, the group $\text{Aut}_0(\mathbb{F}_n, L)$ is 2-dimensional when $l = 1$. This group can also be readily determined in a concrete form. For this, as before let p be the intersection point of L and Γ_0 . (By (2.15) L and Γ_0 intersect transversally at a unique point.)

Proposition 2.4. *Suppose $n > 0$ and let L be any section of $\pi : \mathbb{F}_n \rightarrow \mathbb{C}\mathbb{P}^1$ belonging to the system $|\Gamma_0 + (n+1)f|$. Then the 2-dimensional group $\text{Aut}_0(\mathbb{F}_n, L)$ can be naturally identified with the group $\{g \in \text{PGL}(2, \mathbb{C}) \mid g(\pi(p)) = \pi(p)\}$, which is isomorphic to the affine transformation group $\text{Af}(\mathbb{C})$.*

Proof. The kernel sheaf of the restriction of the natural surjection $\Theta_{\mathbb{F}_n} \rightarrow \pi^*\Theta_{\mathbb{C}\mathbb{P}^1}$ to the subsheaf $\Theta_{\mathbb{F}_n, L}$ can be obtained in a similar way to the first column of the commutative diagram in the proof of Proposition 2.2, and consequently we obtain the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{F}_n}(\Gamma_0 - f) \longrightarrow \Theta_{\mathbb{F}_n, L} \longrightarrow \pi^*\Theta_{\mathbb{C}\mathbb{P}^1} \longrightarrow 0. \quad (2.18)$$

Clearly we have $H^0(\mathcal{O}_{\mathbb{F}_n}(\Gamma_0 - f)) = 0$. Hence we obtain that the natural homomorphism $H^0(\Theta_{\mathbb{F}_n, L}) \rightarrow H^0(\pi^*\Theta_{\mathbb{C}\mathbb{P}^1}) \simeq H^0(\Theta_{\mathbb{C}\mathbb{P}^1})$ is injective. Therefore, unlike $\text{Aut}_0\mathbb{F}_n$, the subgroup $\text{Aut}_0(\mathbb{F}_n, L)$ can be regarded as a subgroup of $\text{PGL}(2, \mathbb{C})$. Moreover any $g \in \text{Aut}_0(\mathbb{F}_n, L)$ has to fix the point $p = \Gamma_0 \cap L$, since Γ_0 is $\text{Aut}\mathbb{F}_n$ -invariant as $n > 0$. Hence under the above inclusion $\text{Aut}_0(\mathbb{F}_n, L) \subset \text{PGL}(2, \mathbb{C})$, $\text{Aut}_0(\mathbb{F}_n, L)$ is included in the subgroup of $\text{PGL}(2, \mathbb{C})$ in the proposition. But since we already know $\dim \text{Aut}_0(\mathbb{F}_n, L) = 2$ by Proposition 2.3, we obtain the coincidence. \blacksquare

Thus the computations for $h^i(\Theta_{\mathbb{F}_n, L})$ and $\text{Aut}_0(\mathbb{F}_n, L)$ is over for arbitrary sections when $l \in \{0, 1\}$. Next we consider the case $l > 1$. In this case, the situation is not completely homogeneous:

Proposition 2.5. *Suppose $n > 0$, $l > 1$, and let L be any section of $\mathbb{F}_n \rightarrow \mathbb{C}\mathbb{P}^1$ belonging to the system $|\Gamma_0 + (n+l)f|$. Then we have $H^2(\Theta_{\mathbb{F}_n, L}) = 0$. Further one of the following holds:*

(i) $h^0(\Theta_{\mathbb{F}_n, L}) = 1$, $h^1(\Theta_{\mathbb{F}_n, L}) = n + 2l - 4$, and there is an exact sequence

$$0 \longrightarrow \mathbb{C}^{2l-3} \longrightarrow H^1(\Theta_{\mathbb{F}_n, L}) \longrightarrow H^1(\Theta_{\mathbb{F}_n}) \longrightarrow 0.$$

(ii) $h^0(\Theta_{\mathbb{F}_n, L}) = 0$, $h^1(\Theta_{\mathbb{F}_n, L}) = n + 2l - 5$, and there is an exact sequence

$$0 \longrightarrow \mathbb{C}^{2l-4} \longrightarrow H^1(\Theta_{\mathbb{F}_n, L}) \longrightarrow H^1(\Theta_{\mathbb{F}_n}) \longrightarrow 0.$$

Furthermore, as long as the section L satisfies (i), the complex structure of the pair (\mathbb{F}_n, L) is independent of the choice of L , and we have $\text{Aut}_0(\mathbb{F}_n, L) \simeq \mathbb{C}^*$.

Proof. The vanishing $H^2(\Theta_{\mathbb{F}_n, L}) = 0$ is an immediate consequence of the exact sequence

$$0 \longrightarrow \Theta_{\mathbb{F}_n, L} \longrightarrow \Theta_{\mathbb{F}_n} \longrightarrow N_{L/\mathbb{F}_n} \longrightarrow 0 \quad (2.19)$$

since as $N_{L/\mathbb{F}_n} \simeq \mathcal{O}(n+2l)$ we have $H^1(N_{L/\mathbb{F}_n}) = 0$ and also $H^2(\Theta_{\mathbb{F}_n}) = 0$.

The ingredient is to show $h^0(\Theta_{\mathbb{F}_n, L}) \leq 1$. For this we first note that, in the same way to the first column of the commutative diagram in the proof of Proposition 2.2 or the exact sequence (2.18) in the case $l = 1$, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{F}_n}(\Gamma_0 - lf) \longrightarrow \Theta_{\mathbb{F}_n, L} \longrightarrow \pi^* \Theta_{\mathbb{C}\mathbb{P}^1} \longrightarrow 0.$$

This again means that the natural map $H^0(\Theta_{\mathbb{F}_n, L}) \rightarrow H^0(\Theta_{\mathbb{C}\mathbb{P}^1})$ is injective, and hence $\text{Aut}_0(\mathbb{F}_n, L)$ may be considered as a subgroup of $\text{PGL}(2, \mathbb{C})$. Moreover, since Γ_0 is $\text{Aut}\mathbb{F}_n$ -invariant, elements of $\text{Aut}_0(\mathbb{F}_n, L)$ fix any point of the intersection $\Gamma_0 \cap L$. Since we have $(L, \Gamma_0) = (\Gamma_0 + (n+l)f, \Gamma_0) = l \geq 2$, $L \cap \Gamma_0$ is non-empty. If it consists of more than two points, then the image of $\text{Aut}_0(\mathbb{F}_n, L) \rightarrow \text{PGL}(2, \mathbb{C})$ is clearly identity, and so $\text{Aut}_0(\mathbb{F}_n, L)$ is trivial, meaning $h^0(\Theta_{\mathbb{F}_n, L}) = 0$.

If $L \cap \Gamma_0$ consists of two points, the image of $\text{Aut}_0(\mathbb{F}_n, L) \rightarrow \text{PGL}(2, \mathbb{C})$ is included in the \mathbb{C}^* -subgroup determined by the two points. Suppose that the image is actually the \mathbb{C}^* -subgroup, and let $T_{\mathbb{C}}$ be the maximal torus of $\text{Aut}_0(\mathbb{F}_n)$ which contains $\text{Aut}_0(\mathbb{F}_n, L) (\simeq \mathbb{C}^*)$. Then $T_{\mathbb{C}}$ determines on \mathbb{F}_n a structure of toric surface, and singles out a $(+n)$ -section Γ_{∞} by $T_{\mathbb{C}}$ -invariance. Moreover L cannot intersect Γ_{∞} since L minus the two fixed points $L \cap \Gamma_0$ forms an orbit of the \mathbb{C}^* -subgroup of $T_{\mathbb{C}}$, and Γ_{∞} is disjoint from the unique 2-dimensional orbit of the $T_{\mathbb{C}}$ -action. This contradicts $(L, \Gamma_{\infty}) = n + l (> 0)$. Therefore if $L \cap \Gamma_0$ consists of two points, $\text{Aut}_0(\mathbb{F}_n, L)$ is trivial.

If $L \cap \Gamma_0$ consists of one point, since $(L, \Gamma_0) = l$, as in the same way to the proof of Proposition 2.3, we can find coordinates (u, ζ) on the line bundle $\mathcal{O}(-n) \subset \mathbb{F}_n$ such that the point $L \cap \Gamma_0$ corresponds to the origin and L is defined by an equation $\zeta = u^l$. In these coordinates the $(+n)$ -section defined by the equation $\zeta = \infty$ intersects L at the unique point (∞, ∞) by the biggest multiplicity $(n+l)$. Then using the computations in Proposition 2.3, by writing a vector field $\theta \in H^0(\Theta_{\mathbb{F}_n})$ as in (2.16), we have (2.17) for the derivative $\theta F|_L$ of the defining equation $F = \zeta - u^l$ of L . Now as $l > 1$ we have $2l > l + 1$. Hence looking the powers to u in (2.17), the vanishing $\theta F|_L = 0$ is equivalent to the equations

$$b_1 = b_2 = \dots = b_{n+1} = 0, \quad c - la_1 = a_2 = a_3 = 0.$$

From these we obtain $h^0(\Theta_{\mathbb{F}_n, L}) = 1$. Thus we have seen that if $l > 1$ we always have $h^0(\Theta_{\mathbb{F}_n, L}) \leq 1$ and the equality holds exactly when L touches the section Γ_0 at a point by the biggest multiplicity.

Once this is obtained, the assertions (i) and (ii) are readily obtained from the exact sequence (2.19). We omit the detail. The final assertion is clear from the above argument since we have seen that if $h^0(\Theta_{\mathbb{F}_n, L}) = 1$, equation for L can be taken as $\zeta = u^l$ in the coordinates (u, ζ) on the line bundle $\mathcal{O}(-n)$. \blacksquare

As a corollary to the results in this subsection, we obtain the following result on the existence and uniqueness up to isomorphisms of \mathbb{C}^* -invariant pair (\mathbb{F}_n, L) for each n and l :

Corollary 2.6. *For each integers $n \geq 0$ and $l \geq 0$, there exists a section L of $\mathbb{F}_n \rightarrow \mathbb{C}\mathbb{P}^1$ which satisfies the following two properties:*

- (i) $L \in |\Gamma_0 + (n+l)f|$,
- (ii) L is \mathbb{C}^* -invariant, where \mathbb{C}^* is a subgroup of $\text{Aut}\mathbb{F}_n$ which acts non-trivially on L .

Moreover, for each n and l the complex structure of the pair (\mathbb{F}_n, L) is independent of the choice of such a section L .

