Special Functions of Hypercomplex Variable on the Lattice Based on SU(1,1)

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Abstract. Based on the representation of a set of canonical operators on the lattice $h\mathbb{Z}^n$, which are Clifford-vector-valued, we will introduce new families of special functions of hypercomplex variable possessing $\mathfrak{su}(1,1)$ symmetries. The Fourier decomposition of the space of Clifford-vector-valued polynomials with respect to the $\mathrm{SO}(n)\times\mathfrak{su}(1,1)$-module gives rise to the construction of new families of polynomial sequences as eigenfunctions of a coupled system involving forward/backward discretizations $E_h^{\pm}$ of the Euler operator $E=\sum\limits_{j=1}^{n} x_j\partial_{x_j}$. Moreover, the interpretation of the one-parameter representation $\mathbb{E}_h(t)=\exp(tE_h^- - tE_h^+)$ of the Lie group $\mathrm{SU}(1,1)$ as a semigroup $(\mathbb{E}_h(t))_{t\geq 0}$ will allows us to describe the polynomial solutions of an homogeneous Cauchy problem on $[0,\infty)\times h\mathbb{Z}^n$ involving the differencial-difference operator $\partial_t + E_h^+ - E_h^-$.

Key words: Clifford algebras; finite difference operators; Lie algebras

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1 Introduction

In the investigation of special functions, the representation theory through Lie groups allows to compute families of orthogonal polynomials in terms of hypergeometric series expansions (see, e.g., [17]). Recent approaches towards discrete quantum mechanics, as for instance in [14], reveals that the representation of finite difference operators as canonical generators of a certain Lie algebra provides a general scheme to construct sequences of polynomials as eigenfunctions of a discrete Hamiltonian operator. These sequences of polynomials that appear on the literature under the name of Sheffer sequences (cf. [6, 12]) or Appell sequences (cf. [16]) give rise to families of Bernoulli and Euler polynomials beyond the classical rising/falling factorial polynomials (see also [1, Section 3]).

Seen the fact that Clifford algebras of signature $(0,n)$ encode the structure of the special orthogonal group $\mathrm{SO}(n)$ of $n\times n$ matrices (cf. [4, Subsection I.1]), it remains natural to study multi-variable extensions of the above approaches to the Euclidean space $\mathbb{R}^n$ in terms of hypercomplex variables. In [13, Section 2] and in [2, Section 3] the authors obtained the hypercomplex extension of Bernoulli and Euler polynomials, respectively; In [3, Section 4] the authors considered discrete versions of Fueter polynomials as an alternative hypercomplex extension for the raising/lowering Clifford-vector-valued polynomials considered in [8, Section 3]. In [9, Section 2] the authors shown that such families of Clifford-vector-valued polynomials of discrete variable may be realized from Lie algebraic representations of an algebra of endomorphisms analogue to the radial algebra representation obtained in *continuum* by Sommen (cf. [15]).

This paper is organized as follows: In Section 2 it will be given the motivation to study, in the framework of Clifford algebras, sequences of polynomials generated from a set of finite difference operators. In Section 3 the construction of irreducible representations for the spaces of Clifford-vector-valued polynomials on the lattice $h\mathbb{Z}^n$ based on the Howe dual pair $(\mathrm{SO}(n),\mathfrak{su}(1,1))$ (cf. [11]) will be considered.

To find the Fourier decomposition for the space of Clifford-vector-valued polynomials on the lattice $h\mathbb{Z}^n$, we will start to determine the positive and negative series representations for the Lie group $\mathrm{SU}(1,1)$ in interconnection with the forward/backward discretizations $E_h^{\pm}$ of the classical Euler operator $E=\sum_{j=1}^n x_j\partial_{x_j}$. Afterwards, the action of the $\mathrm{SO}(n)\times\mathfrak{su}(1,1)$-module on the lattice $h\mathbb{Z}^n$ will produce a sequence of invariant and irreducible subspaces under the discrete series representations of $\mathrm{SU}(1,1)$.

The results obtained in Section 3 will be used in Section 4 to describe the space of Clifford-vector-valued polynomials on the lattice $h\mathbb{Z}^n$ as hypergeometric series expansions and to characterize the homogeneous solutions of the differential-difference operator $\partial_t+E_h^+-E_h^-$ in terms of the semigroup $(\mathbb{E}_h(t))_{t\geq 0}$ carrying the one-parameter representation $\mathbb{E}_h(t)=\exp(tE_h^- - tE_h^+)$ of $\mathrm{SU}(1,1)$.

2 Scope of problems

Let $\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n$ be an orthogonal basis of $\mathbb{R}^n$. The Clifford algebra of signature $(0,n)$, which we will denote by $C\ell_{0,n}$, corresponds to the algebra generated from the set of graded anti-commuting relations

$$\mathbf{e}_j\mathbf{e}_k+\mathbf{e}_k\mathbf{e}_j=-2\delta_{j,k} \qquad \text{for any} \quad j,k=1,2,\ldots,n. \tag{1}$$

Under the linear space isomorphism given by the mapping $\mathbf{e}_{j_1}\mathbf{e}_{j_2}\cdots\mathbf{e}_{j_r}\mapsto dx_{j_1}dx_{j_2}\cdots dx_{j_r}$, with $1\leq j_1<j_2<\cdots<j_r\leq n$, the resulting algebra with dimension 2^n is isomorphic to the exterior algebra $\bigwedge(\mathbb{R}^n)$.

This allows us to represent any vector $x=(x_1,x_2,\ldots,x_n)$ of $\mathbb{R}^n$ as an element $x=\sum_{j=1}^n x_j\mathbf{e}_j\in C\ell_{0,n}$ and the translations $(x_1,x_2,\ldots,x_j\pm h,\ldots,x_n)$ on the grid $h\mathbb{Z}^n\subset\mathbb{R}^n$ with mesh width $h>0$ by the displacements $x\pm h\mathbf{e}_j$ over $C\ell_{0,n}$. Moreover, the Clifford-vector-valued functions correspond to linear combinations in terms of the r-multivector basis $\mathbf{e}_{j_1}\mathbf{e}_{j_2}\cdots\mathbf{e}_{j_r}$ labeled by subsets $J=\{j_1,j_2,\ldots,j_r\}$ of $\{1,2,\ldots,n\}$, i.e.

$$\mathbf{f}(x)=\sum_{r=0}^n\sum_{|J|=r} f_J(x)\mathbf{e}_J \qquad \text{with} \quad \mathbf{e}_J=\mathbf{e}_{j_1}\mathbf{e}_{j_2}\cdots\mathbf{e}_{j_r}.$$

Let us now recall some basic facts about finite difference operators. The forward/backward finite differences $\partial_h^{\pm j}$ defined on the grid $h\mathbb{Z}^n$ by

$$\left(\partial_h^{+j}\mathbf{f}\right)(x)=\frac{\mathbf{f}(x+h\mathbf{e}_j)-\mathbf{f}(x)}{h} \qquad \text{and} \qquad \left(\partial_h^{-j}\mathbf{f}\right)(x)=\frac{\mathbf{f}(x)-\mathbf{f}(x-h\mathbf{e}_j)}{h}$$

are interrelated by the translation operators $\left(T_h^{\pm j}\mathbf{f}\right)(x)=\mathbf{f}(x\pm h\mathbf{e}_j)$, i.e.

$$T_h^{-j}\left(\partial_h^{+j}\mathbf{f}\right)(x)=\left(\partial_h^{-j}\mathbf{f}\right)(x) \qquad \text{and} \qquad T_h^{+j}\left(\partial_h^{-j}\mathbf{f}\right)(x)=\left(\partial_h^{+j}\mathbf{f}\right)(x). \tag{2}$$

Also, they satisfy the product rules

$$\begin{aligned}
\partial_h^{+j}(\mathbf{g}(x)\mathbf{f}(x))&=\left(\partial_h^{+j}\mathbf{g}\right)(x)\mathbf{f}(x+h\mathbf{e}_j)+\mathbf{g}(x)\left(\partial_h^{+j}\mathbf{f}\right)(x),\\
\partial_h^{-j}(\mathbf{g}(x)\mathbf{f}(x))&=\left(\partial_h^{-j}\mathbf{g}\right)(x)\mathbf{f}(x-h\mathbf{e}_j)+\mathbf{g}(x)\left(\partial_h^{-j}\mathbf{f}\right)(x).
\end{aligned} \tag{3}$$

Along the paper we will use the bold letters $\mathbf{f}, \mathbf{g}, \ldots, \mathbf{m}, \ldots, \mathbf{w}, \ldots$ and so on, when we refer to functions of the above form. The finite difference Dirac operators $D_h^\pm$ defined *viz*

$$D_h^+ = \sum_{j=0}^n \mathbf{e}_j \partial_h^{+j} \qquad \text{and} \qquad D_h^- = \sum_{j=0}^n \mathbf{e}_j \partial_h^{-j}$$

are Clifford-vector-valued and correspond to finite difference approximations of the classical gradient operator in a coordinate-free way.

Now let $\mathcal{P} = \mathbb{R}[x] \otimes C\ell_{0,n}$ be the space of Clifford-vector-valued polynomials. We say that $\{\mathbf{m}_s(x;\tau) : s \in \mathbb{N}_0\} \subset \mathcal{P}$ is an Appell set carrying D_h^+, resp. D_h^-, if for $\tau = \pm h$ we have $D_h^+ \mathbf{m}_0(x,-h) = D_h^- \mathbf{m}_0(x;h) = 0$ and

$$\begin{aligned} &D_h^+ \mathbf{m}_s(x;-h) = s\mathbf{m}_{s-1}(x;-h), \\ &D_h^- \mathbf{m}_s(x;h) = s\mathbf{m}_{s-1}(x;h), \qquad \text{for every} \quad s \in \mathbb{N}. \end{aligned} \tag{4}$$

Iterating r-times the operator D_h^+, resp. D_h^-, it turns out that the action of the semigroup $\left(\exp(tD_h^+)\right)_{t\geq 0}$, resp. $\left(\exp(tD_h^-)\right)_{t\geq 0}$, on each $\mathbf{m}_s(x;-h)$, resp. $\mathbf{m}_s(x;h)$, gives rise to a binomial expansion. Indeed, the iterated relations $\left(D_h^\pm\right)^r \mathbf{m}_s(x;\mp h) = \frac{s!}{(s-r)!}\mathbf{m}_{s-r}(x;\mp h)$, resp. $\left(D_h^\pm\right)^r \mathbf{m}_s(x;\mp h) = 0$, that hold for $s \geq r$, resp. $s < r$, leads to

$$\exp\left(tD_h^+\right)\mathbf{m}_s(x;-h) = \sum_{r=0}^\infty \frac{t^r}{r!}\left(D_h^+\right)^r \mathbf{m}_s(x;-h) = \sum_{r=0}^s \binom{s}{r} t^r \mathbf{m}_{s-r}(x;-h),$$

and analogously, to $\exp\left(tD_h^-\right)\mathbf{m}_s(x;h) = \sum\limits_{r=0}^s \binom{s}{r} t^r \mathbf{m}_{s-r}(x;h)$.

