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Covariant Fields of C^*-Algebras under Rieffel Deformation

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Abstract. We show that Rieffel's deformation sends covariant $\mathcal{C}(T)$-algebras into $\mathcal{C}(T)$-algebras. We also treat the lower semi-continuity issue, proving that Rieffel's deformation transforms covariant continuous fields of C^*-algebras into continuous fields of C^*-algebras. Some examples are indicated, including certain quantum groups.

Key words: pseudodifferential operator; Rieffel deformation; C^*-algebra; continuous field; noncommutative dynamical system

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1 Introduction

Let T be a locally compact topological space, always assumed to be Hausdorff. We denote by $\mathcal{C}(T)$ the Abelian C^*-algebra of all complex continuous functions on T that decay at infinity (are arbitrarily small outside large compact subsets). A $\mathcal{C}(T)$-algebra [4, 15, 24] is a C^*-algebra $\mathcal{B}$ together with a nondegenerated injective morphism from $\mathcal{C}(T)$ to the center of $\mathcal{B}$ (multipliers are used if $\mathcal{B}$ is not unital). The main role of the concept of $\mathcal{C}(T)$-algebra consists in translating in a simple and efficient way the idea that $\mathcal{B}$ is fibred in the sense of C^*-algebras over the base T [7, 23]. Actually $\mathcal{C}(T)$-algebras can be seen as upper-semi-continuous fields of C^*-algebras over the base T. Lower-semi-continuity can also be put in this setting if one also uses the space of all primitive ideals [13, 15, 18, 22, 24]; we refer to [3] for deeper results involving semi-continuity. We intend to put these concepts in the perspective of Rieffel quantization.

Rieffel's calculus [21, 22] is a machine that applies to C^*-dynamical systems and their morphisms. The necessary ingredients are an action of the vector group $\Xi := \mathbb{R}^d$ by automorphisms of a C^*-algebra as well as a skew symmetric linear operator of Ξ. When morphisms are involved, they are always assumed to intertwine the existing actions.

Rieffel's machine is actually meant to be a quantization. The initial data define a natural Poisson structure, regarded as a mathematical modelization of the observables of a classical physical system. After applying the machine to this classical data one gets a deformed C^*-algebra seen as the family of observables of the same system, but written in the language of quantum mechanics. By varying a convenient parameter (Planck's constant $\hbar$) one can recover the Poisson structure (at $\hbar = 0$) from the C^*-algebras defined at $\hbar \neq 0$ in a way that satisfies certain natural axioms [12, 21, 22].

The spirit of this deformation procedure is that of a pseudodifferential theory [8]. At least in simple situations the multiplication in the initial C^*-algebra is just point-wise multiplication of functions defined on some locally compact topological space, on which Ξ acts by homeomorphisms. The noncommutative product in the quantized algebra can be interpreted as a symbol composition of a pseudodifferential type. Actually the concrete formulae generalize and are motivated by the usual Weyl calculus.

In a setting where all the relevant concepts make sense, we prove in Theorem 4.3 and Proposition 4.4 their compatibility: by Rieffel quantization an upper-semi-continuous fields of C^*-algebras is turned into an upper-semi-continuous fields of C^*-algebras with fibers which are easy to identify; the proof uses $\mathcal{C}(T)$-algebras. Finally, using primitive ideals techniques, we show the analog of this result for lower-semi-continuity; the key technical result is Proposition 5.1. Putting everything together one gets

Theorem 1.1. *Rieffel quantization transforms covariant continuous fields of C^*-algebras into covariant continuous fields of C^*-algebras.*

We illustrate the result by some examples in Section 6. Most of them involve an Abelian initial algebra $\mathcal{A}$. In this case the information is encoded in a topological dynamical system with locally compact space Σ and the upper-semi-continuous field property can be read in the existence of a continuous covariant surjection $q : \Sigma \to T$; if this one is open, then lower-semi-continuity also holds. If the orbit space of the dynamical system is Hausdorff, it serves as a good space T over which the Rieffel deformed algebra can be decomposed, with easily identified fibers. This can be used to show that the C^*-algebras of some compact quantum groups constructed in [19] can be written as continuous fields, some of the fibers being isomorphic to certain noncommutative tori.

Eventual connections of the present work to the model of quantum spacetime presented in [6] and [17] will be investigated elsewhere, hopefully.