Proof. The assertion for the case $n > 0$ follows from Propositions 2.2, 2.3 and 2.5. The assertion for the case $n = 0$ is immediate to see. \blacksquare

Finally in this subsection, we discuss variation of the complex structures of the affine bundles in the family $\mathcal{A}_n \rightarrow \mathbb{C}^{n-1}$. As in the beginning of this subsection, by identifying the total space of the line bundle $\mathcal{O}(-n)$ with the minimal resolution of $\mathbb{C}^2/\mathbb{Z}_n$, the line bundle $\mathcal{O}(-n)$ admits a $\mathrm{GL}(2, \mathbb{C})$ -action. This naturally gives rise to a $\mathrm{GL}(2, \mathbb{C})$ -action on the cohomology group $H^1(\mathbb{CP}^1, \mathcal{O}(-n))$. Recalling that this cohomology group is exactly the base space of the family $\mathcal{A}_n \rightarrow \mathbb{C}^{n-1}$, fibers of the family are mutually biholomorphic if they are over the same orbit of the $\mathrm{GL}(2, \mathbb{C})$ -action.

From the results in [3], the $\mathrm{GL}(2, \mathbb{C})$ -action on the base space \mathbb{C}^{n-1} is identified with the tensor product

$$S_1^{n-2}\mathbb{C}^2 := S^{n-2}\mathbb{C}^2 \otimes \mathbb{C}_1,$$

where $S^{n-2}\mathbb{C}^2$ is the $(n-2)$ -th symmetric product of the natural $\mathrm{GL}(2, \mathbb{C})$ -action on \mathbb{C}^2 , and \mathbb{C}_1 is the 1-dimensional representation of $\mathrm{GL}(2, \mathbb{C})$ which is just the multiplication of the determinant. (See Section 3, especially the isomorphisms (3.14).) It follows that if $n \in \{2, 3\}$ the $\mathrm{GL}(2, \mathbb{C})$ -action on the base space $\mathbb{C}^{n-1} \setminus \{0\}$ is transitive. Therefore when $n \in \{2, 3\}$, any member of the family $\mathcal{A}_n \rightarrow \mathbb{C}^{n-1}$ is biholomorphic except the central fiber $\mathcal{O}(-n)$. These can also be seen by just noting that, if $n \in \{2, 3\}$, any fiber except the central fiber is identified with $\mathbb{F}_{n-2} \setminus L$ with $L \in |\Gamma_0 + (n-1)f|$, and also that the complex structure of the pair (\mathbb{F}_{n-2}, L) is independent of the choice of a non-singular member $L \in |\Gamma_0 + (n-1)f|$ by Proposition 2.3.

On the other hand, if $n \geq 4$, the $\mathrm{GL}(2, \mathbb{C})$ -action on the base space $S_1^{n-2}\mathbb{C}^2 = \mathbb{C}^{n-1}$ minus the origin is not transitive, and so the quotient space $(\mathbb{C}^{n-1} \setminus \{0\})/\mathrm{GL}(2, \mathbb{C})$ consists of more than two elements. As the total spaces of the affine bundles are not only open but also do not have compact holomorphic curves, it seems difficult to determine when two affine surfaces lying over different $\mathrm{GL}(2, \mathbb{C})$ -orbits are mutually biholomorphic (if $n \geq 4$).

2.5 Computations for surfaces of smooth normal crossing

In this subsection we first construct a variety of smooth normal crossing from two copies of the Hirzebruch surface \mathbb{F}_n by identifying the same sections, and then compute cohomology groups for them. In the next section these varieties will be included as a subvariety in twistor spaces of the 4-dimensional orbifold $\widehat{\mathcal{O}(-n)}$.

For this let D be a non-singular complex surface and $L \subset D$ a non-singular rational curve. Denoting J for the complex structure on D , we denote by \overline{D} the complex surface obtained from D by changing the complex structure J to $-J$. Let $\mathrm{id} : D \rightarrow \overline{D}$ be the identity map. This is an *anti*-holomorphic map. Write $\overline{L} := \mathrm{id}(L) \subset \overline{D}$. Let $\tau : L \rightarrow L$ be an anti-holomorphic involutions of $L \simeq \mathbb{CP}^1$, and we define a map $\phi : L \rightarrow \overline{L}$ by $\phi := \mathrm{id}|_L \circ \tau$. Since both $\mathrm{id}|_L$ and τ are anti-holomorphic, ϕ is a holomorphic map. Let $D \cup_{L, \tau} \overline{D}$ be the space obtained from the disjoint union $D \sqcup \overline{D}$ by identifying L and \overline{L} by ϕ . By the holomorphicity of ϕ , $D \cup_{L, \tau} \overline{D}$ is naturally equipped with the structure of a complex variety which is smooth normal crossing.

Let $\sigma : D \sqcup \overline{D} \rightarrow D \sqcup \overline{D}$ be the map defined by

$$\sigma(p) = \begin{cases} \mathrm{id}(p) & \text{if } p \in D, \\ \mathrm{id}^{-1}(p) & \text{if } p \in \overline{D}. \end{cases}$$

This is clearly an involution which flips the two components (as the map id flips from the definition), and is an anti-holomorphic map since id and id^{-1} are. If the two points $p \in L$ and $q \in \overline{L}$ satisfy $q = \phi(p)$, we have

$$\phi(\sigma(q)) = \phi(\mathrm{id}^{-1}(q)) = \mathrm{id} \circ \tau \circ \mathrm{id}^{-1}(q) = \mathrm{id} \circ \tau \circ \mathrm{id}^{-1}(\mathrm{id} \circ \tau(p)) = \mathrm{id}(p) = \sigma(p).$$

Namely we have $\phi(\sigma(q)) = \sigma(p)$. Hence σ descends to an endomorphism of $D \cup_{L,\tau} \bar{D}$. We use the same letter σ for this map. This is an anti-holomorphic involution since the original σ is. Thus the variety $D \cup_{L,\tau} \bar{D}$ is naturally equipped with a real structure. The structure of $D \cup_{L,\tau} \bar{D}$ as a complex variety with a real structure depends not only on the rational curve L but also on the involution τ . Further, if $p \in L$, we have

$$\phi^{-1}(\sigma(p)) = (\text{id} \circ \tau)^{-1}(\text{id}(p)) = \tau^{-1}(p) = \tau(p).$$

This means that on the intersection $D \cap \bar{D} \subset D \cup_{L,\tau} \bar{D}$, the involution σ may be identified with the involution τ on L .

We apply this construction to the pair $(D, L) = (\mathbb{F}_n, L)$, where $n > 0$ and L is a section of $\mathbb{F}_n \rightarrow \mathbb{C}\mathbb{P}^1$ with a positive self-intersection number and an anti-holomorphic involution $\tau : L \rightarrow L$ without a fixed point. As above the structure of the resulting variety $\mathbb{F}_n \cup_{L,\tau} \bar{\mathbb{F}}_n$ depends on the choice of the involution τ . But if the section L is supposed to be invariant under a \mathbb{C}^* -action on \mathbb{F}_n that acts non-trivially on L , then the choice of τ is naturally constrained to be \mathbb{C}^* -equivariant, and consequently if p and q denote the fixed points of the \mathbb{C}^* -action on L , we have $\tau(p) \in \{p, q\}$. But since τ is supposed to have no fixed point, we obtain $\tau(p) = q$. This means that in an affine coordinate u on L for which the \mathbb{C}^* -action is given by $u \mapsto tu$ for $t \in \mathbb{C}^*$, we can write $\tau(u) = -a/\bar{u}$ for some $a > 0$. Therefore the effect of varying τ (namely varying the number $a > 0$) is absorbed in the \mathbb{C}^* -action on \mathbb{F}_n , and moreover by Corollary 2.6, the complex structure of the pair (\mathbb{F}_n, L) is independent of the choice of such a section L . Consequently the variety $\mathbb{F}_n \cup_{L,\tau} \bar{\mathbb{F}}_n$ makes a unique sense. Further the \mathbb{C}^* -actions on (\mathbb{F}_n, L) and $(\bar{\mathbb{F}}_n, \bar{L})$ are naturally glued and the variety is equipped with a \mathbb{C}^* -action. As we are particularly interested in these varieties, we introduce notation for them:

Definition 2.7. For integers $n \geq 0$ and $l \geq 0$, let $L \in |\Gamma_0 + (n+l)f|$ be any \mathbb{C}^* -invariant section on \mathbb{F}_n , and we denote by $\mathbb{F}_n \cup_l \bar{\mathbb{F}}_n$ for the variety of simple normal crossing with \mathbb{C}^* -action, which is obtained from the two copies of the pair (\mathbb{F}_n, L) by identifying two L -s by an anti-holomorphic involution τ without a fixed point in the above way.

The notation $\mathbb{F}_n \cup_l \bar{\mathbb{F}}_n$ reflects the independency from the choices of L and τ . Thus the complex structure of this variety is solely determined by two non-negative integers n and l . For these varieties we have the following.

Proposition 2.8. *Let $n > 0$ and $l \geq 0$. Then for the tangent sheaf Θ of the variety $\mathbb{F}_n \cup_l \bar{\mathbb{F}}_n$ above, we have the following:*

(i) *If $l = 0$, we have*

$$h^0(\mathbb{F}_n \cup_l \bar{\mathbb{F}}_n, \Theta) = 5, \quad h^1(\mathbb{F}_n \cup_l \bar{\mathbb{F}}_n, \Theta) = 2(n-1), \quad h^2(\mathbb{F}_n \cup_l \bar{\mathbb{F}}_n, \Theta) = 0.$$

(ii) *If $l \geq 1$, we have*

$$h^0(\mathbb{F}_n \cup_l \bar{\mathbb{F}}_n, \Theta) = 1, \quad h^1(\mathbb{F}_n \cup_l \bar{\mathbb{F}}_n, \Theta) = 2(n+2l-3), \quad h^2(\mathbb{F}_n \cup_l \bar{\mathbb{F}}_n, \Theta) = 0.$$