The action of the semigroups $\left(\exp(tD_h^+)\right)_{t\geq 0}$ and $\left(\exp(tD_h^-)\right)_{t\geq 0}$ on $\mathcal{P}$ then correspond to the hypercomplex extension of the Taylor series expansion for polynomials (see, e.g., [1, Subsection 3.3]) that gives rise e.g. to Clifford-vector-valued polynomials of Bernoulli type analogous to the ones obtained in [13, Section 2].

This approach corresponds to the discrete counterpart of the Cauchy–Kovaleskaya extension described in [4, Subsection III.2]. For an alternative application of this approach in interrelationship with discrete versions of Fueter polynomials, one refer to [3, Sections 3, 4].

The Fock space formalism carrying Hilbert spaces (cf. [5]) reveals that the problem of constructing polynomial sets $\{\mathbf{m}_s(x;\tau) : s \in \mathbb{N}_0\}$, with $\tau = \pm h$, possessing the Appell set property is equivalent to the construction of operator endomorphisms $M_h^+, M_h^- \in \mathrm{End}(\mathcal{P})$ in such way that the elements of the form

$$\mathbf{m}_s(x;\pm h) = \lambda_s \big(M_h^\pm\big)^s \mathbf{m}_0(x;\pm h), \qquad s \in \mathbb{N}_0$$

yield a basis for $\mathcal{P}$. Hereby, the constants λ_s are chosen under the condition $\lambda_0 = 1$ and the constraints (4).

In terms of the umbral calculus formalism (see [6, 12] and the references therein), from the identity operator $I : \mathbf{f}(x) \mapsto \mathbf{f}(x)$ and the commuting bracket $[A,B]$ defined as

$$[A,B]\mathbf{f}(x) = A(B\mathbf{f}(x)) - B(A\mathbf{f}(x)),$$

one can start to construct the set of Clifford-vector-valued polynomials $\{\mathbf{m}_s(x;\tau) : s \in \mathbb{N}_0\}$ from Weyl–Heisenberg algebra symmetries. In order to proceed, we will define for a given linear polynomial $w(t) \in \mathbb{R}[t]$ of degree 1 satisfying $(\partial_h^{+j} w)(x_k) = (\partial_h^{-j} w)(x_k) = \delta_{jk}\mu$, the following set of multiplication operators:

$$\begin{aligned}(W_j\mathbf{f})(x) &= \mu^{-1}w(x_j)\mathbf{f}(x),\\ \left(W_h^{+j}\mathbf{f}\right)(x) &= \mu^{-1}w\left(x_j+\frac{h}{2}\right)\mathbf{f}(x+h\mathbf{e}_j),\\ \left(W_h^{-j}\mathbf{f}\right)(x) &= \mu^{-1}w\left(x_j-\frac{h}{2}\right)\mathbf{f}(x-h\mathbf{e}_j).\end{aligned} \tag{5}$$

It is now straightforward from the product rules (3) that the set of operators

$$\left\{W_h^{-j},\partial_h^{+j},I : j=1,2,\ldots,n\right\} \qquad \text{and} \qquad \left\{W_h^{+j},\partial_h^{-j},I : j=1,2,\ldots,n\right\}$$

span the Weyl–Heisenberg algebra of dimension $2n+1$. The remainder graded commuting relations are given by

$$\left[\partial_h^{+j},\partial_h^{+k}\right]=0, \qquad \left[W_h^{-j},W_h^{-k}\right]=0, \qquad \left[\partial_h^{+j},W_h^{-k}\right]=\delta_{jk}I, \tag{6}$$

$$\left[\partial_h^{-j},\partial_h^{-k}\right]=0, \qquad \left[W_h^{+j},W_h^{+k}\right]=0, \qquad \left[\partial_h^{-j},W_h^{+k}\right]=\delta_{jk}I. \tag{7}$$

In the discrete Clifford analysis setting (cf. [9, Subsection 1.3]), it is precisely the Weyl–Heisenberg relations (6), resp. (7), that allows us to determine in a unique way M_h^-, resp. M_h^+, as

$$M_h^- = \sum_{j=1}^n \mathbf{e}_j W_h^{-j}, \qquad \text{resp.} \qquad M_h^+ = \sum_{j=1}^n \mathbf{e}_j W_h^{+j},$$

providing in this way a *Fourier duality* between D_h^+ and M_h^-, resp. D_h^- and M_h^+.

For discretizations of the Dirac operator $D=\sum_{j=1}^n \mathbf{e}_j\partial_{x_j}$, written as a superposition of forward and backward differences, the *Fourier duality* may be constructed by means of a set of skew-Weyl relations (cf. [2]).

Notice that the *Fourier duality* terminology comes from invariant theory (cf. [11]). In the language of Clifford analysis this is nothing else than the so-called *Fischer duality* (cf. [15]).

Since $M_h^{\pm}$ maps $\mathbf{m}_s(x;\pm h)$ into $\mathbf{m}_{s+1}(x;\pm h)$, it remains clear that each Clifford-vector-valued polynomial $\mathbf{m}_s(x;h)$, resp. $\mathbf{m}_s(x;-h)$, is an eigenfunction for the factorized Hamiltonian $M_h^+D_h^-+D_h^-M_h^+$, resp. $M_h^-D_h^++D_h^+M_h^-$. The Weyl–Heisenberg character between the operators ∂_h^{+j} and $W_h^{-j}=\mu^{-1}w\left(x_j-\frac{h}{2}\right)T_h^{-j}$, resp. ∂_h^{-j} and $W_h^{+j}=\mu^{-1}w\left(x_j+\frac{h}{2}\right)T_h^{+j}$, combined with the orthogonality constraint (1) provided by the Clifford basis of $C\ell_{0,n}$ allows us to rewrite the factorized Hamiltonians $M_h^+D_h^-+D_h^-M_h^+$ and $M_h^-D_h^++D_h^+M_h^-$ in terms of the following forward/backward discretizations of the Euler operator $E=\sum_{j=1} x_j\partial_{x_j}$:

$$E_h^+ = \sum_{j=1}^n \mu^{-1}w\left(x_j+\frac{h}{2}\right)\partial_h^{+j}, \qquad E_h^- = \sum_{j=1}^n \mu^{-1}w\left(x_j-\frac{h}{2}\right)\partial_h^{-j}. \tag{8}$$

Indeed, from (2) one can rewrite E_h^+ and E_h^- as

$$E_h^+ = \sum_{j=1}^n W_h^{+j}\partial_h^{-j} \qquad \text{and} \qquad E_h^- = \sum_{j=1}^n W_h^{-j}\partial_h^{+j}.$$

Then $M_h^+ D_h^- + D_h^- M_h^+$ and $M_h^- D_h^+ + D_h^+ M_h^-$ admit the following decompositions:

$$M_h^+ D_h^- + D_h^- M_h^+ = \sum_{j=1}^{n} \left(-2W_h^{+j}\partial_h^{-j} - I\right) = -2E_h^+ - nI,$$

$$M_h^- D_h^+ + D_h^+ M_h^- = \sum_{j=1}^{n} \left(-2W_h^{-j}\partial_h^{+j} - I\right) = -2E_h^- - nI.$$

The second quantization approach through the set of Weyl–Heisenberg generators (6) and (7) (cf. [5]) reveals that $E_h^\pm$ are number-type operators with spectrum $\mathbb{N}_0$. This leads to

$$E_h^+ \mathbf{m}_s(x;h) = s\mathbf{m}_s(x;h) \qquad \text{and} \qquad E_h^- \mathbf{m}_s(x;-h) = s\mathbf{m}_s(x;-h).$$

From the above construction, it turns out that the Weyl–Heisenberg algebra character provided from the graded commuting relations (6), resp. (7), gives the required ladder structure to construct also discrete analogues for Gegenbauer polynomials as a series expansion written in terms of the Appell set $\{\mathbf{m}_s(x;\tau) : s \in \mathbb{N}_0\}$ (cf. [7]).

Since the eigenfunctions of E_h^+ and E_h^- do not coincide, in general, it remains natural to ask which polynomial subsets $\{\mathbf{m}_s(x;h) : s \in \mathbb{N}_0\}$ of $\mathcal{P}$ give rise to solutions of the coupled eigenvalue system

$$E_h^+ \mathbf{m}_s(x;h) = s\mathbf{m}_s(x;h), \qquad E_h^- \mathbf{m}_s(x;h) = s\mathbf{m}_s(x;h), \qquad s \in \mathbb{N}_0.$$

From the relations (2), it remains clear that forward and backward differences, ∂_h^{+j} and ∂_h^{-j} respectively, commute. In contrast, the operators W_h^{-j}, W_h^{+j} do not commute in general. This means that for each $j = 1, 2, \ldots, n$ the set of operators ∂_h^{+j}, ∂_h^{-j}, W_h^{-j}, W_h^{+j} and I do not endow a canonical realization of an Weyl–Heisenberg type algebra.

Although the solutions of the above coupled system of eigenvalue equations may not be represented in terms of Weyl–Heisenberg algebra symmetries, the set of generators W_h^{-j}, W_h^{+j}, W_j itself (see the coordinate expressions (5)) will be the departure point of this paper to describe the hidden Lie algebraic symmetries encoded by the solutions of the above coupled system.

