On the Linearization of Second-Order Ordinary Differential Equations to the Laguerre Form via Generalized Sundman Transformations

M. Tahir MUSTAFA, Ahmad Y. AL-DWEIK and Raed A. MARA'BEH

Department of Mathematics & Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

E-mail: *tmustafa@kfupm.edu.sa*, *aydweik@kfupm.edu.sa*, *raedmaraabeh@kfupm.edu.sa*

URL: http://faculty.kfupm.edu.sa/math/tmustafa/,
http://faculty.kfupm.edu.sa/MATH/aydweik/

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Abstract. The linearization problem for nonlinear second-order ODEs to the Laguerre form by means of generalized Sundman transformations (S-transformations) is considered, which has been investigated by Duarte et al. earlier. A characterization of these S-linearizable equations in terms of first integral and procedure for construction of linearizing S-transformations has been given recently by Muriel and Romero. Here we give a new characterization of S-linearizable equations in terms of the coefficients of ODE and one auxiliary function. This new criterion is used to obtain the general solutions for the first integral explicitly, providing a direct alternative procedure for constructing the first integrals and Sundman transformations. The effectiveness of this approach is demonstrated by applying it to find the general solution for geodesics on surfaces of revolution of constant curvature in a unified manner.

Key words: linearization problem; generalized Sundman transformations; first integrals; nonlinear second-order ODEs

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1 Introduction

The mathematical modeling of many physical phenomena leads to such nonlinear ordinary differential equations (ODEs) whose analytical solutions are hard to find directly. Therefore, the approach of investigating nonlinear ODEs via transforming to simpler ODEs becomes important and has been quite fruitful in analysis of physical problems. This includes the classical linearization problem of finding transformations that linearize a given ODE. For the linearization problem of second-order ODEs via point transformations, it is known that these must be at most cubic in the first-order derivative and its coefficients should satisfy the Lie linearization test [8, 9, 10, 11]. The implementation of the Lie linearization method requires solving systems of partial differential equations (PDEs). It is also well known that only second-order ODEs admitting 8 dimensional Lie symmetry algebra pass the Lie linearization test, which makes it a restricted class of ODEs. In order to consider a larger class of ODEs, linearization problem via nonlocal transformations has been investigated in [3, 4, 6]. Many of these transformations are of the form

$$u(t) = \psi(x, y), \qquad dt = \phi(x, y, y')dx, \qquad \psi_y \phi \neq 0, \tag{1.1}$$

and the linearization problem via transformations (1.1), in general, is an open problem. In case that $\phi = \phi(x, y)$, the transformations of type (1.1) are called generalized Sundman transformations [7] and equations that can be linearized by means of generalized Sundman transformations to the Laguerre form $u_{tt} = 0$ are called S-linearizable [13]. These transformations have also been utilized to define Sundman symmetries of ODEs [5, 6, 7]. It should be mentioned that another special classes of nonlocal transformations of type (1.1) with polynomials of first degree in y' for $\phi(x, y, y')$ have also been studied in [2, 15].

Duarte et al. [4] showed that the S-linearizable second-order equations

$$y'' = f(x, y, y') \tag{1.2}$$

are at most quadratic in the first derivative, i.e. belong to the family of equations of the form

$$y'' + F_2(x, y){y'}^2 + F_1(x, y)y' + F(x, y) = 0. \tag{1.3}$$

Precisely, the free particle equation

$$u_{tt} = 0$$

can be transformed by an arbitrary generalized Sundman transformation

$$u(t) = \psi(x, y), \qquad dt = \phi(x, y)\, dx, \qquad \psi_y \phi \neq 0, \tag{1.4}$$

to the family of equations of the form (1.3) with the coefficients $F(x, y)$, $F_1(x, y)$ and $F_2(x, y)$ satisfying the following system of partial differential equations

$$AF_2 = A_y, \qquad AF_1 = B_y + A_x, \qquad AF = B_x, \tag{1.5}$$

where

$$A = \frac{\psi_y}{\varphi}, \qquad B = \frac{\psi_x}{\varphi}. \tag{1.6}$$

They also gave a characterization of these S-linearizable equations in terms of the coefficients. Muriel and Romero [13] further studied S-linearizable equations and proved that these must admit first integrals that are polynomials of first degree in the first-order derivative.