Proof. Though these can be shown in a standard way by using Propositions 2.2, 2.3 and 2.5, we write a proof as there is a subtle point that relies on our construction of the variety $\mathbb{F}_n \cup_l \bar{\mathbb{F}}_n$. We have the standard exact sequence $0 \rightarrow \Theta_{\mathbb{F}_n \cup_l \bar{\mathbb{F}}_n} \rightarrow \Theta_{\mathbb{F}_n, L} \oplus \Theta_{\bar{\mathbb{F}}_n, \bar{L}} \rightarrow \Theta_L \rightarrow 0$, where $L \in |\Gamma_0 + (n+l)f|$ is a \mathbb{C}^* -invariant section identified by ϕ . For the case $l = 0$, the natural map $H^0(\Theta_{\mathbb{F}_n, L}) \rightarrow H^0(\Theta_L)$ is surjective from that of $\text{Aut}_0(\mathbb{F}_n, L) \rightarrow \text{Aut } L$. Therefore from the above exact sequence and Proposition 2.2 (i), (ii) we obtain the required value for $h^i(\mathbb{F}_n \cup_0 \bar{\mathbb{F}}_n, \Theta)$ for any i as well as natural isomorphisms

$$H^1(\mathbb{F}_n \cup_0 \bar{\mathbb{F}}_n, \Theta) \simeq H^1(\mathbb{F}_n, \Theta_{\mathbb{F}_n, L}) \oplus H^1(\bar{\mathbb{F}}_n, \Theta_{\bar{\mathbb{F}}_n, \bar{L}}) \quad (2.20)$$

$$\simeq H^1(\mathbb{F}_n, \Theta_{\mathbb{F}_n}) \oplus H^1(\bar{\mathbb{F}}_n, \Theta_{\bar{\mathbb{F}}_n}). \quad (2.21)$$

Next for the case $l = 1$, the natural homomorphism $\text{Aut}_0(\mathbb{F}_n, L) \rightarrow \text{Aut } L$ is not surjective and the image is the affine transformation group as in Proposition 2.4. Namely it consists of elements of $\text{Aut } L$ which fixes the point $p = \Gamma_0 \cap L$. As our involution τ is supposed to interchange the two fixed points p and q of the \mathbb{C}^* -action, it follows that the image of the natural map $H^0(\mathbb{F}_n, \Theta_{\mathbb{F}_n, L}) \oplus H^0(\overline{\mathbb{F}}_n, \Theta_{\overline{\mathbb{F}}_n, \overline{L}}) \rightarrow H^0(\Theta_L)$ is again surjective since the two affine groups generate $\text{Aut } L$. Hence the cohomology exact sequence takes the same form as the case $l = 0$, and by using Proposition 2.3 (i) and (ii), we obtain the required value of $h^i(\mathbb{F}_n \cup_1 \overline{\mathbb{F}}_n, \Theta)$ as well as the natural isomorphisms (2.20) and (2.21).

Finally if $l > 1$, by Proposition 2.5 the image of the natural injection $\text{Aut}_0(\mathbb{F}_n, L) \rightarrow \text{Aut } L$ is the \mathbb{C}^* -subgroup that fixes the two points p and q . Therefore from our choice of τ , the image of the natural map $H^0(\mathbb{F}_n, \Theta_{\mathbb{F}_n, L}) \oplus H^0(\overline{\mathbb{F}}_n, \Theta_{\overline{\mathbb{F}}_n, \overline{L}}) \rightarrow H^0(\Theta_L)$ is 1-dimensional. Therefore from the cohomology sequence we obtain $H^0(\Theta_{\mathbb{F}_n \cup_l \overline{\mathbb{F}}_n}) \simeq \mathbb{C}$, the exact sequence

$$0 \longrightarrow \mathbb{C}^2 \longrightarrow H^1(\mathbb{F}_n \cup_l \overline{\mathbb{F}}_n, \Theta) \longrightarrow H^1(\mathbb{F}_n, \Theta_{\mathbb{F}_n, L}) \oplus H^1(\overline{\mathbb{F}}_n, \Theta_{\overline{\mathbb{F}}_n, \overline{L}}) \longrightarrow 0$$

and the isomorphism $H^2(\mathbb{F}_n \cup_l \overline{\mathbb{F}}_n, \Theta) \simeq H^2(\mathbb{F}_n, \Theta_{\mathbb{F}_n, L}) \oplus H^2(\overline{\mathbb{F}}_n, \Theta_{\overline{\mathbb{F}}_n, \overline{L}})$. From Proposition 2.5, we finish the proof of the assertion (ii). \blacksquare

3 Computations for twistor spaces

In this section, based on the results in the previous section, we intensively study small deformations of the LeBrun metric on $\mathcal{O}(-n)$ which preserve ALE SFK properties, and in particular show that any affine \mathbb{C} -bundle over $\mathbb{C}\mathbb{P}^1$ of negative degree admits an ALE SFK metric. Next we investigate small deformations of the metrics on the affine bundles again as ALE SFK metrics, and in particular show that even if we fix the complex structure on the affine bundles, they admit a 1-parameter deformation for which the conformal classes are not constant.

3.1 Some generalities on twistor spaces of ALE SFK metrics

Before starting actual computations, we briefly recall basic properties of the twistor spaces of ALE SFK metrics, including its natural compactification. These will be used for investigating deformations of metrics which preserve ALE SFK property. For more precise treatment on compactifications of ALE ASD 4-manifolds, we refer the paper [14].

Let (X, J) be a complex surface, g an ASD Hermitian metric on it, Z the twistor space of the ASD conformal class $[g]$, and $F := K_Z^{-1/2}$ the natural square root of the anticanonical line bundle of Z , which is available on any twistor space. Then the complex structure J determines a section of the twistor projection $Z \rightarrow X$ in a tautological way, and its image becomes a non-singular divisor D on Z . D is biholomorphic to X by the projection $Z \rightarrow X$. Let \overline{D} be the divisor determined by the conjugate complex structure $-J$ on X . We always have $D \cap \overline{D} = \emptyset$ as $J \neq -J$. Then Pontecorvo's theorem [11] means that the ASD Hermitian metric g is Kähler with respect to J if and only if $D + \overline{D} \in |F|$.

If X is non-compact with one end and the ASD metric g is asymptotically Euclidean at infinity, then $(X, [g])$ can be compactified as an ASD manifold by adding a point at infinity. Let $(\hat{X}, [\hat{g}])$ be the resulting compact ASD manifold, and \hat{Z} the twistor space of $(\hat{X}, [\hat{g}])$, which is smooth. Then the closure $\text{Cl}(D)$ of the above divisor $D \subset Z$ is a divisor in \hat{Z} , and from the ALE SFK property of the metric [9, proof of Proposition 6, p. 312], the divisor $\text{Cl}(D)$ satisfies the following properties: (i) $\text{Cl}(D)$ is still non-singular and $\text{Cl}(D) = D \sqcup L$, where L is the twistor line over the point at infinity, (ii) $\text{Cl}(D) \cap \text{Cl}(\overline{D}) = L$, and the intersection is transverse, and (iii) the normal bundle of L in $\text{Cl}(D)$ (and also in $\text{Cl}(\overline{D})$) is of degree one. Conversely a divisor

in \hat{Z} satisfying these properties determines, up to overall constants, an ASD Kähler metric on (X, J) which is asymptotically Euclidean at infinity.

When the SFK surface (X, J, g) is ALE in a strict sense (i.e. asymptotic to the flat Euclidean orbifold \mathbb{C}^2/Γ at infinity, where Γ is a non-trivial finite subgroup of $U(2)$ acting freely on the unit sphere), the pair (X, g) has a natural compactification (\hat{X}, \hat{g}) as an ASD orbifold, which means that \hat{X} is an orbifold of the form $X \cup \{\infty\}$ with ∞ being an orbifold point of \hat{X} , and \hat{g} is an ASD orbifold metric on \hat{X} whose conformal class on X remains to be equal to g . Also the twistor space Z of (X, g) has a natural compactification, for which we again denote by \hat{Z} . This is of course the twistor space of the ASD orbifold (\hat{X}, \hat{g}) in a natural sense, and we again have $\hat{Z} = Z \sqcup L$, where L is the twistor line over the orbifold point ∞ . We have $\text{Sing } \hat{Z} \subset L$, and all singularities are quotient singularity by the group which is orientation-reversing conjugate (namely conjugate after reversing the orientation; see [14, Definition 1.4] for the precise definition) to the above group Γ . Especially, denoting $U(1) \subset U(2)$ for the subgroup of consisting of scalar matrices, if Γ is a cyclic subgroup of $U(1)$ with order $n \geq 2$, then \hat{Z} has A_{n-1} -singularities along L . (This is particular to these subgroups, and for other subgroup $\Gamma \subset U(2)$, singular points of \hat{Z} are isolated.) Moreover if $D \subset Z$ again denotes the divisor determined by the complex structure J on X and $\text{Cl}(D)$ means its closure in \hat{Z} , then $\text{Cl}(D)$ itself (and therefore $\text{Cl}(\overline{D})$ also) is a non-singular (but non-Cartier) divisor on \hat{Z} . Moreover we have $\text{Cl}(D) \cap \text{Cl}(\overline{D}) = L$, and the normal bundle of L in $\text{Cl}(D)$ (and also in $\text{Cl}(\overline{D})$) is of degree n . Furthermore the union $\text{Cl}(D) \cup \text{Cl}(\overline{D})$ itself is smooth normal crossing. We also note that in this situation the natural extension of the line bundle F over Z to \hat{Z} is not just an orbifold bundle but an ordinary line bundle; in other words the sum $\text{Cl}(D) + \text{Cl}(\overline{D})$ is a Cartier divisor on \hat{Z} , while $\text{Cl}(D)$ and $\text{Cl}(\overline{D})$ are not.

Because $\mathcal{O}(-n)$ is obtained as the minimal resolution of the quotient space \mathbb{C}^2/Γ where $\Gamma \subset U(2)$ is the cyclic subgroup of scalar matrices of order n , ALE SFK metrics on $\mathcal{O}(-n)$ give rise to the last situation where the compactified twistor space \hat{Z} has A_{n-1} -singularities along the twistor line L at infinity. Here, we do not suppose that the complex structure on $\mathcal{O}(-n)$ is the natural one and we will also consider complex structures which support the affine bundles in Section 2.2. Let $\widehat{\mathcal{O}(-n)}$ be the one-point compactification of the 4-manifold $\mathcal{O}(-n)$, and in the following, instead of the letters \hat{Z} and $\text{Cl}(D)$, we use the letters Z and D respectively to mean the twistor space of the conformal compactification of an ALE SFK metric on the 4-manifold $\mathcal{O}(-n)$ and the (non-Cartier) divisor on Z determined by the complex structure on the 4-manifold $\mathcal{O}(-n)$. In this situation D is biholomorphic to \mathbb{F}_{n-2k} for some $k \geq 0$ satisfying $n - 2k \geq 0$. This is because D contains the twistor line L at infinity as a $(+n)$ -curve as above, which means the rationality of D ; further the decomposition $D = \mathcal{O}(-n) \sqcup L$ as a smooth manifold means $b_2(D) = 2$, and hence $D \simeq \mathbb{F}_m$ for some $m \geq 0$; but $\mathbb{F}_m \rightarrow \mathbb{C}P^1$ has a $(+n)$ -section iff $m = n - 2k$ for some $k \geq 0$. Of course we have $k = 0$ if the complex structure on $\mathcal{O}(-n)$ is the natural one. We also remark that if τ is an anti-holomorphic involution of L without a fixed point, the union $D \cup \overline{D}$, which is a Cartier divisor in Z as above, is isomorphic to the surface $D \cup_{L, \tau} \overline{D}$ constructed in the first half of Section 2.5, as a complex variety with a real structure.