3 Clifford-vector-valued polynomials related with $\mathrm{SU}(1,1)$

3.1 Discrete series represention of $\mathrm{SU}(1,1)$

Accordingly to [17, Section 6.4], the Lie group $\mathrm{SU}(1,1)$ has two families of discrete series representations. In order to determine it algebraically, let us take a close look for the graded commuting relations involving the number operators E_h^+ and E_h^- defined in (8).

For this purpose, one starts to show that the set of operators W_h^+, W_h^- and W defined from the left endomorphisms (5) acting on the space $\mathcal{P}$:

$$W_h^+ = \sum_{j=1}^{n} W_h^{+j}, \qquad W_h^- = \sum_{j=1}^{n} W_h^{-j} \qquad \text{and} \qquad W = \sum_{j=1}^{n} W_j \tag{9}$$

generate a Lie algebra isomorphic to $\mathfrak{su}(1,1)$. In order to proceed, we will start with the following lemma which interrelates the set of generators

$$W_h^{+j} = \mu^{-1} w\left(x_j + \frac{h}{2}\right) T_h^{+j}, \qquad W_h^{-j} = \mu^{-1} w\left(x_j - \frac{h}{2}\right) T_h^{-j}, \qquad W_j = \mu^{-1} w(x_j) I.$$

Lemma 1. *For every $j,k=1,2,\ldots,n$ we have the following set of graded commuting rules:*

(1) *The operators W_h^{+j} and W_h^{-k} satisfy $\left[W_h^{+j},W_h^{-k}\right]=2h\delta_{jk}W_k$.*

(2) *The operators W_h^{+k}, resp. W_h^{-k}, and W_j are interrelated by*

$$\left[W_h^{+k},W_j\right]=h\delta_{jk}W_h^{+k}, \qquad \textit{resp.} \qquad \left[W_j,W_h^{-k}\right]=h\delta_{jk}W_h^{-k}.$$

Proof. (1) From the conditions $T_h^{+k}w\left(x_k-\frac{h}{2}\right)=w\left(x_k+\frac{h}{2}\right)$, $T_h^{-k}w\left(x_k+\frac{h}{2}\right)=w\left(x_k-\frac{h}{2}\right)$ and $w\left(x_k\mp\frac{h}{2}\right)=T_h^{\pm j}w\left(x_k\mp\frac{h}{2}\right)$, for $j\neq k$, one obtain for every $j,k=1,2,\ldots,n$, the set of graded commuting relations

$$\left[w\left(x_j+\frac{h}{2}\right)T_h^{+j},w\left(x_k-\frac{h}{2}\right)T_h^{-k}\right]=\delta_{jk}\left(w\left(x_k+\frac{h}{2}\right)^2-w\left(x_k-\frac{h}{2}\right)^2\right)I.$$

Now recall that $w(t)\in\mathbb{R}[t]$ is a polynomial of degree 1. Combination of linearity arguments with the condition $\partial_h^{\pm k}w(x_k)=\mu$ lead to the set of equations $w\left(x_k+\frac{h}{2}\right)+w\left(x_k-\frac{h}{2}\right)=2w(x_k)$ and $w\left(x_k+\frac{h}{2}\right)-w\left(x_k-\frac{h}{2}\right)=h\mu$, and hence, to the set of equations

$$w\left(x_k+\frac{h}{2}\right)^2-w\left(x_k-\frac{h}{2}\right)^2=2\mu h w(x_k) \qquad \text{with} \quad k=1,2,\ldots,n.$$

Thus, for all $j,k=1,2,\ldots,n$ the above set of graded commuting relations are equivalent to $\left[W_h^{+j},W_h^{-k}\right]=2\delta_{jk}hW_k$.

(2) Since $W_k=\mu^{-1}w(x_k)I$ commutes with

$$W_h^{+j}=\mu^{-1}w\left(x_j+\frac{h}{2}\right)T_h^{+j}, \qquad \text{resp.} \qquad W_h^{-j}=\mu^{-1}w\left(x_j-\frac{h}{2}\right)T_h^{-j},$$

for every $j\neq k$, it remains to show for every $j=1,2,\ldots,n$ the graded commuting relations

$$\left[W_h^{+j},W_j\right]=hW_h^{+j} \qquad \text{and} \qquad \left[W_j,W_h^{-j}\right]=hW_h^{-j}.$$

From a direct computation

$$\begin{aligned}
\left[w\left(x_j+\frac{h}{2}\right)T_h^{+j},w(x_j)I\right]&=w\left(x_j+\frac{h}{2}\right)^2T_h^{+j}-w(x_j)w\left(x_j+\frac{h}{2}\right)T_h^{+j}\\
&=h\big(\partial_h^{+j}w_j\big)(x)w\left(x_j+\frac{h}{2}\right)T_h^{+j},\\
\left[w(x_j)I,w\left(x_j-\frac{h}{2}\right)T_h^{-j}\right]&=w(x_j)w\left(x_j-\frac{h}{2}\right)T_h^{-j}-w\left(x_j-\frac{h}{2}\right)^2T_h^{-j}\\
&=h\big(\partial_h^{-j}w_j\big)(x)w\left(x_j-\frac{h}{2}\right)T_h^{-j}.
\end{aligned}$$

Combination of the conditions $\big(\partial_h^{+j}w\big)(x_j)=\big(\partial_h^{-j}w\big)(x_j)=\mu$ with the coordinate expressions (5) yields $\left[W_h^{+j},W_j\right]=hW_h^{+j}$, resp. $\left[W_j,W_h^{-j}\right]=hW_h^{-j}$, as desired. ■

From Lemma 1, one obtain for the set of multiplication operators (5), the Lie algebra isomorphism

$$\text{span}\left\{\frac{1}{h}W_h^{+j},\frac{1}{h}W_h^{-j},\frac{1}{h}W_j:j=1,2,\ldots,n\right\}\cong\mathfrak{sl}(2n,\mathbb{R}).$$

Thus, one can infer that $\frac{1}{h}W_h^+$, $\frac{1}{h}W_h^-$ and $\frac{1}{h}W$ are the canonical generators of the three-dimensional Lie algebra $\mathfrak{su}(1,1) \cong \mathfrak{sl}(2,\mathbb{R})$. The remaining commuting relations are given by

$$\left[\frac{1}{h}W_h^+, \frac{1}{h}W\right] = \frac{1}{h}W_h^+, \qquad \left[\frac{1}{h}W_h^-, \frac{1}{h}W\right] = -\frac{1}{h}W, \qquad \left[\frac{1}{h}W_h^+, \frac{1}{h}W_h^-\right] = \frac{2}{h}W. \tag{10}$$

Now let's turn our attention to the coordinate expressions of $E_h^\pm$ given by (8). Recall that the conditions $\left(\partial_h^{+j}w\right)(x_k) = \left(\partial_h^{-j}w\right)(x_k) = \delta_{jk}\mu$ provided from construction allows us to recast E_h^+ and E_h^- in terms of the multiplication operators defined in (5). Indeed, the basic identities

$$\begin{aligned} w\left(x_j + \frac{h}{2}\right)\partial_h^{+j} &= \frac{1}{h}\left(w\left(x_j + \frac{h}{2}\right)T_h^{+j} - w(x_j)I\right) - \frac{\mu}{2}I, \\ w\left(x_j - \frac{h}{2}\right)\partial_h^{-j} &= \frac{1}{h}\left(w(x_j)I - w\left(x_j - \frac{h}{2}\right)T_h^{-j}\right) - \frac{\mu}{2}I \end{aligned}$$

lead to the following coordinate expressions:

$$\begin{aligned} E_h^+ &= \sum_{j=1}^n \left(\frac{1}{h}W_h^{+j} - \frac{1}{h}W_j - \frac{1}{2}I\right) = \frac{1}{h}W_h^+ - \frac{1}{h}W - \frac{n}{2}I, \\ E_h^- &= \sum_{j=1}^n \left(\frac{1}{h}W_j - \frac{1}{h}W_h^{-j} - \frac{1}{2}I\right) = \frac{1}{h}W - \frac{1}{h}W_h^- - \frac{n}{2}I, \end{aligned} \tag{11}$$

and consequently, to the following coordinate expressions involving the sum/difference between E_h^+ and E_h^-:

$$E_h^+ + E_h^- = \frac{1}{h}W_h^+ - \frac{1}{h}W_h^- - nI, \qquad E_h^+ - E_h^- = \frac{1}{h}W_h^+ + \frac{1}{h}W_h^- - \frac{2}{h}W. \tag{12}$$

The following set of results are also straightforward and will give the key ingredients to construct the positive/negative series representations encoded by SU(1,1).

Lemma 2. *The operators $E_h^+ - E_h^-$, $\frac{1}{h}W_h^+$ and $E_h^+ + \frac{n}{2}I$, resp. $E_h^+ - E_h^-$, $\frac{1}{h}W_h^-$ and $E_h^- + \frac{n}{2}I$ are the canonical generators of the Lie algebra $\mathfrak{su}(1,1)$. The remainder commuting relations are given by*

$$\begin{aligned} &\left[E_h^\pm + \frac{n}{2}I, E_h^+ - E_h^-\right] = E_h^- - E_h^+, \qquad \left[E_h^\pm + \frac{n}{2}I, \frac{1}{h}W_h^\pm\right] = \frac{1}{h}W_h^\pm, \\ &\left[E_h^+ - E_h^-, \frac{1}{h}W_h^\pm\right] = 2\left(E_h^\pm + \frac{n}{2}I\right). \end{aligned}$$

Proof. First, notice that direct combination of relations (10) with the coordinate expressions (11) and (12) results into the following set of graded commuting relations carrying the operators $E_h^+ + \frac{n}{2}I$, $E_h^- + \frac{n}{2}I$ and $E_h^+ - E_h^-$:

$$\left[E_h^+ + \frac{n}{2}I, E_h^- + \frac{n}{2}I\right] = \left[\frac{1}{h}W_h^+ - \frac{1}{h}W, \frac{1}{h}W - \frac{1}{h}W_h^-\right] = E_h^+ - E_h^-.$$