Theorem 1.1 ([13]). *The ODE (1.2) is S-linearizable if and only if it admits a first integral of the form $w(x, y, y') = A(x, y)y' + B(x, y)$. In this case ODE has the form (1.3). If a linearizing S-transformation (1.4) is known then a first integral $w(x, y, y') = A(x, y)y' + B(x, y)$ of (1.3) is defined by (1.6). Conversely, if a first integral $w(x, y, y') = A(x, y)y' + B(x, y)$ of (1.3) is known then a linearizing S-transformation can be determined by*

$$\psi(x, y) = \eta(I(x, y)),$$
$$\phi(x, y) = \frac{\psi_y}{A} \qquad \text{or} \qquad \phi(x, y) = \frac{\psi_x}{B} \quad \text{if} \quad B \neq 0,$$

where $I(x, y)$ is the first integral of

$$y' = -\frac{B}{A}. \tag{1.7}$$

Moreover, Muriel and Romero in [13, 14] revisited Duarte results [4], presented the following equivalent characterization of S-linearizable ODE of the form (1.3), and also provided constructive methods, as given in Theorem 1.3, to derive the linearizing S-transformations.

Theorem 1.2 ([13]). *Let us consider an equation of the form (1.3) and let S_1 and S_2 be the functions defined by*

$$S_1(x, y) = F_{1y} - 2F_{2x},$$
$$S_2(x, y) = (FF_2 + F_y)_y + (F_{2x} - F_{1y})_x + (F_{2x} - F_{1y})F_1. \tag{1.8}$$

The following alternatives hold:

- *If* $S_1 = 0$ *then equation* (1.3) *is S-linearizable if and only if* $S_2 = 0$
- *If* $S_1 \neq 0$*, let* S_3 *and* S_4 *be the functions defined by*

$$S_3(x,y) = \left(\frac{S_2}{S_1}\right)_y - (F_{2x} - F_{1y}),$$
$$S_4(x,y) = \left(\frac{S_2}{S_1}\right)_x + \left(\frac{S_2}{S_1}\right)^2 + F_1\left(\frac{S_2}{S_1}\right) + FF_2 + F_y. \tag{1.9}$$

Equation (1.3) *is S-linearizable if and only if* $S_3 = 0$ *and* $S_4 = 0$.

Theorem 1.3 ([13])**.** *Consider an equation of the form* (1.3) *and let* S_1 *and* S_2 *be the functions defined by* (1.8)*. The following alternative hold:*

- *If* $S_1 = 0$ *then the equation has a first integral of the form* $w = A(x,y)y' + B(x,y)$ *if and only if* $S_2 = 0$*. In this case* A *and* B *can be given as* $A = q\mathrm{e}^P$, $B = Q$*, where* P *is a solution of the system*

$$P_x = \frac{1}{2}F_1, \qquad P_y = F_2, \tag{1.10}$$

q is a nonzero solution of

$$q''(x) + f(x)q(x) = 0, \tag{1.11}$$

where

$$f(x) = FF_2 + F_y - \frac{1}{2}F_{1x} - \frac{1}{4}{F_1}^2$$

and Q *is a solution of the system*

$$Q_x = Fq\mathrm{e}^P, \qquad Q_y = \left(\frac{1}{2}F_1 - \frac{q'}{q}\right)q\mathrm{e}^P.$$