Thus if (Z, D) is a pair of a compact but singular twistor space and a divisor determined by an ALE SFK metric on the 4-manifold $\mathcal{O}(-n)$, deformations of the metric preserving ALE SFK property are equivalent to locally trivial deformations of the pair $(Z, D \cup \overline{D})$ preserving the real structure. For details on locally trivial deformations for complex spaces and pairs of a complex space and a complex subspace of it, we refer a book [12, Section 3.4]. In particular, if we define the subsheaf $\Theta_{Z, D \cup \overline{D}}$ of the tangent sheaf Θ_Z by

$$\Theta_{Z, D \cup \overline{D}} := \{v \in \Theta_Z \mid v(f) \in \mathcal{I}_{D \cup \overline{D}} \text{ if } f \in \mathcal{I}_{D \cup \overline{D}}\},$$

then first order deformations of the pair $(Z, D \cup \overline{D})$ which are locally trivial are in one to one correspondence with the cohomology group $H^1(\Theta_{Z, D \cup \overline{D}})$, and obstructions are in $H^2(\Theta_{Z, D \cup \overline{D}})$.

In particular if $H^2(\Theta_{Z, D \cup \bar{D}}) = 0$, the Kuranishi family for locally trivial deformations of the pair $(Z, D \cup \bar{D})$ is constructed over a neighborhood of the origin in $H^1(\Theta_{Z, D \cup \bar{D}})$.

3.2 Deformations of the LeBrun metric

Having recalled these basic materials, we start to investigate deformations of the LeBrun's ALE SFK metric on $\mathcal{O}(-n)$ as an ALE SFK metrics, by investigating locally trivial deformations of the pair $(Z, D \cup \bar{D})$ of compactified singular twistor space and the divisor. The following proposition provides basic information about such deformations.

Proposition 3.1. *Suppose $n \geq 3$ and let Z be the twistor space on the orbifold $\widehat{\mathcal{O}(-n)}$, which is associated to the conformal compactification of the LeBrun's ALE-SFK metric on $\mathcal{O}(-n)$ with negative mass. Let D be the divisor on Z which is the closure of the section of the twistor fibration that is determined by the complex structure of $\mathcal{O}(-n)$. (D is biholomorphic to \mathbb{F}_n .) Then we have*

$$H^i(\Theta_Z(-D - \bar{D})) = 0, \quad i \neq 1, \quad H^1(\Theta_Z(-D - \bar{D})) \simeq \mathbb{C}, \quad (3.1)$$

$$H^2(\Theta_{Z, D \cup \bar{D}}) = H^2(\Theta_{D \cup \bar{D}}) = 0. \quad (3.2)$$

Moreover there is a natural isomorphism

$$H^1(\Theta_{Z, D \cup \bar{D}}) \simeq H^1(D, \Theta_{D \cup \bar{D}}), \quad (3.3)$$

and these are $2(n-1)$ -dimensional. Furthermore the natural map

$$H^1(\Theta_{Z, D \cup \bar{D}}) \longrightarrow H^1(\Theta_Z) \quad (3.4)$$

is injective, and if $n = 3$, this is moreover surjective.

The isomorphism (3.3) will be of fundamental importance in the rest of this article.

Proof. The vanishing $H^2(\Theta_Z(-D - \bar{D})) = 0$ immediately follows from [3, Proposition 3.1] since S in the proposition is a divisor in the system $|F|$ and hence $\Theta_Z(-D - \bar{D}) \simeq \Theta_Z(-S)$. In order to compute $h^i(\Theta_Z(-D - \bar{D}))$ for $i \in \{0, 1, 3\}$, we use computations in the proof of the above proposition in [3]. Noting $\Theta_Z(-D - \bar{D}) \simeq \Theta_Z \otimes F^{-1}$, the isomorphisms (3.5), (3.6) and (3.8) in the proof of [3, Proposition 3.1] are valid not only for H^2 but also for H^i for any i because we have $H^i(\mathcal{O}_{\mathbb{C}\mathbb{P}^1}(-1)) = H^i(\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1, \mathcal{O}(-1, -1)) = 0$ for any i . Therefore in the notation of that proof, we have $H^i(\Theta_Z \otimes F^{-1}) \simeq H^i(X, \mathcal{L}')$ for any i . Further from the exact sequence (3.9) there, we have $H^i(\mathcal{L}') \simeq H^i(\mathcal{F}')$ for any i . Furthermore from the exact sequence (3.10) there, we obtain

$$H^i(\mathcal{F}') = 0, \quad i \neq 1, \quad H^1(\mathcal{F}') \simeq H^0(\Delta, \mathcal{O}) (\simeq \mathbb{C}).$$

These in particular imply (3.1).

Next in order to deduce (3.2) and (3.3) we consider the standard exact sequence

$$0 \longrightarrow \Theta_Z(-D - \bar{D}) \longrightarrow \Theta_{Z, D \cup \bar{D}} \longrightarrow \Theta_{D \cup \bar{D}} \longrightarrow 0. \quad (3.5)$$

Since the isometry group of the LeBrun metric on $\mathcal{O}(-n)$ is $U(2)/\mathbb{Z}_n$ [8], where \mathbb{Z}_n is the cyclic subgroup consisting of scalar matrices of order n , we have $h^0(\Theta_{Z, D \cup \bar{D}}) = 4$. On the other hand, as $D \cup \bar{D} \simeq \mathbb{F}_n \cup_0 \bar{\mathbb{F}}_n$ biholomorphically, by Proposition 2.8 (i) we have

$$h^0(\Theta_{D \cup \bar{D}}) = 5, \quad h^1(\Theta_{D \cup \bar{D}}) = 2(n-1), \quad h^2(\Theta_{D \cup \bar{D}}) = 0. \quad (3.6)$$

Therefore using (3.1) the cohomology exact sequence of (3.5) implies

$$\begin{aligned} 0 &\longrightarrow H^0(\Theta_{Z, D \cup \bar{D}}) (\simeq \mathbb{C}^4) \longrightarrow H^0(\Theta_{D \cup \bar{D}}) (\simeq \mathbb{C}^5) \\ &\longrightarrow H^1(\Theta_Z(-D - \bar{D})) (\simeq \mathbb{C}) \longrightarrow H^1(\Theta_{Z, D \cup \bar{D}}) \longrightarrow H^1(\Theta_{D \cup \bar{D}}) (\simeq \mathbb{C}^{2(n-1)}) \\ &\longrightarrow 0 \longrightarrow H^2(\Theta_{Z, D \cup \bar{D}}) \longrightarrow 0. \end{aligned} \quad (3.7)$$

From this we obtain $H^2(\Theta_{Z, D \cup \bar{D}}) = 0$, an exact sequence

$$0 \longrightarrow H^0(\Theta_{Z, D \cup \bar{D}}) (\simeq \mathbb{C}^4) \longrightarrow H^0(\Theta_{D \cup \bar{D}}) (\simeq \mathbb{C}^5) \longrightarrow H^1(\Theta_Z(-D - \bar{D})) (\simeq \mathbb{C}) \longrightarrow 0, \quad (3.8)$$

and also the isomorphism (3.3). From the last isomorphism we obtain $h^1(\Theta_{Z, D \cup \bar{D}}) = 2(n-1)$ by (3.6).

Finally we show that the map (3.4) is injective. For this let N' be the cokernel sheaf of the natural injection $\Theta_{Z, D \cup \bar{D}} \rightarrow \Theta_Z$. We have an exact sequence $0 \rightarrow \Theta_{Z, D \cup \bar{D}} \rightarrow \Theta_Z \rightarrow N' \rightarrow 0$, and so for the injectivity it suffices to show $H^0(N') = 0$. Let $N := \mathcal{O}_Z(D + \bar{D})|_{D \cup \bar{D}}$ be the normal sheaf of the divisor $D \cup \bar{D}$ in Z . Since $D + \bar{D} \in |F|$ and F is an ordinary line bundle on Z , the sheaf N is an invertible $\mathcal{O}_{D \cup \bar{D}}$ -module, and isomorphic to $F|_{D + \bar{D}}$. Then by computing local generators of the sheaves Θ_Z and $\Theta_{Z, D \cup \bar{D}}$ in coordinates, and then comparing the resulting generators of the cokernel sheaf N' with local generators of N , we obtain a natural isomorphism

$$N' \simeq N \otimes_{\mathcal{O}_{D \cup \bar{D}}} \mathcal{I}_L, \quad (3.9)$$

where $L = D \cap \bar{D}$ is the twistor line over the point at infinity as before, and \mathcal{I}_L is the ideal sheaf of L in $D \cup \bar{D}$. On the other hand, by the adjunction formula we have $K_{D \cup \bar{D}} \simeq K_Z + [D + \bar{D}]|_{D \cup \bar{D}} \simeq -2F + F|_{D + \bar{D}} \simeq -F|_{D \cup \bar{D}}$. Hence from (3.9) we obtain $N' \simeq -K_{D \cup \bar{D}} \otimes \mathcal{I}_L$. Further for the canonical sheaf of $D \cup \bar{D}$, as this itself is smooth normal crossing, we have

$$K_{D \cup \bar{D}}|_D \simeq K_D + [\bar{D}]|_D \simeq K_D + \mathcal{O}_D(L),$$

and similar for $K_{D \cup \bar{D}}|_{\bar{D}}$. Hence by taking the inverse for these and taking a tensor product with $\mathcal{O}(-L)$, we obtain

$$N'|_D \simeq -K_D - \mathcal{O}_D(2L), \quad N'|_{\bar{D}} \simeq -K_{\bar{D}} - \mathcal{O}_{\bar{D}}(2L). \quad (3.10)$$

Now as $D \simeq \mathbb{F}_n$ we have $-K_D \simeq 2\Gamma_0 + (n+2)f$, and as L is a $(+n)$ -section we have $\mathcal{O}_D(L) \simeq \Gamma_0 + nf$. Hence we have

$$-K_D - \mathcal{O}_D(2L) \simeq 2\Gamma_0 + (n+2)f - 2(\Gamma_0 + nf) \simeq -(n-2)f. \quad (3.11)$$

Thus as $n-2 > 0$ from the assumption $n > 2$ we obtain $H^0(-K_D - \mathcal{O}_D(2L)) = 0$. With reality, this means $H^0(D \cup \bar{D}, N') = 0$. Thus the injectivity of (3.4) follows. If $n = 3$, the map is also surjective since we have $h^1(\Theta_Z) = 4(n-2) = 4$ by [3, Proposition 2.1], which coincides with $2(n-1) = 4$. \blacksquare

Remark 3.2. The computations and the conclusions in the proposition are valid also for the case $n = 2$ except the injectivity of the map (3.4). For the case $n = 2$, as in (3.11), we have $N'|_D \simeq \mathcal{O}_D$. With reality this means $N' \simeq \mathcal{O}_{D \cup \bar{D}}$, and hence we have $H^0(N') \simeq \mathbb{C}$. Further from the cohomology exact sequence this is mapped to $H^1(\Theta_{Z, D \cup \bar{D}})$ injectively. Thus the map (3.4) has a 1-dimensional kernel.