In order to prove the graded commuting relations $\left[E_h^+ + \frac{n}{2}I, \frac{1}{h}W_h^+\right] = \frac{1}{h}W_h^+$ and $\left[E_h^- + \frac{n}{2}I, \frac{1}{h}W_h^-\right] = \frac{1}{h}W_h^-$ one starts to rewrite $\left[E_h^+ + \frac{n}{2}I, \frac{1}{h}W_h^+\right]$ and $\left[E_h^- + \frac{n}{2}I, \frac{1}{h}W_h^-\right]$ and based on the coordinate expressions (11). In concrete

$$\begin{aligned} \left[E_h^+ + \frac{n}{2}I, \frac{1}{h}W_h^+\right] &= \left[\frac{1}{h}W_h^+ - \frac{1}{h}W, \frac{1}{h}W_h^+\right], \\ \left[E_h^- + \frac{n}{2}I, \frac{1}{h}W_h^-\right] &= \left[\frac{1}{h}W - \frac{1}{h}W_h^-, \frac{1}{h}W_h^-\right]. \end{aligned} \tag{13}$$

The graded commuting relations provided from (10) yield

$$\left[\frac{1}{h}W_h^+ - \frac{1}{h}W, \frac{1}{h}W_h^+\right] = \frac{1}{h}W_h^+ \qquad \text{and} \qquad \left[\frac{1}{h}W - \frac{1}{h}W_h^-, \frac{1}{h}W_h^-\right] = \frac{1}{h}W_h^-,$$

and therefore, the relations (13) are equivalent to

$$\left[E_h^+ + \frac{n}{2}I, \frac{1}{h}W_h^+\right] = \frac{1}{h}W_h^+ \qquad \text{and} \qquad \left[E_h^- + \frac{n}{2}I, \frac{1}{h}W_h^-\right] = \frac{1}{h}W_h^-.$$

Finally, the relations $\left[E_h^+ - E_h^-, \frac{1}{h}W_h^+\right] = 2\left(E_h^+ + \frac{n}{2}I\right)$ and $\left[E_h^+ - E_h^-, \frac{1}{h}W_h^-\right] = 2\left(E_h^- + \frac{n}{2}I\right)$ follow straightforwardly from direct combination of the coordinate expression obtained in (12) for $E_h^+ - E_h^-$ with the graded commutators

$$\left[\frac{1}{h}W_h^+ + \frac{1}{h}W_h^- - \frac{2}{h}W, \frac{1}{h}W_h^+\right] = 2\left(\frac{1}{h}W_h^+ - \frac{1}{h}W\right),$$
$$\left[\frac{1}{h}W_h^+ + \frac{1}{h}W_h^- - \frac{2}{h}W, \frac{1}{h}W_h^-\right] = 2\left(\frac{1}{h}W - \frac{1}{h}W_h^-\right). \qquad \blacksquare$$

In the proof of Proposition 1 and in the subsequent results, we will make use of the following lemma which follows straightforwardly from induction on $\mathbb{N}$.

Lemma 3. *For every A, B and for every $s \in \mathbb{N}$, the graded commutator $[A, B^s]$ satisfies the summation formula*

$$[A, B^s] = \sum_{r=0}^{s-1} B^r [A, B] B^{s-1-r}.$$

Proposition 1. *For any $s \in \mathbb{N}$ we have the following graded commuting relations:*

$$\left[E_h^\pm + \frac{n}{2}I, \left(E_h^+ - E_h^-\right)^s\right] = -s\left(E_h^+ - E_h^-\right)^s,$$
$$\left[E_h^\pm + \frac{n}{2}I, \left(\frac{1}{h}W_h^\pm\right)^s\right] = s\left(\frac{1}{h}W_h^\pm\right)^s,$$
$$\left[E_h^+ - E_h^-, \left(\frac{1}{h}W_h^\pm\right)^s\right] = s\left(2E_h^\pm + (n-s+1)I\right)\left(\frac{1}{h}W_h^\pm\right)^{s-1}.$$

Proof. Recall that when $[A, B] = \pm B$, Lemma 3 reduces to

$$[A, B^s] = \sum_{r=0}^{s-1} \pm B^r B B^{s-1-r} = \pm s B^s.$$

Combination of Lemma 2 with the above identity carrying the substitutions $A = E_h^\pm + \frac{n}{2}I$ and $B = E_h^+ - E_h^- / B = \frac{1}{h}W_h^\pm$ yield

$$\left[E_h^\pm + \frac{n}{2}I, \left(E_h^+ - E_h^-\right)^s\right] = -s\left(E_h^+ - E_h^-\right)^s,$$
$$\left[E_h^\pm + \frac{n}{2}I, \left(\frac{1}{h}W_h^\pm\right)^s\right] = s\left(\frac{1}{h}W_h^\pm\right)^s.$$

For the proof of

$$\left[E_h^+ - E_h^-, \left(\frac{1}{h}W_h^\pm\right)^s\right] = s\left(2E_h^\pm + (n-s+1)I\right)\left(\frac{1}{h}W_h^\pm\right)^{s-1}$$

recall that the relations $[E_h^\pm + \frac{n}{2}I, \frac{1}{h}W_h^\pm] = \frac{1}{h}W_h^\pm$ provided from Lemma 2 are equivalent to the intertwining properties $\frac{1}{h}W_h^\pm \left(E_h^\pm + \frac{n}{2}I\right) = \left(E_h^\pm + \left(\frac{n}{2}-1\right)I\right)\frac{1}{h}W_h^\pm$.

Induction over $r = 1, \dots, s-1$ gives

$$\left(\frac{1}{h}W_h^\pm\right)^r \left(E_h^\pm + \frac{n}{2}I\right) = \left(E_h^\pm + \left(\frac{n}{2}-r\right)I\right)\left(\frac{1}{h}W_h^\pm\right)^r.$$

Finally, the direct application of Lemma 3 carrying the substitutions $A = E_h^+ - E_h^-$ and $B = \frac{1}{h}W_h^+$ results into

$$\left[E_h^+ - E_h^-, \left(\frac{1}{h}W_h^\pm\right)^s\right] = \sum_{r=0}^{s-1}\left(\frac{1}{h}W_h^\pm\right)^r 2\left(E_h^\pm + \frac{n}{2}I\right)\left(\frac{1}{h}W_h^\pm\right)^{s-1-r}$$
$$= \sum_{r=0}^{s-1} 2\left(E_h^\pm + \left(\frac{n}{2}-r\right)I\right)\left(\frac{1}{h}W_h^\pm\right)^{s-1} = s\left(2E_h^\pm + (n-s+1)I\right)\left(\frac{1}{h}W_h^\pm\right)^{s-1}. \quad \blacksquare$$

Now let $\mathbf{m}_0(x;h)$ be a Clifford-vector-valued polynomial that satisfies the set of equations $E_h^+\mathbf{m}_0(x;h) = E_h^-\mathbf{m}_0(x;h) = 0$. The intertwining relations $E_h^\pm\left(\frac{1}{h}W_h^\pm\right)^s = \left(\frac{1}{h}W_h^\pm\right)^s\left(E_h^\pm + sI\right)$ that yield from $\left[E_h^\pm + \frac{n}{2}I, \left(\frac{1}{h}W_h^\pm\right)^s\right] = s\left(\frac{1}{h}W_h^\pm\right)^s$ (see Proposition 1) lead to

$$E_h^+\left[\left(\frac{1}{h}W_h^+\right)^s \mathbf{m}_0(x;h)\right] = s\left(\frac{1}{h}W_h^+\right)^s \mathbf{m}_0(x;h),$$
$$E_h^-\left[\left(\frac{1}{h}W_h^-\right)^s \mathbf{m}_0(x;h)\right] = s\left(\frac{1}{h}W_h^-\right)^s \mathbf{m}_0(x;h).$$

Then it is straightforward to see that the basis functions of the form $\mathbf{w}_s(x;h) = \left(\frac{1}{h}W_h^+\right)^s\mathbf{m}_0(x;h)$ and $\mathbf{w}_s(x;-h) = \left(\frac{1}{h}W_h^-\right)^s\mathbf{m}_0(x;h)$ satisfy the following set of ladder operator relations:

$$\begin{aligned}&\frac{1}{h}W_h^+\mathbf{w}_s(x;h) = \mathbf{w}_{s+1}(x;h),\\ &(E_h^+ - E_h^-)\mathbf{w}_s(x;h) = s(s+n+1)\mathbf{w}_{s-1}(x;h),\\ &\left(E_h^+ + \frac{n}{2}I\right)\mathbf{w}_s(x;h) = \left(s+\frac{n}{2}\right)\mathbf{w}_s(x;h);\end{aligned} \tag{14}$$

$$\begin{aligned}&\frac{1}{h}W_h^-\mathbf{w}_s(x;-h) = \mathbf{w}_{s+1}(x;-h),\\ &(E_h^+ - E_h^-)\mathbf{w}_s(x;-h) = s(s+n+1)\mathbf{w}_{s-1}(x;-h),\\ &\left(E_h^- + \frac{n}{2}I\right)\mathbf{w}_s(x;-h) = \left(s+\frac{n}{2}\right)\mathbf{w}_s(x;-h).\end{aligned} \tag{15}$$