- *If* $S_1 \neq 0$ *then the equation has a first integral has a first integral of the form* $w = A(x,y)y' + B(x,y)$ *if and only if* $S_3 = S_4 = 0$*, where* S_3 *and* S_4 *are the functions defined by* (1.9)*. In this case* A *and* B *can be given as* $A = \mathrm{e}^P$, $B = Q$*, where* P *is a solution of the system*

$$P_x = F_1 + \frac{S_2}{S_1}, \qquad P_y = F_2, \tag{1.12}$$

and Q *is a solution of the system*

$$Q_x = F\mathrm{e}^P, \qquad Q_y = -\left(\frac{S_2}{S_1}\right)\mathrm{e}^P.$$

In this paper, a new characterization of S-linearizable equations in terms of the coefficients and one auxiliary function is given, and the equivalence with the old criteria is proved. This criterion is used to provide explicit general solutions for the auxiliary functions A and B given in (1.6) which can be directly utilized to obtain the first integral of (1.3). So, using Theorem 1.1, the linearizing generalized Sundman transformations can be constructed by solving the first-order ODE (1.7). The method is illustrated in examples where we recover the Sundman transformations of Muriel and Romero in [13].

As an application, we express the system of geodesic equations for surfaces of revolution as a single second-order ODE and use our method to find the general solution for geodesics on surfaces of revolution of constant curvature in a unified manner.

In this paper, we have focused on S-linearization to the Laguerre form $u_{tt} = 0$. For an account of S-linearization to any linear second-order ODE, the reader is referred to [16].

2 The method for constructing the first integrals and Sundman transformations

When the ODE (1.3) is S-linearizable, Theorem 1.2 does not give a method to construct the linearizing generalized Sundman transformations. In order to derive a method to obtain linearizing generalized Sundman transformations (1.4) of a given S-linearizable equation (1.3), Muriel and Romero [13] found additional relationships between the functions ϕ and ψ in (1.4) and the functions $F(x,y)$, $F_1(x,y)$ and $F_2(x,y)$, in (1.3), and used these to provide constructive methods to derive the linearizing S-transformations for the case $S_1 = S_2 = 0$ and the case $S_1 \neq 0$ but $S_3 = S_4 = 0$. This section provides an alternative procedure for constructing the first integrals and Sundman transformations for S-linearizable equations, which can be applied to both of the cases.

The key idea in this paper is that instead of finding additional relationships between the functions ϕ and ψ, we find additional relationship between the functions A and B in (1.6) and the functions $F(x,y)$, $F_1(x,y)$ and $F_2(x,y)$, in (1.3). Equation (1.6) implies

$$\left(\frac{B_y - A_x}{A}\right)_y = F_{1y} - 2F_{2x},$$

which leads to the following missing relationship

$$B_y - A_x = A(F_1 - 2h_x), \tag{2.1}$$

where

$$h = \int F_2(x,y)dy + g(x), \tag{2.2}$$

and $g(x)$ can be determined using Theorem 2.1 in case that the ODE is S-linearizable.

This missing equation jointly with (1.5) give a new compact S-linearizability criterion for ODE of the form (1.3), given in Theorem 2.1. The S-linearizability criterion is used to provide explicit general solutions for the auxiliary functions A and B which can be directly utilized to obtain the first integral of (1.5), given in Theorem 2.3, and hence the Sundman transformations can be constructed using Theorem 1.1. Thus an alternative procedure for constructing the first integral and Sundman transformation is obtained.

Theorem 2.1. *Let us consider an equation of the form* (1.3) *and let* $h = \int F_2(x,y)dy + g(x)$. *Equation* (1.3) *is S-linearizable if and only if*

$$F_{1x} + F_1 h_x - h_x^2 - h_{xx} - F_y - FF_2 = 0, \tag{2.3}$$

for some auxiliary function $g(x)$.

Proof. Using the new relationship (2.1) with (1.5) one can get the following equations

$$A_x = Ah_x, \qquad A_y = AF_2, \qquad B_x = AF, \qquad B_y = A(F_1 - h_x). \tag{2.4}$$

The compatibility of the system (2.4), i.e. $A_{xy} = A_{yx}$ and $B_{xy} = B_{yx}$, leads to the criteria (2.3).

