

Mathematical Structure of Loop Quantum Cosmology: Homogeneous Models

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Abstract. The mathematical structure of homogeneous loop quantum cosmology is analyzed, starting with and taking into account the general classification of homogeneous connections not restricted to be Abelian. As a first consequence, it is seen that the usual approach of quantizing Abelian models using spaces of functions on the Bohr compactification of the real line does not capture all properties of homogeneous connections. A new, more general quantization is introduced which applies to non-Abelian models and, in the Abelian case, can be mapped by an isometric, but not unitary, algebra morphism onto common representations making use of the Bohr compactification. Physically, the Bohr compactification of spaces of Abelian connections leads to a degeneracy of edge lengths and representations of holonomies. Lifting this degeneracy, the new quantization gives rise to several dynamical properties, including lattice refinement seen as a direct consequence of state-dependent regularizations of the Hamiltonian constraint of loop quantum gravity. The representation of basic operators – holonomies and fluxes – can be derived from the full theory specialized to lattices. With the new methods of this article, loop quantum cosmology comes closer to the full theory and is in a better position to produce reliable predictions when all quantum effects of the theory are taken into account.

Key words: loop quantum cosmology; symmetry reduction

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1 Introduction

Loop quantum cosmology [22, 26] aims to develop and analyze cosmological models by incorporating crucial guidance from the full theory of loop quantum gravity [4, 87, 93]. Even though its systems cannot yet be derived completely, they constitute more than a set of minisuperspace models. Characteristic effects of quantum geometry have been found in this setting, and contact with a potential full framework of quantum gravity has allowed one to fix some choices left open in traditional models of quantum cosmology. Nevertheless, ambiguities remain in loop quantum cosmology and loop quantum gravity, to be described by sufficiently general parameterizations that might be restricted by phenomenological analysis. Also for these parameterizations, contact with the full theory is essential: many different features collapse on one single parameter when geometry is restricted to exact homogeneity. Disentangling different contributors to one effect is important for estimates of typical ranges of parameters.

Ambiguities notwithstanding, some general features have been found, foremost among them quantum hyperbolicity [14, 21], a mechanism of singularity avoidance based on discrete structures of evolution operators. However, effective geometrical pictures of resolution mechanisms are difficult to derive since several different quantum effects contribute, in addition to higher-curvature corrections also quantum-geometry modifications. (Space-time as we know it may not even exist in extreme quantum phases [37, 39].) Specific details of physics in deep quantum

regimes, for instance the values of upper bounds on the energy density of matter, as another possible indicator of singularity resolution, are often unreliable because the setting remains too reduced and too ambiguous. An appropriate viewpoint is one akin to effective field theories, where one uses proposals for full (but possibly incomplete) theories to derive generic low-energy properties. With such a view, the framework is empirically testable because it can give rise to potentially observable phenomena, even if they cannot be predicted with certainty in all their parameterized details.

Developments in loop quantum cosmology have not always followed a general view, especially since the publication of [5], which stimulated a line of research focusing with minutest detail on pure minisuperspace models. Strenuous contact with the full theory has been replaced by ad-hoc assumptions (for instance related to degrees of freedom and scaling behaviors¹ in discrete structures); ambiguities (such as the so-called area gap and its postulated dynamical role) have been fixed by hand. Valuable results have been produced, chiefly of mathematical interest, showing what discrete features and non-standard quantum representations may imply; see e.g. [68, 69, 78]. For physical statements, however, the viewpoint espoused likely contains too many artefacts to be reliable. This article adds additional items to the list of known minisuperspace limitations.

The main aim, however, is to provide a general description of homogeneous reduced systems for quantum cosmology, focusing on but not restricted to loop quantum gravity. Since quantum cosmological models are beginning to be developed in approaches closely related to loop quantum gravity, such as spin foams [81, 84, 86], it is important to state the general setting of quantum cosmological models, and to point out limitations, dangers, and promises. The following section begins with a description and classification of symmetric models within full (classical or quantum) theories, amended in later sections by specific details and discussions within loop quantum cosmology and, briefly, the spin-foam setting. The final section will put these models in the general framework of effective theory. Along the road, we will be led to several mathematical features overlooked so far. Most importantly, Hilbert space representations based on functions on the Bohr compactification of the real line do not properly capture all aspects of homogeneous connections; rather, they give rise to a degeneracy of two important parameters corresponding, in the full theory, to the edge length and representation label of holonomies. The origin of the degeneracy is identified here as a mathematical coincidence realized in Abelian models only. To solve these problems, we will present a new non-Abelian construction, work out a detailed relation to the full theory, and arrive at a new viewpoint on dynamical difference equations with a natural implementation of lattice refinement.

2 Reduction

A classical reduced model, realizing a given symmetry, uses an embedding $\mathcal{M} \rightarrow \mathcal{S}$ from the set \mathcal{M} of symmetric geometries into full superspace \mathcal{S} . A minisuperspace geometry as an element of \mathcal{M} is specified by finitely many parameters a_I (which, to be specific, one may think of as the three scale factors of a diagonal Bianchi model), mapped to a full metric $g_{ab}(a_I)$ by the classification of invariant metric tensors. (A classification of invariant connections and triads is mathematically more well-developed [44, 71]; see [25, 36] for applications in the present context.) Inserting $g_{ab}(a_I)$ in equations of motion, the action or constraints of the full theory then produces corresponding equations for the finitely many a_I .²

¹See Sections 3.5.5 and 4.1.

²The procedure of inserting symmetric tensors may not commute with variations used to compute equations of motion [1, 94]: symmetric actions or constraints do not always produce the correct symmetric equations of motion. In our following discussions we make use of constraints, and therefore must assume that variation and symmetrization commute. In the Bianchi classification of homogeneous models, for instance, we are restricted to class A [76].

We distinguish between a minisuperspace model (a definition of minisuperspace \mathcal{M} with a dynamical flow on it) and the stronger notion of a reduced model (a minisuperspace model including also an embedding $\mathcal{M} \rightarrow \mathcal{S}$, making contact with the full theory). There is not much of a difference classically, but there is a big one at the quantum level. In quantum cosmology, most models remain in the minisuperspace setting, defining some dynamical flow on a system with finitely many degrees of freedom such that the dynamics of general relativity follows in a semiclassical limit, perhaps with inspiration by but no derivation from some full theory that quantizes \mathcal{S} . The key ingredient of reductions – making contact with a full theory of quantum gravity, if only in a weak sense – most often is missing. This article deals with the problem of reduction at the quantum level, going beyond pure minisuperspace models.

2.1 Reduction, selection, projection

In addition to minisuperspace versus reduced models, it will be useful to distinguish between three classical procedures of deriving symmetric systems beyond a mere minisuperspace prescription:

Reduction: The definition of a reduced model, as already stated, contains two parts: embedding \mathcal{M} in \mathcal{S} and deriving a dynamical flow by inserting minisuperspace metrics into the full equations. After a reduced model has been defined, one can proceed to solve its equations and evaluate solutions for potential predictions³.

Selection: Instead of starting with a set of minisuperspace geometries and deriving a reduced flow from the full theory, one could first solve the full equations and then select symmetric ones that admit a set of Killing vector fields with an algebra of the desired symmetry type. In general relativity with its complicated and largely unknown solution space, this procedure is rather impractical.

Projection: Or, again starting with solutions to the full equations, one may derive a symmetric geometry for every full solution by some kind of averaging process. In addition to the problem of a selection procedure, one would have to face the daunting averaging problem well-known from cosmology [49, 58].

While reduction is a standard classical procedure, selection or projection cannot be performed by current means. In quantum cosmology⁴, the best one may attempt is therefore a quantum version of reduction, going beyond pure minisuperspace quantizations but not directly accessing the full solution space. This is the topic of the present article.

2.2 Quantum cosmology

Several difficulties arise when one tries to extend classical reductions to quantum theory. Unlike classically, symmetric solutions can no longer be exact: Inhomogeneous degrees of freedom are set to zero, both for configuration and momentum variables. The spatial metric at any time must be invariant under the given symmetry, and so do its rate of change or extrinsic curvature for the metric to be able to remain invariant. Setting non-symmetric modes of both canonical fields to zero violates the uncertainty principle, and symmetric quantum solutions cannot be exact solutions of the full theory. Reduced quantum models can at best be approximations, but making sense of the approximation scheme in a clear way remains one of the outstanding challenges to be faced.

³“Embedding” might be a better name for this prescription, but “reduction” is much more standard.

⁴A great danger in quantum cosmology is that its procedures amount to neither reduction nor selection nor projection, but rather to *production* – not a combination of *projection* and *reduction* as the word might suggest, but just the presentation of an artificial model of unknown pedigree.

Mimicking classical constructions, the kinematical structure of quantum gravity can be reduced by making use of a mapping $\sigma: \mathcal{H}_{\text{hom}} \rightarrow \mathcal{D}_{\text{full}}$ from the kinematical Hilbert space \mathcal{H}_{hom} of a homogeneous minisuperspace model, quantizing the degrees of freedom a_I , to the space of distributional states in the full theory [36]. This mapping is analogous to the classical $\mathcal{M} \rightarrow \mathcal{S}$, but the distributional nature of the target space spells additional complications. Moreover, with uncertainty relations violated, symmetric quantum evolution is not exact in the full theory. Starting with a homogeneous full (but distributional) state $\psi_{\text{hom}} \in \sigma(\mathcal{H}_{\text{hom}}) \subset \mathcal{D}_{\text{full}}$, the distributional extension of (the dual action of) the full constraints \hat{C}_{full} or their gauge flows $\exp(i\delta\hat{C}_{\text{full}})$ does not leave the state in the image of homogeneous states. In loop quantum cosmology, methods exist to define and analyze maps σ ,⁵ but the derivation of a dynamical flow from the full theory remains difficult even though candidates do exist. Loop quantum cosmology therefore realizes an incomplete quantum reduction. Wheeler–DeWitt quantum cosmology, on the other hand, is not a reduction but a pure minisuperspace quantization since no analog of σ exists. (There are also models of loop quantum cosmology which do not go beyond pure minisuperspace models, disregarding proper considerations of σ .)

In order to restrict or truncate full quantum evolution (or gauge flows) to homogeneous states, one must specify a projection of $\hat{C}_{\text{full}}\psi_{\text{hom}}$ or $\exp(i\delta\hat{C}_{\text{full}})\psi_{\text{hom}}$ back to the image of \mathcal{H}_{hom} in $\mathcal{D}_{\text{full}}$, for all states ψ_{hom} and all full constraints – some part of the averaging problem plays a role even for reduction when quantum effects are involved, another indication of more-severe problems. No such projection has been provided so far, and therefore the dynamics of reduced models, let alone minisuperspace models, remains incomplete.

The problem is challenging not just owing to quantum issues, such as the distributional nature of symmetric states. Even classically, the question of how to project a non-symmetric metric to a homogeneous one is complicated, and unresolved in its generality; it constitutes the averaging problem of cosmology. Since a complete derivation of reduced quantum models from the full theory of some form would, in its semiclassical limit, include a solution to the averaging problem, one cannot expect progress on the dynamical front of quantum reduction unless the classical averaging problem is better understood.

The averaging problem remaining open, the only way at present to go beyond minisuperspace models is to use properties of a homogeneous background for an implementation of inhomogeneity, perhaps by perturbation theory. In classical cosmology, one commonly makes use of this perspective when inhomogeneous fields are expanded by Fourier transformation with respect to the modes on a homogeneous background. Classically, the approximation is well-understood. In quantum cosmology, the procedure suffers from the same problems encountered for homogeneous models, and adds new ones related to the quantization of inhomogeneous modes. Also the question whether results may depend sensitively on the background (and often gauge) chosen before quantization remains thorny, related to the complicated anomaly issue of quantum gravity. (Some anomaly-free realizations with partial quantum effects exist [34, 41, 50, 65, 72, 83, 85]. The anomaly problem is to be faced in canonical and covariant approaches alike: in canonical approaches it appears in the constraint algebra; in covariant ones, in the path-integral measure or in the correct choice of face amplitudes of spin foams [40].)

Facing these difficulties, it is an effective viewpoint which allows progress, making use of sufficiently general parameterizations of quantum effects, but disregarding fine details. The viewpoint is half-way between minisuperspace models and a complete dynamical embedding in the full theory: One avoids the averaging problem by using inhomogeneous model states adapted to the symmetry, much like classical Fourier modes on a given background. In practice, it is often lattice states with links along the symmetry generators that allow one to include

⁵More precisely, as detailed below, instead of $\mathcal{D}_{\text{full}}$ a distributional space based on lattices, and therefore fully inhomogeneous but not the most general states, are used. Otherwise, if edges not adapted to the symmetry appear, there are obstructions to embeddings of states [45].

a discrete version of inhomogeneity at the quantum level [19]. A background structure is then built into the framework, but it becomes possible to deal with inhomogeneity, going beyond pure minisuperspace models and escaping their limitations and artefacts. A background or some gauge fixing has entered, possibly giving rise to new problems. But at this stage, effects sufficiently general and parameterized can give reliable access to the physics involved.

In loop quantum cosmology, this procedure has been developed to the degree that cosmological phenomenology can be done. The kinematical structure – the basic algebra of holonomy and flux operators – can be derived from the full one. Evolution and the dynamics, facing the classical averaging problem and the anomaly problem of quantum gravity, remain much less understood, but here parameterizations have been developed that capture interesting effects. Especially the interplay of various quantum corrections, significant in different regimes of curvature, puts restrictions on possible phenomena.

We will now go back to the basics of these constructions to clarify and generalize several mathematical objects involved. We will point out one major problem due to oft-used Abelianizations of cosmological models, overlooked so far. Its solution has several ramifications even at the level of formulating the dynamics.

3 Loop quantum cosmology

Loop quantum gravity is of the type of a quantized co-tangent bundle of the space of connections, with additional constraints that restrict solutions to covariant dynamics. One usually follows Dirac’s quantization procedure, in which one first finds a representation of connection components and their conjugate momenta as operators on a suitable state space, and then requires physical states to be invariant under the flow generated by constraint operators. The same separation of kinematics (representing the connection and its momentum) and dynamics (implementing the constraints) appears in homogeneous models.

In order to formulate the relevant expressions, the abstract index notation is useful and common. The local connection 1-form is then denoted by A_a^i in component form, while its momentum is E_i^a . Indices a, b, c, \dots refer to the tangent space of the base manifold, while indices i, j, k, \dots refer to the Lie algebra of the structure group, in this context $SU(2)$. (Geometrically, the momentum plays the role of a densitized triad, or an orthonormal frame whose components are multiplied with the absolute value of its determinant.) The positions of indices indicate the dual nature of the fields (such as 1-forms dual to vector fields), and in products with pairs of mutually dual indices contraction, or the summation over all values of the paired indices, is understood. For instance, the densitized triad is related to the inverse spatial metric q^{ab} by $E_i^a E^{bi} = q^{ab} \det q$. As used here, indices i, j, k, \dots are raised or lowered by contraction with the Killing metric of the structure group (or an application of its associated musical morphism). Indices a, b, c, \dots could be raised or lowered with the spatial metric q_{ab} , but since it depends on the E_i^a , the field to be quantized, such operations are usually written explicitly.

In order to quantize the theory, one represents the basic geometric fields, connections A_a^i and densitized triads E_i^a , by integrated versions: holonomies (or parallel transports) of the connection and fluxes of the densitized triad. (A flux is a surface integration of E_i^a , which is well-defined if one uses the Hodge-type duality of E_i^a to an $\mathfrak{su}(2)$ -valued 2-form $\epsilon_{abc} E_i^a$.) In homogeneous models, one tries to find versions of these quantities that respect some transitive symmetry acting on space. Instead of all curves for parallel transports and all surfaces for fluxes, one is then led to a restricted set.

As a first step toward symmetry reduction, mathematical theorems are available to classify and construct different types of connections invariant under the action of some symmetry group [71]. When specialized to homogeneity [11], or a transitive group action on space, a set of models equivalent to the usual Bianchi classification results: For every Bianchi type, there

is a set of three left-invariant 1-forms ω_a^I , $I = 1, 2, 3$, obtained as standard invariant 1-forms on the corresponding transitive symmetry group, for instance by expanding the Maurer–Cartan form $\omega_{\text{MC}} = \omega^I T_I$ in terms of Lie-algebra generators T_I . These 1-forms serve as a basis of the space of invariant connections: All invariant connections are $A_a^i = \tilde{c}_I^i \omega_a^I$ with spatial constants (but time-dependent) \tilde{c}_I^i . Invariant densitized triads take the dual form, $E_i^a = \tilde{p}_I^I X_I^a |\det \omega_a^I|$ with invariant vector fields X_I^a dual to ω_a^I (that is, $X_I^a \omega_a^J = \delta_I^J$). This choice of densitized-triad components ensures that $(8\pi\gamma G)^{-1} \int_{\mathcal{V}} \tilde{A}_a^i E_i^a d^3x = (8\pi\gamma G)^{-1} V_0 \tilde{c}_I^i \tilde{p}_I^I$, integrating the symplectic term of an action over some region \mathcal{V} of coordinate volume $V_0 = \int_{\mathcal{V}} d^3x$. Up to constant factors, \tilde{c}_I^i and \tilde{p}_I^I are therefore canonically conjugate:

$$\{\tilde{c}_I^i, \tilde{p}_J^J\} = \frac{8\pi\gamma G}{V_0} \delta_j^i \delta_I^J. \quad (3.1)$$

A Bianchi metric of the given type is $q_{ab} = |\det(\tilde{p}_k^K)| \tilde{p}_I^i \tilde{p}_J^j \omega_a^I \omega_b^J$ with inverse matrices \tilde{p}_I^i of \tilde{p}_i^I . The metric is invariant under rotations $\tilde{p}_i^I \mapsto R_i^J \tilde{p}_J^I$ with $R \in \text{SO}(3)$, generated as gauge transformations by the Gauss constraint $\epsilon_{ij}{}^k \tilde{c}_I^j \tilde{p}_k^I$. The diffeomorphism constraint is not relevant for homogeneous models, and the Hamiltonian constraint is [13]

$$\begin{aligned} H = & -\frac{1}{8\pi G \sqrt{|\det(\tilde{p}_i^I)|}} \left(\epsilon_{ijk} C_{IJ}^K \tilde{c}_K^i \tilde{p}_j^J \tilde{p}_k^J - \tilde{c}_I^j \tilde{c}_J^k \tilde{p}_j^I \tilde{p}_k^J + \tilde{c}_I^k \tilde{c}_J^j \tilde{p}_j^I \tilde{p}_k^J \right. \\ & \left. + 2(1 + \gamma^{-2})(\tilde{c}_I^j - \Gamma_I^j)(\tilde{c}_J^k - \Gamma_J^k) \tilde{p}_j^I \tilde{p}_k^J \right) \end{aligned} \quad (3.2)$$

with the structure constants C_{IJ}^K of the Bianchi group and the spin connection Γ_I^i , depending on C_{IJ}^K and \tilde{p}_i^I . In diagonal models, where $\tilde{c}_I^i = \tilde{c}_{(I)} \delta_I^i$ and $\tilde{p}_i^I = \tilde{p}^{(I)} \delta_i^I$ (no summation over I), (3.2) reads [17]

$$\begin{aligned} H = & \frac{1}{8\pi G} \left(((c_2\Gamma_3 + c_3\Gamma_2 - \Gamma_2\Gamma_3)(1 + \gamma^{-2}) - n^1 c_1 - \gamma^{-2} c_2 c_3) \sqrt{|p^2 p^3 / p^1|} \right. \\ & + ((c_1\Gamma_3 + c_3\Gamma_1 - \Gamma_1\Gamma_3)(1 + \gamma^{-2}) - n^2 c_2 - \gamma^{-2} c_1 c_3) \sqrt{|p^1 p^3 / p^2|} \\ & \left. + ((c_1\Gamma_2 + c_2\Gamma_1 - \Gamma_1\Gamma_2)(1 + \gamma^{-2}) - n^3 c_3 - \gamma^{-2} c_1 c_2) \sqrt{|p^1 p^2 / p^3|} \right) \end{aligned} \quad (3.3)$$

with spin-connection components

$$\Gamma_I = \frac{1}{2} \left(\frac{p^K}{p^J} n^J + \frac{p^J}{p^K} n^K - \frac{p^J p^K}{(p^I)^2} n^I \right) \quad \text{for} \quad \epsilon_{IJK} = 1$$

and coefficients n^I of the Bianchi classification.