We are now in conditions to construct the positive, resp. negative, part for the discrete series representation carrying the group $SU(1,1)$ in the same order of ideas of [17, Section 6.4]. Recall that for the given set of generators $\frac{1}{h}W_h^+$, $\frac{1}{h}W_h^-$ and $\frac{1}{h}W$ of $\mathfrak{su}(1,1)$, the graded commuting relations (10) endow a $*$-structure defined viz $\left(\frac{1}{h}W\right)^* = \frac{1}{h}W$ and $\left(\frac{1}{h}W_h^+\right)^* = -\frac{1}{h}W_h^-$. A direct computation shows that the Casimir operator

$$K_h = \left(\frac{1}{h}W\right)^2 - \frac{1}{2}\left(\frac{1}{h}W_h^+\frac{1}{h}W_h^- + \frac{1}{h}W_h^-\frac{1}{h}W_h^+\right) \tag{16}$$

determines all irreducible unitary representations π_λ of $SU(1,1)$ on the enveloping algebra $U(\mathfrak{su}(1,1))$ through its eigenvalues λ. Moreover, the positive series representation π_λ^+ labelled by λ is thus determined by the set of ladder operators

$$\pi_\lambda^+\left(\frac{1}{h}W_h^-\right) = E_h^+ - E_h^-, \qquad \pi_\lambda^+\left(\frac{1}{h}W_h^+\right) = \frac{1}{h}W_h^+, \qquad \pi_\lambda^+\left(\frac{1}{h}W\right) = E_h^+ + \frac{n}{2}I,$$
$$\pi_\lambda^+(K_h) = \left(E_h^+ + \frac{n}{2}I\right)\left(E_h^+ + \left(\frac{n}{2}-1\right)I\right) - \frac{W_h^+}{h}(E_h^+ - E_h^-)$$

underlies the representation space $\ell^2(\mathbb{N}_0)$ carrying the family of subspaces $(\mathcal{H}_{s;h})_{s\in\mathbb{N}_0}$ of the form

$$\mathcal{H}_{s;h} = \left\{ \mathbf{w}_s(x;h) = \left(\frac{1}{h}W_h^+\right)^s \mathbf{m}_0(x;h) \in \mathcal{P} : E_h^+\mathbf{m}_0(x;h) = E_h^-\mathbf{m}_0(x;h) = 0 \right\},$$

while the negative ones is thus determined by the set of ladder operators

$$\pi_\lambda^-\left(\frac{1}{h}W_h^-\right) = \frac{1}{h}W_h^-, \qquad \pi_\lambda^-\left(\frac{1}{h}W_h^+\right) = E_h^+ - E_h^-, \qquad \pi_\lambda^-\left(\frac{1}{h}W\right) = -E_h^- - \frac{n}{2}I,$$
$$\pi_\lambda^-(K_h) = \left(E_h^- + \frac{n}{2}I\right)\left(E_h^- + \left(\frac{n}{2}-1\right)I\right) - \frac{W_h^-}{h}(E_h^+ - E_h^-)$$

and underlies the representation space $\ell^2(\mathbb{N}_0)$ carrying the family of subspaces $(\mathcal{H}_{s;-h})_{s\in\mathbb{N}_0}$ of the form

$$\mathcal{H}_{s;-h} = \left\{ \mathbf{w}_s(x;-h) = \left(\frac{1}{h}W_h^-\right)^s \mathbf{m}_0(x;h) \in \mathcal{P} : E_h^+\mathbf{m}_0(x;h) = E_h^-\mathbf{m}_0(x;h) = 0 \right\}.$$

A short computation based on the ladder operator properties (14) and (15) shows that the positive/negative discrete series representations $\pi_\lambda^+/\pi_\lambda^-$ of the Lie group $\mathrm{SU}(1,1)$ are labeled by the constants $\lambda = \frac{n^2}{4} - \frac{n}{2} - 2s$, with $s \in \mathbb{N}_0$.

Remark 1. In the context of quantum mechanical systems, this framework may be derived as consequence of a more general result – the so-called Crum's theorem (cf. [14, Subsections 2.2, 2.3]).

3.2 The action of SO(n) on the lattice

With the aim of understanding the action of $\mathrm{SO}(n)$ on the lattice $h\mathbb{Z}^n$ as a representation theory carrying canonical generators which are invariant with respect to the orthogonal Lie algebra $\mathfrak{so}(n)$, we will first recall some basic concepts and observations about $\mathrm{SO}(n)$ and $\mathfrak{so}(n)$ in the context of Clifford algebras.

Let us denote by $\mathcal{B}(x,y) = -\frac{1}{2}(xy+yx)$ the bilinear form generated by the Clifford vector representations $x = \sum_{j=1}^n x_j\mathbf{e}_j$ and $y = \sum_{j=1}^n y_j\mathbf{e}_j$ of $\mathbb{R}^n$. The set of matrices $T : \mathbb{R}^n \to \mathbb{R}^n$ with determinant equals 1 for which $\mathcal{B}(Tx,Ty) = \mathcal{B}(x,y)$ forms a group under the operation of composition. This group is called the *special orthogonal group of rotations* and it is denoted by $\mathrm{SO}(n)$.

Let us take a close look for the $\mathrm{SO}(n)$-action on the lattice $h\mathbb{Z}^n$ given by the left regular representation

$$\Lambda(T)\mathbf{f}(x) = \mathbf{f}\big(T^{-1}x\big) \qquad \text{with} \quad T \in \mathrm{SO}(n), \quad \mathbf{f} \in \mathcal{P} \quad \text{and} \quad x \in h\mathbb{Z}^n. \tag{17}$$

In order to proceed, one can select from $\mathrm{SO}(n)$ the 1-parameter subgroups elements $T_{jk}^{\pm h}(\theta) = \exp\big(\theta S_{jk}^{\pm h}\big)$ generated by exponentiation from the Lie algebra elements $S_{jk}^{\pm h} \in \mathfrak{so}(n)$ $(1 \leq j < k \leq n)$ in such way that each $S_{jk}^{\pm h}$ is skew-symmetric, i.e. $S_{jk}^{\pm h} = -S_{kj}^{\pm h}$.

In particular, the canonical elements of the form

$$S_{jk}^{+h} = \mu^{-1}w\left(x_j + \frac{h}{2}\right)\partial_h^{+k} - \mu^{-1}w\left(x_k + \frac{h}{2}\right)\partial_h^{+j},$$
$$S_{jk}^{-h} = \mu^{-1}w\left(x_j - \frac{h}{2}\right)\partial_h^{-k} - \mu^{-1}w\left(x_k - \frac{h}{2}\right)\partial_h^{-j} \tag{18}$$

that correspond to the discrete counterparts of the classical angular momentum operators $L_{jk} = x_j\partial_{x_k} - x_k\partial_{x_j}$, endow the left representations of SO(n) acting on each subspace $\mathcal{H}_{s;h}$, resp. $\mathcal{H}_{s;-h}$ of $\mathcal{P}$, that is $\Lambda\big(T^{+h}_{jk}(\theta)\big) = \exp\big(\theta S^{+h}_{jk}\big)$, resp. $\Lambda\big(T^{-h}_{jk}(\theta)\big) = \exp\big(\theta S^{-h}_{jk}\big)$.

It is straightforwardly to see from (8) that the operator $E_h^+ - E_h^-$ commutes with the skew-symmetric elements (18) belonging in this way to the center of the enveloping algebra $U(\mathfrak{so}(n))$. Then $E_h^+ - E_h^-$ commutes with all left regular representations $\Lambda\big(T^{+h}_{jk}(\theta)\big)$, resp. $\Lambda\big(T^{-h}_{jk}(\theta)\big)$, of SO$(n)$ and so $E_h^+ - E_h^-$ commutes with all left regular representations of the form (17).

By virtue of the above Lie algebraic representation we have shown that the family of subspaces $(\mathcal{H}_{s;\pm h})_{s\in\mathbb{N}_0}$ of $\mathcal{P}$ are SO(n)-invariant on $h\mathbb{Z}^n$ with respect to $E_h^+ - E_h^-$, and so, to all the operators encoded by left-regular representations of SO(n) on $h\mathbb{Z}^n$.

Remark 2. The left regular representations $\Lambda\big(T^{\pm h}_{jk}(\theta)\big)$ of SO(n) on the lattice $h\mathbb{Z}^n$ are canonically isomorphic to standard left regular representations $\Lambda(T_{jk}(\theta))$ of SO(n) on $\mathbb{R}^n$ given in terms of rotations on the 2-dimensional plane with coordinates (x_j, x_k) (cf. [17, Chapter 9.1]).

Indeed, the Clifford-vector-valued extension of the classical Sheffer map Ψ_x from $\mathbb{R}[x]$ to $\mathcal{P}$ defined by linearity from the mapping

$$\Psi_x : \prod_{j=1}^n x_j^{\alpha_j} \mapsto \prod_{j=1}^n \left(W_h^{+j}\right)^{\alpha_j} 1, \qquad \text{resp.} \qquad \Psi_x : \prod_{j=1}^n x_j^{\alpha_j} \mapsto \prod_{j=1}^n \left(W_h^{-j}\right)^{\alpha_j} 1,$$

satisfies the intertwining relations $\Psi_x x_j = W_h^{+j}\Psi_x$ and $\Psi_x\partial_{x_j} = \partial_h^{-j}\Psi_x$, resp. $\Psi_x x_j = W_h^{-j}\Psi_x$ and $\Psi_x\partial_{x_j} = \partial_h^{+j}\Psi_x$.

This leads to the intertwining relations $\Psi_x L_{jk} = S^{+h}_{jk}\Psi_x$, resp. $\Psi_x L_{jk} = S^{-h}_{jk}\Psi_x$, with $L_{jk} = x_j\partial_{x_k} - x_k\partial_{x_j}$, and hence to the intertwining property below at the level of SO(n):

$$\Psi_x \Lambda\left(T_{jk}(\theta)\right) = \Lambda\big(T^{\pm h}_{jk}(\theta)\big)\Psi_x.$$

This in turn shows that the 1-parameter representation $T^{+h}_{jk}(\theta)$, resp. $T^{-h}_{jk}(\theta)$ of SO(n), on $h\mathbb{Z}^n$ is canonically isomorphic to to the 1-parameter representation $T_{jk}(\theta)$ of SO(n) on $\mathbb{R}^n$.

3.3 The Howe dual pair $(\mathrm{SO}(n), \mathfrak{su}(1,1))$

Along this section we will study subspaces of Clifford-vector-valued polynomials which are invariant under the action of the SO$(n) \times \mathfrak{su}(1,1)$-module. The ladder properties (14) and (15) reveal that for each $s \in \mathbb{N}_0$ the representation $\pi^+_\lambda\left(\frac{1}{h}W_h^+\right) = \frac{1}{h}W_h^+$, resp. $\pi^-_\lambda\left(\frac{1}{h}W_h^-\right) = \frac{1}{h}W_h^-$, maps $\mathcal{H}_{s;h}$, resp. $\mathcal{H}_{s;-h}$, into $\mathcal{H}_{s+1;h}$, resp. $\mathcal{H}_{s+1;-h}$, while $\pi^+_\lambda\left(\frac{1}{h}W_h^-\right) = \pi^-_\lambda\left(\frac{1}{h}W_h^+\right) = E_h^+ - E_h^-$ maps $\mathcal{H}_{s;h} \cap \mathcal{H}_{s;-h}$ onto the trivial space $\{0\}$. Also, the positive/negative representations $\pi^\pm_\lambda\left(\frac{1}{h}W\right)$, resp. $\pi^\pm_\lambda(K_h)$, leave the the subspace $\mathcal{H}_{s;h}$, resp. $\mathcal{H}_{s;-h}$, invariant. Here W and K_h denote the multiplication and the Casimir operator labeled by (9) and (16), respectively.