In order to show that the new criterion (2.3) is equivalent to the one given in Theorem 1.2, we note that the system consisting of equation (2.3) and the second derivatives of h given by (2.2)

$$h_{xx} = F_{1x} + F_1 h_x - h_x^2 - F_y - FF_2, \qquad h_{yx} = F_{2x}, \qquad h_{yy} = F_{2y}. \tag{2.5}$$

is compatible, i.e. $h_{xy} = h_{yx}$, $h_{xxy} = h_{yxx}$ and $h_{yyx} = h_{yxy}$, when the following equation holds

$$h_x(F_{1y} - 2F_{2x}) + F_{2x}F_1 + F_{1xy} - F_{2xx} - F_{yy} - F_yF_2 - FF_{2y} = 0. \tag{2.6}$$

Now, using S_1 and S_2 defined by (1.8), equation (2.6) can be rewritten as

$$S_1 h_x = S_2 + F_1 S_1.$$

Then clearly if $S_1 = 0$, then $S_2 = 0$ and if $S_1 \neq 0$, then

$$h_x = \frac{S_2}{S_1} + F_1. \tag{2.7}$$

Finally, substituting (2.7) in (2.5), gives

$$S_3(x,y) = \left(\frac{S_2}{S_1}\right)_y - (F_{2x} - F_{1y}) = 0,$$
$$S_4(x,y) = \left(\frac{S_2}{S_1}\right)_x + \left(\frac{S_2}{S_1}\right)^2 + F_1\left(\frac{S_2}{S_1}\right) + FF_2 + F_y = 0. \qquad \blacksquare$$

Remark 2.2. In the case $S_1 = S_2 = 0$, the criteria (2.3) can be transformed by the change of variable

$$g(x) = \ln q(x) + k(x), \tag{2.8}$$

where $k'(x) = \frac{1}{2}F_1 - \int F_{2x}dy$ to the well-defined ODE, equation (1.11), in Theorem 1.3 and $P = h - \ln q$ verifies the system (1.10) in Theorem 1.3.

Moreover, in the case $S_3 = S_4 = 0$, the criteria (2.3) implies equation (2.7) which shows that $P = h$ verifies the system (1.12) in Theorem 1.3 and hence solving this system provides a well-defined ODE

$$g'(x) = k(x), \tag{2.9}$$

where $k(x) = \frac{S_2}{S_1} + F_1 - \int F_{2x}dy$.

Hence when equation (1.3) is S-linearizable, one can solve the criteria (2.3) for a function $g(x)$ by considering both of x and y as independent variables. Or equivalently one can get the function $g(x)$ by equation (2.8) when $S_1 = S_2 = 0$ whereas when $S_3 = S_4 = 0$, the function $g(x)$ can be obtained from equation (2.9).

In the next theorem, the general solution of the first integral is given explicitly in terms of the function $h(x)$ where $h = \int F_2(x,y)dy + g(x)$. It can be verified that this solution coincides with the solution of the systems given in Theorem 1.3. Hence, it provides an alternate direct procedure for constructing the first integrals and the S-transformations.

Theorem 2.3. *Let us assume that equation (1.3) is S-linearizable. Then (1.3) has the first integral $w(x,y,y') = A(x,y)y' + B(x,y)$ where A and B are given by*

$$A(x,y) = \mathrm{e}^h, \qquad B(x,y) = \int F\mathrm{e}^h dx + \int\left(\mathrm{e}^h(F_1 - h_x) - \int \mathrm{e}^h(F_y + FF_2)dx\right)dy,$$

and h is given by (2.2).

Proof. The functions A and B defined by (1.6) can be given explicitly by finding the general solution of the system (2.4) where the second and the third equations of the system (2.4) have general solution

$$A = v(x)\mathrm{e}^h, \qquad B = \int FA dx + z(y), \tag{2.10}$$

for arbitrary functions $v(x)$ and $z(y)$.

Substituting (2.10) in the first and the fourth equations of the system (2.4) gives

$$v(x) = c_1, \qquad z(y) = \int \left(A(F_1 - h_x) - \int (FA_y + F_y A)dx \right) dy + k(x), \tag{2.11}$$

for arbitrary functions $k(x)$.