For Bianchi I with $C_{IJ}^K = 0$, the constraint reduces to

$$H = -\frac{1}{8\pi\gamma^2 G} \frac{\tilde{c}_I^j \tilde{c}_J^k \tilde{p}_j^I \tilde{p}_k^J - \tilde{c}_I^k \tilde{c}_J^j \tilde{p}_j^I \tilde{p}_k^J}{\sqrt{|\det(\tilde{p}_i^I)|}}.$$

In a spatially flat isotropic model with $\tilde{c}_I^i = \tilde{c} \delta_I^i$ and $\tilde{p}_i^I = \tilde{p} \delta_i^I$ with $\tilde{c} = \gamma \dot{a}$ and $|\tilde{p}| = a^2/4$ – see [11, 22] for the origin of the factor 1/4 – this expression reduces correctly to the gravitational contribution $H = -3(4\pi\gamma^2 G)^{-1} \tilde{c}^2 \sqrt{|\tilde{p}|}$ of the Friedmann equation.

3.1 Abelian artefacts

One of the first steps of loop quantization consists in replacing connection components \tilde{c}_I^i with holonomies or exponentials $\exp(\tilde{c}_I^i \tau_i) \in \text{SU}(2)$. The vast majority of investigations in homogeneous loop quantum cosmology, however, deals with Abelian models in which the original $\text{SU}(2)$

is replaced by $U(1)$, thanks either to additional isotropy symmetries [16] or a diagonalization assumption [17]. With isotropy or diagonalization, a classical invariant connection automatically becomes Abelian, and the use of $U(1)$ is not ad hoc but required. However, in quantum cosmology, Abelian structures turn out to allow specific choices of Hilbert space representations not possible in non-Abelian ones. Such quantizations, based essentially on spaces of functions on the Bohr compactification of the real line, are therefore in danger of introducing additional artefacts – structural properties that cannot be met in non-Abelian models, let alone the full theory. For instance, an isotropic connection has the form $A_a^i = \tilde{c}\delta_a^i$ with just one phase-space component \tilde{c} [16]. Mimicking matrix elements of holonomies along straight lines such as the edges of the integration cube \mathcal{V} (or some other fixed set of edges), one first represents \tilde{c} by $U(1)$ -holonomies $h = \exp(iV_0^{1/3}\tilde{c})$, or $h^n = \exp(in\tilde{c})$ with an integer $U(1)$ -representation label n and $c := V_0^{1/3}\tilde{c}$. By spanning a function space with superpositions of h^n for all integer n , all continuous functions on the group $U(1)$ are realized.

From $U(1)$ -holonomies h^n one can reconstruct the connection component c only modulo 2π . One gains full control over the connection if one considers holonomies along all pieces of the edges of the integration cube of lengths $\ell_0 \leq V_0^{1/3}$, such that holonomies $h^\mu = \exp(i\mu c)$ with $\mu \in \mathbb{R}$ result, where μ may be considered as a product λn of the fractional edge length $\lambda = \ell_0/V_0^{1/3}$ with the representation label n . Allowing for superpositions of all h^μ as an orthonormal basis, the Hilbert space of all integrable functions on the Bohr compactification $\overline{\mathbb{R}}_{\text{Bohr}}$ of the real line is obtained [2], rather than a function space on some periodification of \mathbb{R} .

In this procedure, which has become standard, one implicitly makes use of an identity realized for representations of $\overline{\mathbb{R}}_{\text{Bohr}}$ but lacking a non-Abelian analog. In the Abelian case, we start with the $U(1)$ -holonomy $\exp(i\lambda c)$ and evaluate it in the ρ_n -representation: $\rho_n(\exp(i\lambda c)) = \exp(i\lambda n c)$. It so happens that this is the same function of c as obtained from $\rho_{\lambda n}(\exp(ic))$, now using a representation of $\overline{\mathbb{R}}_{\text{Bohr}}$. Holonomies in the n -representation have led us to the first expression, which is then identified with the latter. Since they agree as functions, one may base Hilbert space constructions on functions on $\overline{\mathbb{R}}_{\text{Bohr}}$. However, this step is not available for non-Abelian models, in which case there is no relationship between $\rho_j(\exp(\lambda A))$ and $\rho_{\lambda j}(\exp(A))$ for A in the Lie algebra of the group, usually $SU(2)$. The second expression is not even defined unless λ is an integer (or a half-integer if j is integer), but even then, the two matrices are unrelated.

If functions on $\overline{\mathbb{R}}_{\text{Bohr}}$ are used, one must proceed with care to avoid artefacts in Abelian models, a problem which has not been realized in existing constructions. Mathematically, one would confuse $\rho_n(\exp(i\lambda c))$ with $\rho_{\lambda n}(\exp(ic))$, which are identical as functions of c but have different meanings and are elements of different function spaces. Physically, merging λ and n to one number $\mu = \lambda n$, as done when the Bohr compactification is used, eliminates important information because the edge length λ and representation label (or geometrical excitation level) n are then indistinguishable. In operators, however, λ and n should play rather different roles according to what is known from the full theory: Full holonomy operators act on (spin-network) states labeled by graphs with representations assigned to their edges. In the connection representation, such a state can be written as a function on the space of connections, obtained by multiplying matrix elements of all parallel transports along the edges, taken in the respective representations. A single holonomy operator adds its curve as a new edge to the graph if it had not been present before, and it changes the representation if its curve overlaps with one of the original edges. Since λ in the reduction corresponds to the edge length, operators with different λ should change the underlying graph in different ways, while operators with different n but the same λ change the graph in the same way but modify the labels differently. These different types of actions cannot be modeled faithfully if reduced operators depend only on the product λn . In this section, we present a new quantization of homogeneous models in which λ and n are kept separate – lifting their degeneracy and assigning to them distinct roles – in a way that extends to non-Abelian models.

3.2 Homogeneous holonomies

A local homogeneous connection A_{hom} is a 1-form on the translational symmetry group S underlying some Bianchi model, taking values in the Lie algebra $\mathcal{L}G$ of the structure group G , $G = \text{SU}(2)$ for gravity in Ashtekar–Barbero variables. If the symmetry group acts freely, without any isotropy subgroups, there is a one-to-one correspondence [71] between homogeneous connections according to S and linear maps $\tilde{\phi}: \mathcal{L}S \rightarrow \mathcal{L}G$ (or elements of $\mathcal{L}S^* \times \mathcal{L}G$), not required to be Lie algebra homomorphisms. Given $\tilde{\phi}$, the corresponding homogeneous connection is the pull-back $A_{\text{hom}} = \tilde{\phi}^* \omega_{\text{MC}}$ under $\tilde{\phi}$ of the Maurer–Cartan form, which latter can be written as $\omega_{\text{MC}} = \omega^I T_I$ in terms of left-invariant 1-forms $\omega^I = \omega_a^I dx^a$ on S and its generators T_I . The homogeneous connection components introduced before are the coefficients in $\tilde{\phi}(T_I) = \tilde{c}_I^i \tau_i$ with generators τ_i of $\mathcal{L}G$ (here, $\tau_j = -\frac{1}{2}i\sigma_j$ in terms of Pauli matrices).

A minisuperspace model quantizes the components \tilde{c}_I^i , or rather the linear maps $\tilde{\phi}$. Both ingredients are in one-to-one correspondence, but the additional structure shown by the linear maps is useful to decide how different quantum numbers, such as λ and n in Section 3.1 should be related to properties of S (space) and G (internal space). We will therefore derive a quantization based on the mathematical structure of $\tilde{\phi}$. To extract independent degrees of freedom, we fix a set of generators T_I of $\mathcal{L}S$ and understand a homogeneous G -connection for a given symmetry group S as a set of maps $\tilde{\phi}_I: \langle T_I \rangle \rightarrow \mathcal{L}G$ with the scaling condition $\tilde{\phi}_I(rX) = r\tilde{\phi}_I(X)$ for all $r \in \mathbb{R}$.

Following the methods of loop quantum gravity, we quantize connection components in terms of holonomies. According to the structure of homogeneous connections, we introduce the notion of homogeneous holonomies by exponentiation – maps $h_\phi: \mathcal{L}S \rightarrow G$, $h_\phi(X) = \exp(\phi(X))$ with the scaling condition $h_\phi(rX) = h_{r\phi}(X)$. The maps $\phi_I = L_I \tilde{\phi}_I$ used here differ from $\tilde{\phi}_I$ by factors of L_I , side lengths of the integration region \mathcal{V} of volume $V_0 = L_1 L_2 L_3$, assumed cubic (spanned by the three generators $T_I \cong X_I^a \partial / \partial x^a$ of the S -action). If the sides of \mathcal{V} are aligned with the three symmetry generators, one may think of $h_\phi(T_I)$ as the holonomy along the corresponding side. This relationship will be made more precise below.

Elements $X \in \mathcal{L}S$ to which ϕ is applied carry information about the edge used to compute holonomies $h_\phi(X)$. Referring to the Killing metric on $\mathcal{L}S$, we decompose $X = \lambda v \neq 0$ into its norm $\lambda = |X|$ and the unit vector $v = X/|X|$, corresponding to the coordinate length of the edge and its direction. With the scaling condition, we then have $h_\phi(X) = \exp(\lambda\phi(v))$. We can compute all information about ϕ from derivatives $\phi(T_I) = dh_\phi(\lambda T_I)/d\lambda|_{\lambda=0}$. We are indeed representing the space of all homogeneous connections, not some periodic identification.

The dependence of homogeneous holonomies on $\lambda = |X|$ will play an important role in our constructions. If we consider an edge e_I of coordinate length ℓ_I along the generator X_I^a , the holonomy, in general a path-ordered exponential $h_e = \mathcal{P} \exp(\int_e ds A_a^i \tau_i \dot{e}^a)$ of the connection integrated over a spatial curve $e(s)$, is $h_{e_I} = \exp(\ell_I \tilde{c}_I^i \tau_i) = \exp(\ell_I L_I^{-1} \phi(T_I)) = h_\phi(\lambda_I T_I)$ with $\lambda_I = \ell_I/L_I$. If all edges are contained in the integration region, we always have $\lambda_I \leq 1$. More generally, we can allow all real values, but for $\text{SU}(2)$, given periodicity of the exponential function, may restrict to $0 \leq \lambda_I < 4\pi$ without loss of generality.

For a given connection, the three choices for I , or three directions of space, give rise to three independent $\text{SU}(2)$ -elements $h_\phi(T_I)$. For fixed λ_I , one can therefore describe the space of homogeneous connections in terms of $\text{SU}(2)^3$,⁶ but the connection used can be reconstructed completely from the holonomies only if different choices for λ_I , or curves of different lengths,

⁶Thanks to homogeneity, each holonomy transforms as $h_{e_I} \mapsto g h_{e_I} g^{-1}$ under an internal gauge transformation, with the same $g \in \text{SU}(2)$ for all three edges and on all their endpoints. These transformations are identical to those obtained in the full theory for three closed loops intersecting in one 6-valent vertex [11]. One may picture homogeneous spin-network states as such vertices, but with homogeneity, the vertex corresponds to all of space – homogeneous states are distributional and not given by single spin networks; see Section 3.4.2.

are considered. If the curves and their lengths are fixed, as in the original constructions of [11, 16, 17], only a certain periodic identification of the space of connections is realized.

In order to generalize homogeneous holonomies $h_\phi(T_I)$, we consider a set \mathcal{F} of functions $g_I: \mathbb{R} \times \mathcal{L}G \rightarrow G$ that fulfill $g_I(rL_I, r^{-1}\tilde{c}_I) = g_I(L_I, \tilde{c}_I)$ for all $r \in \mathbb{R}$ and $I = 1, 2, 3$ (the scaling condition). In this way, we can drop the reference to particular edges as appropriate for a minisuperspace model, but, as demonstrated in what follows, will still be able to distinguish a length parameter from a spin label. The choice $r = 1/L_I$ shows that any such function can be written as $g_I(L_I, \tilde{c}_I) = \tilde{g}_I(L_I \tilde{c}_I)$ with a function \tilde{g}_I of just one variable $A \in \mathcal{L}G$. If L_I and r are fixed, g_I is simply the group exponential; setting r free allows for different scalings or different sizes L_I of the integration region within one model.

3.3 Representation

For homogeneous models of loop quantum cosmology, we turn the function space based on holonomies into a Hilbert space with an action of holonomies and fluxes as basic operators, such that their commutator corresponds to the classical Poisson bracket (3.1). One immediate problem caused by the non-Abelian nature of general connections in combination with homogeneity regards the way of exponentiating connection components to holonomies and obtaining a closed basic algebra for $\{\exp(\lambda_I c_I^i \tau_i), p_j^j\}$. Once the path-ordering of inhomogeneous holonomies is no longer available, derivatives of $\exp(\lambda_I c_I^i \tau_i)$ by c_j^j will produce extra factors of τ_j between products of $c_j^j \tau_i$ in a power-series expansion of the matrix exponential:

$$\frac{\partial \exp(\lambda_I c_I^i \tau_i)}{\partial c_j^j} = \delta_I^j \sum_{n=0}^{\infty} \frac{\lambda_I^n}{n!} \sum_{k=1}^n (c_I^i \tau_i)^{k-1} \tau_j (c_I^i \tau_i)^{n-k}, \quad (3.4)$$

but they do not automatically factor into products of exponentials with τ_j to mimic the full holonomy-flux algebra. (While the cotangent bundle T^*G defines a natural phase space with group-valued configuration variables, it does not necessarily model the correct relation to inhomogeneous holonomies.)

For a closed basic algebra to result, the factors in derivations of basic holonomy-like functions of c_j^j may have to be re-ordered, but within a pure minisuperspace model, there is no guideline, no trace of the path-ordering left by which one could construct a natural ordering. By looking more closely at the relation between basic operators in models and the full theory (or at least extended curves in holonomies), we will be led to one distinguished choice.

3.3.1 Hilbert space

We first construct a suitable C^* -algebra \mathcal{A} of functions on homogeneous connections, making use of our generalized homogeneous holonomies g_I : We consider a function ψ on the space of homogeneous connections as a function on the domain of definition $\mathbb{R} \times \mathcal{L}G$ of g_I which factorizes through g_I , that is a function $\psi(L_I, \tilde{c}_I)$ which can be written as $\bar{\psi}(g_I(L_I, \tilde{c}_I))$ with a function $\bar{\psi}$ on G . The scaling condition for g_I then translates into an analogous condition for ψ .

If we fix L_I , considering $g_I(L_I, \tilde{c}_I)$ simply as an element of G , and refer to the Peter–Weyl theorem, the general dependence on \tilde{c}_I can be realized by superpositions of functions $\langle m | \rho_j(g_I(L_I, \tilde{c}_I)) | n \rangle$ with all irreducible representations ρ_j of G and elements $|m\rangle$ and $|n\rangle$ of an orthonormal basis of the representation space of ρ_j . Setting L_I free, with $g_I(\cdot, \tilde{c}_I)$ as a 1-parameter family of G -elements, a larger class of functions is possible. However, the scaling condition can be realized only if our functions are superpositions of $\rho_{\lambda,j}(g_I)_n^m := \langle m | \rho_j(g_I(\lambda L_I, \tilde{c}_I)) | n \rangle$ for $\lambda \in \mathbb{Q}$. (The restriction to rational as opposed to real λ will be motivated later on. The labels λ, j, m and n may depend on I , but we will often suppress the dependence for notational simplicity.) Note that, perhaps in a slight abuse of notation, $\rho_{\lambda,j}$ is not a representation of the

group $\mathbb{R} \times G$. It does, however, provide elements of a suitable function space, whose elements are labeled by λ , j , m , and n and on which we can represent holonomies as constructed in what follows. (The \mathbb{R} -factor is related to curve lengths and not part of the structure group, and therefore should not be expected to be represented in the standard way of gauge theories.)

We multiply two functions $\rho_{\lambda_1, j_1}(g_I)_{n_1}^{m_1}$ and $\rho_{\lambda_2, j_2}(g_J)_{n_2}^{m_2}$ as follows: If $I \neq J$, we simply take the product function depending on g_I and g_J as independent variables, thereby generating a tensor-product space. If $I = J$, we write $\lambda_1 = N_1 z$ and $\lambda_2 = N_2 z$, with integers N_1 and N_2 and z the largest rational number that obeys the two relationships (so that N_1 and N_2 are relatively prime), and define

$$\begin{aligned} \rho_{\lambda_1, j_1}(g_I)_{n_1}^{m_1} \cdot \rho_{\lambda_2, j_2}(g_I)_{n_2}^{m_2} := & \sum_{h_1, \dots, h_{N_1-1}, k_1, \dots, k_{N_2-1}} \rho_{z, j_1}(g_I)_{h_1}^{m_1} \rho_{z, j_1}(g_I)_{h_2}^{h_1} \cdots \rho_{z, j_1}(g_I)_{n_1}^{h_{N_1-1}} \\ & \times \rho_{z, j_2}(g_I)_{k_1}^{m_2} \rho_{z, j_2}(g_I)_{k_2}^{k_1} \cdots \rho_{z, j_2}(g_I)_{n_2}^{k_{N_2-1}}. \end{aligned} \quad (3.5)$$

One may decompose the products of matrix elements on the right-hand side into superpositions of irreducible contributions to the tensor product $\rho_{j_1}^{\otimes N_1} \otimes \rho_{j_2}^{\otimes N_2}$, akin to a spin-network decomposition in the full theory [89]. The product is then again a superposition of $\rho_{\lambda, j}(g_I)_n^m$.

Multiplication as defined is commutative and associative because these properties are respected by the conditions defining z . (For associativity, the condition that z be maximal is crucial. There is then a unique z so that $\lambda_i = N_i z$ for any given number n of λ_i , $i = 1, \dots, n$.) There is a unit element given by $\lambda = 0$ (in which case the value of j does not matter). For vanishing j , having the trivial representation of G , one may expect a trivial action, too. However, according to (3.5), multiplication with $\rho_{\lambda, 0}(g_I)$ for $\lambda \neq 0$ may still give rise to decompositions of factors, providing a different form of the function product even though the values taken by the original function and its product with $\rho_{\lambda, 0}(g_I)$ do not differ. The functions $\rho_{\lambda, 0}(g_I)$ (matrix indices are not required in the trivial representation) play the role of refinement operators, decomposing a holonomy into pieces whose length is determined by λ in relation to the corresponding parameter of the state acted on. Since $\rho_0(h_\phi(\lambda T_I)) = 1$ classically, these refinement operators have no classical analog, as one may expect for a classical theory knowing nothing about the underlying discreteness.

We define a star operation by $(\rho_{\lambda, j}(g_I)_n^m)^* := \overline{\rho_{\lambda, j}(g_I)_n^m}$ (related to matrix elements of the dual representation ρ_j^* of ρ_j for unitary groups). The space of functions turns into an Abelian C^* -algebra with the supremum norm, assuming G to be compact. The supremum is obtained by evaluating $\rho_{\lambda, j}(g_I)_n^m$ on $\mathcal{L}G$. (Thanks to the scaling condition, the supremum does not depend on λ .) The C^* -identity $\|\rho^* \cdot \rho\| = \|\rho^*\| \|\rho\|$ then follows as it does for the standard example of functions on G .

A Hilbert space structure is obtained by combining the product rule with the Haar measure on G . We define the inner product

$$(\rho_{\lambda_1, j_1}(g_I)_{n_1}^{m_1}, \rho_{\lambda_2, j_2}(g_J)_{n_2}^{m_2}) := \prod_K \int_G d\mu_H(g_K(zL_K, \tilde{c}_K)) (\rho_{\lambda_1, j_1}(g_I)_{n_1}^{m_1})^* \cdot \rho_{\lambda_2, j_2}(g_J)_{n_2}^{m_2}, \quad (3.6)$$

where z is defined as before for given λ_1 and λ_2 . Equation (3.6) is just the standard inner product on G if we realize, using (3.5), that the relevant degree of freedom of $(\rho_{\lambda_1, j_1}(g_I)_{n_1}^{m_1})^* \cdot \rho_{\lambda_2, j_2}(g_J)_{n_2}^{m_2}$ is $g_K(zL_K, \tilde{c}_K)$. On the right-hand side of this equation, the value of z no longer matters because we integrate over all group elements $g_K(zL_K, \tilde{c}_K)$, the sole arguments of $(\rho_{\lambda_1, j_1}(g_I)_{n_1}^{m_1})^* \cdot \rho_{\lambda_2, j_2}(g_J)_{n_2}^{m_2}$.

3.3.2 Flux operators

Components $p_i^I = L_J L_K \tilde{p}_i^I$ ($\epsilon_{IJK} = 1$) of the densitized triad, canonically conjugate to $c_I^i = L_I \tilde{c}_I^i$ via $\{c_I^i, p_j^J\} = 8\pi\gamma G \delta_I^J \delta_j^i$, are quantized to operators with action

$$\hat{p}_i^I \rho_{\lambda,j}(gJ)_n^m := -8\pi i \gamma \ell_{\text{P}}^2 \lambda \delta_I^J \rho_{\lambda,j}(\tau_i gJ)_n^m \quad (3.7)$$

on our Hilbert space, where we define the short form $\rho_{\lambda,j}(\tau_i gJ)_n^m := \sum_k \rho_j(\tau_i)_k^m \rho_{\lambda,j}(gJ)_n^k$.