In addition, for any $r = 0, 1, \ldots, s-1$, the Clifford-vector-valued polynomial spaces

$$\left(\frac{1}{h}W_h^\pm\right)^r (\mathcal{H}_{s-r;h} \cap \mathcal{H}_{s-r;-h}) = \left\{\left(\frac{1}{h}W_h^\pm\right)^r \mathbf{m}_{s-r}(x;h) : \mathbf{m}_{s-r}(x;h) \in \mathcal{H}_{s-r;h} \cap \mathcal{H}_{s-r;-h}\right\}$$

are also invariant under the action of $\pi^+_\lambda\left(\frac{1}{h}W\right)$ and $\pi^+_\lambda(K_h)$, resp. $\pi^-_\lambda\left(\frac{1}{h}W\right)$ and $\pi^-_\lambda(K_h)$.

So, each SO(n)-invariant subspace $\mathcal{H}_{s;h}$, resp. $\mathcal{H}_{s;-h}$, of $\mathcal{P}$ is isomorphic to the family of subspaces $\left(\frac{1}{h}W_h^+\right)^r (\mathcal{H}_{s-r;h} \cap \mathcal{H}_{s-r;-h})$, resp. $\left(\frac{1}{h}W_h^-\right)^r (\mathcal{H}_{s-r;h} \cap \mathcal{H}_{s-r;-h})$, with $r = 0, \ldots, s$. This means that each $\mathcal{H}_{s;h}$, resp. $\mathcal{H}_{s;-h}$, appears with infinite multiplicity.

In addition, since $\left(\frac{1}{h}W_h^+\right)^r (\mathcal{H}_{s-r;h} \cap \mathcal{H}_{s-r;-h}) = \{0\} = \left(\frac{1}{h}W_h^-\right)^r (\mathcal{H}_{s-r;h} \cap \mathcal{H}_{s-r;-h})$ if and only if $s = r$, by virtue of ladder operator relations (14), resp. (15), one can infer that the direct sum decomposition

$$\left(\frac{1}{h}W_h^+\right)^r(\mathcal{H}_{s-r;h}\cap\mathcal{H}_{s-r;-h})=\mathcal{V}_h\oplus\mathcal{W}_h,$$
$$\text{resp.}\qquad \left(\frac{1}{h}W_h^-\right)^r(\mathcal{H}_{s-r;h}\cap\mathcal{H}_{s-r;-h})=\mathcal{V}_h\oplus\mathcal{W}_h,$$

only fulfils for the subspaces $\mathcal{V}_h=\left(\frac{1}{h}W_h^+\right)^r(\mathcal{H}_{s-r;h}\cap\mathcal{H}_{s-r;-h})$ and $\mathcal{W}_h=\{0\}$, resp. $\mathcal{V}_h=\left(\frac{1}{h}W_h^-\right)^r(\mathcal{H}_{s-r;h}\cap\mathcal{H}_{s-r;-h})$ and $\mathcal{W}_h=\{0\}$. This means that the SO(n)-invariant subspaces $\left(\frac{1}{h}W_h^+\right)^r(\mathcal{H}_{s-r;h}\cap\mathcal{H}_{s-r;-h})$, resp. $\left(\frac{1}{h}W_h^-\right)^r(\mathcal{H}_{s-r;h}\cap\mathcal{H}_{s-r;-h})$, of $\mathcal{P}$ are also irreducible.

In order to collect the infinite multiplicities of $\pi_\lambda^+/\pi_\lambda^-$ carrying the eigenvalues $\lambda=\frac{n^2}{4}-\frac{n}{2}-2s$ of the Casimir operator (16), it remains to investigate the $\mathfrak{su}(1,1)$-action on $\mathcal{P}$ regarded as a $\mathrm{SO}(n)\times\mathfrak{su}(1,1)$-module that yields the Howe dual pair $(\mathrm{SO}(n),\mathfrak{su}(1,1))$. For a sake of readability, the Howe dual pair construction will be only sketched. Further details arising this construction can be found in [10, Chapters 4 & 5].

First, notice that the above set of properties produces the following infinite triangle as a chain diagram carrying the families of subspaces $(\mathcal{H}_{s;h})_{s\in\mathbb{N}_0}$, resp. $(\mathcal{H}_{s;-h})_{s\in\mathbb{N}_0}$:

$$\begin{array}{cccccccc}
\{0\} & \mathcal{H}_{0;\pm h} & \leftarrow & \mathcal{H}_{1;\pm h} & \leftarrow & \mathcal{H}_{2;\pm h} & \leftarrow & \dots \\
\{0\} & \mathcal{H}_{0;h}\cap\mathcal{H}_{0;-h} & \leftarrow & \frac{1}{h}W_h^+(\mathcal{H}_{0;h}\cap\mathcal{H}_{0;-h}) & \leftarrow & (\frac{1}{h}W_h^\pm)^2(\mathcal{H}_{0;h}\cap\mathcal{H}_{0;-h}) & \leftarrow & \dots \\
 & \oplus & & \oplus & & \oplus & & \\
 & \{0\} & & \mathcal{H}_{1;h}\cap\mathcal{H}_{1;-h} & \leftarrow & \frac{1}{h}W_h^+(\mathcal{H}_{1;h}\cap\mathcal{H}_{1;-h}) & \leftarrow & \dots \\
 & & & \oplus & & \oplus & & \\
 & & & \{0\} & & \mathcal{H}_{2;h}\cap\mathcal{H}_{2;-h} & \leftarrow & \dots \\
 & & & & & \oplus & & \\
 & & & & & \{0\} & \leftarrow & \dots \\
 & & & & & & & \\
 & & & & & & & \dots
\end{array}$$

In the above triangle diagram, the representations $\pi_\lambda^+\left(\frac{1}{h}W_h^-\right)=\pi_\lambda^-\left(\frac{1}{h}W_h^+\right)=E_h^+-E_h^-$ – the Fourier duals of $\pi_\lambda^\pm\left(\frac{1}{h}W_h^\pm\right)$ – act as isomorphisms that shift each individual summand from the right to the left. The first line gives the direct sum decomposition of $\mathcal{P}$ in terms of the SO(n)-invariant pieces $\mathcal{H}_{s;h}$, resp. $\mathcal{H}_{s;-h}$, through $h\mathbb{Z}^n$, i.e.

$$\mathcal{P}=\bigoplus_{s=0}^{\infty}\mathcal{H}_{s;h}=\bigoplus_{s=0}^{\infty}\mathcal{H}_{s;-h}.$$

Also, for each $s\in\mathbb{N}_0$ the $(s+1)$-row (which is infinite-dimensional) give rise to $\mathfrak{su}(1,1)$-modules isomorphic to the SO(n)-module $\mathcal{H}_{s;h}$, resp. $\mathcal{H}_{s;-h}$, while the $(s+1)$-column provides the splitting of the subspace $\mathcal{H}_{s;h}$, resp. $\mathcal{H}_{s;-h}$, as a direct sum in terms of the irreducible pieces $\left(\frac{1}{h}W_h^+\right)^r(\mathcal{H}_{s-r;h}\cap\mathcal{H}_{s-r;-h})$, resp. $\left(\frac{1}{h}W_h^-\right)^r(\mathcal{H}_{s-r;h}\cap\mathcal{H}_{s-r;-h})$:

$$\mathcal{H}_{s;h}=\bigoplus_{r=0}^{s}\left(\frac{1}{h}W_h^+\right)^r(\mathcal{H}_{s-r;h}\cap\mathcal{H}_{s-r;-h}),$$
$$\mathcal{H}_{s;-h}=\bigoplus_{r=0}^{s}\left(\frac{1}{h}W_h^-\right)^r(\mathcal{H}_{s-r;h}\cap\mathcal{H}_{s-r;-h}).$$

These chain decompositions lead, in a multiplicity free way, to the following Fourier decompositions of $\mathcal{P}$:

$$\mathcal{P} = \bigoplus_{s=0}^{\infty}\bigoplus_{r=0}^{s}\left(\frac{1}{h}W_h^+\right)^r(\mathcal{H}_{s-r;h}\cap\mathcal{H}_{s-r;-h}),$$
$$\mathcal{P} = \bigoplus_{s=0}^{\infty}\bigoplus_{r=0}^{s}\left(\frac{1}{h}W_h^-\right)^r(\mathcal{H}_{s-r;h}\cap\mathcal{H}_{s-r;-h}). \tag{19}$$

Remark 3. The algebra of endomorphisms End($\mathcal{P}$) encoded on the Fourier decompositions (19) is the so-called Weyl algebra of polynomial operators (cf. [10, Chapter 4]).