Now, differentiating (2.11) with respect to x and using the criterion (2.3) gives

$$k_x = -\int A\big(F_{1x} + F_1 h_x - h_x^2 - h_{xx} - F_y - FF_2\big)dy = 0.$$

So $k(x) = c_2$, and finally, from Theorem 1.1, equation (1.3) has the first integral $w(x, y, y') = A(x, y)y' + B(x, y)$ and without loss of generality, one can choose $c_1 = 1$ and $c_2 = 0$ by relabeling of $\frac{w(x,y,y')-c_2}{c_1}$. ■

Remark 2.4. An algorithmic implementation of our approach can be carried out as summarized below. Given a S-linearizable ODE of the form (1.3). Use Theorem 2.1 to determine an auxiliary function $g(x)$. Find the first integrals using $g(x)$ and Theorem 2.3. Construct the Sundman transformations (1.1) using the first integral and Theorem 1.1. Since the free particle equation $u_{tt} = 0$ has the general solution $u(t) = c_1 + c_2 t$, finally using the Sundman transformations leads to the second integral of the ODE (1.3)

$$\psi(x, y) = c_1 + c_2\mu(x),$$

where $t = \mu(x)$ is a solution of the first-order ODE

$$\frac{dt}{dx} = \phi(x, \gamma(x, t)),$$

and $y = \gamma(x, t)$ can be obtained by solving $c_1 + c_2 t = \psi(x, y)$ for y.

But in case that $\phi = \phi(x)$, $\mu(x) = \int \phi(x)dx$ and so the Sundman transformation is a point transformation and leads to the general solution [16].

In the next two examples, we apply our approach to construct the first integrals and use these to recover the Sundman transformations of Muriel and Romero in [13]. In addition, we provide the two-parameter family of solution in the first example.

Example 2.5. Consider the ODE for the variable frequency oscillator [12]

$$y'' + yy'^2 = 0, \tag{2.12}$$

Theorem 1.2 shows that the coefficients of the equation satisfy $S_1 = 0, S_2 = 0$. By Theorem 2.1, ODE (2.12) is S-linearizable if and only if

$$g'' + g'^2 = 0 \tag{2.13}$$

for some auxiliary function $g(x)$. A particular solution of (2.13) is $g(x) = \ln x$ so by (2.2) we have $h = \frac{y^2}{2} + \ln x$ and hence using Theorem 2.3, we can get the first integral $w(x, y, y') = A(x, y)y' + B(x, y)$ where

$$A(x, y) = x \exp\left(\frac{y^2}{2}\right), \qquad B(x, y) = -\int \exp\left(\frac{y^2}{2}\right) dy.$$

Finally, the Sundman transformations can be constructed using Theorem 1.1 as follows

$$\psi(x, y) = \eta(I(x, y)), \qquad \phi(x, y) = \frac{1}{x^2}\eta'(I(x, y)),$$

where $I(x, y) = x^{-1} \int \exp\left(\frac{y^2}{2}\right) dy$.

Now choosing $\eta(I) = I$ makes $\phi(x, y) = \phi(x)$ and so using Remark 2.4 gives the two-parameter family of solutions of the ODE (2.12)

$$\operatorname{erfi}\left(\frac{y}{\sqrt{2}}\right) = C_1 x + C_2,$$

where $\operatorname{erfi}(y) = \frac{2}{\sqrt{\pi}} \int_0^y \mathrm{e}^{t^2} dt$ is the imaginary error function.

Example 2.6. Consider the equation

$$y'' - \left(\tan y + \frac{1}{y}\right) y'^2 + \left(\frac{1}{x} - \frac{\tan y}{xy}\right) y' - \frac{\tan y}{x^2} = 0. \tag{2.14}$$

Theorem 1.2 shows that the coefficients of this equation satisfy $S_1 \neq 0$ but $S_3 = S_4 = 0$. In addition, it follows from Theorem 2.1 that ODE (2.14) is S-linearizable if and only if

$$g'' + g'^2 - \left(\frac{1}{x} - \frac{\tan y}{xy}\right) g' = 0 \tag{2.15}$$

for some auxiliary function $g(x)$. The only solution of (2.15) is $g(x) = C$ so by (2.2) we have $h = \ln\left(\frac{\cos y}{y}\right)$ and hence using Theorem 2.3, we can get the first integral $w(x, y, y') = A(x, y)y' + B(x, y)$ where

$$A(x, y) = \frac{\cos y}{y}, \qquad B(x, y) = \frac{\sin y}{xy}.$$

Finally, the Sundman transformations can be constructed using Theorem 1.1 as follows

$$\psi(x, y) = \eta(I(x, y)), \qquad \phi(x, y) = xy\eta'(I(x, y)),$$

where $I(x, y) = x \sin y$.