Non-Abelian flux operators as defined are *not* symmetric because the product (3.5) in (3.6) in general includes a decomposition. (One may think of the measure factor as including refinement operators $\rho_{z,0}(g_K)$.) While a flux operator acting on either entry in the inner product inserts a τ_i to the left of g_J according to (3.7), the integration required to evaluate the inner product splits holonomies according to the rational number z depending on λ_1 and λ_2 . Integration by parts, performed after multiplying according to (3.5) and decomposing, would then insert τ_i in each decomposed contribution, not just to the left of the whole g_K .

A convenient set of states in the homogeneous Hilbert space is given by a form of spin-network functions, depending on the connection via finitely many holonomies,

$$\begin{aligned} \psi(g_1, g_2, g_3) = & \sum_{\lambda_j^{(k)}, j_J^{(k)}, m_j^{(k)}, n_j^{(k)}} \psi_{\lambda_1^{(k)}, \lambda_2^{(k)}, \lambda_3^{(k)}, j_1^{(k)}, j_2^{(k)}, j_3^{(k)}, m_1^{(k)}, m_2^{(k)}, m_3^{(k)}, n_1^{(k)}, n_2^{(k)}, n_3^{(k)}} \\ & \times \prod_{I=1}^3 \rho_{\lambda_I^{(k)}, j_I^{(k)}}(g_I)_{n_I^{(k)}}^{m_I^{(k)}} \end{aligned}$$

with coefficients $\psi_{\lambda_1^{(k)}, \lambda_2^{(k)}, \lambda_3^{(k)}, j_1^{(k)}, j_2^{(k)}, j_3^{(k)}, m_1^{(k)}, m_2^{(k)}, m_3^{(k)}, n_1^{(k)}, n_2^{(k)}, n_3^{(k)}}$ for k in some finite index set. Acting on the contribution $\rho_{\lambda_I, j_I}(g_I)$, $\hat{J}_i^I = (8\pi\gamma\ell_{\text{P}}^2)^{-1} \lambda_I^{-1} \hat{p}_i^I$ satisfies the \mathcal{LG} -algebra by the definition in (3.7). For $\text{SU}(2)$, as used in loop quantum gravity, the flux spectrum is therefore given by all numbers $8\pi\gamma\ell_{\text{P}}^2 \lambda m$ with half-integer m , and the area spectrum (or the spectrum of $\sqrt{\hat{p}_i^I \hat{p}^{(I)i}}$) by $8\pi\gamma\ell_{\text{P}}^2 \lambda \sqrt{j(j+1)}$. Although these eigenvalues are real, one can see the non-symmetric nature of non-Abelian flux operators: Eigenstates with different eigenvalues (specifically, states with the same j but different λ) are not necessarily orthogonal, again owing to the decomposition in (3.6). With all rational λ allowed, these spectra form continuous sets, but all eigenstates are normalizable. The spectra are pure point.

3.3.3 Heuristics and properties

To summarize so far, the non-Abelian nature of connections requires care in the proper ordering of holonomy-type variables to be used in lieu of connections for a loop quantization. We refer to edge-integrated holonomies rather than pointwise exponentials, even in homogeneous models. But since the connection is still homogeneous, we must specify the product rule (3.5) – the first place in our constructions where different holonomies may be compared – so that the correct reduction of degrees of freedom is realized. This requirement leads us to the decomposition rule, which then further motivates definitions of compatible inner products and derivative operators, including non-selfadjoint features of the latter. (Decomposition is not an issue in Abelian models because they obey $\rho_{\lambda,n}(\exp(iLc)) = \rho_n(\exp(iLc/r))^p$ for $\lambda = p/r$ with integer p and r , an identity that trivially brings all holonomies to the same distance L . However, this feature makes use of the Abelian coincidence discussed in Section 3.1. See below for more on Abelian models.)

The multiplication rule (3.5) observes homogeneity: One can interpret the law as a decomposition of two initial holonomies of different lengths as products of equal-length pieces. Without homogeneity, these pieces at different places would be independent, but homogeneity makes them identical. We therefore take the tensor product of all small pieces, split as illustrated in



Figure 1. Evaluated in homogeneous connections, different pieces of equal length in one holonomy amount to the same function (left). To avoid overcounting degrees of freedom, multiplying a holonomy with a shorter one requires a decomposition into pieces of maximal common length (right), giving rise to the rule (3.5).

Fig. 1, and sum over indices according to the product form. At this stage, it becomes clear why λ should be rational: For incommensurate λ_1 and λ_2 , the product $\rho_{\lambda_1, j_1}(g_I)_{n_1}^{m_1} \cdot \rho_{\lambda_2, j_2}(g_I)_{n_2}^{m_2}$ would have to be split into infinitely many infinitesimally small pieces⁷.

In this picture, non-symmetric flux operators (3.7) arise because reduction entails averaging and decomposition, and what may appear as a simple quantization of the densitized triad in a pure minisuperspace model turns out to be a more complicated operator when inhomogeneous degrees of freedom are taken into account. We will see this in more detail in Section 3.4.3. Even non-symmetric flux operators (quantizing the densitized triad combined with the action of holonomy-dependent decomposition, realized for instance by $\rho_{\lambda, 0}(g_I)$) model the behavior of the full theory, in which fluxes are self-adjoint.

Also the commutator $[\rho_{\lambda, j}(g_J)_n^m, \hat{p}_i^I]$ of basic operators in the non-Abelian holonomy-flux algebra requires care due to decomposition. Up to ordering, it equals $8\pi i \gamma \ell_P^2 \lambda \delta_J^I \rho_{\lambda, j}(\tau_i g_J)_n^m$ and corresponds to $i\hbar$ times the Poisson bracket of $\rho_j(\exp(\lambda c_j^i \tau_i))_n^m$, the classical analog of $\rho_{\lambda, j}(g_J)_n^m$, and p_i^I . For an example of ordering issues, as indicated in the beginning of Section 3.3, look at the commutator $[\rho_{1/2, j}(g_I)_N^M, \hat{p}_i^I]$ acting on the state $\psi(g_I) = \rho_{1, j}(g_I)_n^m$. Acting with \hat{p}_i^I first produces a single insertion of τ_i to the left of g_I . Acting with \hat{p}_i^I after the action of $\rho_{1/2, j}(g_I)$ produces two insertions because we first decompose $\rho_{1/2, j}(g_I)_{n'}^{m'} \cdot \rho_{1, j}(g_I)_n^m = \sum_k \rho_{1/2, j}(g_I)_{n'}^{m'} \rho_{1/2, j}(g_I)_k^m \rho_{1/2, j}(g_I)_n^k$ according to (3.5) and use the Leibniz rule. The result,

$$4\pi i \gamma \ell_P^2 \rho_{1/2, j}(\tau_i g_I)_N^M \psi(g_I) + 4\pi i \gamma \ell_P^2 \rho_{1/2, j}(g_I)_N^M \left(\sum_k \rho_{1/2, j}(g_I)_k^m \rho_{1/2, j}(\tau_i g_I)_n^k - \rho_{1, j}(\tau_i g_I)_n^m \right)$$

is as expected up to the second term, a contribution that can be made to vanish by reordering.

Similar calculations for arbitrary λ_1 and λ_2 in $[\rho_{\lambda_1, j_1}(g_I)_N^M, \hat{p}_i^I]$ acting on a state $\psi(g_I) = \rho_{\lambda_2, j_2}(g_I)_n^m$ (J -contributions with $J \neq I$ do not matter) result in

$$[\rho_{\lambda, j}(g_I)_N^M, \hat{p}_i^I] = 8\pi i \gamma \ell_P^2 \lambda \rho_{\lambda, j}(\tau_i g_I)_N^M + 8\pi i \gamma \ell_P^2 \hat{R}_i^I \rho_{\lambda, j}(g_I)_N^M.$$

The specific ordering introduces an extra contribution from the reordering operator \hat{R}_i^I defined by

$$\begin{aligned} \hat{R}_i^I(\rho_{\lambda_1, j_1}(g_I)_{n_1}^{m_1} \cdots \rho_{\lambda_N, j_N}(g_I)_{n_N}^{m_N}) &= z \sum_{n=1}^N \rho_{\lambda_1, j_1}(g_I) \cdots \\ &\times \left(\sum_{p=1}^{N_n} \sum_{k_1, \dots, k_{N_n-1}} \rho_{z, j_n}(g_I)_{k_1}^{m_n} \cdots \rho_{z, j_n}(\tau_i g_I)_{k_{p+1}}^{k_p} \cdots \rho_{z, j_n}(g_I)_{n_n}^{k_{N_n-1}} \right) \cdots \rho_{\lambda_N, j_N}(g_I) \end{aligned}$$

⁷Projective-limit constructions, in which states with a given denominator r would play the analog of fixed-state cylindrical states in the full theory, may be used to allow for incommensurate parameters λ , but we will not need this in the present article.

$$- \sum_{n=1}^N \lambda_n \rho_{\lambda_1, j_1}(g_I)_{n_1}^{m_1} \cdots \rho_{\lambda_n, j_n}(\tau_i g_I)_{n_n}^{m_n} \cdots \rho_{\lambda_N, j_N}(g_I)_{n_N}^{m_N}$$

with z defined as before but for all λ_n involved: $\lambda_n = N_n z$ with integers N_n and rational z maximal. Up to ordering, $\hat{R}_i^I(\rho_{\lambda_1, j_1}(g_I)_{n_1}^{m_1} \cdots \rho_{\lambda_N, j_N}(g_I)_{n_N}^{m_N})$ vanishes. For more compact notation, one may express the refinement included in the action of \hat{R}_i^I in terms of $\rho_{\lambda, 0}(g_I)$ with suitable λ . Therefore, \hat{R}_i^I is not independent of the operators already introduced. Specifically, we can write $\hat{R}_i^I = -(8\pi i \gamma \ell_P^2)^{-1} \hat{p}_i^I(\rho_{z, 0}(g_I) - 1)$ where z is determined as before for all λ -parameters of holonomy factors to the right of $\rho_{z, 0}(g_I)$. The basic commutator then reads

$$[\rho_{\lambda, j}(g_I)_N^M, \hat{p}_i^I] = -(\hat{p}_i^I \rho_{\lambda, j}(g_I)_N^M) - \hat{p}_i^I(\rho_{z, 0}(g_I) - 1)\rho_{\lambda, j}(g_I)_N^M = \{\rho_j(\widehat{h_\phi(\lambda T_I)})_N^M, p_i^I\}.$$

Since $\rho_{\lambda, 0}(g_I) = 1$ classically, the commutator can indeed play the role of a quantization of the Poisson bracket, $\{\rho_j(\widehat{h_\phi(\lambda T_I)})_N^M, p_i^I\} = 8\pi i \gamma \ell_P^2 \lambda \rho_{\lambda, j}(\tau_i g_I)_N^M$. Note that it does not quantize (3.4). In addition to a quantum representation, we have to choose an ordering of non-Abelian terms when we define (3.7), corresponding to our realization of connection degrees of freedom. We do not represent pointwise exponentials, which as argued after (3.4) would not give rise to a closed algebra, but rather, as we will confirm later, use re-ordered classical exponentials closer to integrated holonomies. The basic algebra is more complicated than one may have expected, but we do obtain a closed algebra of basic operators with the correct classical limit.

Each operator \hat{p}_j^I can be viewed as a component of the flux operator for a surface given by the full side of a cubic \mathcal{V} , transversal to the direction X_j^a . These minisuperspace fluxes refer just to the artificial integration region \mathcal{V} and its coordinate volume $V_0 = L_1 L_2 L_3$, not to actual edge lengths or areas of dual surfaces, for instance in a lattice – we are still quantizing homogeneous connections. One can easily rescale the linear flux by using $\lambda_J \lambda_K p_i^I = \ell_J \ell_K \tilde{p}_i^I$ for $\epsilon_{IJK} = 1$, now corresponding to the flux through a surface of area related to edge lengths $\ell_J = \lambda_J L_J$ and $\ell_K = \lambda_K L_K$. Without a lattice construction to show how dual surfaces are related to links, however, a strict minisuperspace quantization does not provide a satisfactory, \mathcal{V} -independent definition of fluxes. We will complete the construction in Section 3.4 when considering the relation to the full theory.

The construction provided here is certainly more complicated than a traditional minisuperspace quantization, but it has several convenient features:

- Because ϕ is required to be a linear map, we need consider holonomies only along linear combinations of the generators T_I , identified with tangent vectors along “straight lines”. No path-ordering is required to compute these holonomies, and they have simple forms. (For examples of more complicated holonomies, see [45].) In particular, their matrix elements span a non-Abelian version of the space of almost-periodic functions: superpositions of periodic functions with different periodicities given by λ .
- The space of homogeneous holonomies is compact if G is compact, as the spectrum of a unital Abelian C^* -algebra. Homogeneous loop quantum cosmology makes use of function spaces on compactifications of the classical spaces of homogeneous connections, which contain all classical connections as a dense subset.
- If G is Abelian, the homogeneous Hilbert space is (non-unitarily) related to a product of the Hilbert space $\mathcal{H}_{\text{Bohr}}$ of square-integrable functions on the Bohr compactification of the real line, with $\dim(S) \times \dim(G)$ factors. More details will be given below.
- If S does not act freely, its isotropy subgroup requires additional identifications of the components of $\tilde{\phi}_I^i$: Linear maps $\tilde{\phi}$ must then satisfy $\tilde{\phi} \circ \text{ad}_f = \text{ad}_{F(f)} \tilde{\phi}$ for any f in the isotropy subgroup and a corresponding element $F(f) \in G$; see [25, 36, 44] for details.

Accordingly, we restrict homogeneous holonomies by $h_\phi(\text{ad}_f X) = h_{\text{ad}_{F(f)} \circ \phi}(X)$ in addition to the scaling condition. These functions, and therefore g_I , take values in a subgroup of G , the centralizer of the subset of all $F(f)$ in G , which is Abelian if the isotropy subgroup is sufficiently large⁸.

3.3.4 Diagonalization

Detailed ordering prescriptions make some formulas of representations of non-Abelian models look rather complicated. It is more straightforward to describe the space of connections by holonomies when diagonal Bianchi models are used, implying Abelianization. In diagonal models, the relevant parameters appear by writing a homogeneous connection as $A_a^i = \tilde{c}_{(I)} \omega_a^I \Lambda_I^i$ with left-invariant 1-forms ω_a^I of the given symmetry type and $\text{SO}(3)$ -matrices Λ_I^i which can easily be fixed to equal δ_I^i . (Writing $\tilde{c}_I^i = \tilde{c}_{(I)} \Lambda_I^i$, not summing over I , makes use of the polar decomposition of matrices [17].) The conjugate field, the densitized triad, has a similar decomposition, $E_i^a = \tilde{p}^{(I)} X_I^a \Lambda_i^I |\det \omega_a^I|$. All ingredients except \tilde{c}_I and \tilde{p}^I are determined by the symmetry type or gauge choices, and \tilde{c}_I and \tilde{p}^I are the canonical degrees of freedom. As before, the symplectic term $(8\pi\gamma G)^{-1} \int_{\mathcal{V}} d^3x \dot{A}_a^i E_i^a = V_0 (8\pi\gamma G)^{-1} \dot{\tilde{c}}_I \tilde{p}^I$, integrated over some bounded region \mathcal{V} of volume V_0 , provides Poisson brackets

$$\{\tilde{c}_I, \tilde{p}^J\} = \frac{8\pi\gamma G}{V_0} \delta_I^J. \quad (3.8)$$

Holonomies $h_{e_I} = \exp(\ell_I \tilde{c}_{(I)} \Lambda_I^i \tau_i)$ of diagonal connections still take values in $\text{SU}(2)$. However, the relations between h_{e_I} for different I are not arbitrary because their generators $Y_I := \ell_I \tilde{c}_{(I)} \Lambda_I^i \tau_i$ satisfy the condition $\text{tr}(Y_I Y_J) = 0$ for $I \neq J$. (Different rows or columns of the $\text{SO}(3)$ -matrix Λ_I^i are orthogonal with respect to the $\text{SU}(2)$ -Killing metric $\eta_{ij} = -2\text{tr}(\tau_i \tau_j) = \delta_{ij}$.) Although the h_{e_I} do not commute with one another, any pair of them obeys $gh = hg + h^{-1}g + hg^{-1} - \text{tr}(hg)$ [17]: A product of diagonal holonomies with h appearing to the right of g can be reordered so that h appears on the left in all terms. The gauge structure of diagonal models is essentially Abelian.

To make Abelianization manifest, one usually works with matrix elements $h_I = \exp(i\ell_I \tilde{c}_I/2) \in \text{U}(1)$, completing the reduction of the theory to an Abelian one. For diagonal models, all phase-space information is indeed captured by these matrix elements because $\exp(\ell_I \tilde{c}_{(I)} \Lambda_I^i \tau_i) = \cos(\ell_I \tilde{c}_I/2) + 2\Lambda_I^i \tau_i \sin(\ell_I \tilde{c}_I/2)$. Fluxes computed for surfaces normal to X_K^a are $F^K = \ell_I \ell_J \tilde{p}^K$ ($\epsilon_{IJK} = 1$). Holonomies as multiplication operators on states in the connection representation and fluxes as derivatives simplify significantly by Abelianization. For instance, the volume operator, notorious for its complicated spectrum in the full theory [46, 47] and also on the 6-valent vertices of non-diagonal homogeneous models [12], is a simple product $\hat{V} = \sqrt{|\hat{F}^1 \hat{F}^2 \hat{F}^3|}$ of derivative operators $\hat{F}^K = -8\pi i \gamma \ell_p^2 \lambda_I \lambda_J \partial / \partial c_K$ ($\epsilon_{IJK} = 1$) on $\text{U}(1)$. Abelianization also implies that a triad representation becomes available.

3.3.5 Abelian homogeneous connections

In diagonalized or isotropic models we encounter Abelian homogeneous connections with $G = \text{U}(1)$. In this case, the structures introduced for general homogeneous connections simplify: Our function space consists of superpositions of functions $\rho_{\lambda,n}(g)$ of a single variable $g(L, \tilde{c}) = \exp(iL\tilde{c})$ per independent direction, with $\lambda \in \mathbb{Q}$ and $n \in \mathbb{N}$, using $\text{U}(1)$ -representations $\rho_n(g) = g^n$. Multiplication (3.5) now reads

$$\rho_{\lambda_1, n_1}(g) \cdot \rho_{\lambda_2, n_2}(g) = \rho_{z, n_1}(g)^{N_1} \rho_{z, n_2}(g)^{N_2} = \rho_{z, N_1 n_1 + N_2 n_2}(g), \quad (3.9)$$

⁸Formally, classical Abelianization may also be implemented via second-class constraints [53, 54].

again with integers N_1, N_2 and maximal z so that $\lambda_1 = N_1 z$ and $\lambda_2 = N_2 z$. The star relation is $\rho_{\lambda,n}(g)^* = \rho_{\lambda,-n}(g)$, and the inner product (3.6) evaluates to

$$\begin{aligned} (\rho_{\lambda_1, n_1}(g), \rho_{\lambda_2, n_2}(g)) &= \frac{1}{2\pi} \int_0^{2\pi/z} d(zc) \rho_{\lambda_1, -n_1}(\exp(ic)) \cdot \rho_{\lambda_2, n_2}(\exp(ic)) \\ &= \frac{1}{2\pi} \int_0^{2\pi} dx \rho_{-N_1 n_1 + N_2 n_2}(\exp(ix)) = \delta_{N_1 n_1, N_2 n_2} = \begin{cases} 0, & \lambda_1 n_1 \neq \lambda_2 n_2, \\ 1, & \lambda_1 n_1 = \lambda_2 n_2 \end{cases} \end{aligned} \quad (3.10)$$

after substitution. Finally, the derivative operator (3.7) is

$$\hat{p}\rho_{\lambda,n}(g) = 8\pi\gamma\ell_{\text{P}}^2 \lambda n \rho_{\lambda,n}(g)$$

with eigenvalues $8\pi\gamma\ell_{\text{P}}^2 \lambda n$. (This Abelian derivative operator is self-adjoint.)

These equations bear some semblance with representations on function spaces on the Bohr compactification of the real line, but they are not identical.