4 Families of special functions

4.1 Hypergeometric series representations

Proposition 2. *Any sequence of polynomials* $\{\mathbf{m}_s(x;h) : s\in\mathbb{N}_0\}$ *satisfying the set of eigenvalue equations* $E_h^+\mathbf{m}_s(x;h) = E_h^-\mathbf{m}_s(x;h) = s\mathbf{m}_s(x;h)$ *is determined in a unique way as*

$$\mathbf{m}_s(x;h) = \gamma_s\mathbf{w}_s(x;\pm h) \qquad \textit{with} \quad \mathbf{w}_s(x;\pm h)\in\mathcal{H}_{s;\pm h} \quad \textit{and} \quad \gamma_s\in\mathbb{R}.$$

Moreover, the constants γ_s *are given by*

$$\gamma_s = \sum_{r=0}^{s}\frac{(-1)^r(-s-n-1)_r}{(-2s-n+2)_r}\binom{s}{r}.$$

Hereby $(a)_r = a(a+1)\cdots(a+r-1)$ *denotes the Pochhammer symbol.*

Proof. To prove this, let us consider the family $\{\mathbf{m}_s(x;h) : s\in\mathbb{N}_0\}$ of Clifford-vector-valued polynomials, each of them given as a solution of the eigenvalue equations

$$E_h^+\mathbf{m}_s(x;h) = E_h^-\mathbf{m}_s(x;h) = s\mathbf{m}_s(x;h). \tag{20}$$

From the Fourier decompositions labeled by (19) each $\mathbf{w}_s(x;\pm h)\in\mathcal{H}_{s;h}$ may be written in a unique way as

$$\mathbf{w}_s(x;\pm h) = \sum_{r=0}^{s}\left(\frac{1}{h}W_h^\pm\right)^r\mathbf{m}_{s-r}(x;h).$$

From the ladder operator relations (14) and (15), one obtain, for any $r=0,1,\dots,s$, the mapping property $\left(\frac{1}{h}W_h^\pm\right)^r\left(E_h^+-E_h^-\right)^r : \mathcal{H}_{s;\pm h}\to\mathcal{H}_{s;\pm h}$. Then, for a given polynomial sequence $\{\mathbf{w}_s(x;\tau) : s\in\mathbb{N}_0\}$ of $\mathcal{P}$, one can compute each $\mathbf{m}_s(x;h)$ from the formulae

$$\mathbf{m}_s(x;h) = \sum_{r=0}^{s}c_{r,s}\left(\frac{1}{h}W_h^\pm\right)^r\widetilde{\mathbf{w}}_{s-r}(x;\pm h). \tag{21}$$

where $\widetilde{\mathbf{w}}_{s-r}(x;\pm h) = \left(E_h^+-E_h^-\right)^r\mathbf{w}_s(x;\pm h)$ and the constants $c_{r,s}\in\mathbb{R}$ are determined from the constraint (20).

Combination of the graded commuting relation provided from Proposition 1:

$$\left[E_h^+-E_h^-,\left(\frac{1}{h}W_h^\pm\right)^r\right] = r(2E_h^\pm+(n-r+1)I)\left(\frac{1}{h}W_h^\pm\right)^{r-1}$$

with the eigenvalue properties

$$E_h^{\pm}\left[\left(\frac{1}{h}W_h^{\pm}\right)^{r-1}\widetilde{\mathbf{w}}_{s-r}(x;\pm h)\right]=(s-1)\left(\frac{1}{h}W_h^{\pm}\right)^{r-1}\widetilde{\mathbf{w}}_{s-r}(x;\pm h)$$

lead to the recursive relations

$$\begin{aligned}&\left(E_h^{+}-E_h^{-}\right)\left[\left(\frac{1}{h}W_h^{\pm}\right)^{r}\widetilde{\mathbf{w}}_{s-r}(x;\pm h)\right]\\&\quad=r(2s+n-r-1)\left(\frac{1}{h}W_h^{\pm}\right)^{r-1}\widetilde{\mathbf{w}}_{s-r}(x;\pm h)+\left(\frac{1}{h}W_h^{\pm}\right)^{r}\widetilde{\mathbf{w}}_{s-r-1}(x;\pm h)\end{aligned}$$

and moreover, to the following linear expansions

$$E_h^{+}\mathbf{m}_s(x;h)-E_h^{-}\mathbf{m}_s(x;h)=\sum_{r=1}^{s}d_{r,s}\left(\frac{1}{h}W_h^{\pm}\right)^{r}\widetilde{\mathbf{w}}_{s-r-1}(x;\pm h),$$

where the coefficients $d_{r,s}$ are given by $d_{r,s}=(r+1)(2s+n-r-2)c_{r+1,s}+c_{r,s}$.

Hence $\mathbf{m}_s(x;h)$ satisfies the constraint (20) if and only if $d_{r,s}=0$ holds for every $r=0,1,\ldots,s$, that is, each $c_{r,s}$ is determined from the condition $c_{0,s}=1$ and from the constraint

$$c_{r+1,s}=\frac{c_{r,s}}{(r+1)(-2s-n+r+2)}.$$

Therefore

$$c_{r,s}=\prod_{q=1}^{r}\frac{1}{q(-2s-n+q+1)}=\frac{1}{r!(-2s-n+2)_r}.$$

Now it remains to show the relations $\mathbf{m}_s(x;h)=\gamma_s\mathbf{w}_s(x;\pm h)$. Iterating r times the operator $E_h^{+}-E_h^{-}$, one obtain from (14) the recursive relations

$$\begin{aligned}&\left(E_h^{+}-E_h^{-}\right)^{r}\mathbf{w}_s(x;h)=(-1)^r\frac{s!}{(s-r)!}(-s-n)_r\mathbf{w}_{s-r}(x;h),\\&\left(\frac{1}{h}W_h^{+}\right)^{r}\mathbf{w}_{s-r}(x;h)=\mathbf{w}_s(x;h).\end{aligned}$$

Then we have $\left(\frac{1}{h}W_h^{+}\right)^{r}\widetilde{\mathbf{w}}_{s-r}(x;\pm h)=(-1)^r\frac{s!}{(s-r)!}(-s-n-1)_r\mathbf{w}_s(x;\pm h)$, and therefore, the set of relations (21) are equivalent to $\mathbf{m}_s(x;h)=\gamma_s\mathbf{w}_s(x;\pm h)$, with

$$\gamma_s=\sum_{r=0}^{s}\frac{(-1)^r(-s-n-1)_r}{(-2s-n+2)_r}\binom{s}{r}.$$ ■

Remark 4. A simple computation involving the binomial identity $(-1)^r\binom{s}{r}=\frac{(-s)_r}{r!}$ shows that the constant γ_s corresponds to the s-term truncation of the hypergeometric function ${}_2F_1(a,b;c;z)=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c)_r}\frac{z^r}{r!}$ labelled by the parameters $a=-s-n-1$, $b=-s$, $c=-2s-n+2$ and $z=1$.

4.2 Application to Cauchy problems

On this section it will be studied families of Clifford-vector-valued polynomials given as solutions of the following homogeneous Cauchy problem in $[0,\infty)\times h\mathbb{Z}^n$:

$$\begin{array}{ll} \partial_t \mathbf{g}(t,x) + E_h^+\mathbf{g}(t,x) - E_h^-\mathbf{g}(t,x) = 0, & t>0,\\ \mathbf{g}(0,x)=\mathbf{f}(x), & t=0,\\ E_h^+\mathbf{g}(t,x) = E_h^-\mathbf{g}(t,x), & t\geq 0. \end{array} \tag{22}$$

The solution of the Cauchy problem (22) may be written formally as $\mathbf{g}(t,x)=\mathbb{E}_h(t)\mathbf{f}(x)$ with $\mathbf{f}(x)\in\bigoplus_{s=0}^{\infty}\mathcal{H}_{s;h}\cap\mathcal{H}_{s;-h}$. Hereby $\mathbb{E}_h(t)=\exp(tE_h^- - tE_h^+)$ is a one-parameter representation of the Lie group $\mathrm{SU}(1,1)$.

Note that the graded commuting property $\left[E_h^+ - E_h^-, \mathbb{E}_h(t)\right]=0$ for each $t\geq 0$ assures that $(\mathbb{E}_h(t))_{t\geq 0}$ is a semigroup, i.e.

$$\mathbb{E}_h(0)=I \qquad \text{and} \qquad \mathbb{E}_h(t+\tau)=\mathbb{E}_h(t)\mathbb{E}_h(\tau) \quad \text{for all} \;\; t,\tau\geq 0.$$

The next sequence of results will makes clear that, for any $t\geq 0$, the operator $\mathbb{E}_h(t)$ leaves the space of Clifford-vector-valued polynomials $\mathcal{P}$ invariant.

Lemma 4. *We have the following intertwining properties carrying the semigroup operator* $\mathbb{E}_h(t)=\exp(tE_h^- - tE_h^+)$*:*

$$\begin{aligned} &\left(tE_h^- + (1-t)E_h^+\right)\mathbb{E}_h(t)=\mathbb{E}_h(t)E_h^+,\\ &\left(\frac{1}{h}W_h^+ - t\left(E_h^+ + E_h^- + nI\right)\right)\mathbb{E}_h(t)=\mathbb{E}_h(t)\frac{1}{h}W_h^+. \end{aligned}$$

Proof. In order to start proving the intertwining property

$$\left(tE_h^- + (1-t)E_h^+\right)\mathbb{E}_h(t)=\mathbb{E}_h(t)E_h^+,$$

one can start to compute, for each $s\in\mathbb{N}$, the graded commutator $\left[2E_h^+ + nI, \left(E_h^- - E_h^+\right)^s\right]$. A short computation based on Lemma 2 shows that

$$\left[E_h^+, E_h^- - E_h^+\right] = \left[E_h^+ + \frac{n}{2}I, E_h^- - E_h^+\right] = E_h^+ - E_h^-.$$

Thus, direct application of Lemma 3 for $A=E_h^+$ and $B=E_h^- - E_h^+$ results into

$$\left[E_h^+, \left(E_h^- - E_h^+\right)^s\right] = s\left(E_h^+ - E_h^-\right)\left(E_h^- - E_h^+\right)^{s-1}.$$

This leads to

$$\begin{aligned} \left[E_h^+, \mathbb{E}_h(t)\right] &= \sum_{r=0}^{\infty}\frac{t^r}{r!}\left[E_h^+, \left(E_h^- - E_h^+\right)^r\right]\\ &= \sum_{r=1}^{\infty}\frac{t^r}{(r-1)!}\left(E_h^+ - E_h^-\right)\left(E_h^- - E_h^+\right)^{r-1} = \left(tE_h^+ - tE_h^-\right)\mathbb{E}_h(t). \end{aligned}$$

Therefore, the above graded commuting relation is equivalent to the intertwining property

$$\mathbb{E}_h(t)E_h^+ = (tE_h^- + (1-t)E_h^+)\mathbb{E}_h(t).$$

For the proof of $\left(\frac{1}{h}W_h^+ - t\left(E_h^+ + E_h^- + nI\right)\right)\mathbb{E}_h(t)=\mathbb{E}_h(t)\frac{1}{h}W_h^+$, one needs to compute, for every $r\in\mathbb{N}$, the graded commutator $\left[\frac{1}{h}W_h^+, \left(E_h^+ - E_h^-\right)^r\right]$ based on Lemma 3. First, recall that the relation $[E_h^- - E_h^+, E_h^+ + \frac{n}{2}I] = E_h^- - E_h^+$ that follows from Lemma 2 is equivalent to the intertwining property

$$\left(E_h^- - E_h^+\right)\left(E_h^+ + \frac{n}{2}I\right) = \left(E_h^+ + \left(\frac{n}{2}+1\right)I\right)\left(E_h^- - E_h^+\right).$$