One can show that there is no $\eta(I)$ which makes $\phi = \phi(x)$ and so using Remark 2.4 for $\eta(I) = I$ gives the two-parameter family of solution of the ODE (2.14)

$$x \sin y = c_1 + c_2 \mu(x),$$

where the function $t = \mu(x)$ is a solution of the equation

$$\frac{dt}{dx} = x \sin^{-1}\left(\frac{c_1 + c_2 t}{x}\right).$$

For example, if $c_2 = 0$, then one obtains the solution of ODE (2.14) as

$$x \sin y = c_1.$$

As another application, we solve geodesic equations for surfaces of revolution of constant curvature in a unified manner. Consider a surface of revolution with parameterization $(f(y)\cos x, f(y)\sin x, g(y))$ obtained by revolving the unit speed curve $(f(y), g(y))$. The geodesic equations are [17]

$$\ddot{y} = f(y)f'(y)\dot{x}^2, \qquad \frac{d}{dt}\big(f(y)^2\dot{x}\big) = 0,$$

where $\dot{y} = \frac{dy}{dt}$ and $\dot{x} = \frac{dx}{dt}$.

Using the formulas $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$ and $\frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$ gives

$$y'' - 2\frac{f'(y)}{f(y)} y'^2 - f'(y)f(y) = 0, \tag{2.16}$$

which as special case for $f(y) = \sin y$ includes the equation for geodesics on unit sphere given in [1, 18].

Example 2.7. We consider the nonlinear second-order ODE (2.16) for $f(y) = y$, $f(y) = b + y$, $f(y) = \sin y$ and $f(y) = \sinh y$ describing the geodesics on cone, plane, sphere and surface of conic type respectively.

For the surfaces under consideration we have $f'^2(y) - f(y)f''(y) = 1$. It can be checked from Theorem 1.2 that the coefficients of the equation satisfy $S_1 = 0$, $S_2 = 0$. In addition, it follows from Theorem 2.1 that ODE (2.16) for each of $f(y) = y$, $f(y) = b + y$, $f(y) = \sin y$ and $f(y) = \sinh y$ is S-linearizable if and only if

$$g'' + g'^2 + 1 = 0, \tag{2.17}$$

for some auxiliary function $g(x)$. A particular solution of (2.17) is $g(x) = \ln(\sin x)$, so by (2.2) we have $h = \ln\left(\frac{\sin x}{f^2(y)}\right)$ and hence using Theorem 2.3, we can get the first integral $w(x, y, y') = A(x, y)y' + B(x, y)$ where

$$A(x,y) = \frac{\sin x}{f^2(y)}, \qquad B(x,y) = \frac{f'(y)}{f(y)}\cos x.$$

Finally, the Sundman transformations can be constructed using Theorem 1.1 as follows

$$\psi(x,y) = \eta(I(x,y)), \qquad \phi(x,y) = \left(\frac{f(y)}{f'(y)}\right)^2 \eta'(I(x,y)),$$

where $I(x, y) = \frac{f(y)}{f'(y)}\sin x$.

Now choosing $\eta(I) = \frac{1}{I}$ makes $\phi(x, y) = \phi(x)$ and so using Remark 2.4 gives the two-parameter family of solution of the ODE (2.16)

$$c_1 f(y)\sin x + c_2 f(y)\cos x = f'(y).$$

Example 2.8. We consider the nonlinear second-order ODE

$$y'' - 2\tanh y y'^2 - \cosh y \sinh y = 0, \tag{2.18}$$

that describes the geodesics on hyperboloid of one sheet.