3.3.6 Relation to the Bohr compactification of the real line

As recalled in Section 3.1, traditional loop-based minisuperspace quantizations, for instance in isotropic models, combine the length parameter ℓ_0 of edges with discrete representation labels as one real number, giving exponentials $h(\tilde{c})^n = \exp(in\ell_0\tilde{c}) = \exp(in\lambda c)$ with $c = V_0^{1/3}\tilde{c}$, $\lambda = \ell_0/V_0^{1/3}$, and $\mu = n\lambda \in \mathbb{R}$. In this Abelian case, homogeneous connections are often viewed as elements of the Bohr compactification $\overline{\mathbb{R}}_{\text{Bohr}}$ of the real line [2] (rather than $U(1)$, which is obtained for fixed ℓ_0 [16]). The Bohr compactification of the real line is a compact Abelian group with representations in one-to-one correspondence with those of \mathbb{R} : they are given by $z \mapsto z^\mu$ for all real μ .⁹ Functions on $\overline{\mathbb{R}}_{\text{Bohr}}$ form a Hilbert space using the Haar measure

$$\int d\mu_{\text{H}}(c) = \lim_{C \rightarrow \infty} \frac{1}{2C} \int_{-C}^C dc. \quad (3.11)$$

As per the Peter–Weyl theorem, all continuous functions on $\overline{\mathbb{R}}_{\text{Bohr}}$ can be written as countable superpositions

$$\psi(c) = \sum_{\mu} \psi_{\mu} \exp(i\mu c),$$

states $\exp(i\mu c)$ forming an orthonormal basis for all real μ . These are eigenstates of the derivative operator $\hat{p} = -8\pi i\gamma\ell_{\text{P}}^2 d/dc$ with eigenvalues $8\pi\gamma\ell_{\text{P}}^2 \mu$. (See [61, 97, 98] for more details on the Bohr compactification of the real line.)

The form of states suggests a map between the spaces of functions on Abelian homogeneous connections and functions on the Bohr compactification of the real line: $B: \rho_{\lambda,n}(g) \mapsto \exp(i\lambda n c)$ onto the subspace spanned by all $\exp(i\mu c)$ with rational μ . With the formulas for inner products, (3.10) and (3.11), it follows that this map is an isometry, and it is a $*$ -algebra morphism and commutes with the action of \hat{p} . However, it is not invertible, and therefore not unitary: one can easily find $(\lambda_1, n_1) \neq (\lambda_2, n_2)$ such that $\lambda_1 n_1 = \lambda_2 n_2$.¹⁰

⁹Starting from $U(1)$ instead of \mathbb{R} , we make the family of representations continuous by enlarging the group manifold while keeping it compact. This procedure has no analog for the non-Abelian $SU(2)$, in which non-trivial Lie brackets determine the representations and discrete spectra of its generators, as well-known from angular momentum in quantum mechanics.

¹⁰There is a bijection between suitable subspaces of the Bohr–Hilbert space and the Abelian homogeneous Hilbert space with structure group $G = U(1)$. If we take the subspace restricted by $0 \leq \lambda < 1$, the map $\rho_{\lambda,n}(g) \mapsto |\mu\rangle := |\lambda + n\rangle$ is a one-to-one transformation to the subspace of the Bohr–Hilbert space with rational μ . With the restriction on λ , one can, given μ , uniquely determine n as the integer part of μ and λ as $\mu - n$. Moreover, if $\lambda_1 + n_1 \neq \lambda_2 + n_2$, $\lambda_1 \neq \lambda_2$ or $n_1 \neq n_2$. However, choices $(\lambda_1, n_1, \lambda_2, n_2)$ exist for which $\lambda_1 + n_1 \neq \lambda_2 + n_2$ but $\lambda_1 n_1 = \lambda_2 n_2$; the inner product is therefore not preserved and the map is not unitary.

Not all features of the Bohr compactification are realized in homogeneous models even of Abelian type; care is therefore required if only the Bohr compactification is used:

1. The label μ is a degenerate version of a pair (λ, n) of state parameters, playing distinct roles in holonomies and discrete dynamics. The degeneracy of λ and n in μ is lifted by a direct quantization of homogeneous connections as linear maps $\phi: \mathcal{L}S \rightarrow \mathcal{L}G$.
2. Our new quantization of homogeneous connections easily applies to non-Abelian models, while the Bohr compactification of \mathbb{R}^3 does not properly display non-Abelian features of general anisotropic models. Via the spectrum of our C^* -algebra, we obtain a compactification of the space of non-Abelian connections unrelated to the Bohr compactification.

The Bohr compactification was introduced to loop quantum cosmology in [2] by way of a pure minisuperspace quantization of the isotropic connection component c . Compared to using a periodification of the real line to $U(1)$, as originally done in [16], this procedure has the advantage of faithfully representing all values of the connection component: c can be computed if exponentials $\exp(i\mu c)$ are known for all real μ (irreducible representations of $\overline{\mathbb{R}}_{\text{Bohr}}$), while knowing $\exp(in c)$ with integer n (irreducible representations of $U(1)$), allows one to compute c only up to adding integer multiples of 2π . Still, this alteration of the original quantization is inadequate, as shown here. An isotropic connection is not a number c , and a diagonal homogeneous connection is not a triple of numbers c_I , just as an inhomogeneous connection is not a collection of scalar fields. A homogeneous connection is a linear map from $\mathcal{L}S$ to $\mathcal{L}G$, or an element of $\mathcal{L}S^* \times \mathcal{L}G$. The factor of $\mathcal{L}S^*$ is crucial to relate the nature of a connection as a 1-form, but it is overlooked if one takes only the components c_I , or a single c for isotropic models. The new quantization of homogeneous models provided here takes into account the correct mathematical structure of homogeneous connections, leading to inequivalent Hilbert space representations. In some of the following sections, we will see that these differences are crucial for realizing a relation to the full theory and for some dynamical aspects.

3.4 Minisuperspace operators and averaging

Minisuperspace quantizations allow a large set of choices regarding quantum representations, kinematical operators, and, most of all, the dynamics. The dynamics is the most difficult to derive from the full theory, requiring detailed projection maps to ensure that one stays on the space of homogeneous states; no strict derivation is available as of now. Fortunately, however, quantum geometry implies several general effects in the dynamics, for instance in Hamiltonian constraint operators of loop quantum cosmology, deviating from classical expressions by the use of holonomies and inverse-triad operators. The form of holonomies and inverse triads, in turn, is dictated by properties of the kinematical quantum representation used. If one can derive the simpler setting of kinematical representations and basic operators, properties that imply characteristic dynamics in the full theory are realized in reduced models as well. Reliable qualitative effects can be predicted even if the dynamics is not directly derived but rather constructed by analogy with the full theory, using reduced operators. Given that the full dynamics so far appears to be ambiguous, too, only generic effects are reliable, anyway. Details of the reduction of dynamics may not matter much, provided one is asking the right questions. Relating models to the full theory helps one decide which questions can (and should) be asked.

3.4.1 Lattice subalgebras and spin-lattice states

For any fixed triple of integers \mathcal{N}_I , the operators $\rho_{k_I/\mathcal{N}_I, j_I}(g_I)_{n_I}^{m_I}$, for all integer k_I , together with \hat{p}_i^J form a subalgebra of the homogeneous holonomy-flux algebra, which we call a lattice subalgebra or, more specifically, the $(1/\mathcal{N}_1, 1/\mathcal{N}_2, 1/\mathcal{N}_3)$ -lattice subalgebra. Any state

$\rho_{k_I/\mathcal{N}_I, j_I}(g_I)_{n_I}^{m_I}|0\rangle$, obtained by acting with a homogeneous lattice-subalgebra holonomy on the cyclic state $|0\rangle$ independent of g_I , can be written as a superposition

$$\rho_{k_I/\mathcal{N}_I, j_I}(g_I)_{n_I}^{m_I} = \sum_{h_1, \dots, h_{k_I-1}} \rho_{1/\mathcal{N}_I, j_I}(g_I)_{h_1}^{m_I} \rho_{1/\mathcal{N}_I, j_I}(g_I)_{h_2}^{h_1} \cdots \rho_{1/\mathcal{N}_I, j_I}(g_I)_{n_I}^{h_{k_I-1}}$$

of products of elementary excitations $\rho_{1/\mathcal{N}_I, j_I}(g_I)_{n_I}^{m_I}$. It can be viewed as the evaluation of a lattice-based spin-network state in a homogeneous connection – a cylindrical state whose graph is a lattice with straight edges and regular spacings $\ell_I = L_I/\mathcal{N}_I$.

In order to make contact with inhomogeneous states, we use a spatial lattice of the form just introduced, with uniform spacing ℓ_I in direction X_I^a as measured in coordinates, from links along the three invariant vector fields X_I^a of a Bianchi I model. We require the region \mathcal{V} of coordinate size $V_0 = L_1 L_2 L_3$ to be sufficiently large, to allow many lattice links of the chosen spacings. We must restrict attention to Bianchi I at this stage to obtain closed lattices. For non-Abelian symmetry groups, such as those of Bianchi models other than type I, different generators do not form closed square loops by their integral curves, and therefore no lattice can be constructed¹¹. As seen in Section 3.5.1, such Bianchi models can still be quantized at least as far as their dynamics is concerned: One would refer to Bianchi I lattices to define basic homogeneous variables, and implement the different dynamics by an additional potential in the Hamiltonian constraint. Lacking closed lattices, the spatial manifold structure of other Bianchi models cannot be realized in a quantum model. However, this classical ingredient should not be taken too seriously, anyway, because from inhomogeneous models the effects of loop quantum gravity are known to modify the classical space-time structure as a consequence of a quantum-corrected hypersurface-deformation algebra [37, 38]. Lattices can be seen as a crutch or a helpful visualization to construct suitable state spaces on which one can represent operators with the correct classical limit. Once such state spaces have been obtained, they can be extended to different kinds of dynamics even if lattices are no longer available.

Fixing an orientation for each of the three directions, we label lattice links by pairs (v, I) of a vertex v as the starting point of a link $e_{v, I}$ in direction X_I^a (as in [35]). For a connection A_a^i (not assumed homogeneous at this stage), each link gives rise to a holonomy $h_{v, I} = \mathcal{P} \exp(\int_{e_{v, I}} A_a^i \tau_i \dot{e}^a ds)$. We will work with spin-network states of the underlying lattice (not required to be gauge invariant), or spin-lattice states. Each link holonomy appears in some irreducible $SU(2)$ -representation with spin $j_{v, I}$. In the matrix representation $\rho_{j_{v, I}}(h_{v, I})$, we pick matrix elements $\langle m_{v, I} | \rho_{j_{v, I}}(h_{v, I}) | n_{v, I} \rangle$, with two eigenstates $|m_{v, I}\rangle$

¹¹For two generators X_1 and X_2 , a single closed loop is obtained if one uses integral curves of the left-invariant vector field of X_1 and the right-invariant vector field of X_2 , as proposed in [7]: left-invariant vector fields commute with right-invariant ones. However, no complete lattice can be formed from these integral curves in three spatial dimensions: To generate lattice sites, one would have to fix one type of vector field, left- or right-invariant, for each spatial direction. If X_1^a is taken as left-invariant, X_2^a must be right-invariant for a closed 2-dimensional lattice in the 1–2-surface. For a closed lattice in the 1–3-surface, also X_3^a would have to be right-invariant, but then, with both X_2^a and X_3^a right-invariant, there is no closed lattice in the 2–3-surface – unless X_2^a and X_3^a happen to commute. Lattice constructions based on the interplay of left- and right-invariant vector fields cannot be performed for all Bianchi types, making those constructions in the available cases (Bianchi I and II) non-generic. Attempts at such constructions in anisotropic models show some of the pitfalls of ad-hoc assumptions, as illustrated by the series [8, 9, 100] of papers where most initial claims of [8], for instance regarding averaging or a possible relation to lattice constructions, had to be withdrawn or weakened in later installments. Initially simple-looking constructions became more and more contrived. Instead, it is more general (while still not free of assumptions) to use lattices according to the kinematical structure of Bianchi I, and then implement other Bianchi models by suitable curvature terms in the dynamics [33]. In this way, all Bianchi class A models can be quantized with one and the same scheme. One may worry about an inconsistency in using Bianchi I lattices for other Bianchi models. However, at the inhomogeneous lattice level, no strict Bianchi models can be realized. The symmetry type just provides guidelines along the way to consistent dynamics, which can well be realized for all class A Bianchi models.

and $|n_{v,I}\rangle$ of $\rho_{j_{v,I}}(\tau_3)$ (or any other component). The function $\langle m_{v,I}|\rho_{j_{v,I}}(h_{v,I})|n_{v,I}\rangle$ is then an eigenstate with eigenvalues $m_{v,I}$ and $n_{v,I}$, respectively, of the 3-components of right-invariant and left-invariant derivatives by $h_{v,I}$. Our spin-lattice states are therefore functions $\psi_{(j_{v,I},m_{v,I},n_{v,I})}(h) = \prod_{v,I} \langle m_{v,I}|\rho_{j_{v,I}}(h_{v,I})|n_{v,I}\rangle$ depending on the connection via link holonomies, with an inner product defined as usual by integrating over all $h_{v,I}$ using the Haar measure [3]. This defines the Hilbert space $\mathcal{H}_{\text{lattice}}$. For short, we will write these states as $|(j_{v,I}, m_{v,I}, n_{v,I})\rangle$. We have the usual action of holonomies and fluxes.

3.4.2 Homogeneous distributions

A homogeneous analog of spin-lattice states, depending on holonomies $h_\phi(\lambda_I T_I) = \exp(\lambda_I \phi(T_I))$, is $\psi_{(\lambda_I, j_I, m_I, n_I)}(g_I) = \prod_I \langle m_I|\rho_{j_I}(h_\phi(\lambda_I T_I))|n_I\rangle = \prod_I \langle m_I|\rho_{\lambda_I, j_I}(g_I)|n_I\rangle$ with $\lambda_I \mathcal{N}_I$ integer, written for short as $|(\lambda_I, j_I, m_I, n_I)\rangle$. There is an additional label λ_I , replacing the edge or link dependence of inhomogeneous states and representing the \mathcal{LS} -part $X = \lambda_I T_I$ of a homogeneous connection $\phi \in \mathcal{LS}^* \otimes \mathcal{LSU}(2)$, subject to the scaling condition. The set of these states is fixed by holonomies in the $(1/\mathcal{N}_1, 1/\mathcal{N}_2, 1/\mathcal{N}_3)$ -lattice subalgebra, with elementary holonomies acting by multiplication, changing the $\text{SU}(2)$ -representations j_I according to recoupling rules, and flux operators having eigenvalues $8\pi\gamma\ell_P^2 m_I$ (for right-invariant vector fields) and $8\pi\gamma\ell_P^2 n_I$ (for left-invariant ones). No decomposition as in (3.5) is required since we have a fixed common denominator \mathcal{N}_I for all holonomies considered in direction X_I .

So far, homogeneous and inhomogeneous lattice states are defined separately from each other. We relate them by introducing a map $\sigma: \mathcal{H}_{\text{hom}} \rightarrow \mathcal{D}_{\text{lattice}}, |(\lambda_I, j_I, m_I, n_I)\rangle \mapsto (|\lambda_I, j_I, m_I, n_I\rangle|$ from the homogeneous Hilbert space to distributions on the lattice Hilbert space. This map is the key ingredient of quantum symmetry reduction, as described in Section 2. Following [36], we define homogeneous distributions by their evaluations

$$\begin{aligned} & ((\lambda_I, j_I, m_I, n_I)| (j_{v,I}, m_{v,I}, n_{v,I})\rangle \\ &= \langle (\lambda_I, j_I, m_I, n_I)| (j_{v,I}, m_{v,I}, n_{v,I})\rangle|_{h_{v,I}=\exp(\phi(T_I)/\mathcal{N}_I)} \end{aligned} \quad (3.12)$$

on all basis states of $\mathcal{H}_{\text{lattice}}$. On the right-hand side, the inner product is taken in \mathcal{H}_{hom} , with $|(j_{v,I}, m_{v,I}, n_{v,I})\rangle|_{h_{v,I}=\exp(\phi(T_I)/\mathcal{N}_I)}$ obtained by restricting the connection dependence of the spin-lattice state to homogeneous ϕ . The distributional evaluation vanishes unless the representation j_I appears in the tensor product $\bigotimes_{v,I} j_{v,I}$, and $m_I = \sum_{v,I} m_{v,I}$, $n_I = \sum_{v,I} n_{v,I}$.¹² The reduction of states depends on the size of the region \mathcal{V} via \mathcal{N}_I , just like the classical reduction of the phase space.

3.4.3 Averaged operators

An operator \hat{O} can be reduced from the lattice theory to the homogeneous Hilbert space if its dual action fixes the space of homogeneous distributions: If there is a $|\psi\rangle \in \mathcal{H}_{\text{hom}}$ such that $((\lambda_I, j_I, m_I, n_I)| \hat{O}^\dagger | (j_{v,I}, m_{v,I}, n_{v,I})\rangle = \langle \psi | (j_{v,I}, m_{v,I}, n_{v,I})\rangle$ for all $|(j_{v,I}, m_{v,I}, n_{v,I})\rangle$, we define $\hat{O}|(\lambda_I, j_I, m_I, n_I)\rangle = |\psi\rangle$. All link holonomies $\langle m_{v,J}|\rho_{j_{v,J}}(h_{v,J})|n_{v,J}\rangle$ along symmetry generators, taken as multiplication operators, satisfy this condition. They act on distributional homogeneous states by $\langle m_{v,J}|\rho_{j_{v,J}}(h_{v,J})|n_{v,J}\rangle \langle g_I | (\lambda_I, j_I, m_I, n_I)\rangle = \rho_{1/\mathcal{N}_J, j_{v,J}}(g_J)_{n_{v,J}}^{m_{v,J}} \cdot \rho_{\lambda_I, j_I}(g_I)_{n_I}^{m_I}$, just as in the reduced space of homogeneous states.

Flux operators require additional constructions. A single lattice flux $\hat{F}_{v,I}$ associated with a surface dual to link $e_{v,I}$ does not map a homogeneous distribution to another such state: Take

¹²In (3.12), we restrict to holonomies $h_{v,I} = \exp(\lambda_I \phi(T_I))$ with v -independent $\lambda_I = 1/\mathcal{N}_I$, or a regular aligned lattice of uniform link lengths. At this stage, we could allow irregular lattices with varying $\ell_I(v)$, as long as all links are still along symmetry generators $\sum_I \lambda_I T_I$. Different lattice sectors would then contribute to the reduction, and refinement would be necessary in the multiplication and action of holonomies. This option will be discussed in more detail below.

a set of states $|\psi_{v,I}\rangle := |0, \dots, 0, (1/2, 1/2, 1/2), 0, \dots, 0\rangle$, each with non-zero labels only on one lattice link $e_{v,I}$. We have

$$\langle (1/\mathcal{N}_I, j_I, m_I, n_I) | \hat{F}_{v,I} |\psi_{v,I}\rangle = 4\pi\gamma\ell_P^2 \delta_{j_I, 1/2} \delta_{m_I, 1/2} \delta_{n_I, 1/2}$$

and $\langle (\lambda_I, j_I, m_I, n_I) | \hat{F}_{v,I} |\psi_{v',I'}\rangle = 0$ if $v \neq v'$ or $I \neq I'$. Therefore,

$$\langle (1/\mathcal{N}_I, 1/2, 1/2, 1/2) | \hat{F}_{v,I} |\psi_{v,I}\rangle \neq \langle (1/\mathcal{N}_I, 1/2, 1/2, 1/2) | \hat{F}_{v',I'} |\psi_{v',I'}\rangle \quad \text{for } v \neq v'.$$

However, we must have $\langle \Psi | \psi_{v,I}\rangle = \langle \Psi | \psi_{v',I'}\rangle$ for any homogeneous state $|\Psi\rangle \in \mathcal{H}_{\text{hom}}$ since $\psi_{v,I} |_{h_{w,I}=\exp(\lambda_I \phi(T_I))} = \psi_{v',I'} |_{h_{w',I'}=\exp(\lambda_I \phi(T_I))}$. The state $\langle (1/\mathcal{N}_I, 1/2, 1/2, 1/2) |$ cannot be contained in a decomposition of $\langle (1/\mathcal{N}_I, j_I, m_I, n_I) | \hat{F}_{v,I}$ in our basis, and we can repeat the arguments with arbitrary values of the non-zero label in $|\psi_{v,I}\rangle$ to show that no homogeneous state can be contained in the decomposition. Therefore, the distribution $\langle (1/\mathcal{N}_I, j_I, m_I, n_I) | \hat{F}_{v,I}$ cannot be a superposition of homogeneous distributional states: flux operators associated with a single link do not map the space of homogeneous states to itself.

Even classically, the flow $\{\cdot, F_S\}$ generated by a flux operator is not everywhere tangent to the submanifold of homogeneous connections and *unrestricted* triads in the inhomogeneous phase space, but it is tangent to the subspace on which both the connection *and* the triad are homogeneous. In the quantized theory, using distributional states, we have ensured states to be restricted to homogeneous connections, but no such condition has yet been implemented for the densitized triad or fluxes.