Induction over $r \in \mathbb{N}$ gives

$$\left(E_h^- - E_h^+\right)^r\left(E_h^+ + \frac{n}{2}I\right) = \left(E_h^+ + \left(\frac{n}{2}+r\right)I\right)\left(E_h^- - E_h^+\right)^r.$$

On the other hand, from $\left[\frac{1}{h}W_h^+, E_h^- - E_h^+\right] = 2E_h^+ + nI$ (see Lemma 2) and from direct application of Lemma 3 results into the following set of relations carrying the generators $A = \frac{1}{h}W_h^+$ and $B = E_h^- - E_h^+$:

$$\begin{aligned}\left[\frac{1}{h}W_h^+, \left(E_h^- - E_h^+\right)^s\right] &= \sum_{r=0}^{s-1}\left(E_h^- - E_h^+\right)^r\left(2E_h^+ + nI\right)\left(E_h^- - E_h^+\right)^{s-1-r} \\ &= \sum_{r=0}^{s-1}\left(2E_h^+ + (n+2r)I\right)\left(E_h^- - E_h^+\right)^{s-1} \\ &= s\left(2E_h^+ + (n+s-1)I\right)\left(E_h^+ - E_h^-\right)^{s-1}.\end{aligned}$$

This leads to

$$\begin{aligned}\left[\frac{1}{h}W_h^+, \mathbb{E}_h(t)\right] = \sum_{s=0}^{\infty}\frac{t^s}{s!}\left[\frac{1}{h}W_h^+, \left(E_h^- - E_h^+\right)^s\right] = \sum_{s=1}^{\infty}\frac{t^s}{(s-1)!}\left(2E_h^+ + nI\right)\left(E_h^- - E_h^+\right)^{s-1} \\ + \sum_{s=2}^{\infty}\frac{t^s}{(s-2)!}\left(E_h^- - E_h^+\right)^{s-1} = t\left(E_h^+ + E_h^- + nI\right)\mathbb{E}_h(t).\end{aligned}$$

and hence, to the intertwining property

$$\left(\frac{1}{h}W_h^+ - t\left(E_h^+ + E_h^- + nI\right)\right)\mathbb{E}_h(t) = \mathbb{E}_h(t)\frac{1}{h}W_h^+. \qquad \blacksquare$$

Proposition 3. *For any $t \geq 0$ and $s \in \mathbb{N}_0$, we have the mapping property*

$$\mathbb{E}_h(t): \ \mathcal{H}_{s;h} \cap \mathcal{H}_{s;-h} \to \mathcal{H}_{s;h} \cap \mathcal{H}_{s;-h}.$$

Moreover the semigroup $(\mathbb{E}_h(t))_{t\geq 0}$ leaves invariant the space of Clifford-vector-valued polynomials $\mathcal{P}$.

Proof. Recall that from the Fourier decomposition (19) any $\mathbf{f}(x) \in \mathcal{P}$ may be written uniquely as

$$\mathbf{f}(x) = \sum_{s=0}^{\infty}\sum_{r=0}^{s}\left(\frac{1}{h}W_h^+\right)^s \mathbf{m}_{s-r}(x;h),$$

where $\{\mathbf{m}_s(x;h) : s \in \mathbb{N}\}$ is a sequence of polynomials satisfying the set of eigenvalue equations $E_h^+\mathbf{m}_s(x;h) = E_h^-\mathbf{m}_s(x;h) = s\mathbf{m}_s(x;h)$.

From Lemma 4, one can easily see that the operator $\mathbb{E}_h(t)$ intertwines E_h^+ and $tE_h^- + (1-t)E_h^+$. Since $E_h^+ - E_h^-$ commutes with $\mathbb{E}_h(t)$, it follows that the set of functions $\mathbf{m}_s(t,x;h) := \mathbb{E}_h(t)\mathbf{m}_s(x;h)$ satisfy the coupled system of equations

$$\left(tE_h^- + (1-t)E_h^+\right)\mathbf{m}_s(t,x;h) = s\mathbf{m}_s(t,x;h), \qquad \left(E_h^+ - E_h^-\right)\mathbf{m}_s(t,x;h) = 0.$$

From the above system of equations, one can infer the eigenvalue relations $E_h^+\mathbf{m}_s(t,x;h) = E_h^-\mathbf{m}_s(t,x;h) = s\mathbf{m}_s(t,x;h)$. Thus we have shown the mapping property $\mathbb{E}_h(t) : \mathcal{H}_{s;h}\cap\mathcal{H}_{s;-h} \to \mathcal{H}_{s;h}\cap\mathcal{H}_{s;-h}$.

Finally, since the operator $\frac{1}{h}W_h^+ - t\left(E_h^+ + E_h^- + nI\right)$ is canonically isomorphic to $\frac{1}{h}W_h^+$, one can see that

$$\mathbb{E}_h(t)\left[\left(\frac{1}{h}W_h^+\right)^r \mathbf{m}_{s-r}(x;h)\right] = \left(\frac{1}{h}W_h^+ - t\left(E_h^+ + E_h^- + nI\right)\right)^r \mathbf{m}_{s-r}(t,x;h).$$

From the representation of $E_h^+ + E_h^- + nI$ given by (12), one can conclude that the right-hand side of the above relation is a Clifford-vector-valued polynomial but also an eigenfunction of $tE_h^- + (1-t)E_h^+$ carrying the eigenvalue $s \in \mathbb{N}_0$.

Therefore $\mathbb{E}_h(t)\mathbf{f}(x) = \sum\limits_{s=0}^{\infty}\sum\limits_{r=0}^{s} \mathbb{E}_h(t)\left[\left(\frac{1}{h}W_h^+\right)^r \mathbf{m}_{s-r}(x;h)\right] \in \mathcal{P}$. ∎

Remark 5. One can see from a direct combination of Lemma 4 with (11) that for $t=0$, the polynomial solution provided from the initial condition $\mathbf{g}(0,x) = \mathbf{f}(x)$ belongs to $\bigoplus_{s=0}^{\infty}\mathcal{H}_{s;h}$ while for $t=1$ the solution $\mathbf{g}(1,x) = \mathbb{E}_h(1)\mathbf{f}(x)$ belongs to $\bigoplus_{s=0}^{\infty}\mathcal{H}_{s;-h}$. So, the semigroup $(\mathbb{E}_h(t))_{t\geq 0}$ gives, in particular, a direct link between the positive series representation of SU(1,1) with the negative ones.

Remark 6. One can see from Lemma 4 that $\frac{1}{2}\left(\frac{1}{h}W_h^+ + \frac{1}{h}W_h^-\right)$ is the Fourier dual of $E_h^+ - E_h^-$ carrying the parameter $t = \frac{1}{2}$.

One can also deduce from Lemma 4 that the solutions of the eigenvalue problem

$$E_h^+\mathbf{g}_s(t,x) + E_h^-\mathbf{g}_s(t,x) = 2s\mathbf{g}_s(t,x)$$

correspond to $\mathbf{g}_s\left(\frac{1}{2},x\right) = \mathbb{E}_h\left(\frac{1}{2}\right)\mathbf{f}(x)$, with $\mathbf{f}(x)\in\mathcal{H}_{s;h}$. Moreover, given $\mathbf{g}_s(1,x)\in\mathcal{H}_{s;-h}$, one can compute $\mathbf{g}_s\left(\frac{1}{2},x\right)$ by letting act the inverse of $\mathbb{E}_h\left(\frac{1}{2}\right)$ on $\mathbf{g}_s(1,x)$, i.e.

$$\mathbf{g}_s\left(\frac{1}{2},x\right) = \mathbb{E}_h\left(-\frac{1}{2}\right)\mathbf{g}_s(1,x).$$

The next corollary, that follows straightforwardly from the combination of the above proposition with Proposition 2, fully characterize the Clifford-vector-valued polynomial solutions of the Cauchy problem (22) as hypergeometric ${}_0F_1$-series expansions.

Corollary 1. *Let $\mathbf{f}(x)\in\bigoplus_{s=0}^{\infty}\mathcal{H}_{s,h}\cap\mathcal{H}_{s,-h}$. If there is a sequence of polynomials $\{\mathbf{w}_s(x;h) : s\in\mathbb{N}_0\}$, each of them belonging to $\mathcal{H}_{s,h}$, then the solutions $\mathbf{g}(t,x) = \mathbb{E}_h(t)\mathbf{f}(x)$ of the Cauchy problem (22) are given explicitly in terms of the hypergeometric ${}_0F_1$-series expansion*

$$\mathbf{f}(x) = \sum_{s=0}^{\infty}\left[{}_0F_1(-2s-n+2;\partial_t)t^s\right]_{t=1}\mathbf{w}_s(x;h),$$

where ${}_0F_1(c;z)$ denotes the hypergeometric function ${}_0F_1(c;z) = \sum\limits_{r=0}^{\infty}\frac{1}{(c)_r}\frac{z^r}{r!}$.

Proof. From Proposition 2 it follows that each $\mathbf{f}(x)\in\bigoplus_{s=0}^{\infty}\mathcal{H}_{s,h}\cap\mathcal{H}_{s,-h}$ may be written as

$$\mathbf{f}(x) = \sum_{s=0}^{\infty}\gamma_s\mathbf{w}_s(x;h) \qquad \text{with} \quad \gamma_s = \sum_{r=0}^{s}\frac{(-1)^r(-s-n-1)_r}{(-2s-n+2)_r}\binom{s}{r}.$$

A short computation based on the lowering properties $(\partial_t)^r t^s = \frac{s!}{(s-r)!}t^{s-r}$ for $s\geq r$ and $(\partial_t)^r t^s = 0$ for $s<r$ yields $\gamma_s = \left[{}_0F_1(-2s-n+2;\partial_t)t^s\right]_{t=1}$. ∎

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