Theorem 1.2 shows that the coefficients of the equation satisfy $S_1 = 0$, $S_2 = 0$. It follows from Theorem 2.1 that ODE (2.18) is S-linearizable if and only if

$$g'' + g'^2 - 1 = 0, \tag{2.19}$$

for some auxiliary function $g(x)$. A particular solution of (2.19) is $g(x) = \ln(\sinh x)$, so by (2.2) we have $h = \ln\left(\frac{\sinh x}{\cosh^2 y}\right)$ and hence using Theorem 2.3, we can get the first integral $w(x, y, y') = A(x, y)y' + B(x, y)$ where

$$A(x,y) = \frac{\sinh x}{\cosh^2 y}, \qquad B(x,y) = -\frac{\sinh y}{\cosh y}\cosh x.$$

Finally, the Sundman transformations can be constructed using Theorem 1.1 as follows

$$\psi(x,y) = \eta(I(x,y)), \qquad \phi(x,y) = \operatorname{csch}^2 x \eta'(I(x,y)),$$

where $I(x, y) = \frac{\tanh y}{\sinh x}$.

Now choosing $\eta(I) = I$ makes $\phi(x, y) = \phi(x)$ and so using Remark 2.4 gives the two-parameter family of solution of the ODE (2.18)

$$c_1 \cosh y \sinh x - c_2 \cosh y \cosh x = \sinh y.$$

Example 2.9. We consider the nonlinear second-order ODE

$$y'' - 2y'^2 - e^{2y} = 0, \tag{2.20}$$

that describes the geodesics on pseudosphere.

Theorem 1.2 shows that the coefficients of the equation satisfy $S_1 = 0$, $S_2 = 0$. It follows from Theorem 2.1 that ODE (2.20) is S-linearizable if and only if

$$g'' + g'^2 = 0, \tag{2.21}$$

for some auxiliary function $g(x)$. A particular solution of (2.21) is $g(x) = \ln x$, so by (2.2) we have $h = \ln x - 2y$ and hence using Theorem 2.3, we can get the first integral $w(x, y, y') = A(x, y)y' + B(x, y)$ where

$$A(x, y) = xe^{-2y}, \qquad B(x, y) = \frac{1}{2}\left(e^{-2y} - x^2\right).$$

Finally, the Sundman transformations can be constructed using Theorem 1.1 as follows

$$\psi(x, y) = \eta(I(x, y)), \qquad \phi(x, y) = -\frac{2}{x^2}\eta'(I(x, y)),$$

where $I(x, y) = \frac{e^{-2y}}{x} + x$.

Now choosing $\eta(I) = I$ makes $\phi(x, y) = \phi(x)$ and so using Remark 2.4 gives the two-parameter family of solutions of the ODE (2.20)

$$e^{-2y} + x^2 = c_1x + 2c_2.$$

3 Conclusion

The recent Muriel–Romero characterization, Theorem 1.1, of the class of S-linearizable equations identifies these as the class of equations that admit first integrals of the form $A(x, y)y' + B(x, y)$. In this paper, a new characterization of S-linearizable equations in terms of the coefficients and one auxiliary function is given in Theorem 2.1. This criterion is used to directly provide explicit general solutions for the auxiliary functions A and B Theorem 2.3. So, using Theorem 1.1, the linearizing generalized Sundman transformations can be constructed by solving the first-order ODE (1.7). Finally, it is shown in [13] that an equation of the form (1.3) is S-linearizable and linearizable via a point transformation if and only if $S_1 = S_2 = 0$. It is also known that the generalized Sundman transformation is a point transformation if and only if $\phi = \phi(x)$. So, by Remark 2.4, the generalized Sundman transformation leads to the general solution $\psi(x, y) = c_1 + c_2 \int \phi(x)dx$ if and only if $S_1 = S_2 = 0$.

Our method is illustrated in examples where we recover the Sundman transformations of Muriel and Romero in [13]. Furthermore, the system of geodesic equations for surfaces of revolution is expressed as a single second-order ODE. It is noticed that this ODE is S-linearizable for surfaces of revolution with constant curvature. The method is applied to find the general solution of these geodesics in a unified manner.

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