Flux operators must be averaged to generate a flow that keeps the space of homogeneous states invariant. However, non-Abelian gauge properties prevent us from simply adding $\sum_n \int_{S_n} E_i^a n_a d^2y$ for a family of surfaces S_n translated along the generators of the symmetry group. Instead, we must relate the fibers of the $SU(2)$ -bundle in which E_i^a takes values, using parallel transport between the S_n . (This problem seems to be related to issues encountered in constructions of a non-Abelian triad representation [55]. Here, homogeneity will help us to propose a solution.)

To describe the specific construction, we assume an aligned state, consisting only of holonomies $h_{v,I}$ in the three independent directions but not necessarily forming a regular lattice. For an averaged p_i^I , we choose families of surfaces $S_{n,I}$ transversal to the symmetry generators X_I^a , such that they have co-normals $n_a^I = \delta_a^I$, layered at regular intervals across the region \mathcal{V} . Eventually, we will send the number N of surfaces to infinity. Before doing so, we define a gauge-covariant averaging by $\bar{p}_i^I = N^{-1} \sum_{n=1}^N \int_{S_{n,I}} \text{ad}_{h_I(y)}(E_i^a(y) n_a^I) d^2y$ where $h_I(y)$ is the connection-dependent parallel transport from some base point, chosen for each integral curve in direction I , to a point y on the surface.

The base points will be chosen in a state-dependent way because the state determines how the connection is excited, usually in a discontinuous way at lattice vertices. We decompose a state as a superposition of contributions $\Psi = \psi(h_{w,J}, h_{w',K}) \prod_{v_I} \rho_{j_{v,I}}(h_{v_I,I})$ where the dependence on holonomies along directions J and K will not matter. The set of all v_I then gives us all vertices where parallel transport in the I -direction changes discontinuously. We will average with these vertices chosen as base-points, so that only the continuous parts of parallel transport are taken into account. We first decompose surfaces $S_{n,I} = \bigcup_k S_{n,I,k}$ so that each piece $S_{n,I,l}$ intersects at most one edge. In the action of the flux operator, instead of summing over k we will then be summing over edges intersecting the surface: We write

$$\frac{1}{N} \sum_{n=1}^N \int_{S_{n,I}} d^2y \text{ad}_{h_I(y)}(\hat{E}_i^a(y) n_a^I) \Psi = \frac{1}{N} \sum_{n=1}^N \sum_{e_{v_I, I} \cap S_{n,I} \neq \emptyset} \text{ad}_{h_{v_I, I}(v_n)}(\hat{F}_{v_I, I}(S_{n,I,k})) \Psi$$

with $h_{v_I, I}(v_n)$ the parallel transport from v_I to the intersection point v_n of $e_{v_I, I}$ with $S_{n,I}$.

If a piece of the surface $S_{n,I}$ intersects an edge $e_{v_I,I}$, we choose the base point to be v_I (for a right-invariant vector field quantizing the flux, or the other endpoint of $e_{v_I,I}$ for a left-invariant one). The adjoint action of $h_{v_I,I}(v_n)$ in the averaged flux then implies that a flux operator does not insert just τ_i in the holonomy, at the intersection point $\{v_n\} = S_{n,I} \cap e_{v_I,I}$ with the surface, but $h_{v_I,I}(v_n)^{-1} \tau_i h_{v_I,I}(v_n)$. For a single edge e , splitting the holonomy $h_e := h_e^{(1)}(v_n) h_e^{(2)}(v_n)$ in two pieces $h_e^{(1)}(v_n)$ and $h_e^{(2)}(v_n)$ at an intersection point v_n , we thus have

$$\begin{aligned} \hat{p}_i^I \langle m' | \rho_j(h_e) | n' \rangle &= -8\pi i \gamma \ell_{\mathbb{P}}^2 \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \langle m' | \rho_j(h_e^{(1)}(v_n) \text{ad}_{h_e^{(1)}(v_n)}(\tau_i) h_e^{(2)}(v_n)) | n' \rangle \\ &= -8\pi i \gamma \ell_{\mathbb{P}}^2 \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \langle m' | \rho_j(\tau_i h_e) | n' \rangle \delta_{S_{n,I} \cap e \neq \emptyset}, \end{aligned}$$

where only terms such that $S_{n,I} \cap e \neq \emptyset$ contribute. For large N , the number of non-zero contributions divided by N approaches λ_e , the ratio of the length spanned by h_e relative to L_I . We obtain eigenvalues $8\pi \gamma \ell_{\mathbb{P}}^2 \lambda_e m'$. Not surprisingly, in this homogeneous context the averaged flux does not refer to any point on the edge where it acts, but it picks up the relative length of the edge by the number of intersection points. For multiple edges, the flux acts by the product rule.

If all edges involved form a regular lattice, with the number $\mathcal{N}_I = L_I/\ell_I$ of lattice links, it follows that

$$\hat{p}_i^I = \frac{1}{\mathcal{N}_I} \sum_v \hat{F}_{v,I,i}. \quad (3.13)$$

The factor of $1/\mathcal{N}_I = \ell_I/L_I$ eliminates over-counting by adding fluxes of all lattice sites along direction I . In the other two directions, on the other hand, we sum rather than average because the minisuperspace p_i^I is defined for a surface stretching through the region \mathcal{V} , as in (3.7). Indeed, heuristically, the eigenvalues of \hat{p}_i^I , $8\pi \gamma \ell_{\mathbb{P}}^2 \mathcal{N}_I^{-1} \sum_v m_{v,I}$ can be written as $L_J L_K$ multiplying the average value of the densitized triad: $8\pi \gamma \ell_{\mathbb{P}}^2 \mathcal{N}_I^{-1} \sum_v m_{v,I} = 8\pi \gamma \ell_{\mathbb{P}}^2 \mathcal{N}_J \mathcal{N}_K \overline{m_I} = \mathcal{N}_J \mathcal{N}_K \overline{E_3^I}$, where $\overline{m_I} = (\mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3)^{-1} \sum_v m_{v,I}$ is the lattice average, quantizing the average of the plaquette flux $\int E_3^I dx^J dx^K / 8\pi \gamma \ell_{\mathbb{P}}^2$.

3.4.4 Kinematical quantization commutes with symmetry reduction

Holonomy operators in \mathcal{H}_{hom} are directly obtained from their dual action on $\mathcal{D}_{\text{lattice}}$. It does not matter whether we act with a holonomy operator first and then symmetry-reduce, or first reduce and then act with the corresponding homogeneous operator:

$$\sigma(\rho_{1/\mathcal{N}_J, j_{v,J}}(g_J) | (\lambda_I, j_I, m_I, n_I) \rangle) = \rho_{j_{v,J}}(h_{v,J}) \sigma(| (\lambda_I, j_I, m_I, n_I) \rangle).$$

After averaging, the same commutation relationship is realized for lattice flux operators. We have

$$\begin{aligned} &((1/\mathcal{N}_I, j_I, m_I, n_I) | \hat{p}_3^J | (j_{v,I}, m_{v,I}, n_{v,I}) \rangle) \\ &= \frac{1}{\mathcal{N}_J} \sum_v ((1/\mathcal{N}_I, j_I, m_I, n_I) | \hat{F}_{v,J,3} | (j_{v,I}, m_{v,I}, n_{v,I}) \rangle) \\ &= \frac{8\pi \gamma \ell_{\mathbb{P}}^2}{\mathcal{N}_J} \sum_v m_{v,J} \delta_{j_I \in \otimes_v j_{v,I}} \delta_{m_I, \sum_v m_{v,I}} \delta_{n_I, \sum_v n_{v,I}} \\ &= 8\pi \gamma \ell_{\mathbb{P}}^2 \lambda_J m_J ((\lambda_I, j_I, m_I, n_I) | (j_{v,I}, m_{v,I}, n_{v,I}) \rangle) \end{aligned}$$

using the δ -identifications and $\lambda_J = 1/\mathcal{N}_J$ in the last step. On reduced states, on the other hand, we have $\hat{p}_3^J |(\lambda_I, j_I, m_I, n_I)\rangle = 8\pi\gamma\ell_P^2 \lambda_J m_J$ from the right-invariant vector field (3.7) in the λ_J -sector. Comparing these equations, we see that $\sigma(\hat{p}_3^J |(\lambda_I, j_I, m_I, n_I)\rangle) = \hat{p}_3^J \sigma(|(\lambda_I, j_I, m_I, n_I)\rangle)$, with analogous calculations for other components of \hat{p}_i^J . (As remarked in Section 3.3.2, the non-Abelian \hat{p}_i^J is not a symmetric operator unless we are in the lattice setting of fixed λ_I for all states involved. Similarly, \hat{p}_i^J is not symmetric in this situation due to averaging, in particular with a state-dependent choice of base points for parallel transport.)

For basic operators, it does not matter whether we quantize or reduce first. We obtain the same representation properties as in the full theory, and the same qualitative quantum-geometry effects. But a quantitative correspondence is more complicated for composite operators, especially the Hamiltonian constraint crucial for dynamics.

3.4.5 Holonomy-flux algebra in reduced models

The previous calculations have shown how holonomy-flux representations of homogeneous models are derived from the full algebra. Using the minisuperspace embedding σ , we obtain basic operators – holonomies and fluxes – from their action on inhomogeneous lattice states. Since the holonomy-flux representation of the full theory is unique [62, 74], the minisuperspace representation derived here, by restriction of the full algebra to lattices followed by taking the dual action on homogeneous distributions, enjoys the same distinction for a given lattice.

So far, we have written all formulas for the general case of non-Abelian homogeneous models. Using the same techniques of restriction of states and reduction of operators, it is straightforward to implement diagonalization or isotropy: We restrict states to $\phi_I^i = c_{(I)} \Lambda_I^i$ for diagonal models, or $\phi_I^i = c \Lambda_I^i$ for isotropic ones. Flux operators $\hat{p}^I = \Lambda_{(I)}^i \hat{p}_i^I$ then leave diagonal states invariant (while $\Lambda_{(J)}^i \hat{p}_i^I$ with $J \neq I$ would not), and the averaged $\hat{p} = \frac{1}{3} \sum_I \hat{p}^I$ leaves isotropic states invariant. These situations are covered in [19].

3.5 Dynamics

From holonomy and flux operators, we construct more complicated ones such as the volume or the Hamiltonian constraint. The volume on spin-lattice states can be defined as in the full theory, using $|\frac{1}{6} \epsilon^{ijk} e^{IJK} \hat{F}_{v,I,i} \hat{F}_{v,J,j} \hat{F}_{v,K,k}|^{1/2}$, just restricted to 6-valent vertices as encountered in a lattice. The complete spectrum is unknown in the non-Abelian case. For simpler algebraic relations, we may replace the cubic SU(2)-invariant with a product of quadratic invariants, $\hat{V}_v := \prod_{I=1}^3 (\hat{F}_{v,I,i} \hat{F}_{v,I}^i)^{1/4}$ with eigenvalues $(8\pi\gamma)^{3/2} \ell_P^3 \prod_{I=1}^3 (j_{v,I}(j_{v,I}+1))^{1/4}$. In what follows, details and differences of these spectra will not play a major role, and we will make use of the simpler version.

As part of the formulation of dynamics, we will be interested in reducing the volume operator. An important question for non-linear combinations of basic operators is whether the average is taken before or after reduction. The minisuperspace volume

$$\prod_{I=1}^3 (\hat{p}_i^I \hat{p}^{I,i})^{1/4} = (\mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3)^{-1/2} \prod_{I=1}^3 \sqrt[4]{\hat{J}_I^2},$$

using $\hat{J}_I = \sum_v \hat{F}_{v,I}$ and (3.13), has eigenvalues

$$\frac{(8\pi\gamma)^{3/2} \ell_P^3}{\sqrt{\mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3}} \prod_{I=1}^3 \sqrt[4]{j_I(j_I+1)} = (8\pi\gamma)^{3/2} \ell_P^3 \prod_{I=1}^3 \sqrt{\lambda_I} \sqrt[4]{j_I(j_I+1)}.$$

The spectrum can be computed using \hat{p}_i^I in (3.7) or $\hat{\tilde{p}}_i^I$ in (3.13), but it does not equal that of the averaged $(\mathcal{N}_1\mathcal{N}_2\mathcal{N}_3)^{-1}\sum_v V_v$, which would be the reduced volume operator. The distinction is important for the correct size of quantum-geometry effects, as we will see below. Pure minisuperspace models make use of \hat{p}_i^I or $\hat{\tilde{p}}_i^I$; correctly capturing quantum effects requires an averaged volume operator.

3.5.1 Hamiltonian constraint

The classical Hamiltonian constraint contains curvature components, to be represented in homogeneous models by holonomies $\rho_{\lambda,j}(g_I)_n^m$, non-polynomial functions of the connection which differ from connection or curvature components by higher-order terms. In Abelian models, it is easy to see that $\rho_{\lambda,n}(g_I)$ as an operator is not weakly continuous in λ at $\lambda = 0$, and it not possible to define a connection operator via $d/d\lambda|_{\lambda=0}$: The diagonal matrix elements $\langle(\lambda'_I, n'_I)|\rho_{\lambda,n}(g_I)|(\lambda'_I, n'_I)\rangle = \delta_{0,\lambda n}$, using (3.10), are not continuous in λ at $\lambda = 0$. In the non-Abelian case, the argument is more complicated because $\langle(\lambda'_I, j'_I, m'_I, n'_I)|\rho_{\lambda,j}(g_I)_n^m|(\lambda'_I, j'_I, m'_I, n'_I)\rangle$ need not be zero for $\lambda \neq 0$. The value rather depends on the multiplicity of the trivial representation in the tensor product $\rho_{j'_I}^{\otimes 2q} \otimes \rho_j^{\otimes p}$ if $\lambda'_I = q/r$ and $\lambda = p/r$ with their least common denominator r . For $\lambda = 0$, we pick out the trivial representation in $\rho_{j_I}^{\otimes 2q}$; for $\lambda \neq 0$ we pick out the coefficients of all irreducible representations of $\rho_j^{\otimes p}$ in $\rho_{j_I}^{\otimes 2q}$. In general, these coefficients are not continuously related at $\lambda = 0$.

Instead of using derivatives and connection operators, we are required to use holonomies $\rho_{\delta,j}(g_I)_n^m$ with some finite δ to construct the Hamiltonian constraint operator, appealing to the standard relationship between curvature components and holonomies around small closed loops. Since the same relationship is used for the full constraint [88, 91], there is at least a plausible connection between models and the full theory. We have

$$h_\Delta = 1 + \ell^2 s_1^a s_2^b F_{ab}^i \tau_i + O(\ell^4) \quad (3.14)$$

if the loop Δ , spanned by two unit vectors s_1^a and s_2^a , is of coordinate area ℓ^2 . In lattice constructions, one may use loops around elementary plaquettes, although consistency issues of the constraint algebra may require more complicated routings (see e.g. [65]). The coordinate area of a loop in the (I, J) -plane is then close to $\ell_I \ell_J$, and we are led to use homogeneous holonomies $\rho_{\delta_I, j_I}(g_I)$ with $\delta_I = \ell_I/L_I = 1/\mathcal{N}_I$.

In Bianchi models, we have the Hamiltonian constraint (3.2), or (3.3) when diagonalized. Putting holonomies along square loops and suitable constructions of triad operators together as in the full theory [91], we obtain, following [13, 17],

$$\begin{aligned} \hat{H} = & -\frac{1}{(8\pi)^2 \gamma^3 G^2 \hbar \delta_1 \delta_2 \delta_3} \sum_{I,J,K} \epsilon^{IJK} \text{tr} \left(\rho_{\delta_I, 1/2}(g_I) \rho_{\delta_J, 1/2}(g_J) \rho_{\delta_I, 1/2}(g_I)^{-1} \rho_{\delta_J, 1/2}(g_J)^{-1} \right. \\ & \left. \times \left| \rho_{\delta_K, 1/2}(g_K) [\rho_{\delta_K, 1/2}(g_K)^{-1}, \hat{V}] \right| \right) + \hat{H}_\Gamma \end{aligned} \quad (3.15)$$

with matrix products of all holonomies involved. Instead of $j = 1/2$ one may use other irreducible representations, or add different such contributions [63, 82, 96]. Also the values of δ_I are subject to choices whose implications we will discuss in more detail below. One may choose fixed values, or relate them to properties of the state acted on by the constraint operator. In the latter case, the state dependence of regularized constraints in the full theory would be modeled. For now, however, we will assume fixed δ_I so that \hat{H} preserves a lattice subalgebra.

The term \hat{H}_Γ in (3.15) vanishes for the Bianchi I model and incorporates spin-connection terms for other models, as in [33]. The commutator $|\rho_{\delta_K, 1/2}(g_K) [\rho_{\delta_K, 1/2}(g_K)^{-1}, \hat{V}]|$ in the second line quantizes the classical combination $\epsilon^{ijk} E_i^a E_j^b / \sqrt{|\det E|}$ which diverges for degenerate

triads, at classical singularities of collapse type¹³. Writing this operator on the right in \hat{H} , in the ordering chosen in (3.15) as in the full theory [91], implies that singular states decouple from the dynamics: They are automatically annihilated by the constraint. (There is a similar factor in \hat{H}_Γ , also ordered to the right.) In this way, quantum hyperbolicity [21] is realized even if no difference equation is available as an explicit evolution equation.

3.5.2 Abelian models and difference equations

Difference equations are obtained in Abelian models after transforming to the triad representation, provided the Hamiltonian constraint fixes a lattice subalgebra. First, writing Abelian holonomies of a diagonal Bianchi model as $\rho_{\delta_I, 1/2}(g_I) = \cos(\delta_I c_I/2) + 2\Lambda_I^i \tau_i \sin(\delta_I c_I/2)$, one can compute all matrix products and the trace in (3.15). Since the final result is lengthy, we define

$$\hat{K}_3 := \sin(\delta_1 c_1/2) \cos(\delta_1 c_1/2) \sin(\delta_2 c_2/2) \cos(\delta_2 c_2/2) = \frac{1}{4} \sin(\delta_1 c_1) \sin(\delta_2 c_2)$$

and cyclic permutations thereof, as well as

$$\hat{I}_J = \left| 2i \left(\sin(\delta_J c_J/2) \hat{V} \cos(\delta_J c_J/2) - \cos(\delta_J c_J/2) \hat{V} \sin(\delta_J c_J/2) \right) \right|.$$

The Hamiltonian constraint is then

$$\hat{H} = -\frac{1}{8\pi^2 \gamma^3 G^2 \hbar \delta_1 \delta_2 \delta_3} \sum_{J=1}^3 \hat{K}_J \hat{I}_J.$$

It is straightforward to compute the action of \hat{K}_J and the eigenvalues of \hat{I}_J on the $(\delta_1, \delta_2, \delta_3)$ -lattice subalgebra (now dropping the fixed δ_I from the notation of states):

$$\begin{aligned} \hat{K}_3 |n_1, n_2, n_3\rangle &= -\frac{1}{16} (|n_1 + 2\delta_1, n_2 + 2\delta_2, n_3\rangle - |n_1 - 2\delta_1, n_2 + 2\delta_2, n_3\rangle \\ &\quad - |n_1 + 2\delta_1, n_2 - 2\delta_2, n_3\rangle + |n_1 - 2\delta_1, n_2 - 2\delta_2, n_3\rangle) \end{aligned}$$

and cyclic permutations, and

$$\begin{aligned} \hat{I}_1 |n_1, n_2, n_3\rangle &= |V_{n_1+\delta_1, n_2, n_3} - V_{n_1-\delta_1, n_2, n_3}| |n_1, n_2, n_3\rangle \\ &= (8\pi\gamma)^{3/2} \ell_P^3 \left| \sqrt{|n_1 + \delta_1|} - \sqrt{|n_1 - \delta_1|} \right| \sqrt{|n_2 n_3|} |n_1, n_2, n_3\rangle. \end{aligned}$$

Since $\hat{H}|n_I\rangle$ must vanish (or equal the action of a matter Hamiltonian \hat{H}_{matter} on $|n_I\rangle$), a difference equation is obtained in the triad representation of coefficients ψ_{n_1, n_2, n_3} in $|\psi\rangle = \sum_{n_I} \psi_{n_1, n_2, n_3} |n_I\rangle$. Introducing $s_{n_1, n_2, n_3} := \sqrt{|n_1 n_2 n_3|} \psi_{n_1, n_2, n_3}$ to shorten the expression¹⁴, we have

$$-A_{\delta_1}(n_1) (s_{n_1, n_2+2\delta_2, n_3+2\delta_3} - s_{n_1, n_2-2\delta_2, n_3+2\delta_3} - s_{n_1, n_2+2\delta_2, n_3-2\delta_3} + s_{n_1, n_2-2\delta_2, n_3-2\delta_3})$$

¹³The absolute value around the commutator is necessary because the classical analog $\{A_a^i, V\} = 2\pi\gamma G \text{sgn}(\det E) \epsilon^{ijk} E_i^a E_j^b / \sqrt{|\det E|}$ carries a sign factor. The absolute value avoids parity violation (see [31] for a detailed discussion of parity). Classically, the sign of $\det E$ changes whenever the sign of $\{A_a^i, V\}$ changes, and one could multiply the latter with $\text{sgn}(\det E)$ to avoid parity violation. However, when quantized, $\rho_{\delta_K, 1/2}(g_K) [\rho_{\delta_K, 1/2}(g_K)^{-1}, \hat{V}]$ and $\widehat{\det E}$ do not commute in non-Abelian models. Since the dynamics identifies states annihilated by the commutator as degenerate ones corresponding to classical singularities, we refer to its own sign instead of multiplying it with the non-commuting operator $\widehat{\text{sgn} \det E}$ [18].