Next, letting Z and D be as in Proposition 3.1, we collect basic results on versal families of locally trivial deformations of Z , $(Z, D \cup \overline{D})$ and $D \cup \overline{D}$ and their relationship, which are readily derived from Proposition 3.1 and the results in Section 2.

First, for the the twistor space Z of the LeBrun structure on $\widehat{\mathcal{O}(-n)}$, as showed in [3, Proposition 2.1], we have $H^2(\Theta_Z) = 0$ and $h^1(\Theta_Z) = 4n - 8$. Hence the parameter space of the Kuranishi family of locally trivial deformations of Z may be identified with a neighborhood of the origin in $H^1(\Theta_Z) \simeq \mathbb{C}^{4n-8}$. Versal family of Z as twistor spaces is obtained as the restriction of the Kuranishi family onto the real locus of the neighborhood. We denote the last real locus by \mathcal{U}_{ASD} , which is clearly smooth and real $(4n - 8)$ -dimensional. As in [3] we call the corresponding family of ASD conformal structures on $\widehat{\mathcal{O}(-n)}$ (parameterized by \mathcal{U}_{ASD}) as the *versal family of ASD structures* for the LeBrun structure. If $n > 3$, not all these ASD structures preserve the Kähler representative. From the construction we have a canonical isomorphism

$$T_0 \mathcal{U}_{\text{ASD}} \simeq H^1(\Theta_Z)^\sigma$$

as real vector spaces, where the upper-script means the real subspace.

Second, for the pair $(Z, D \cup \overline{D})$, in a similar way to the above argument, since $H^2(\Theta_{Z, D \cup \overline{D}}) = 0$ and $h^1(\Theta_{Z, D \cup \overline{D}}) = 2(n - 1)$ as in Proposition 3.1, the parameter space of the Kuranishi family for locally trivial deformations of the pair $(Z, D \cup \overline{D})$ is identified with a neighborhood of the origin in $H^1(\Theta_{Z, D \cup \overline{D}}) \simeq \mathbb{C}^{2(n-1)}$. Restricting this to the real locus, we obtain a deformation of Z preserving not only a structure of twistor space but also the Kähler representative in the conformal class. Let $\mathcal{K}' \subset H^1(\Theta_{Z, D \cup \overline{D}})^\sigma$ be the parameter space of this family. We have a natural isomorphism $T_0 \mathcal{K}' \simeq H^1(\Theta_{Z, D \cup \overline{D}})^\sigma$.

For a relationship between the families over \mathcal{U}_{ASD} (of twistor spaces) and \mathcal{K}' (of pairs of twistor spaces and Cartier divisors), by versality, after a possible shrinking of the domain, there is an induced map, for which we denote by ψ_1 , from \mathcal{K}' to \mathcal{U}_{ASD} , such that the pullback by ψ_1 of the family over \mathcal{U}_{ASD} is isomorphic to the Z -portion of the family of pairs over \mathcal{K}' . Though ψ_1 is not uniquely determined, the derivative $\psi_1'(0)$ is exactly the restriction of the map (3.4) to the real locus. By the proposition the last map is injective, and moreover isomorphism if $n = 3$. So if $n = 3$ we may think $\mathcal{K}' \simeq \mathcal{U}_{\text{ASD}}$ by ψ_1 . If $n > 3$, since $h^1(\Theta_{Z, D \cup \overline{D}}) = 2(n - 1)$ by Proposition 3.1, $\psi_1 : \mathcal{K}' \rightarrow \mathcal{U}_{\text{ASD}}$ is an embedding as a real submanifold of dimension $2(n - 1)$ in \mathcal{U}_{ASD} (and $\dim \mathcal{U}_{\text{ASD}} = 4(n - 2)$ as above). We call the image $\psi_1(\mathcal{K}')$ the *Kähler locus* in \mathcal{U}_{ASD} and denote it by \mathcal{K} . If $n = 3$, we may think $\mathcal{K} = \mathcal{U}_{\text{ASD}}$ as above. From the construction we have a natural isomorphism

$$\psi_1'(0)^{-1} : T_0 \mathcal{K} \xrightarrow{\sim} H^1(\Theta_{Z, D \cup \overline{D}})^\sigma$$

as real vector spaces, where the upper-script means the real subspace.

Next for locally trivial deformations of the variety $D \cup \overline{D}$, since $H^2(\Theta_{D \cup \overline{D}}) = 0$ by Proposition 2.8 (i), the Kuranishi family is parameterized by a neighborhood of the origin in $H^1(\Theta_{D \cup \overline{D}})$. Denote $\mathcal{J} \subset H^1(\Theta_{D \cup \overline{D}})^\sigma$ for the real locus of the neighborhood. Then again by versality, after a possible shrinking of the domain, there is an induced map, for which we denote by ψ_2 , from \mathcal{K}' to \mathcal{J} that induces an isomorphism between the two families. Similarly to ψ_1 , while ψ_2 is not uniquely determined, the derivative $\psi_2'(0)$ is identified with the real part of the natural map $H^1(\Theta_{Z, D \cup \overline{D}}) \rightarrow H^1(\Theta_{D \cup \overline{D}})$. The last map is an isomorphism by (3.3), and therefore ψ_2 is isomorphic in a neighborhood of the origin in \mathcal{K}' . Hence the composition $\psi_2 \circ \psi_1^{-1}$ gives an isomorphism from the Kähler locus $\mathcal{K} \subset \mathcal{U}_{\text{ASD}}$ to \mathcal{J} , and the $D \cup \overline{D}$ -portion of the families of pairs over \mathcal{K}' and the family over \mathcal{J} are isomorphic by ψ_2 . The situation is summarized as in

the following diagram:

$$\begin{array}{ccccc}
 H^1(\Theta_{D \cup \bar{D}})^\sigma & \xleftarrow{\sim} & H^1(\Theta_{Z, D \cup \bar{D}})^\sigma & \xrightarrow{\text{inj.}} & H^1(\Theta_Z)^\sigma \\
 \text{incl.} \uparrow & & \text{incl.} \uparrow & & \text{incl.} \uparrow \\
 \mathcal{J} & \xleftarrow[\psi_2]{\sim} & \mathcal{K}' & \xrightarrow[\psi_1]{\sim} & \mathcal{K} \subset \mathcal{U}_{\text{ASD}}
 \end{array} \tag{3.12}$$

(Note again that $\mathcal{K} = \mathcal{U}_{\text{ASD}}$ when $n = 3$.) Thus in order to understand the complex structures on $\mathcal{O}(-n)$ determined by points on \mathcal{K} , it is enough to understand the complex structures on fibers of the family over \mathcal{J} . For this purpose we recall from Sections 2.2 and 2.3 that the Kuranishi family $\mathcal{F}_n \rightarrow \mathbb{C}^{n-1}$ of \mathbb{F}_n is obtained from the family $\mathcal{A}_n \rightarrow \mathbb{C}^{n-1}$ of affine bundles by taking a simultaneous compactification. Let $\mathcal{L}_n := \mathcal{F}_n \setminus \mathcal{A}_n$ be the family of sections at infinity. We now apply the construction in Section 2.5 to all fibers of $(\mathcal{F}_n, \mathcal{L}_n) \rightarrow \mathbb{C}^{n-1}$ simultaneously. For this, we need to give an involution τ on each section to make the variety. For this purpose we note that since all fibers of $\mathcal{F}_n \rightarrow \mathbb{C}^{n-1}$ have a common projection to $\mathbb{C}\mathbb{P}^1$ (equipped with the coordinates u and v as before), all fibers of $\mathcal{L}_n \rightarrow \mathbb{C}^{n-1}$ are naturally identified each other. Let $\tau_0 : \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^1$ be an anti-holomorphic involution defined by $\tau_0(u) = -1/\bar{u}$, and through the identification we regard τ_0 as an anti-holomorphic involution which is defined on each fiber of $\mathcal{L}_n \rightarrow \mathbb{C}^{n-1}$. Then taking this τ_0 as the involution τ in the construction of Section 2.5 for any $(t_1, \dots, t_{n-1}) \in \mathbb{C}^{n-1}$, we obtain a family of smooth normal crossing surfaces, whose parameter space is \mathbb{C}^{n-1} . We write this family as

$$\mathcal{F}_n \cup \overline{\mathcal{F}_n} \rightarrow \mathbb{C}^{n-1}. \tag{3.13}$$

From the construction in Section 2.5 each fiber of this family has a canonical real structure that interchanges the two components. In the notation of Definition 2.7, the fiber over the origin of this family is isomorphic to $\mathbb{F}_n \cup_0 \overline{\mathbb{F}_n}$ as a complex variety with real structure, while on the l -th coordinate axis $\mathbb{C}(t_l)$, fibers are isomorphic to $\mathbb{F}_{n-2l} \cup_l \overline{\mathbb{F}_{n-2l}}$ except over the origin.