After the present work was completed, we learnt of the recent articles [9, 10] in which a statement comparable with our Theorem 4.3 is included (essentially without proof) with interesting applications. These papers reply on Kasprzak's reformulation [11] of the Rieffel deformation (based on Landstad's characterization of crossed products). However, in [9, 10] there is no comment about how nondegenerecy is proved, and this is crucial in the definition of a $\mathcal{C}(T)$-algebra (cf. Definition 3.1). It is easy to give counterexamples showing how important this condition is for the theory. Moreover, for us this was the main technical problem in proving Theorem 4.3; we solved it using the highly nontrivial Dixmier–Malliavin theorem [5]. Our detailed proof based on the more traditional approach is intended to support future work on spectral theory of pseudodifferential operators (see also [1, 14]), as well as applications to quantum mechanics. The lower-semi-continuity part (contained in Section 4), often described as difficult, is not treated in [9, 10] and seems not to have correspondence in the literature.

Let us mention that a different proof of the main result (Theorem 1.1) has been given in [1], where this result is needed as an important ingredient in proving spectral continuity properties of quantum Hamiltonians defined as phase-space anisotropic pseudodifferential operators. That paper has been submitted after the completion of the present article and refers to it explicitly. In [1] the proof relies on a recently found connection between Rieffel deformation and twisted crossed products [2] and on the deep analysis from [20], therefore being less direct or self-contained than the present one.

2 Rieffel's pseudodifferential calculus: a short review

We start by describing briefly Rieffel deformation [21, 22] in a slightly restricted setting. The initial object, containing *the classical data*, is a quadruplet $(\mathcal{A}, \Theta, \Xi, [\![\cdot,\cdot]\!])$. The pair $(\Xi, [\![\cdot,\cdot]\!])$ will be taken to be a $2n$-dimensional symplectic vector space. On the other hand $(\mathcal{A}, \Theta, \Xi)$ is a C^*-*dynamical system*, meaning that the vector group acts strongly continuously by automorphisms of the (possibly noncommutative) C^*-algebra $\mathcal{A}$. Let us denote by $\mathcal{A}^\infty$ the family of elements f such that the mapping $\Xi \ni X \mapsto \Theta_X(f) \in \mathcal{A}$ is C^∞. It is a dense *-algebra of $\mathcal{A}$ and also a Fréchet algebra with the family of semi-norms

$$\|f\|_{\mathcal{A}}^{(k)} := \sum_{|\alpha|\le k} \frac{1}{|\alpha|!} \|\partial_X^\alpha \left[\Theta_X(f)\right]_{X=0}\|_{\mathcal{A}} \equiv \sum_{|\alpha|\le k} \frac{1}{|\alpha|!} \|\delta^\alpha(f)\|_{\mathcal{A}}, \qquad k \in \mathbb{N}. \tag{1}$$

To quantize the above structure, one keeps the involution unchanged but introduces on $\mathcal{A}^\infty$ the product

$$f \# g := \pi^{-2n} \int_\Xi \int_\Xi dY\, dZ\, e^{2i[\![Y,Z]\!]} \Theta_Y(f)\Theta_Z(g),$$

suitably defined by oscillatory integral techniques [8, 21]. One gets a *-algebra $(\mathcal{A}^\infty, \#, {}^*)$, which admits a C^*-completion $\mathfrak{A}$ in a C^*-norm $\|\cdot\|_{\mathfrak{A}}$ defined by Hilbert module techniques [21]. The action Θ leaves $\mathcal{A}^\infty$ invariant and extends to a strongly continuous action of the C^*-algebra $\mathfrak{A}$, that will also be denoted by Θ. The space $\mathfrak{A}^\infty$ of C^∞-vectors coincides with $\mathcal{A}^\infty$ and it is a Fréchet space with semi-norms

$$\|f\|_{\mathfrak{A}}^{(k)} := \sum_{|\alpha|\le k} \frac{1}{|\alpha|!} \|\partial_X^\alpha \left[\Theta_X(f)\right]_{X=0}\|_{\mathfrak{A}} \equiv \sum_{|\alpha|\le k} \frac{1}{|\alpha|!} \|\delta^\alpha(f)\|_{\mathfrak{A}}, \qquad k \in \mathbb{N}. \tag{2}$$