¹⁴Dividing by $n_1 n_2 n_3$ is well-defined for the evolution equation because $|n_1, n_2, n_3\rangle$ is annihilated by the constraint whenever $n_1 n_2 n_3 = 0$, as part of the property of quantum hyperbolicity. The coefficients ψ_{n_1, n_2, n_3} with $n_1 n_2 n_3 = 0$ decouple from the rest and can safely be ignored.

$$\begin{aligned}
& - A_{\delta_2}(n_2) (s_{n_1+2\delta_1, n_2, n_3+2\delta_3} - s_{n_1-2\delta_1, n_2, n_3+2\delta_3} - s_{n_1+2\delta_1, n_2, n_3-2\delta_3} + s_{n_1-2\delta_1, n_2, n_3-2\delta_3}) \\
& - A_{\delta_3}(n_3) (s_{n_1+2\delta_1, n_2+2\delta_2, n_3} - s_{n_1-2\delta_1, n_2+2\delta_2, n_3} - s_{n_1+2\delta_1, n_2-2\delta_2, n_3} + s_{n_1-2\delta_1, n_2-2\delta_2, n_3}) \\
& = 128\pi^2\gamma^3 G\ell_{\text{P}}^2 \delta_1 \delta_2 \delta_3 \frac{\hat{H}_{\text{matter}}(n_1, n_2, n_3)}{V_{n_1, n_2, n_3}} s_{n_1, n_2, n_3} \tag{3.16}
\end{aligned}$$

with volume eigenvalues inserted. We used partial eigenvalues of the matter Hamiltonian¹⁵

$$\hat{H}_{\text{matter}}|\psi\rangle = \sum_{n_I} \left(\hat{H}_{\text{matter}}(n_I) \psi_{n_1, n_2, n_3} \right) |n_I\rangle$$

which may still act on a matter-field dependence of s_{n_1, n_2, n_3} . The other coefficients refer to eigenvalues of \hat{I}_I , defining $A_{\delta_I}(n_I) := I_I(n_1, n_2, n_3)/V_{n_1, n_2, n_3}$ such that $A_{\delta}(n) = |\sqrt{|n+\delta|} - \sqrt{|n-\delta|}|/\sqrt{|n|}$. Equation (3.16) correctly quantizes the classical terms such as $c_2 c_3 a_1 = c_2 c_3 \sqrt{|p^2 p^3/p^1|}$ in the Hamiltonian constraint (3.3), using the ordering $|p^1|^{-1} c_2 c_3 \sqrt{|p^1 p^2 p^3|}$.

Equations such as (3.16) have been derived long ago [17], and reproduced since then many times, in slightly different forms. We have presented the derivation here with some detail because, lacking the proper notion of homogeneous connections, it had not been realized before that the form is valid only for a Hamiltonian constraint operator fixing a lattice subalgebra of the homogeneous model. If the δ_I are not fixed once and for all but depend on the state acted on (and even its graph), or if the δ_I used in the operator are not the same as those of the lattice subalgebra, the decomposition rule (3.5) or its Abelian analog (3.9) must be used, implying refinement. The operator (3.15) will still be valid, but its action must be re-derived, and does not easily give rise to a difference equation, certainly not one of constant step-size. We emphasize that refinement is realized even if the operator (3.15) is not modified, provided only one applies it to states not in the lattice subalgebra fixed by it.

3.5.3 Lattice refinement: a toy model

So far, we have presented the minisuperspace quantization of Hamiltonians. For contact with the full theory, we must try to reduce an inhomogeneous operator and face the averaging problem. This task, at present, cannot be done in detail, but its outcome will affect the choice of δ_I . Instead of deriving these values and their relation to states, we must resort to sufficiently general parameterizations to model different possible reductions.

If the δ_I are not adapted to a lattice or to the common denominator of all holonomies involved, \hat{H} will not fix any lattice subalgebra, even if the δ_I are fixed and not state-dependent. Lattice refinement then occurs by multiplying holonomies of different edge lengths and obeying the decomposition rule (3.5). Evaluations of the Hamiltonian constraint, especially the non-Abelian version, become more involved and difference equations no longer are readily available, but the property of lattice refinement can be seen already in an admittedly rough toy model. The model is certainly far from an actual derivation from some full Hamiltonian constraint. Instead, it is meant to illustrate how the behavior of δ_I could follow from some discrete dynamics. Although we use the language of inhomogeneous lattice states and operators, the actual dynamics is chosen only for illustrative purposes. Nevertheless, there are some interesting features which may be promising for a more faithful representation of the loop dynamics.

Let us assume that each δ_I is always half the maximum λ_I encountered in a lattice-subalgebra state acted on. In terms of an inhomogeneous lattice, this means that every new edge generated by a vertex contribution of the Hamiltonian constraint would go half-way to the next vertex. In an inhomogeneous lattice, the presentation of refinement depends on the order in which

¹⁵For simplicity, we assume the absence of connection couplings; see [32] for a fermionic model in which the assumption does not hold. No qualitative changes to the present statements occur in such a case.

individual holonomies or vertex contributions of the Hamiltonian constraint act. If refinement proceeds regularly, staying close to cubic lattices of nearly constant link lengths, one would expect that all plaquettes will first be split half-way along edges, and when this has happened for all of them, one would proceed to quarters and so on. However, in the Hamiltonian constraint all vertex contributions appear in superposition, not in simultaneous action on a single lattice. To realize an ordering, one may assume that a physical state annihilated by the constraint is expanded in spin-lattice states according to the eigenspace of some operator such as the total volume, by the maximum spin on all edges, or by the number of plaquettes. Ordering spin-lattice contributions in a physical state with respect to any of these values, plaquettes will be filled in a certain arrangement, such as the one described.

Back in our homogeneous model, starting with a state in some lattice subalgebra with values $\lambda_I^{(0)}$, the first action of the Hamiltonian constraint, multiplying with holonomies of lengths $\lambda_I^{(0)}/2$ in different directions, requires a decomposition (3.5) of the whole state, refining edge lengths to halves. After a single multiplication, no homogeneous holonomy of the original length $\lambda_I^{(0)}$ will occur explicitly, but such edges are still present in a corresponding homogeneous lattice because there are several matrix products $\sum_k \rho_{\lambda_I^{(0)}/2,j}(gI)_k^m \rho_{\lambda_I^{(0)}/2,j}(gI)_n^k$ of two $\lambda_I^{(0)}/2$ -holonomies without intermediate factors. Taking these holonomies into account, we keep acting with $\lambda_I^{(0)}/2$ -holonomies until all those products disappear. (In inhomogeneous lattice language, we fill all $\lambda_I^{(0)}$ -plaquettes with new edges and vertices at all midpoints of the original edges.) Once these options have been exhausted, the next refinement step is due, going to $\lambda_I^{(0)}/4$ until all the previously refined plaquettes have been filled. In the process just described, we have assumed a certain ordering of the actions of individual vertex contributions, first filling all the $\lambda_I^{(0)}$ -plaquettes, then moving to $\lambda_I^{(0)}/2$ -plaquettes, and so on, as we would do on an inhomogeneous lattice.

We can relate the number of plaquettes, or the degree of refinement, to geometrical quantities of the whole lattice. Starting with a nearly homogeneous lattice with all edge spins j_0 equal, the initial area in the J, K -plane is approximately $A_0 = (\lambda_I^{(0)} j_0)^2 \mathcal{N}_J \mathcal{N}_K$, with $\mathcal{N}_J \mathcal{N}_K$ plaquettes in this plane and transversal links of the size $\lambda_I^{(0)} = L_I / \mathcal{N}_I$. Thus, $A_0 = L_I^2 j_0^2 \mathcal{N}_J \mathcal{N}_K / \mathcal{N}_I^2$, or $A_0 \approx V_0^{2/3} j_0^2$ for a nearly isotropic lattice with equal edge numbers in the three directions. When all these plaquettes have been refined after the first stage, the maximum spins have changed to $j_1 = 2j_0 + 1/2$: multiplying j_0 with two because we decompose holonomies halfway according to (3.5), doubling them over, and adding $1/2$ from the action of a new holonomy in the fundamental representation. The added $1/2$ will soon be irrelevant when j becomes larger by repeated doubling. The area has then increased to $A_1 = (\lambda_I^{(1)})^2 (2j_0)^2 (2\mathcal{N}_J) (2\mathcal{N}_K)$ with $\lambda_I^{(1)} = \lambda_I^{(0)}/2$. Combining these equations, $A_1 = 4A_0$, in which only the increased spin due to refinement contributes. After N steps, the same arguments show that the area has increased to $A_N = 2^N A_0$, and $\lambda_I^{(N)} = \lambda_I^{(0)}/2^N = \lambda_I^{(0)} \sqrt{A_0/A_N}$. The spin quantum numbers increase by $j_N \approx 2^N j_0$.

It is an interesting feature that the spin of a single action (here $1/2$), an ambiguity parameter of the full constraint, becomes progressively less important in the model as refinement proceeds, increasing j_N . The large-scale behavior is insensitive to details of the microscopic dynamics and associated ambiguities, a property that makes effective and mean-field viewpoints meaningful.

In this model, the edge lengths $\lambda_I^{(N)}$ are inversely proportional to the square root of the area, or to a linear measure of the extension of the lattice. With near isotropy, this scaling, $\lambda = \lambda_0/a$, is of advantage for holonomy-modified dynamics, in which holonomies $\exp(i\lambda c) = \exp(i\lambda_0 c/a)$ depend on the isotropic connection component c only in the combination $c/a \propto \mathcal{H}$ proportional to the Hubble parameter; the same behavior has been proposed in [6] as an ad-hoc choice for the actual dynamics of loop quantum cosmology. While c may grow large even at small curvature, for

instance if there is a positive cosmological constant, \mathcal{H} remains small in low-curvature regimes. The refined dynamics, with a non-constant λ , is more well-behaved in semiclassical regimes.

In diagonal anisotropic models, the refinement behavior described here implies that holonomies depend on the connection by the combinations $c_I/\sqrt{|p^I|}$, as in [51], $|p^I|$ being proportional to the area A of the plane transversal to the I -direction. This refinement is problematic in terms of stability properties of the difference equation it implies [30]. A different refinement scheme in which λ_I is inversely proportional to the length of the I -direction is preferable [8, 30, 52]; a more advanced action of the Hamiltonian constraint not fixing a lattice subalgebra would be required, a refinement scheme in which δ_I depends not only on the length λ_I of its own direction but also on the other two links meeting at a vertex.

3.5.4 Difference equations with mean-field refinement

Lattice refinement is the homogeneous realization of discrete dynamical processes in the full theory; ideally, its form would be derived by reducing a full Hamiltonian constraint. A dynamical state of quantum gravity should in general be expected to have different lattice structures and spacings at different times, or on different spatial slices, especially in loop quantum gravity whose Hamiltonians are generically graph-changing. The number of lattice sites is then a dynamical parameter. Indeed, if the \mathcal{N}_I or the ℓ_I are kept constant – we always assume L_1 , L_2 and L_3 to be constant as these are classical auxiliary parameters – cosmic expansion would quickly blow up the discreteness scale, $\ell_I\sqrt{|p^1p^2p^3|}/|p^1|$ as measured in a diagonal Bianchi geometry, to macroscopic sizes. Lattice refinement must be a key feature of quantum-gravity dynamics. Dynamical minisuperspace operators such as the Hamiltonian constraint should not refer to constant \mathcal{N}_I or δ_I , as assumed so far in (3.15) and (3.16), but to parameters that depend on the total volume or the scale factor via an evolving discrete state.

Strict difference equations of loop quantum cosmology then do not exist, even in Abelian models, and approximations cannot always be derived easily. The correct evolution equation in a triad representation would rather have to implement the changing number of degrees of freedom, a problem studied in other contexts as well [56, 57, 66]. Instead of working with such complicated equations, there are two approximation schemes that help to find properties of solutions: Effective equations and difference equations in redefined variables.

Effective equations describe properties of solutions of difference equations in Abelian models via a non-canonical basic algebra, the discreteness implemented by using exponentials of the connection. For instance, we would represent a discrete degree of freedom (c, p) by a non-canonical basic pair $(\exp(i\delta c), p)$ with a closed linear algebra under Poisson brackets. If δ depends on p by a power law $\delta(p) = \delta_0|p|^x$ as a form of lattice refinement, $(\exp(i\delta_0|p|^x c), |p|^{1-x})$ still satisfies a closed linear algebra [20]. We then generate evolution by a Hamiltonian much like (3.15), depending on $\exp(i\delta(p)c)$ according to the regularization chosen. Effective quantum evolution equations then follow the general scheme of [42, 43] and provide approximate information about refining solutions. While strict difference equations are not available in lattice-refining Abelian or in non-Abelian models, effective equations can still be formulated and solved in both cases.

Approximate difference equations in re(de)defined variables model refined quantum evolution by difference equations equally spaced not in the original triad eigenvalues, but rather in some redefined versions obtained as non-linear functions of them, such as power laws. One can derive a suitable equidistant parameter if one knows how the δ_I depend on n_J in the refining case. Instead of an equation (3.16) with n_J -dependent increments, for instance in $s_{n_1, n_2+2\delta_2(n_1, n_2, n_3), n_3+2\delta_3(n_1, n_2, n_3)}$, one can sometimes work with an equidistant difference equation in re(de)defined independent variables. If δ_I depends only on n_I with the same value of I , we define $\bar{n}_I(n_I) := \int_0^{n_I} (\delta_I(z))^{-1} dz$ such that

$\bar{n}_I(n_I + \delta_I(n_I)) = \int_0^{n_I + \delta_I(n_I)} (\delta_I(z))^{-1} dz = \bar{n}_I(n_I) + \int_{n_I}^{n_I + \delta_I(n_I)} (\delta_I(z))^{-1} dz = \bar{n}_I(n_I) + \delta_I(n_I) (\delta_I(n_I))^{-1} (1 + O(\delta'_I(n_I))) = \bar{n}_I(n_I) + 1 + O(\delta'_I(n_I))$, a constant increment in regions in which the derivative $\delta'_I(n_I)$ is sufficiently small. If $\delta_I(n_I) \propto |n_I|^x$ is a power law with $x < 0$ for refinement, the equidistant approximation is good at large n_I but not for small n_I , where the quantum dynamics remains ambiguous, anyway. (For $\delta_I \propto |n_I|^x$, we have an equidistant equation in $\bar{n}_I \propto |n_I|^{1-x}$, corresponding to the new p -dependent variable used for effective equations.) One may also redefine the whole difference equation in terms of \bar{n}_I with constant increments, dropping $O(\delta'_I(n_I))$ -terms as a specific choice of factor ordering [30]. If δ_I depends not just on n_I with the same I , a redefinition is more complicated to derive. If $\delta_1 \delta_2 \delta_3$ is proportional to a power of $|n_1 n_2 n_3|$, $\delta_1 \delta_2 \delta_3 \propto |n_1 n_2 n_3|^x$ such that refinement does not introduce additional anisotropy, one can always find one equidistant variable given by

$$N(n_1, n_2, n_3) := \int_0^{n_1} \int_0^{n_2} \int_0^{n_3} (\delta_1(z_1, z_2, z_3) \delta_2(z_1, z_2, z_3) \delta_3(z_1, z_2, z_3))^{-1} dz_1 dz_2 dz_3 \\ \propto |n_1 n_2 n_3|^{1-x}$$

related to the total volume [30]. This choice resembles Misner variables [79], which refer to the volume (or scale factor) and two anisotropy parameters.

A state dependence of dynamical operators, underlying lattice refinement, may seem unexpected. After all, the dynamical operators are used to derive evolving states; how can properties of such states enter the definition of dynamical operators or the difference equations they imply? Taking reduction seriously, it turns out that state dependence is unavoidable. In minisuperspace models, we cannot formulate a dynamical operator from first principles, or if we do so, the results are fraught with minisuperspace artefacts because full properties of the discreteness are ignored. As described before, reduced dynamics is supposed to model the full dynamics of a symmetric state, to be projected back to the space of symmetric states after each application of the evolution operator, a process that includes the decomposition rule (3.5) used crucially in our toy model of refinement. A minisuperspace evolution operator obtained by reduction must encode both the full Hamiltonian constraint and properties of the projection. The latter depends on the evolving state to be projected back on the symmetric space. While the precise form remains complicated to determine, we see how a state dependence of the end result is obtained.

Without a detailed method to perform dynamical reduction, the phase-space dependence of parameters such as δ_I or \mathcal{N}_I is inserted in equations only after the operator has been formulated, as a kind of mean field describing microscopic properties not directly accessible at the minisuperspace level. Such details and ambiguities are relevant at small scales and at higher curvature, or in strong quantum regimes. These regimes can be understood only by general effects, such as quantum hyperbolicity, but away from deep quantum regimes, effective and mean-field pictures are meaningful and useful.

3.5.5 Ad-hoc modifications in pure minisuperspace models

We have presented a quantization of homogeneous models which has a tight link with the full theory and, unlike previously existing versions, applies in non-Abelian cases. Lattice refinement naturally arises as a consequence of state-dependent regularizations as in the full theory, combined with a reduction of all states, including physical ones, to the space of homogeneous loop quantum cosmology.

Lattice refinement is important for consistent dynamics, for a fixed lattice expanded by cosmic evolution would either be coarse at the present time, or would have to start at tiny spacings, orders of magnitude below the Planck length, to be unnoticeable in current observations. Lattice

refinement as a dynamical process ensures that the discreteness scale does not need to follow cosmic expansion; it can remain small at a constant or slowly-changing value as macroscopic events happen on larger regions. With a concrete realization of lattice refinement, we can look back at minisuperspace modifications that have been proposed in the hope of obtaining appropriate dynamics, and see how justified their assumptions are from the perspective of the new picture. The most commonly used ad-hoc modification is a change of classical basic variables before isotropic minisuperspace quantization, in which $c/a \propto \dot{a}/a$ appears in holonomies, and the role of p is played by the volume.

We first note that fluxes necessarily result as reduced operators in the derived basic algebra, not other powers of densitized-triad components or the volume¹⁶. Basic operators or linear functions of them are directly reduced by reference to the commutation result of Section 3.4.4. Non-linear functions such as the volume, on the other hand, are more complicated to average or reduce exactly, with no currently known procedure to do so¹⁷. The basic holonomy operators therefore act by shifts on the flux spectrum by constant amounts (of p in isotropic models), not the volume spectrum. If constant shifts of the volume spectrum have dynamical advantages, as in the model of [6], they cannot be derived by direct use of basic operators but only after re(de)fining variables as in the preceding subsection. The volume can be used as a basic variable only as a modification within a pure minisuperspace quantization, without reduction and a justified analog in the full theory.

The modifications proposed in [6] have been motivated in holonomy-based expressions for F_{ab}^i in the constraint by referring to geometrical areas $a^2\ell^2$ instead of coordinate ones ℓ^2 , where a is the scale factor of a Friedmann–Lemaître–Robertson–Walker model to be quantized. However, just as tensor components F_{ab}^i depend on coordinates, it is the coordinate area ℓ^2 which should be used in the expansion (3.14), not geometrical areas obtained using the metric or densitized triad. (Contracting F_{ab}^i with the two vector fields provides a scalar. However, for the coordinate area ℓ^2 to be the correct factor in the expansion, the vector fields must be normalized using a background metric. Changing coordinates and retaining normalization then makes the contracted version transform.)

If these and other ad-hoc assumptions, for instance about factor ordering, are dropped, dynamical equations are much more ambiguous than usually realized or admitted. More-involved constructions of lattice refinement are required, which capture necessary projections of the dynamical flow back on the space of symmetric states. Exact projections being largely unknown, the dynamics can be obtained only in parameterized ways, faithfully taking into account ambiguities [19, 23]. At this dynamical stage, the construction of minisuperspace operators currently proceeds by analogy with full operators, not by derivation.