The family (3.13) is in effect isomorphic to the (abstract) family over \mathcal{J} :

Lemma 3.3. *In a sufficiently small neighborhood of the origin, the family (3.13) is isomorphic to the family of smooth normal crossing surfaces over \mathcal{J} .*

Proof. By versality of the Kuranishi family for locally trivial deformations of $D \cup \bar{D} \simeq \mathbb{F}_n \cup_0 \overline{\mathbb{F}_n}$, we have an induced map, for which we denote by α , from a neighborhood of the origin of the parameter space \mathbb{C}^{n-1} of (3.13) to that of the last Kuranishi family, such that the pull-back by α is isomorphic to the family (3.13). Though α is not uniquely determined, from naturality, the derivative $\alpha'(0)$ is nothing but the Kodaira–Spencer map for (3.13) at the origin. On the other hand as in (2.20) and (2.21) we have natural isomorphisms

$$H^1(\mathbb{F}_n \cup_0 \overline{\mathbb{F}_n}, \Theta) \simeq H^1(\mathbb{F}_n, \Theta_{\mathbb{F}_n, L}) \oplus H^1(\overline{\mathbb{F}_n}, \Theta_{\overline{\mathbb{F}_n}, \bar{L}}) \simeq H^1(\mathbb{F}_n, \Theta_{\mathbb{F}_n}) \oplus H^1(\overline{\mathbb{F}_n}, \Theta_{\overline{\mathbb{F}_n}}),$$

and the composition of the Kodaira–Spencer map $\alpha'(0)$ with these two isomorphisms is an injection onto the real locus of the last direct sum, because from the construction of the family (3.13), if we further take the composition with the projection to the first factor $H^1(\mathbb{F}_n, \Theta_{\mathbb{F}_n})$ of the last direct sum, we obviously obtain the Kodaira–Spencer map (2.11), which is an isomorphism. This means that, in the neighborhood of the origin, the family (3.13) is isomorphic to the real locus of the Kuranishi family of $\mathbb{F}_n \cup_0 \overline{\mathbb{F}_n}$. Since the isomorphism $D \cup \bar{D} \simeq \mathbb{F}_n \cup_0 \overline{\mathbb{F}_n}$ respects the real structure, the last real locus is exactly \mathcal{J} , as desired. \blacksquare

Now we are able to prove our main result, concerning extendability of the LeBrun metric on $\mathcal{O}(-n)$ to all nearby fibers of the above family $\mathcal{A}_n \rightarrow \mathbb{C}^{n-1}$ as an ALE SFK metric:

Theorem 3.4. *The LeBrun metric on $\mathcal{O}(-n)$ extends smoothly to all nearby fibers of the family $\mathcal{A}_n \rightarrow \mathbb{C}^{n-1}$ in (2.9) of affine \mathbb{C} -bundles, as an ALE SFK metric.*

Proof. As in Lemma 3.3, via the induced map α , the family $\mathcal{F}_n \cup \overline{\mathcal{F}}_n \rightarrow \mathbb{C}^{n-1}$ is isomorphic to the family over \mathcal{J} . Moreover, as we have already seen, the family over \mathcal{J} is isomorphic to the $(D \cup \overline{D})$ -portion of the deformation of the pair $(Z, D \cup \overline{D})$ parameterized by \mathcal{K}' via the induced map ψ_2 . Furthermore, the Z -portion of the family of pairs over \mathcal{K}' is identified with the family of twistor spaces over the Kähler locus \mathcal{K} via the map ψ_1 . (See the diagram (3.12).) By the theorem of Pontecorvo [11], for any point of \mathcal{K} , the corresponding twistor space determines an SFK metric on the 4-manifold $\mathcal{O}(-n)$ up to overall constants. These SFK metrics can be made to be ALE by multiplying overall constant for each metrics, because the affine bundles we are considering have a compactification by a $(+n)$ -curve. Via the isomorphisms ψ_1 , ψ_2 and α , we conclude that all fibers of the family $\mathcal{A}_n \rightarrow \mathbb{C}^{n-1}$ admit ALE SFK metrics at least as long as the fibers are sufficiently close to the central fiber. The smoothness for the variation of the metrics immediately follows from smoothness for ψ_1 , ψ_2 and α . ■

We note that as in the above proof, the ALE SFK metrics on all nearby fibers of the central fiber are uniquely determined up to overall constants once we fix the maps ψ_1 , ψ_2 and α .

From Theorem 3.4 it is immediate to prove the existence of an ALE SFK metric on any affine \mathbb{C} -bundle over $\mathbb{C}\mathbb{P}^1$ of negative degree (see Definition 2.1). For this, we recall that as we have explained in Section 2.2, any affine \mathbb{C} -bundle over $\mathbb{C}\mathbb{P}^1$ of degree $-n$ (≤ -1) is a member of the family $\mathcal{A}_n \rightarrow \mathbb{C}^{n-1}$. Also, from the \mathbb{C}^* -action on the total space of \mathcal{A}_n which is a lift of the scalar multiplication on \mathbb{C}^{n-1} , any fiber over the same line through the origin is mutually biholomorphic except over the origin. Thus for any sequence $\{U_{-n} \mid n \geq 1\}$ of neighborhoods of the origin in \mathbb{C}^{n-1} , the union $\cup_{n \geq 1} U_{-n}$ contains arbitrary affine \mathbb{C} -bundles over $\mathbb{C}\mathbb{P}^1$ of negative degree. Hence by Theorem 3.4 we obtain

Corollary 3.5. *Any affine \mathbb{C} -bundle over $\mathbb{C}\mathbb{P}^1$ of negative degree (see Definition 2.1) admits an ALE SFK metric.*

Also now it is easy to show the following rigidity result for the LeBrun metric on $\mathcal{O}(-n)$ when the complex structure is fixed:

Proposition 3.6. *Let $k > 0$ and Δ be a unit disk in \mathbb{R}^k around the origin, and let $\{g_t \mid t \in \Delta\}$ be a smooth family of ALE SFK metrics on the complex surface $\mathcal{O}(-n)$ equipped with the natural complex structure as a line bundle. Assume that g_0 is isometric to the LeBrun metric. Then there exists a neighborhood $\Delta' \subset \Delta$ of the origin, such that g_t is isometric to the LeBrun metric up to overall constants for any $t \in \Delta'$.*

Proof. For each $t \in \Delta$ we take a conformal compactification \hat{g}_t of g_t to $\widehat{\mathcal{O}(-n)}$. Let Z_t be the twistor space of \hat{g}_t and F_t be the line bundle $K_{Z_t}^{-1/2}$. Then by the assumption for complex structure on $\mathcal{O}(-n)$, for any $t \in \Delta$, the twistor space Z_t has a Cartier divisor $D_t \cup \overline{D}_t (\simeq \mathbb{F}_n \cup_0 \overline{\mathbb{F}}_n)$ in the system $|F_t|$ which is biholomorphic to the divisor $D \cup \overline{D}$ in Proposition 3.1. Hence the family $\{(Z_t, D_t \cup \overline{D}_t) \mid t \in \Delta\}$ gives a locally trivial deformation of the pair $(Z_0, D_0 \cup \overline{D}_0)$ for which the complex structure of $D_0 \cup \overline{D}_0$ does not vary. By versality of the Kuranishi family for locally trivial deformations of the pair $(Z_0, D_0 \cup \overline{D}_0)$, there exist a neighborhood $\Delta' \subset \Delta$ of the origin and a smooth map $\varphi : \Delta' \rightarrow H^1(\Theta_{Z_0, D_0 \cup \overline{D}_0})$ which satisfies $\varphi(0) = 0$ and whose pullback of the Kuranishi family is isomorphic to the original family $\{(Z_t, D_t \cup \overline{D}_t) \mid t \in \Delta'\}$. But because of the constancy $D_t \cup \overline{D}_t \simeq D_0 \cup \overline{D}_0$ and the natural isomorphism $H^1(\Theta_{Z, D \cup \overline{D}}) \simeq H^1(\Theta_{D \cup \overline{D}})$ in (3.3), φ has to satisfy $\varphi(t) = 0$ for any $t \in \Delta'$. This means that the family $\{Z_t \mid t \in \Delta'\}$ itself is a trivial family. Hence the conformal classes $[\hat{g}_t]$ do not vary. This means the required rigidity of the LeBrun's Kähler metric. ■

Next we take group actions into account for the moduli problem. Since LeBrun's metric on $\mathcal{O}(-n)$ is $U(2)$ -invariant, its twistor space Z admits a $U(2)$ -action and the divisor $D \cup \bar{D}$ is $U(2)$ -invariant. Hence the cohomology group $H^1(\Theta_Z)$ and $H^1(\Theta_{Z, D \cup \bar{D}})$ have natural $U(2)$ -actions. The action on $H^1(\Theta_Z)$ was computed in [3], and if $H^1(\Theta_Z)^\sigma$ denotes the relevant real locus, we have, as a real $U(2)$ -module,

$$H^1(\Theta_Z)^\sigma \simeq S_1^{n-2}\mathbb{C}^2 \oplus S_2^{n-4}\mathbb{C}^2.$$

Here, $S_k^m\mathbb{C}^2 := S^m\mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}_k$, where $S^m\mathbb{C}^2$ denotes the m -th symmetric product of the natural representation on \mathbb{C}^2 , and \mathbb{C}_k is the 1-dimensional representation obtained by multiplying $(\det)^k$. (If $m < 0$, $S^m\mathbb{C}^2$ means 0, and $S^0\mathbb{C}^2$ means the trivial representation on \mathbb{C} .) For the $U(2)$ -action on $H^1(\Theta_{Z, D \cup \bar{D}})$, it is immediate from Proposition 3.1 to derive the following

Proposition 3.7. *As a real $U(2)$ -module, we have $H^1(\Theta_{Z, D \cup \bar{D}})^\sigma \simeq S_1^{n-2}\mathbb{C}^2$.*

Proof. By the injectivity of the natural map (3.4), $H^1(\Theta_{Z, D \cup \bar{D}})^\sigma$ is naturally a subspace of $H^1(\Theta_Z)^\sigma$ which is of course $U(2)$ -invariant. Since both $S_1^{n-2}\mathbb{C}^2$ and $S_2^{n-4}\mathbb{C}^2$ are irreducible $U(2)$ -modules, $H^1(\Theta_{Z, D \cup \bar{D}})^\sigma$ has to coincide with one of these two spaces or the whole space $S_1^{n-2}\mathbb{C}^2 \oplus S_2^{n-4}\mathbb{C}^2$. But it has to be $S_1^{n-2}\mathbb{C}^2$ as $\dim H^1(\Theta_{Z, D \cup \bar{D}})^\sigma = 2(n-1)$ from Proposition 3.1 while $\dim_{\mathbb{R}} S_2^{n-4}\mathbb{C}^2 = 2(n-3) \neq 2(n-1)$. ■

Thus connecting the series of the natural isomorphisms

$$H^1(\Theta_{Z, D \cup \bar{D}})^\sigma \stackrel{(3.3)}{\simeq} H^1(\Theta_{D \cup \bar{D}})^\sigma \stackrel{(2.20)\&(2.21)}{\simeq} H^1(\Theta_D) \stackrel{(2.11)}{\simeq} \mathbb{C}^{n-1}, \quad (3.14)$$

we obtain that the parameter space \mathbb{C}^{n-1} of the family $\mathcal{A}_n \rightarrow \mathbb{C}^{n-1}$ may be identified with $S^{n-2}\mathbb{C}^2$ as a real $U(2)$ -module.

We also have the following result concerning $U(2)$ -action on another cohomology group. Recall from Proposition 3.1 that we have $H^1(\Theta_Z(-D - \bar{D})) \simeq \mathbb{C}$.