By Proposition 4.10 in [21], there exist $k \in \mathbb{N}$ and $C_k > 0$ such that $\|f\|_{\mathfrak{A}} \le C_k \|f\|_{\mathcal{A}}^{(k)}$ for any $f \in \mathcal{A}^\infty = \mathfrak{A}^\infty$. Replacing here f by $\delta^\alpha f$ for every multi-index α, it follows that on $\mathcal{A}^\infty$ the topology given by the semi-norms (1) is finer than the one given by the semi-norms (2). As a consequence of Theorem 7.5 in [21], the role of the C^*-algebras $\mathcal{A}$ and $\mathfrak{A}$ can be reversed: one obtains $\mathcal{A}$ as the deformation of $\mathfrak{A}$ by replacing the skew-symmetric form $[\![\cdot,\cdot]\!]$ by $-[\![\cdot,\cdot]\!]$. Thus $\mathcal{A}^\infty$ *and* $\mathfrak{A}^\infty$ *coincide as Fréchet spaces.*

The deformation transfers to Ξ-morphisms. Let $(\mathcal{A}_j, \Theta_j, \Xi, [\![\cdot,\cdot]\!])$, $j = 1, 2$, be two classical data and let $\mathcal{R} : \mathcal{A}_1 \to \mathcal{A}_2$ be a Ξ-morphism, i.e. a C^*-morphism intertwining the two actions Θ_1, Θ_2. Then $\mathcal{R}$ sends $\mathcal{A}_1^\infty$ into $\mathcal{A}_2^\infty$ and extends to a morphism $\mathfrak{R} : \mathfrak{A}_1 \to \mathfrak{A}_2$ that also intertwines the corresponding actions. In this way, one obtains a covariant functor. The functor is exact [21]: it preserves short exact sequences of Ξ-morphisms. Namely, if $\mathcal{J}$ is a (closed, self-adjoint, two-sided) ideal in $\mathcal{A}$ that is invariant under Θ, then its deformation $\mathfrak{J}$ can be identified with an invariant ideal in $\mathfrak{A}$ and the quotient $\mathfrak{A}/\mathfrak{J}$ is canonically isomorphic to the deformation of the quotient $\mathcal{A}/\mathcal{J}$ under the natural quotient action.

We will refer to *the Abelian case* under the following circumstances: A continuous action Θ of Ξ by homeomorphisms of the locally compact Hausdorff space Σ is given. For $(\sigma, X) \in \Sigma \times \Xi$ we are going to use all the notations $\Theta(\sigma, X) = \Theta_X(\sigma) = \Theta_\sigma(X) \in \Sigma$ for the X-transformed of the point σ. The function Θ is continuous and the homeomorphisms Θ_X, Θ_Y satisfy $\Theta_X \circ \Theta_Y = \Theta_{X+Y}$ for every $X, Y \in \Xi$.

We denote by $\mathcal{C}(\Sigma)$ the Abelian C^*-algebra of all complex continuous functions on Σ that are arbitrarily small outside large compact subsets of Σ. When Σ is compact, $\mathcal{C}(\Sigma)$ is unital. The action Θ of Ξ on Σ induces an action of Ξ on $\mathcal{C}(\Sigma)$ (also denoted by Θ) given by $\Theta_X(f) := f \circ \Theta_X$. This action is strongly continuous, i.e. for any $f \in \mathcal{C}(\Sigma)$ the mapping

$$\Xi \ni X \mapsto \Theta_X(f) \in \mathcal{C}(\Sigma) \tag{3}$$

is continuous; thus we are placed in the setting presented above. We denote by $\mathcal{C}(\Sigma)^\infty \equiv \mathcal{C}^\infty(\Sigma)$ the set of elements $f \in \mathcal{C}(\Sigma)$ such that the mapping (3) is C^∞; it is a dense *-algebra of $\mathcal{C}(\Sigma)$. The general theory supplies a C^*-algebra $\mathfrak{A} \equiv \mathfrak{C}(\Sigma)$ (noncommutative in general), acted continuously by the group Ξ, with smooth vectors $\mathfrak{C}^\infty(\Sigma) = \mathcal{C}^\infty(\Sigma)$.

3 Families of C^*-algebras

Now we give a short review of $\mathcal{C}(T)$-algebras and semi-continuous fields of C^*-algebras (see [4, 12, 13, 15, 20, 24] and references therein), outlining the connection between the two notions.