3.6 Quantum-geometry corrections

Using holonomies instead of curvature or connection components implies quantum-geometry corrections in the dynamics. There is a second type of effect, called inverse-triad correction, which comes from the fact that an inverse of the densitized triad appears in the Hamiltonian constraint of gravity and in matter Hamiltonians, but flux operators have discrete spectra containing zero. No inverse flux operators exist, but the inverse densitized triad can be quantized

¹⁶It is possible to construct flux operators from the volume operator, viewing the latter as some kind of basic operator [64]. However, for the present purposes one cannot substitute the volume for fluxes because no linear basic algebra would result for the definition of quantum representations and their averaging and reduction.

¹⁷Moreover, the volume operator usually used in homogeneous models, and also here, is a simplified version of the cubic SU(2)-invariant of the full theory. The assumed simplification of the much more complicated full spectrum does not follow from reduction but is put in by hand. When details of the eigenvalues are important, for instance when one uses the volume as the independent variable of difference equations, the simplified spectrum could lead to additional artefacts, not covered by the methods of this article.

to a densely defined operator using classical rewritings following [91, 92]. In the Hamiltonian constraint, inverse-triad operators appear in the gravitational part (giving rise to differences of volume eigenvalues in (3.16) and in matter Hamiltonians).

Holonomy corrections are controlled by the parameters λ_I , or by the values δ_I chosen for a constraint operator. Inverse-triad operators entering (3.15) via $|\rho_{\delta_K,1/2}(g_K)[\rho_{\delta_K,1/2}(g_K)^{-1}, \hat{V}]|$ are built using the same type of holonomies, and so their corrections refer to the same lattice scales δ_I as holonomy corrections. Both corrections are therefore linked to each other, and comparing the explicit forms of corrections allows one to estimate which one might be dominant in a given regime.

Like holonomy corrections, the size of inverse-triad corrections depends on the values of δ_I and requires a proper consideration of lattice structures. However, there is an additional operator, the volume \hat{V} , used crucially in the definition of inverse-triad operators as commutators. For this operator, the same question must be asked as for holonomies, namely what lattice scale it refers to. In a local lattice picture, as in the full theory, one should expect the relevant volume to be the one associated with a single lattice site or a spin-lattice vertex, just as the holonomies used correspond to single lattice links. However, incorporating the volume in this way is not as obvious as for holonomies, and so different versions have been considered, making use of macroscopic volumes [101] or even one associated with the artificial integration region \mathcal{V} . In this subsection, we derive in detail the form of inverse-triad operators and the corrections they imply. To simplify commutator calculations involved, we will present the main equations for Abelian models and briefly comment on non-Abelian effects later.

3.6.1 Local and non-local lattice operators

Working with lattice spin-network states, one can define different flux operators which all give rise to the same flux when averaged to minisuperspace operators. This situation complicates constructions in pure minisuperspace models and has led to considerable confusion. Only relating models to the full theory, completing the kinematical reduction, can solve these issues.

Local lattice operators: We begin with the local flux operator, able to show any inhomogeneity realized in the lattice model: $\hat{F}_{v,I}$, taken for a plaquette transversal to a surface X_I^a and intersecting only one edge $e_{v,I}$ starting at the vertex v . We choose the surface to be a square of coordinate area $\ell_J \ell_K = \lambda_J \lambda_K L_J L_K$, so that we can view $\hat{F}_{v,I}$ as a quantization of the classical $\lambda_J \lambda_K p^I(v)$, where $p^I(v)$ is an inhomogeneous diagonal component making the homogeneous variables position dependent. The conjugate variable $\lambda_I c_I$ is quantized via local holonomies $h_{v,I} = \exp(i \int_{e_I} c_I ds)$. (Recall our Abelian simplification in this subsection.) These local lattice operators satisfy the commutator algebra

$$[\hat{h}_{v,I}, \hat{F}_{v',I'}] = -8\pi\gamma\ell_P^2 \delta_{v,v'} \delta_{I,I'} \hat{h}_{v,I}. \quad (3.17)$$

Minisuperspace operators: If each surface used for local flux operators is centered at the intersection point with $e_{v,I}$, the union of all those that have the same I -coordinate as v form a surface stretching through the whole integration region \mathcal{V} , without overlap of non-zero measure. Including an average in the transversal direction, we can view the lattice sum $\hat{p}^I = \mathcal{N}_I^{-1} \sum_v \hat{F}_{v,I}$ according to (3.13) as the flux quantizing the minisuperspace variable $p^I = L_J L_K \tilde{p}^I$. Its conjugate variable $c_I = L_I \tilde{c}_I$ in minisuperspace is quantized by holonomies, $h_I = \exp(ic_I)$ for an edge stretching through the whole integration region in direction X_a^I . We have the commutator

$$[\hat{h}_I, \hat{p}^J] = -8\pi\gamma\ell_P^2 \delta_I^J \hat{h}_I \quad (3.18)$$

for minisuperspace operators, correctly quantizing (3.8).

Non-local operators: There is a version of operators between local lattice and minisuperspace ones. We can average local lattice fluxes $\hat{F}_{v,I}$ over the lattice rather than sum as in \hat{p}^I , or reduce the size of the minisuperspace flux \hat{p}^I by dividing by the number of vertices in a surface, and define

$$\widehat{F}_I = \frac{1}{\mathcal{N}_1\mathcal{N}_2\mathcal{N}_3} \sum_v \hat{F}_{v,I} = \frac{1}{\mathcal{N}_J\mathcal{N}_K} \hat{p}^I. \quad (3.19)$$

This flux operator refers to the lattice spacing but, via averaging, includes all lattice sites in the integration region. We will call it the non-local flux operator. With a local holonomy, it obeys the commutator relation

$$[\hat{h}_{v,I}, \widehat{F}_J] = -\frac{8\pi\gamma\ell_P^2}{\mathcal{N}_1\mathcal{N}_2\mathcal{N}_3} \delta_{IJ} \hat{h}_{v,I}. \quad (3.20)$$

In (3.20), the number $\mathcal{N}_1\mathcal{N}_2\mathcal{N}_3$ of lattice sites in a region \mathcal{V} replaces the coordinate volume V_0 of (3.8). At a technical level, $1/\mathcal{N}_1\mathcal{N}_2\mathcal{N}_3$ comes about as the product of $1/\mathcal{N}_J\mathcal{N}_K$ in the plane average (3.19), and a factor of $1/\mathcal{N}_I$ because only one out of \mathcal{N}_I lattice links along direction I provides a non-zero commutator $[\hat{h}_{v,I}, \hat{F}_{v',I}] \propto \delta_{v,v'}$ according to (3.17).

It may seem questionable to use local holonomies and non-local fluxes within the same setting (3.20), but a consistent and closed algebra of basic operators is obtained in this way (provided the \mathcal{N}_I are fixed). Whether such operators are meaningful physically is another question which we will soon discuss. For now, our motivation for looking at such a mix of local and non-local operators is that it has been used (implicitly or explicitly) in several proposals to formulate inverse-triad corrections.

Properties of basic operators in the different algebras can be translated into one another and are mutually consistent. If $|(\mu_I)\rangle$ denotes non-local flux eigenstates with $\widehat{F}_J|(\mu_I)\rangle = 8\pi\gamma\ell_P^2\mu_J|(\mu_I)\rangle$, the holonomy-flux algebra (3.20) determines the action $\hat{h}_{v,I}|(\mu_I) + 1/\mathcal{N}_1\mathcal{N}_2\mathcal{N}_3\rangle$ of local holonomies. Constant shifts of flux eigenvalues result for a fixed lattice. The quantized densitized-triad component $p^I = L_J L_K \tilde{p}^I$ is obtained from the averaged flux as $\hat{p}^I = \mathcal{N}_J\mathcal{N}_K \widehat{F}_I$. Its eigenvalues change under the action of a basic holonomy operator by $p^I = 8\pi\gamma\ell_P^2\mathcal{N}_J\mathcal{N}_K\mu_I \mapsto 8\pi\gamma\ell_P^2\mathcal{N}_J\mathcal{N}_K(\mu_I + 1/\mathcal{N}_1\mathcal{N}_2\mathcal{N}_3) = p^I + 1/\mathcal{N}_I$. This dependence of the constant shift on \mathcal{N}_I is consistent with the form $\exp(ic_I/\mathcal{N}_I)$ of local holonomies $\exp(i\ell_I\tilde{c}_I)$, with $c_I = L_I\tilde{c}_I$, $\ell_I/L_I = 1/\mathcal{N}_I$, and $\{c_I, p^J\} = 8\pi\gamma G\delta_I^J$. Notice the different behaviors of the non-local averaged flux \widehat{F}_I and the minisuperspace densitized-triad component \tilde{p}^I , corresponding by its factor of $L_J L_K$ to the flux through a complete plane in the region \mathcal{V} .

The general form of the algebra of basic operators does not depend much on whether the local \hat{F}_I , the non-local \widehat{F}_I or the minisuperspace \tilde{p}^I is used; the latter two differ from each other just by constant factors at the kinematical level (disregarding lattice refinement). However, \widehat{F}_I and \tilde{p}^I are much less local than $\hat{F}_{v,I}$ and therefore unsuitable for local expressions such as quantized Hamiltonians or inverse-triad corrections.

3.6.2 Inverse-triad corrections

The gravitational part of the Hamiltonian constraint contains a factor of $\epsilon^{ijk}\epsilon_{abc}E_j^b E_k^c / \sqrt{|\det E|}$ in which one divides by the determinant of E_i^a , and similar terms occur in matter Hamiltonians. Flux operators and the volume operator having zero in their discrete spectra, no densely defined inverse exists to quantize $1/\det E$ directly. Instead, one makes use of the classical identity [91]

$$2\pi\gamma G \text{sgn}(\det E) \frac{\epsilon^{ijk}\epsilon_{abc}E_j^b E_k^c}{\sqrt{|\det E|}} = \left\{ A_a^i, \int \sqrt{|\det E|} d^3x \right\} \quad (3.21)$$

and quantizes the Poisson bracket to a commutator of the form

$$\frac{1}{i\hbar} \left(\hat{h}_e^{-1} [\hat{h}_e, \hat{V}] - \hat{h}_e [\hat{h}_e^{-1}, \hat{V}] \right) = \frac{1}{i\hbar} \left(\hat{h}_e \hat{V} \hat{h}_e^{-1} - \hat{h}_e^{-1} \hat{V} \hat{h}_e \right). \quad (3.22)$$

In a lattice model, holonomies refer to lattice links, or $\rho_{\delta,j}(g_I)$ in their reduction. The volume operator is expressed via fluxes, and here local flux operators are used, given the local form of the classical Poisson bracket in (3.21) and of the commutator in (3.22) which depends only on vertex contributions to \hat{V} lying on the edges used in \hat{h}_e . At this stage, minisuperspace models can easily become misleading because their most immediate flux operators \hat{p}^I or $\widehat{\hat{p}}^I$, proportional to $\widehat{\hat{F}}_I$ are non-local. The wrong form and size of inverse-triad effects then results.

We now present the detailed derivation of inverse-triad corrections based on local flux operators, as in [26, 29], and then show how non-local versions differ. The simplified volume operator of Abelian models is $\hat{V} = \sum_v |\hat{F}_{v,1} \hat{F}_{v,2} \hat{F}_{v,3}|^{1/2}$, summed over all vertices of a spin-lattice state. In expressions such as (3.22), it suffices to look at contributions from all lattice-aligned $\hat{h}_{v,I}$. A single such commutator is then

$$\hat{I}_{v,I} = \frac{|\hat{h}_{v,I}^\dagger \hat{V} \hat{h}_{v,I} - \hat{h}_{v,I} \hat{V} \hat{h}_{v,I}^\dagger|}{8\pi\gamma G \ell_P^2} = \frac{|\hat{h}_{v,I}^\dagger \sqrt{|\hat{F}_{v,I}|} \hat{h}_{v,I} - \hat{h}_{v,I} \sqrt{|\hat{F}_{v,I}|} \hat{h}_{v,I}^\dagger|}{8\pi\gamma G \ell_P^2} \sqrt{|\hat{F}_{v,J} \hat{F}_{v,K}|}$$

to be summed over all I . (Classically, the combination of holonomies and fluxes corresponds to $|F_{v,I}|^{-1/2} \sqrt{|F_{v,J} F_{v,K}|}$. We have used an absolute value around the commutator as in (3.15).) For Abelian holonomies it is easy to simplify the inverse-triad operator, making use of the commutator $[\hat{h}_{v,I}, \hat{F}_{v,I}] = -8\pi\gamma \ell_P^2 \hat{h}_{v,I}$ from (3.17) and the reality condition $\hat{h}_{v,I}^\dagger \hat{h}_{v,I} = 1$. Commuting holonomies past flux operators then gives $\hat{h}_{v,I}^\dagger |\hat{F}_{v,I}|^{1/2} \hat{h}_{v,I} = |\hat{F}_{v,I} + 8\pi\gamma \ell_P^2|^{1/2}$, and therefore

$$\hat{I}_{v,I} = \frac{\left| \sqrt{|\hat{F}_{v,I} + 8\pi\gamma \ell_P^2|} - \sqrt{|\hat{F}_{v,I} - 8\pi\gamma \ell_P^2|} \right|}{8\pi\gamma G \ell_P^2} \sqrt{|\hat{F}_{v,J} \hat{F}_{v,K}|}. \quad (3.23)$$

In strong quantum regimes, non-Abelian features should be relevant [48] and inverse-triad effect compete with holonomy and higher-curvature terms; however, the form (3.23) still plays a characteristic role in effective actions [37]. The expression (3.23) is a good approximation in perturbative settings with $F_{v,I} > 8\pi\gamma \ell_P^2$, where it may be used to estimate qualitative effects or potential observational tests [27, 28, 29].

Since inverse-triad operators are local – commutators $\hat{h}_e[\hat{h}_e^{-1}, \hat{V}]$ provide contributions only for vertices on e even if the volume operator for the full region \mathcal{V} is used – their commutators refer to local $\hat{F}_{v,I}$ in $\hat{V} = \sum_v |\hat{F}_{v,1} \hat{F}_{v,2} \hat{F}_{v,3}|^{1/2}$, not to the minisuperspace operator \hat{p}^I or the non-local $\widehat{\hat{F}}_I$. Inverse-triad corrections therefore depend on $F_{v,I} \pm 8\pi\gamma \ell_P^2$, where the Planckian addition can easily be a significant contribution to the eigenvalue or expectation value of $\hat{F}_{v,I}$, the flux through an elementary lattice site¹⁸. Had we used the average $\widehat{\hat{F}}_I$, the algebra would have led us to $\widehat{\hat{F}}_I \pm 8\pi\gamma \ell_P^2 / \mathcal{N}_I \mathcal{N}_J \mathcal{N}_K$, with corrections not only much suppressed by dividing by the large number of lattice sites but also depending on the size of the arbitrary region \mathcal{V} chosen¹⁹. Such operators would be incorrect; they are based on the confusion of the correct average $\sqrt{\widehat{\hat{F}}_I}$ with the non-local $\sqrt{\widehat{\hat{F}}_I}$.

¹⁸In fact, if $F_{v,I}$ is Planckian, with lattice spins near 1/2 for the fundamental representation, as often assumed, inverse-triad corrections are large. Geometry must be sufficiently excited above fundamental spins (some kind of ground state) for good semiclassical states to result.

¹⁹Sometimes, it is suggested to take a limit of $V_0 \rightarrow \infty$, or $\mathcal{N}_I \rightarrow \infty$, viewing a finite V_0 as a regulator. The procedure removes any \mathcal{V} dependence and makes inverse-triad corrections disappear. However, as discussed in

3.6.3 Local quantum corrections

We have distinguished three types of constructions for composite operators quantizing a symmetric model: the minisuperspace treatment using $\hat{h}_I = \exp(\widehat{ic}_I)$ and \hat{p}^J with algebra (3.18), chimerical constructions with (local) link holonomies $\hat{h}_{v,I}$ but non-local fluxes \widehat{F}_J with algebra (3.20), and finally local lattice operators built from $\hat{h}_{v,I}$ and $\hat{F}_{v,J}$ with algebra (3.17).

Local and minisuperspace treatments differ from each other by the order in which reduction and composition of operators are done. In non-local models, as in traditional minisuperspace versions, one first postulates or derives the reduced basic operators \hat{h}_I and \widehat{F}_J (or \hat{p}^J) and their algebra, and in a second step constructs composite operators of the form $\mathcal{O}_{\text{non-local}}(\hat{h}_I, \widehat{F}_J)$ from them by simple insertions, following analogous steps taken in the full theory. In local quantizations, one first constructs operators $\mathcal{O}_{\text{local}}(\hat{h}_{v,I}, \hat{F}_{w,J})$, adapting the full techniques to lattice states, and then restricts them to a quantized homogeneous model. The second, local method is more complicated because it must deal with the reduction of non-basic, composite operators or their averaging. Tractable techniques exist only in rare cases, and therefore the main effects, for instance in the Hamiltonian constraint, are incorporated by parameterizations or mean-field techniques as in Section 3.5.3 – an unsurprising feature given that local methods are analogous to a transition from microscopic Hamiltonians to tractable models of large-scale effects in condensed-matter physics.

Despite technical difficulties, the local viewpoint has several clear advantages: it produces the correct sizes of quantum corrections and naturally gives rise to lattice refinement. As already noted, the misrepresentation of quantum corrections in non-local models can easily be seen for inverse-triad operators, or the key ingredient $\mathcal{O} = |F_I|^{1/2}$. Non-local operators make use of the averaged flux before taking the square root, quantizing \mathcal{O} as $\hat{\mathcal{O}}_{\text{non-local}} = |\widehat{F}_I|^{1/2}$. A local quantization, by contrast, leads to $\hat{\mathcal{O}}_{\text{local}} = |\widehat{F}_{v,I}|^{1/2}$, the over-line now indicating restriction to homogeneous states after taking the square root. While linear combinations of the basic operators commute with averaging (Section 3.4.4) – producing similar-looking basic algebras in local and non-local versions – non-linear combinations do not. In non-linear combinations of the basic operators, drastic deviations between local and non-local operators can therefore result, but only the local version correctly captures properties of the full theory in which no averaging is done.

Local composite operators can be formulated only when the full lattice structure is taken into account, but after reduction they refer to reduced degrees of freedom as suitable for a quantization of a classically reduced symmetric model. Minisuperspace and non-local operators may be formally consistent without direct reference to a lattice, provided one chooses the length parameter $\ell_I = \delta_I L_I$ of holonomies in some way, for instance related to the Planck length or to full area eigenvalues, but at this stage the models become ad-hoc. Moreover, in spite of their formal consistency, non-local composite operators do not provide the correct form of corrections in operators that refer to fluxes or the volume, most importantly inverse-triad corrections.

In addition to making inverse-triad corrections sizeable and interesting, the local treatment taking into account inhomogeneity, has another implication regarding the physical evaluation of models. There is just one set of parameters, δ_I or equivalently $\mathcal{N}_I = 1/\delta_I$, which determines the magnitude of holonomy *and* inverse-triad corrections. It is not possible to ignore one of the corrections and focus only on the other, unless one can show that a regime of interest leads to values of δ_I that make one correction dominate the other. In general, the two corrections are

more detail in the next section, this reasoning is misguided: V_0 is not a regulator because its value does not at all affect the classical theory. Classical models with different V_0 produce the same physics, and so they should all be quantizable, without an effect of V_0 . Moreover, the limit of $\mathcal{N}_I \rightarrow \infty$ is not consistent with the basic algebra of averaged operators.

not strictly related to each other, because holonomy corrections are sensitive to the classical curvature scale relative to the Planck density, while inverse-triad corrections are sensitive to the local discreteness scale $F_{v,I}$ relative to the Planck area, as seen in (3.23). A detailed, state-dependent analysis, taking into account inhomogeneous quantum geometry, is required to estimate both corrections.

4 Limitations of minisuperspace models

Minisuperspace models of quantum cosmology never provide exact solutions to full quantum gravity. In some cases, deviations can be strong, for instance when unstable dynamics of neglected degrees of freedom (a classical property) enlarges the mismatch between symmetric and less-symmetric solutions, which at an initial time may have been the consequence only of a mild violation of uncertainty relations [73]. The discreteness of loop quantum cosmology shows a much larger class of minisuperspace limitations, for discreteness is not easily reconciled with homogeneity. As always, such limitations should be pointed out and discussed in detail, not to slander but to warn.