Proposition 3.8. *For the LeBrun metric, the natural $U(2)$ -action on $H^1(\Theta_Z(-D - \bar{D})) \simeq \mathbb{C}$ is trivial.*

Proof. As in (3.8) we have the exact sequence

$$0 \longrightarrow H^0(\Theta_{Z, D \cup \bar{D}}) (\simeq \mathbb{C}^4) \xrightarrow{\iota} H^0(\Theta_{D \cup \bar{D}}) (\simeq \mathbb{C}^5) \longrightarrow H^1(\Theta_Z(-D - \bar{D})) \longrightarrow 0, \quad (3.15)$$

and hence the space $H^1(\Theta_Z(-D - \bar{D})) (\simeq \mathbb{C})$ can be identified with the cokernel of the injection ι . In the space $H^0(\Theta_{D \cup \bar{D}})$ we have the 2-dimensional subspace generated by the scalar multiplication on each of D and \bar{D} , where we are viewing these as the compactification of the line bundle $\mathcal{O}(-n)$ for the scalar multiplication. As the scalar multiplications commute with any element of $U(2)$, the group $U(2)$ acts trivially on this 2-dimensional subspace. Moreover the image of ι cannot contain this subspace since any real element of $H^0(\Theta_{Z, D \cup \bar{D}})$ is a lift of a conformal Killing field on $\widehat{\mathcal{O}(-n)}$ and hence on D and \bar{D} the vector field cannot move independently each other. This means that the 2-dimensional subspace of $H^0(\Theta_{D \cup \bar{D}})$ is mapped surjectively to the 1-dimensional space $H^1(\Theta_Z(-D - \bar{D}))$. Since the sequence (3.15) is $U(2)$ -equivariant, the assertion follows. ■

Next we investigate the restrictions of the family of ALE SFK metrics in Theorem 3.4 to the coordinate axes of \mathbb{C}^{n-1} , which provide $U(1)$ -equivariant deformations of the LeBrun metric. Suppose $n \geq 3$ and for each $1 \leq l \leq n-1$ let $\mathbb{C}(t_l)$ be the l -th coordinate axis of \mathbb{C}^{n-1} as before, and let

$$\mathcal{A}_{n,l} \rightarrow \mathbb{C}(t_l) \quad \text{and} \quad \mathcal{F}_{n,l} \rightarrow \mathbb{C}(t_l) \quad (3.16)$$

be the restrictions of the families $\mathcal{A}_n \rightarrow \mathbb{C}^{n-1}$ and $\mathcal{F}_n \rightarrow \mathbb{C}^{n-1}$ respectively to the t_l -axis. From the \mathbb{C}^* -action which is a lift of the scalar multiplication on \mathbb{C}^{n-1} , all fibers of $\mathcal{A}_{n,l} \rightarrow \mathbb{C}(t_l)$ and $\mathcal{F}_{n,l} \rightarrow \mathbb{C}(t_l)$ are mutually isomorphic except the central fiber for each, and as in (2.12), fibers of $\mathcal{F}_{n,l} \rightarrow \mathbb{C}(t_l)$ are isomorphic to \mathbb{F}_{n-2l} except the fiber over the origin. Moreover recalling that \mathcal{A}_n is defined by the equation

$$\zeta_0 = \frac{1}{u^n} \zeta_1 + \sum_{l=1}^{n-1} \frac{t_l}{u^l} \quad \text{on } U_{01},$$

as in (2.8), we obtain that, as an enlargement of the above \mathbb{C}^* -action on \mathcal{A}_n (and \mathcal{F}_n), the total space of the family \mathcal{A}_n (and \mathcal{F}_n) carries a $(\mathbb{C}^* \times \mathbb{C}^*)$ -action defined by

$$(u, \zeta_0, t_l) \xrightarrow{(s_1, s_2)} (s_1 u, s_2 \zeta_0, s_1^l s_2 t_l), \quad (s_1, s_2) \in \mathbb{C}^* \times \mathbb{C}^*. \quad (3.17)$$

On the central fiber this gives a $(\mathbb{C}^* \times \mathbb{C}^*)$ -action of the toric structure on $\mathcal{O}(-n)$ or \mathbb{F}_n . Putting $s_1 = 0$ in (3.17) gives the original \mathbb{C}^* -action on \mathcal{A}_n and \mathcal{F}_n . If we define a \mathbb{C}^* -subgroup G_l of $\mathbb{C}^* \times \mathbb{C}^*$ by

$$G_l := \{(s_1, s_2) \in \mathbb{C}^* \times \mathbb{C}^* \mid s_1^l s_2 = 1\}, \quad (3.18)$$

which acts trivially on $\mathbb{C}(t_l)$, then the two families (3.16) may be regarded as G_l -equivariant deformations of $\mathcal{O}(-n)$ and \mathbb{F}_n respectively. Then basically by restricting the family of ALE SFK metrics in Theorem 3.4 to the coordinate axis $\mathbb{C}(t_l)$, we obtain the following result about existence of $U(1)$ -equivariant deformations of the LeBrun metric:

Proposition 3.9. *Let $n \geq 3$ and $l \geq 1$ be integers satisfying $n - 2l \geq 0$, and $\mathcal{A}_{n,l} \rightarrow \mathbb{C}(t_l)$ be the 1-parameter deformation of $\mathcal{O}(-n)$ to an affine \mathbb{C} -bundle as above. Then the LeBrun metric on the central fiber $\mathcal{O}(-n)$ extends smoothly to any nearby fibers of $\mathcal{A}_{n,l} \rightarrow \mathbb{C}(t_l)$ preserving not only the ALE SFK property but also a $U(1)$ -action, where $U(1)$ is the compact torus of the stabilizer subgroup $G_l (\simeq \mathbb{C}^*)$ of the axis $\mathbb{C}(t_l)$ defined in (3.18).*

Proof. From the proof of Theorem 3.4, we just need to show that the isomorphism between the Kähler locus \mathcal{H} and the parameter space of the family $\mathcal{F}_n \cup \overline{\mathcal{F}}_n \rightarrow \mathbb{C}^{n-1}$ around the origin can be taken to be T^2 -equivariant, where T^2 is the standard maximal torus of $U(2)$ consisting of diagonal matrices. For this it is enough to see that the induced maps ψ_1 , ψ_2 and α , which were used to identify the relevant families in Section 3.2, can be taken to be T^2 -equivariant. This holds for ψ_1 and ψ_2 since Z , $D \cup \overline{D}$ and the pair $(Z, D \cup \overline{D})$ are $U(2)$ -invariant. For T^2 -equivariance of the remaining map α , it is enough to see that the total space of the family $\mathcal{F}_n \cup \overline{\mathcal{F}}_n \rightarrow \mathbb{C}^{n-1}$ has a T^2 -action whose restriction to the central fiber is identified with the T^2 -action on $D \cup \overline{D}$. Each component of $\mathcal{F}_n \cup \overline{\mathcal{F}}_n$ has a T^2 -action which is the restriction of the $(\mathbb{C}^* \times \mathbb{C}^*)$ -action given in (3.17) to the maximal torus, and these are clearly identified with the T^2 -actions on D and \overline{D} respectively. So to complete the proof we just need to see that the gluing map which was used for making the family $\mathcal{F}_n \cup \overline{\mathcal{F}}_n$ is T^2 -equivariant. But this is immediate if we notice that the anti-podal map τ_0 commutes with the T^2 -actions. ■

Remark 3.10. The parameter space $\mathbb{C}(t_l)$ of the above family of the metric is of course real 2-dimensional, but the family is in effect real 1-dimensional by the following reason. As in the above proof, the axis $\mathbb{C}(t_l)$ is naturally identified (via the isomorphisms in (3.14)) with a T^2 -invariant subspace of $H^1(\Theta_{Z, D \cup \overline{D}})^\sigma$ on which the subgroup G_l acts trivially. This T^2 -action has clearly real 1-dimensional orbits, and along each orbit the complex structure of the pairs of twistor spaces and the divisors are constant. We also note that although the group $T_{\mathbb{C}}$ (and $T_{\mathbb{C}}/G_l$) is acting on the axis $\mathbb{C}(t_l) \subset \mathbb{C}^{n-1}$, the corresponding subspace is *not* $T_{\mathbb{C}}$ - (nor $T_{\mathbb{C}}/G_l$ -) invariant, because the isomorphism in (3.14) are not $T_{\mathbb{C}}$ -equivariant and just T^2 -equivariant.

By Corollary 2.6 and Proposition 3.9, we obtain the following

Proposition 3.11. *Let $n \geq 0$, $l \geq 0$, and let $L \in |\Gamma_0 + (n+l)f|$ be any \mathbb{C}^* -invariant section of $\mathbb{F}_n \rightarrow \mathbb{C}\mathbb{P}^1$, where \mathbb{C}^* acts non-trivially on L . Then the complement $\mathbb{F}_n \setminus L$ admits a $U(1)$ -invariant ALE SFK metric, where $U(1)$ is the compact torus of \mathbb{C}^* .*

Proof. By Corollary 2.6, the complex structure of the pair (\mathbb{F}_n, L) satisfying the properties in the proposition is uniquely determined from n and l . If $l = 0$, we have $\mathbb{F}_n \setminus L \simeq \mathcal{O}(-n)$ and the existence of the metric on it is guaranteed by the original LeBrun metric on $\mathcal{O}(-n)$. If $l > 0$, $\mathbb{F}_n \setminus L$ is biholomorphic to general fibers of the 1-parameter family $\mathcal{A}_{n+2l,l} \rightarrow \mathbb{C}(t_l)$, and the existence of the metric is guaranteed by Proposition 3.9, as long as $n+2l \geq 3$. The situation where $n+2l \geq 3$ does not hold is only the case $(n, l) = (0, 1)$. But the existence of an ALE SFK metric on $\mathbb{F}_0 \setminus L$ ($L \in |\mathcal{O}(1, 1)|$) is guaranteed by the Eguchi–Hanson metric. ■

When $l > 1$, if L is the \mathbb{C}^* -invariant section as in the above proposition, then we have $\text{Aut}_0(\mathbb{F}_n, L) \simeq \mathbb{C}^*$ by Proposition 2.5. But when $l = 1$, we have $\text{Aut}_0(\mathbb{F}_n, L) \simeq \text{Af}(\mathbb{C})$ by Proposition 2.4. Thus the affine surface $\mathbb{F}_n \setminus L$ admits an ALE SFK metric with an effective $U(1)$ -action but $\text{Aut}_0(\mathbb{F}_n, L)$ is not reductive. But we do not know whether the holomorphic transformation group of the complex surface $\mathbb{F}_n \setminus L$ itself is reductive, nor even whether it is of finite dimensional. In this respect, for the surface $\mathcal{O}(-n)$, the holomorphic automorphism group is known to be *not* of finite dimensional [4, Remark 2.20].