If $\mathcal{B}$ is a C^*-algebra, we denote by $\mathcal{M}(\mathcal{B})$ its multiplier algebra and by $\mathcal{ZM}(\mathcal{B})$ its center. If $\mathcal{B}_1$, $\mathcal{B}_2$ are two vector subspaces of $\mathcal{M}(\mathcal{B})$, we set $\mathcal{B}_1 \cdot \mathcal{B}_2$ for the vector subspace generated by $\{b_1 b_2 \mid b_1 \in \mathcal{B}_1, b_2 \in \mathcal{B}_2\}$. We are going to denote by $\mathcal{C}(T)$ the C^*-algebra of all complex continuous functions on the (Hausdorff) locally compact space T that decay at infinity.

Definition 3.1. We say that $\mathcal{B}$ is a $\mathcal{C}(T)$*-algebra* if a nondegenerate monomorphism $\mathcal{Q} : \mathcal{C}(T) \to \mathcal{ZM}(\mathcal{B})$ is given.

We recall that nondegeneracy means that the ideal $\mathcal{Q}[\mathcal{C}(T)] \cdot \mathcal{B}$ is dense in $\mathcal{B}$.

Definition 3.2. By *upper-semi-continuous field of C^*-algebras* we mean a family of epimorphisms of C^*-algebras $\left\{\mathcal{B} \xrightarrow{\mathcal{P}(t)} \mathcal{B}(t) \mid t \in T\right\}$ indexed by a locally compact topological space T and satisfying:

1. For every $b \in \mathcal{B}$ one has $\|b\|_{\mathcal{B}} = \sup_{t \in T} \|\mathcal{P}(t)b\|_{\mathcal{B}(t)}$.
2. For every $b \in \mathcal{B}$ the map $T \ni t \mapsto \|\mathcal{P}(t)b\|_{\mathcal{B}(t)}$ is upper-semi-continuous and decays at infinity.
3. There is a multiplication $\mathcal{C}(T) \times \mathcal{B} \ni (\varphi, b) \to \varphi * b \in \mathcal{B}$ such that

$$\mathcal{P}(t)[\varphi * b] = \varphi(t)\mathcal{P}(t)b, \qquad \forall\, t \in T, \quad \varphi \in \mathcal{C}(T), \quad b \in \mathcal{B}.$$

If, in addition, the map $t \mapsto \|\mathcal{P}(t)b\|$ is continuous for every $b \in \mathcal{B}$, we say that

$$\left\{\mathcal{B} \xrightarrow{\mathcal{P}(t)} \mathcal{B}(t) \mid t \in T\right\}$$

is *a continuous field of C^*-algebras.*

The requirement 2 is clearly equivalent with the condition that for every $b \in \mathcal{B}$ and every $\epsilon > 0$ the subset $\{t \in T \mid \|\mathcal{P}(t)b\|_{\mathcal{B}(t)} \geq \epsilon\}$ is compact. One can rephrase 1 as $\cap_t \ker[\mathcal{P}(t)] = \{0\}$, so one can identify $\mathcal{B}$ with a C^*-algebra of sections of the field; this make the connection with other approaches, as that of [15] for example. It will always be assumed that $\mathcal{B}(t) \neq \{0\}$ for all $t \in T$.

We are going to describe briefly in which way the two definitions above are actually equivalent.

First let us assume that $\mathcal{B}$ is a $\mathcal{C}(T)$-algebra and denote by $\mathcal{C}_t(T)$ the ideal of all the functions in $\mathcal{C}(T)$ vanishing at the point $t \in T$. We get ideals $\mathcal{I}(t) := \overline{\mathcal{Q}\left[\mathcal{C}_t(T)\right] \cdot \mathcal{B}}$ in $\mathcal{B}$, quotients $\mathcal{B}(t) := \mathcal{B}/\mathcal{I}(t)$ as well as canonical epimorphisms $\mathcal{P}(t) : \mathcal{B} \to \mathcal{B}(t)$. One also sets

$$\varphi * b := \mathcal{Q}(\varphi)b, \qquad \forall\, \varphi \in \mathcal{C}(T), \quad b \in \mathcal{B}.$$

Then $\left\{\mathcal{B} \xrightarrow{\mathcal{P}(t)} \mathcal{B}(t) \mid t \in T\right\}$ is an upper-semi-continuous field of C^*-algebras with multiplication $*$.