At the level of states and basic operators, homogeneous wave functions can be derived in precise terms, using the distributional constructions of [36], as recalled in Section 3.4.2. Intuitively, averaging a discrete state over a continuous symmetry group cannot result in a normalizable wave function in the original Hilbert space (or even a meaningful density matrix), but it is well-defined as a distribution. At this level, discreteness is not problematic and does not introduce ambiguities. At the dynamical stage, however, discrete space-time structures with possible refinement or (even if there is no refinement) reference to the local discreteness scale are more complicated and more ambiguous, as occasionally pointed out well before loop quantum cosmology was introduced [67, 95, 99]. Loop quantum cosmology has provided means to analyze such situations.

4.1 Parameters

In addition to phase-space variables, a strict minisuperspace model has only the parameter $V_0 = L_1 L_2 L_3$ to refer to, appearing in the symplectic structure (3.1). There is no analog of $\ell_I = \lambda_I L_I = L_I / \mathcal{N}_I$, the discrete lattice scales. And yet, the local quantum dynamics of the full theory, together with the quantum corrections it implies, depend on the parameters λ_I via holonomies around loops used to quantize F_{ab}^i or along edges used in commutators with the volume operator to quantize inverse triads.

It may not be obvious how exactly edge lengths $\lambda_I L_I$ enter quantum corrections, owing to a certain conceptual gap between the coordinate dependent L_I or ℓ_I and geometrical aspects in this background-independent formulation, as well as quantization ambiguities. But quantum corrections certainly cannot depend on V_0 , which is chosen at will (the coordinate size of a region used to reduce the symplectic structure) and knows nothing about the discrete scale. If V_0 or any of its factors L_I appears in quantum corrections, an artificial dependence on coordinates and the chosen region results, as well as wrong sizes of quantum effects.

The authors of [6] proposed to modify the strict minisuperspace dynamics in a fashion that successfully eliminates the V_0 -dependence at least in holonomy corrections. (Inverse-triad corrections could not be represented meaningfully in this scheme.) As a consequence in isotropic models, the Hubble parameter rather than $c = \gamma \dot{a}$ appears in holonomies, and the discrete dynamics proceeds by constant steps of the volume $V_0 a^3$, not of the densitized triad $V_0^{2/3} p$. Heuristically, as recalled in Section 3.5.5, one can argue for this scheme by identifying geometrical areas $a^2 \ell^2$, instead of coordinate areas ℓ^2 , with the Planck area when specifying the size of holonomy modifications.

Holonomy corrections depending on the Hubble parameter \dot{a}/a rather than \dot{a} have the advantage of being easily coordinate independent. If the Planck length – or a parameter close to it such as the smallest non-zero area eigenvalue of the full theory – is chosen as the discrete scale, modifications $\sin(\ell_P \mathcal{H})/\ell_P$ of \mathcal{H} result, independent of coordinates and of V_0 . Holonomy corrections then refer simply to the curvature radius relative to the Planck length (or, via the Friedmann equation, the density scale relative to the Planck density), a parameter which can easily be estimated in regimes of interest.

As a general scheme, however, the procedure suffers from several problems:

1. If the volume is used as a basic variable, following [6], one introduces a sign choice by hand, allowing all real values for $v = \pm V_0 a^3$. (Otherwise, $i\partial/\partial V$ is not essentially self-adjoint, and $\exp(t\partial/\partial V)$ not unitary. One would have to use the methods of affine quantum gravity [70] for acceptable quantizations, but it has not been shown that this can be compatible with the use of holonomies.) In fluxes, the sign appears automatically thanks to the orientation of triads [16], and $\exp(t\partial/\partial p)$ is unitary.
2. The Planck length in $\sin(\ell_P \mathcal{H})/\ell_P$ enters by a mere postulate, which cannot be avoided because the quantization, still at the minisuperspace-level, does not have access to discrete structures. One may use the full area spectrum to guess what the scale might be, but such a procedure leaves open the question of what structure or eigenvalues a dynamical discrete state might give rise to. Moreover, the minisuperspace area or volume spectrum does not have a smallest non-zero eigenvalue. The spectrum, seen in (3.7), is discrete, with all eigenstates normalizable, but on the non-separable kinematical Hilbert space the spectrum still amounts to a continuous set of numbers as eigenvalues. One has to go slightly beyond minisuperspace models by referring to the full area spectrum, which does have a smallest non-zero eigenvalue, but still no reduction is performed. In this way, the scheme becomes improvised, heuristic, and ad-hoc.
3. Going beyond strict minisuperspace quantizations is not a bad thing; in fact, it is required for realistic modeling. However, schemes following [6] do not provide justifications for the detailed way in which one tries to go beyond. Moreover, while they give rise to meaningful results for holonomy corrections, inverse-triad corrections from (3.23) are not modeled properly. These corrections depend on the ratio of the discreteness scale $|\langle \hat{F} \rangle|$ to the Planck area. If one assumes that the discreteness scale is exactly the Planck area, inverse-triad corrections would merely result in a constant factor, not affecting the dynamics much at first sight. But the factor differs from one, and it has dynamical effects even if it changes just slightly. In other attempts, $|\langle \hat{F} \rangle|$ was related to macroscopic areas [101], sometimes even involving V_0 , for instance by using areas related to the size of the region \mathcal{V} . These proposals ignore the fact that there is only one discrete structure that both holonomy and inverse-triad corrections can refer to, as well as their local nature.
4. To counter inappropriate references to V_0 in non-local quantizations, the parameter is sometimes treated as a regulator to be sent to infinity after quantization. Such a formal limit would undo all inverse-triad corrections, leaving only the classical inverse in dynamical equations. However, the limit does not exist at the level of operators – if it existed, it would result in an inverse of the triad operator, which is not densely defined. One can perform the limit at the level of the difference equation for wave functions, or in effective equations. But while this is formally possible, the overall quantization procedure would no longer be coherent. After all, Abelian loop quantum cosmology has difference equations for states because curvature is replaced by holonomies, resulting in a true modification $\sin(\ell_I \mathcal{H})/\ell_I$ for \mathcal{H} . The limit $\ell_I \rightarrow 0$ or $\lambda_I = 1/\mathcal{N}_I \rightarrow 0$, which would send holonomy operators to derivatives by p , does not exist at the operator level. The Hamiltonian constraint is quantized to a difference operator with non-zero step-size. At the level of

wave equations, acting with the operator on states in the triad representation, the limit does exist and produces a version of the Wheeler–DeWitt equation [15]. If one insists on removing the “regulator” V_0 by sending it to infinity at the level of wave equations, one should also remove the true regulators ℓ_I in holonomies at the same level. The dynamics of loop quantum cosmology would then be no different from Wheeler–DeWitt dynamics. In the new homogeneous quantization of this article, the limit $L_I \rightarrow \infty$ or $V_0 = L_1 L_2 L_3 \rightarrow \infty$ is impossible at fixed δ_I , because $g_I = \exp(L_I \tilde{\phi}(T_I))$ has no such limit. For edge lengths $\ell_I = \delta_I L_I$ in $\rho_{\delta_I, j_I}(g_I)$ to remain finite, one would have to take the limit $\delta_I \rightarrow 0$ simultaneously with $L_I \rightarrow \infty$, but then the difference equation would become a differential equation, and loop quantum cosmology would, again, reduce to Wheeler–DeWitt quantum cosmology. Instead, one must be able to derive models for arbitrary values of L_I , such that observables are independent of them.

5. The parameter V_0 , in contrast to δ_I in holonomy modifications, is not a regulator because it does not modify the classical theory. Classical models can be formulated with all finite choices of V_0 , producing the same dynamics and observables. (Different choices of V_0 to some degree resemble different normalizations of the scale factor. Rescaling V_0 is not a canonical transformation as it changes the symplectic structure (3.1). Classically, this is not a problem, but one cannot expect a unitary transformation at the quantum level, making the issue in quantum theory more complicated.) It should then be possible to formulate also quantum dynamics for all possible choices, or else V_0 would acquire more physical meaning than it deserves. Another problematic feature of regularization attempts is a possible topology dependence. If one looks at a model of closed spatial slices, for instance the FLRW model with positive curvature, one cannot send V_0 to infinity. Instead, it may seem natural to use the full spatial coordinate volume as a distinguished value of V_0 . (But again, one may equally well formulate the classical dynamics with different values of V_0 , choosing different coordinates on the unit 3-sphere or integration regions smaller than the whole sphere.) It is sometimes argued that some effects, such as inverse-triad corrections, are meaningful or non-zero only with closed spatial topologies, but not for the flat, non-compactified FLRW model. Not surprisingly for a quantization based on non-local fluxes, quantum dynamics would then suffer from a strong violation of locality, depending on the global spatial topology even in its elementary changes. Such models in cosmology would also be hard to test empirically. One would have to know the spatial volume – and whether it is compact or not – before one can estimate quantum effects and make predictions. A more detailed discussion is given in [26].

The many conflicting comments that can be found in the literature following [6], for instance regarding the size of inverse-triad corrections, attest to the complicated and incomplete state of affairs in this scheme. Sometimes, a single paper may claim that inverse-triad corrections are too small to be significant, and at the same time can be changed at will by tuning the value of V_0 . Although it is not always realized by all authors, such conflicting statements spell out limitations of pure minisuperspace models.

4.2 Parameterizations

By introducing “holonomies” as functions of the Hubble parameter rather than the connection component, one mimics an a -dependent $\ell_I = \lambda_I L_I \propto 1/a = 1/\sqrt{|p|}$. As a or p changes and the universe expands or contracts, the lattice spacing evolves. Although there is no explicit creation of new vertices, the number \mathcal{N} of lattice sites must change, for a fixed V_0 with changing λ_I implies an evolving $\mathcal{N} = 1/\lambda_1 \lambda_2 \lambda_3$.

Deriving a precise functional form for $\lambda_I(p_i^J)$ would require one to formulate a correspondence between full discrete dynamics and reduced minisuperspace dynamics, including projections of

evolved states onto the space of symmetric states. Lacking such complicated constructions, one can use phenomenological input to restrict possible forms of $\lambda_I(p_j^J)$, or $\lambda_I(p^J)$ in diagonal anisotropic models, for instance different exponents of power-laws $\lambda(p) \propto |p|^x$ in isotropic models with a real parameter x [19]. If λ is constant ($x = 0$, corresponding to [16]), the discreteness scale would be magnified by cosmic expansion, presumably making it noticeable in observations. Since no discreteness has been seen, $x = 0$ or values close to it are ruled out. The suggestion of [6] amounts to $x = -1/2$, with constant discreteness scale, and is compatible with observations. However, a constant discreteness scale is not in agreement with full constraint operators changing vertex structures and local volume values, not just the number of vertices. On average over many individual actions of the Hamiltonian constraint and on large scales, cosmic minisuperspace dynamics may be close to $x = -1/2$ as in the toy model presented in Section 3.5.3, but this value cannot be realized precisely.

The choice of $x = -1/2$ (or its generalization to anisotropic models) is compatible with most cases of cosmic evolution, but it has problems with black-hole models [24]. By its construction using geometrical areas of the region \mathcal{V} , the scheme relates the number of vertices to the total volume of spatial regions. Near the horizon in homogeneous coordinates of the Schwarzschild interior, the spatial volume shrinks, making the number of lattice sites small. However, the regime is supposed to be semiclassical for large black-hole mass, which is in conflict with a small number of lattice sites, implying noticeable discreteness. The analysis of different models – cosmological ones and those for black holes – shows that there cannot be a single universal power-law exponent for $\lambda_I(p^J)$ in all regimes. Discrete quantum dynamics and refinement behavior, just as the underlying state, depend on the regime analyzed. The role of coordinate choices hints at another important issue, namely how much the condition of covariance and anomaly freedom restricts possible refinement schemes. This question remains largely unexplored owing to the complicated nature of the quantum constraint algebra, but see [50] for an interesting cosmological example that suggests restrictions, also pointing at a value near $x = -1/2$.

These phenomenological indications notwithstanding, a demonstration that $x = -1/2$ or a value near it is more than an ad-hoc choice in cosmological models would require some kind of derivation from the full theory. For this feat, in turn, one would need to solve the problem of the semiclassical limit of unrestricted loop quantum gravity, which remains one of the most pressing and most complicated problems of the field.

4.3 Reduction

Lattice refinement in difference and effective equations refers to state parameters, most importantly λ_I , depending on a geometrical variable such as the total volume V . One may view the appearance of V as an internal time, on which the evolving state depends. A possible procedure of implementing such a dependence, as alluded to in Section 3.5.3, would be to write a full state as a superposition $\sum_V \psi_V$ of contributions ψ_V belonging to some fixed volume eigenvalue V . One would decompose a dynamical state as an expansion in eigenstates of the internal-time operator, such as the volume. Although the procedure would be difficult owing to the complicated volume spectrum and the fact that one would have to solve for a dynamical state first, it is in line with standard treatments of internal time. The states appearing in the decomposition of ψ_V in the spin-network basis then show what discrete structures are realized at a given volume V , and the spacing as well as the number of vertices might certainly change as one moves from one V to the next.

After decomposing a dynamical state as $\sum_V \psi_V$, one would still have to adapt it to near-homogeneous geometries, that is implement the projection back to symmetric states. Additional state-dependent parameters may arise, all to be modeled by suitable functions $\lambda_I(p_j^J)$, the only parameters that survive with exact homogeneity. Such functions would then be in-

serted in dynamical equations of reduced models, for instance in difference equations of Abelian models.

Notice that one must know some features of the full evolution of a state before defining the reduced Hamiltonian constraint, which in a second step can be used to evolve a reduced state. Since the Hamiltonian constraint is one of the operators to be averaged for reduction, evolution and reduction are not independent processes in the construction of models. As a consequence, reduced Hamiltonian constraints are state-dependent, even more so than the full constraint operator with its state-dependent regularization of [91].

If for a precise reduction we must know how to evolve a full state, why do we not work with the full evolved state rather than its reductions? The advantage of reduced models is that they offer additional approximation schemes, for instance in the derivation of observables or of effective equations. However, reduced models can never provide exact predictions – if their predictions were exact, one would not be dealing with a reduced model. It does not make much sense to derive physical quantities, for instance bounce densities, in exact terms within minisuperspace models because the models themselves are not exact. Only general effects, such as quantum hyperbolicity, the presence of bounces under certain conditions, or qualitative low-curvature corrections may be meaningful predictions, but not specific values of some parameters related to the discreteness scale.

4.4 Spin-foam cosmology?

Spin-foam cosmology [10, 75, 90] attempts to enlist spin-foam techniques to address quantum-cosmological questions by embedding a simple structure with finitely many edges in a spatial (or space-time) manifold Σ . Such a map is clearly different from the (mini-)superspace embedding $\mathcal{M} \rightarrow \mathcal{S}$ used for classical reductions, or a map $\sigma: \mathcal{H}_{\text{hom}} \rightarrow \mathcal{D}_{\text{inhom}}$ of state spaces used for quantum reductions. Looking back at our discussion in Section 2, the question therefore arises what kind of construction spin-foam cosmology can provide.

As usually emphasized in this context, spin-foam cosmology aims at a description of quantum cosmological space-times without making use of reductions, rather describing physics in a full theory of quantum gravity in which inhomogeneous modes are still present and quantum fluctuate. According to the classification in Section 2, such a non-reduction scheme could only be selection or projection, but there is certainly no control over the full non-symmetric solution space, let alone the averaging problem, within spin-foam cosmology. For this reason, spin-foam cosmology does not fall within our classification of reduction schemes. If one wanted to change this conclusion, one would first have to clarify the precise relation between an inhomogeneous amplitude and the proposed isotropic one in spin-foam cosmology. So far, one just obtains a new model from a full one by inserting isotropic labels in the evaluation, which is not enough for a classification of the relationship between different models attempted in this setting.

In fact, it is not clear in which sense – or if at all – spin-foam cosmology describes symmetric models. It is true that inhomogeneous modes have not been truncated but remain present and may fluctuate, potentially a feature that would allow one to go beyond reduced models (see also [59, 60] in the canonical setting). However, spin-foam cosmology at present lacks conditions that would ensure inhomogeneous modes to be sufficiently small for the models to be considered symmetric, not just in their fluctuations but even in their expectation values. The graphs used in spin-foam cosmology refer to finitely many degrees of freedom, often related heuristically to the number of degrees of freedom of homogeneous minisuperspaces. However, counting degrees of freedom is not enough to ensure that a model is good. One might simply define a finite-dimensional “minisuperspace” by picking some point x_0 in space-time and considering only the metric components $g_{ab}(x_0)$ as degrees of freedom, a model which would be meaningless because of its dependence on the space-time gauge. By using spatial embeddings, of graphs instead of

points x_0 , spin-foam cosmology is in danger of producing models close to the one just sketched. (Indeed, the status of covariance remains unclear in spin-foam cosmology as well as full spin foams.)

What is missing in this context is a well-defined analog of the map σ for states used in loop quantum cosmology. Only such an object could tell whether the correct degrees of freedom have been captured. Another question, related to the topics of this article, is how spin foams Abelianize. The final equations often produced in this context resemble difference equations of Abelian loop quantum cosmology, even though the starting point has $SU(2)$ degrees of freedom. No clear Abelianization step has been provided. Finally, working with fixed graphs embedded in space, spin-foam cosmology has not given rise to refinement models.

4.5 Evaluation of models of quantum cosmology

So far, we have discussed the construction of reduced and other models. Their limited nature regarding the dynamics requires care also, and especially, when they are evaluated for physical predictions. In addition, there are caveats which apply to any construction in quantum gravity, and so to model systems as well. To guarantee that models, obtained in a sufficiently parameterized way to ensure their genericness, can indeed be evaluated reliably, one must use evaluation methods or approximations that do not bring in hidden assumptions about the general form of effects. Important questions such as the problem of time, deparameterization²⁰, and potential signature change enter here. Moreover, in spite of the ubiquitous use of “effective equations”, in most cases they are based on a misinterpretation of the classical *limit* presented in [2, Section 4.3], based on [15], as a semiclassical approximation. Accordingly, important quantum corrections have been missed in many analyses: While some \hbar -terms are kept in ℓ_P -related holonomy corrections, quantum back-reaction terms of the same order are dropped. For details on these important issues, we refer to the review [80].

5 Conclusions

Much work remains to be done to establish reduced models of loop quantum gravity as well-defined and controlled approximations, and as reliable sources of detailed predictions in high-curvature regimes. As discussed in Section 2, an analogous problem must be faced even classically in relating non-symmetric geometries to symmetric ones: the averaging problem. A complete understanding of quantum minisuperspace models as approximations of the full theory, even if a reduction mechanism is included, can be obtained only when the classical averaging problem is better understood. Lacking a general solution, no approach to quantum gravity is yet able to produce a complete derivation of reduced models.

Nevertheless, with sufficient care one can make progress and render it at least likely that all crucial effects of the full theory are captured. That non-Abelian effects should play some role in loop quantum cosmology and require caution has been emphasized quite some time ago regarding specific properties of inverse-triad corrections [48] as well as general properties of homogeneous models [18], but it has not often been realized. The present article offers several new observations and constructions to this end: We have pointed out that most considerations made so far in loop quantum cosmology suffer from Abelian artefacts, related to the use of function spaces on the Bohr compactification of the real line, following [2]. To correct this oversight, a new quantization

²⁰Deparameterization has become something of a method of choice in quantum cosmology, and models in which physical Hilbert spaces are derived in this way are often called “complete quantizations”. However, as a quantization of a space-time theory, such constructions can be considered complete only when one has shown that results do not depend on the choice of internal time. No such demonstration has been given in the models proposed so far. See also the discussion in [77].

of homogeneous connections is developed in Section 3, which starts from non-Abelian models and takes into account the complete structure of invariant connections. The resulting Hilbert space representation, when restricted to Abelian variables, is related to Bohr-quantized models by an isometric $*$ -algebra morphism, but one that is not unitary or bijective. We lose information when we map states to the Bohr–Hilbert space, corresponding to the edge-spin degeneracy inherent in previous models.

The edge-spin degeneracy of holonomies is removed by the new quantization in non-Abelian and Abelian models, giving a better handle on lattice structures and the relation to the full theory. We have strictly related basic operators in the full theory and in models, and showed how quantum effects in composite operators can be captured by local quantizations. The averaging required can lead to unexpected features – as seen in detail for flux operators – which one would not endeavor to implement in a pure minisuperspace quantization without being prompted by the relationship with the full theory. Several implications have been demonstrated, especially regarding lattice refinement and the form and sizes of quantum-geometry corrections, most importantly those due to inverse triads.

We emphasize that our constructions started with the realization of deficiencies in current quantizations based solely on Abelian models; seeing lattice refinement or local features was not the main aim but nevertheless resulted as an unavoidable consequence. Section 4 has provided cautionary remarks, detailing the current incomplete status of the field and providing some guidelines for evaluations and the approach to physicality.

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