3.3 Deformations of the metrics on the affine bundles

In the last subsection we have obtained ALE SFK metrics on affine \mathbb{C} -bundles over $\mathbb{C}\mathbb{P}^1$ as small deformations of the LeBrun metric on $\mathcal{O}(-n)$. In this subsection we in turn investigate small deformations of these metrics on affine bundles, which again preserve ALE SFK property. So let $\mathcal{A}_n \rightarrow \mathbb{C}^{n-1}$ ($n \geq 3$) be the family (2.9) of affine bundles over $\mathbb{C}\mathbb{P}^1$ as before, and for each $t \in \mathbb{C}^{n-1}$ write A_t for the affine bundle lying over t . By Theorem 3.4, if t is sufficiently close to the origin, the affine bundle A_t admits an ALE SFK metric. We write g_t for this metric. We recall that these metrics are uniquely determined up to overall constants by the (natural but non-unique) maps ψ_1 , ψ_2 and α . Then these metrics satisfy the following property:

Proposition 3.12. *If $t \neq 0$, the metric g_t on the affine bundle A_t is a member of a non-trivial (see below), real 1-parameter family of ALE SFK metrics, in which the complex structure on A_t does not deform.*

Here, ‘non-trivial family’ means that the complex structures of the corresponding 1-parameter family of twistor spaces actually vary as the parameter moves. Thus the situation is in contrast with the LeBrun metric on the line bundle $\mathcal{O}(-n)$, for which the metric cannot be deformed as an ALE SFK metric when the complex structure is fixed (see Proposition 3.6).

Proof of Proposition 3.12. Let Z_t be the twistor space of a conformal compactification of g_t , and let D_t be the divisor determined by the complex structure of A_t . The sum $D_t + \overline{D}_t$ is a Cartier divisor belonging to $|K_{Z_t}^{-1/2}|$. We first show that the natural map

$$H^0(\Theta_{Z_t, D_t \cup \overline{D}_t}) \longrightarrow H^0(\Theta_{D_t \cup \overline{D}_t}) \quad (3.19)$$

is surjective as long as $t \neq 0$ and t is sufficiently close to 0. (Note that if $t = 0$ this is not surjective as in the sequence (3.2).) This is trivially satisfied if $H^0(\Theta_{D_t \cup \overline{D}_t}) = 0$. If $H^0(\Theta_{D_t \cup \overline{D}_t}) \neq 0$, we clearly have $H^0(\Theta_{D_t, L_t}) \neq 0$, where $L_t = D_t \cap \overline{D}_t$. From Propositions 2.3, 2.4 and 2.5, this is the case only when (D_t, L_t) is a \mathbb{C}^* -invariant pair. Again by the same propositions, the complex structure of such a pair is unique once the two integers $m \geq 0$ and $l \geq 1$ are fixed for which

$D_t \simeq \mathbb{F}_m$ and $L \in |\Gamma_0 + (m+l)f|$ hold. This means that we have $D_t \cup \bar{D}_t \simeq \mathbb{F}_{n-2l} \cup_l \bar{\mathbb{F}}_{n-2l}$ for some $l \geq 1$ (as $t \neq 0$) in the notation of Definition 2.7. For the central fiber $(Z_0, D_0 \cup \bar{D}_0)$, as in (3.3), we have a natural isomorphism

$$H^1(\Theta_{Z_0, D_0 \cup \bar{D}_0}) \simeq H^1(\Theta_{D_0 \cup \bar{D}_0}), \quad (3.20)$$

and this is $U(2)$ -equivariant. Since $H^2(\Theta_{Z_0, D_0 \cup \bar{D}_0}) = H^2(\Theta_{D_0 \cup \bar{D}_0}) = 0$ as in (3.2), (3.20) means that, for any subgroup $G \subset U(2)$, G -action on the surface $D_0 \cup \bar{D}_0$ extends to $D_t \cup \bar{D}_t$ for sufficiently small t if and only if the G -action on the pair $(Z_0, D_0 \cup \bar{D}_0)$ extends to $(Z_t, D_t \cup \bar{D}_t)$. Moreover the restriction of the family $\mathcal{F}_n \cup \bar{\mathcal{F}}_n \rightarrow \mathbb{C}^{n-1}$ given in (3.13) to the coordinate axis $\mathbb{C}(t_l)$ actually connects $D_0 \cup \bar{D}_0$ and $D_t \cup \bar{D}_t$ in a $U(1)$ -equivariant way, where $U(1)$ is the compact torus of G_t defined in (3.18). Thus the $U(1)$ -action on $(Z_0, D_0 \cup \bar{D}_0)$ actually extends to $(Z_t, D_t \cup \bar{D}_t)$. This means that the map (3.19) is surjective for sufficiently small t .

By the upper semi-continuity and the invariance of the Euler characteristic under deformation, we have

$$H^i(Z_t, \Theta_{Z_t}(-D_t - \bar{D}_t)) = 0 \quad \text{if } i \neq 1, \quad \text{and} \quad H^1(Z_t, \Theta_{Z_t}(-D_t - \bar{D}_t)) \simeq \mathbb{C},$$

because these are true for the case of the LeBrun metric as in (3.1). Therefore from the exact sequence (3.5) (which remains obviously valid for $(Z_t, D_t \cup \bar{D}_t)$) and the surjectivity of the map (3.19), we obtain the short exact sequence

$$0 \longrightarrow H^1(\Theta_{Z_t}(-D_t - \bar{D}_t))(\simeq \mathbb{C}) \xrightarrow{\alpha} H^1(\Theta_{Z_t, D_t \cup \bar{D}_t}) \xrightarrow{\beta} H^1(\Theta_{D_t \cup \bar{D}_t}) \longrightarrow 0. \quad (3.21)$$

Then as we have $H^2(\Theta_{Z_t, D_t \cup \bar{D}_t}) = 0$ by (3.2) and the upper semi-continuity again, the parameter space of the Kuranishi family for locally trivial deformations of the pair $(Z_t, D_t \cup \bar{D}_t)$ can be regarded as a small disk (for which we denote by Δ_1) about the origin in $H^1(\Theta_{Z_t, D_t \cup \bar{D}_t})$. Similarly as we have $H^2(\Theta_{D_t \cup \bar{D}_t}) = 0$ from (3.2), the Kuranishi family for $D_t \cup \bar{D}_t$ may be regarded as a small disk (for which we denote by Δ_2) about the origin in $H^1(\Theta_{D_t \cup \bar{D}_t})$. If $\varphi : \Delta'_1 \rightarrow \Delta_2$ denotes a holomorphic map from a possibly smaller disk $\Delta'_1 \subset \Delta_1$ which is induced by the versality of the Kuranishi family for $D_t \cup \bar{D}_t$, then we naturally have $\varphi'(0) = \beta$. Hence from the surjectivity of β in (3.21), φ is a submersion at least in a neighborhood of the origin, and therefore $\varphi^{-1}(0)$ is non-singular and 1-dimensional near the origin, at which the tangent space is exactly the line $\text{Image}(\alpha)$. Thus the Kuranishi family for locally trivial deformations of the pair $(Z_t, D_t \cup \bar{D}_t)$ contains a (complex) 1-parameter subfamily over which the complex structure of $D_t \cup \bar{D}_t$ is constant. By restricting to the real locus of $\varphi^{-1}(0)$, we obtain the real 1-parameter family of deformation of the pair $(Z_t, D_t \cup \bar{D}_t)$ for which the complex structure of $D_t \cup \bar{D}_t$ is constant. By the same reason as in the proof of Theorem 3.4, the ASD structure determined by $(Z_t, D_t \cup \bar{D}_t)$ has a Kähler representative which is ALE at infinity.

It remains to show that the 1-parameter family of ALE SFK metrics on the 4-manifold $\mathcal{O}(-n)$ thus obtained is non-trivial in the sense explained right after the proposition. For this, it suffices to see that the Kodaira–Spencer class of the 1-parameter family of Z_t is non-zero, because this means that the complex structure of Z_t actually deforms. From the above argument the Kodaira–Spencer class of the deformation of the pair $(Z_t, D_t \cup \bar{D}_t)$ is non-zero (belonging to $\text{Image}(\alpha)$), and the genuine Kodaira–Spencer class (belonging to $H^1(\Theta_{Z_t})$) is obtained from this by sending it under the natural map $H^1(\Theta_{Z_t, D_t \cup \bar{D}_t}) \rightarrow H^1(\Theta_{Z_t})$.

We show that the last map is injective in the same way to the last part in the proof of Proposition 3.1. Exactly by the same argument as to deduce (3.10), for the cokernel sheaf N'_t of the natural injection $\Theta_{Z_t, D_t \cup \bar{D}_t} \rightarrow \Theta_{Z_t}$, we obtain

$$N'_t|_{D_t} \simeq -K_{D_t} - \mathcal{O}_{D_t}(2L_t), \quad N'_t|_{\bar{D}_t} \simeq -K_{\bar{D}_t} - \mathcal{O}_{\bar{D}_t}(2L_t).$$

Now as $t \neq 0$ we can write $D_t \simeq \mathbb{F}_{n-2l}$ for some l satisfying $1 \leq l \leq n-1$, and we have $L_t \in |\Gamma_0 + (n-l)f|$. Hence by the same computation to deduce (3.11) we obtain

$$-K_{D_t} - \mathcal{O}_{D_t}(2L_t) \simeq 2\Gamma_0 + (n-2l+2)f - 2\{\Gamma_0 + (n-l)f\} \simeq -(n-2)f.$$

Thus as $n > 2$ we obtain $H^0(-K_{D_t} - \mathcal{O}_{D_t}(2L_t)) = 0$. With reality, this means $H^0(D_t \cup \overline{D}_t, N'_t) = 0$. Hence the injectivity of the map $H^1(\Theta_{Z_t, D_t \cup \overline{D}_t}) \rightarrow H^1(\Theta_{Z_t})$ follows, and we obtained the assertion of the proposition. ■

Proof of Proposition 1.4. Let $\mathcal{Z} \rightarrow B$ be the family of twistor spaces associated to ALE SFK metrics on nearby fibers for the central fiber of the family $\mathcal{A}_n \rightarrow \mathbb{C}^{n-1}$ of affine \mathbb{C} -bundle over $\mathbb{C}\mathbb{P}^1$. For $t \in B$ we write Z_t for the twistor space over the affine bundle A_t . Since the family $\mathcal{Z} \rightarrow B$ is versal at the origin, it is also versal at $t \in B$ as long as t is sufficiently close to the origin. Therefore for a real 1-parameter family of twistor spaces associated to the family of ALE SFK metrics obtained in Proposition 3.12, there exists an induced map from the parameter space of the last family to B . Then if we take the image of the map as the arc, it clearly satisfies the property of the proposition. ■

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