Conversely, if an upper-semi-continuous field $\left\{\mathcal{B} \xrightarrow{\mathcal{P}(t)} \mathcal{B}(t) \mid t \in T\right\}$ is given, also involving the multiplication $*$, we set

$$\mathcal{Q} : \ \mathcal{C}(T) \to \mathcal{ZM}(\mathcal{B}), \qquad \mathcal{Q}(\varphi)b := \varphi * b.$$

In this way one gets a $\mathcal{C}(T)$-algebra and each of the quotients $\mathcal{B}/\mathcal{I}(t)$ is isomorphic to the fiber $\mathcal{B}(t)$.

To discuss lower semi-continuity we need $\mathrm{Prim}(\mathcal{B})$, the space of all the primitive ideals (kernels of irreducible representations) of $\mathcal{B}$. The hull-kernel topology turns $\mathrm{Prim}(\mathcal{B})$ into a locally compact (not necessarily Hausdorff) topological space. We recall that *the hull application* $\mathcal{J} \mapsto h(\mathcal{J}) := \{\mathcal{K} \in \mathrm{Prim}(\mathcal{B}) \mid \mathcal{J} \subset \mathcal{K}\}$ realizes a containment reversing bijection between the family of ideals of $\mathcal{B}$ and the family of closed subsets of $\mathrm{Prim}(\mathcal{B})$. Its inverse is *the kernel map* $\Omega \mapsto k(\Omega) := \cap_{\mathcal{K}\in\Omega}\mathcal{K}$, which is also decreasing.

The Dauns–Hoffman theorem establishes the existence of a unique isomorphism

$$\Gamma : \ BC[\mathrm{Prim}(\mathcal{B})] \to \mathcal{ZM}(\mathcal{B}),$$

where $BC[\mathrm{Prim}(\mathcal{B})]$ is the C^*-algebra of bounded and continuous functions over $\mathrm{Prim}(\mathcal{B})$, such that for each $\mathcal{K} \in \mathrm{Prim}(\mathcal{B})$, $\Psi \in BC[\mathrm{Prim}(\mathcal{B})]$ and $b \in \mathcal{B}$ we have $\Gamma(\Psi)b + \mathcal{K} = \Psi(\mathcal{K})b + \mathcal{K}$. For a detailed study of the space $\mathrm{Prim}(\mathcal{B})$ and a proof of the Dauns–Hoffman theorem, cf. Sections A.2 and A.3 in [18]. Let us suppose that there is a continuous surjective function $q : \mathrm{Prim}(\mathcal{B}) \to T$. Then we can define $\mathcal{Q} : \mathcal{C}(T) \to \mathcal{ZM}(\mathcal{B})$ by $\mathcal{Q}(\varphi) = \Gamma(\varphi \circ q)$ and one can check that $\mathcal{Q}$ endows $\mathcal{B}$ with the structure of a $\mathcal{C}(T)$-algebra.

On the other hand, if we have a nondegenerate monomorphism $\mathcal{Q} : \mathcal{C}(T) \to \mathcal{ZM}(\mathcal{B})$, we can define canonically a continuous map $q : \mathrm{Prim}(\mathcal{B}) \to T$. One has $q(\mathcal{K}) = t$ if and only if $\mathcal{I}(t) \subset \mathcal{K}$, and we can recover $\mathcal{Q}$ from the above construction. Moreover *the map* $T \ni t \to \|b(t)\|_{\mathcal{B}(t)} \in \mathbb{R}_+$ *is continuous for every* $b \in \mathcal{B}$ (*so we have a continuous field of* C^**-algebras*) *if and only if* q *is open*. For the proof of this facts see Propositions C.5 and C.10 in [24].

4 Covariant $\mathcal{C}(T)$-algebras and upper-semi-continuity under Rieffel quantization

Let T be a locally compact Hausdorff space and $(\mathcal{A}, \Theta, \Xi, [\![\cdot,\cdot]\!])$ a classical data. The canonical C^*-dynamical system defined by Rieffel quantization is $(\mathfrak{A}, \Theta, \Xi)$.

Definition 4.1. We say that $\mathcal{A}$ is a *covariant* $\mathcal{C}(T)$*-algebra with respect to the action* Θ if a nondegenerate monomorphism $\mathcal{Q} : \mathcal{C}(T) \to \mathcal{ZM}(\mathcal{A})$ is given (so it is a $\mathcal{C}(T)$-algebra) and in addition one has

$$\Theta_X[\mathcal{Q}(\varphi)f] = \mathcal{Q}(\varphi)\left[\Theta_X(f)\right], \qquad \forall\, f \in \mathcal{A}, \quad X \in \Xi, \quad \varphi \in \mathcal{C}(T). \tag{4}$$

We intend to prove that the Rieffel quantization transforms covariant $\mathcal{C}(T)$-algebras into covariant $\mathcal{C}(T)$-algebras. For this and for a further result identifying the emerging quotient algebras, we are going to need

Lemma 4.2. *Let* I *be an ideal of* $\mathcal{C}(T)$ *and denote by* $\overline{\mathcal{Q}(I)\cdot\mathcal{A}}$ *the closure of* $\mathcal{Q}(I)\cdot\mathcal{A}$ *in the* C^**-algebra* $\mathcal{A}$*. Then* $\mathcal{Q}(I)\cdot\mathcal{A}^\infty$ *is dense in* $\left(\overline{\mathcal{Q}(I)\cdot\mathcal{A}}\right)^\infty \equiv \left(\overline{\mathcal{Q}(I)\cdot\mathcal{A}}\right)\cap\mathcal{A}^\infty$ *for the Fréchet topology inherited from* $\mathcal{A}^\infty$*.*

Proof. By the covariance condition $\overline{\mathcal{Q}(I)\cdot\mathcal{A}}$ is an invariant ideal of $\mathcal{A}$.

The proof uses regularization. Consider *the integrated form of* Θ, i.e. for each $\Phi \in C_c^\infty(\Xi)$ (compactly supported smooth function) and $g \in \mathcal{A}$ define

$$\Theta_\Phi(g) = \int_\Xi dY\,\Phi(Y)\Theta_Y(g).$$

Note that for every $X \in \Xi$ one has

$$\Theta_X\left[\Theta_\Phi(g)\right] = \int_\Xi dY\,\Phi(Y - X)\Theta_Y(g).$$

Then $\Theta_\Phi(g) \in \mathcal{A}^\infty$ and for each multi-index μ we have

$$\delta^\mu\left[\Theta_\Phi(g)\right] = (-1)^{|\mu|}\Theta_{\partial^\mu\Phi}(g) \qquad \text{and} \qquad \|\delta^\mu\left[\Theta_\Phi(g)\right]\|_{\mathcal{A}} \le \|\partial^\mu\Phi\|_{L^1(\Xi)}\|g\|_{\mathcal{A}}.$$

One of the deepest theorems about smooth algebras, the Dixmier–Malliavin theorem [5], say that $\mathcal{A}^\infty$ is generated (algebraically) by the set of all the elements of the form $\Theta_\Phi(g)$ with $\Phi \in C_c^\infty(\Xi)$ and $g \in \mathcal{A}$. Replacing $\mathcal{A}$ with $\overline{\mathcal{Q}(I)\cdot\mathcal{A}}$, for $f \in \left(\overline{\mathcal{Q}(I)\cdot\mathcal{A}}\right)^\infty$ there exist $\Phi_1,\dots,\Phi_m \in C_c^\infty(\Xi)$ and $f_1,\dots,f_m \in \overline{\mathcal{Q}(I)\cdot\mathcal{A}}$ such that $f = \sum_{i=1}^m \Theta_{\Phi_i}(f_i)$. Let $\epsilon > 0$ and fix a multi-index α. Choose $g_1,\dots,g_m \in \mathcal{Q}(I)\cdot\mathcal{A}$ such that for each i

$$\|f_i - g_i\|_{\mathcal{A}} \le \frac{\epsilon}{m\|\partial^\alpha\Phi_i\|_{L^1(\Xi)}}.$$

Then

$$\left\|\delta^\alpha\left(f - \sum_{i=1}^m \Theta_{\Phi_i}(g_i)\right)\right\|_{\mathcal{A}} = \left\|\sum_{i=1}^m \Theta_{\partial^\alpha\Phi_i}(f_i - g_i)\right\|_{\mathcal{A}} \le \sum_{i=1}^m \|\partial^\alpha\Phi_i\|_{L^1(\Xi)}\|f_i - g_i\|_{\mathcal{A}} \le \epsilon.$$

Thus we only need to prove that for each $\Phi \in C_c^\infty(\Xi)$ and $g \in \mathcal{Q}(I)\cdot\mathcal{A}$ the element $\Theta_\Phi(g)$ belongs to $\mathcal{Q}(I)\cdot\mathcal{A}^\infty$. Let $\varphi_1,\dots,\varphi_j \in I$ and $h_1,\dots,h_j \in \mathcal{A}$ such that $g = \sum_{i=1}^j \mathcal{Q}(\varphi_i)h_i$. Then

$$\Theta_\Phi(g) = \sum_{i=1}^j \Theta_\Phi[\mathcal{Q}(\varphi_i)h_i],$$

and by covariance, for each index i one has

$$\Theta_\Phi\left[\mathcal{Q}(\varphi_i)h_i\right] = \int_\Xi dY\,\Phi(Y)\mathcal{Q}(\varphi_i)\Theta_X(h_i) = \mathcal{Q}(\varphi_i)\left[\Theta_\Phi(h_i)\right] \in \mathcal{Q}(I)\cdot\mathcal{A}^\infty. \qquad \blacksquare$$

Theorem 4.3. *Rieffel quantization transforms covariant $\mathcal{C}(T)$-algebras into covariant $\mathcal{C}(T)$-algebras: there exists a nondegenerate monomorphism $\mathfrak{Q} : \mathcal{C}(T) \to \mathcal{ZM}(\mathfrak{A})$ satisfying for all $\varphi \in \mathcal{C}(T)$, $f \in \mathcal{A}$ and $X \in \Xi$ the covariance relation $\Theta_X[\mathfrak{Q}(\varphi)f] = \mathfrak{Q}(\varphi)[\Theta_X(f)]$.*

Proof. The action Θ of Ξ on $\mathcal{A}$ extends canonically to an action by automorphisms of the multiplier algebra $\mathcal{M}(\mathcal{A})$, also denoted by Θ, which is not strongly continuous in general. But, tautologically, it restricts to a strongly continuous action $\Theta : \Xi \to \mathrm{Aut}[\mathcal{M}_0(\mathcal{A})]$ on the C^*-subalgebra

$$\mathcal{M}_0(\mathcal{A}) := \{m \in \mathcal{M}(\mathcal{A}) \mid \Xi \ni X \mapsto \Theta_X(m) \in \mathcal{M}(\mathcal{A}) \text{ is norm continuous}\}.$$

In these terms, the covariance condition on $\mathcal{Q}$ says simply that for any $\varphi \in \mathcal{C}(T)$ the multiplier $\mathcal{Q}(\varphi)$ is a fixed point for all the automorphisms Θ_X (take $f = 1$ in (4)). As a very weak consequence one has $\mathcal{Q}[\mathcal{C}(T)] \subset \mathcal{M}_0(\mathcal{A})^\infty$, with an obvious notation for the smooth vectors.

Proposition 5.10 from [21] applied to the unital C^*-algebra $\mathcal{M}_0(\mathcal{A})$ says that the Rieffel quantization of $\mathcal{M}_0(\mathcal{A})$ is a C^*-subalgebra of $\mathcal{M}(\mathfrak{A})$. Consequently one has $\mathcal{Q}[\mathcal{C}(T)] \subset \mathcal{M}_0(\mathcal{A})^\infty \subset \mathcal{M}(\mathfrak{A})$ and this supplies a candidate $\mathfrak{Q} : \mathcal{C}(T) \to \mathcal{M}(\mathfrak{A})$. This is obviously an injective map and the range is only composed of fixed points, which insures covariance.

Let us set for a moment $\mathcal{M} := \mathcal{M}_0(\mathcal{A})$, with multiplication $\cdot$, and denote by $\mathfrak{M} \subset \mathcal{M}(\mathfrak{A})$ its Rieffel quantization, with multiplication legitimately denoted by $\#$. For smooth elements $m, n \in \mathcal{M}^\infty = \mathfrak{M}^\infty$, one of them being a fixed point central in $\mathcal{M}$, one has $m\#n = m\cdot n = n\cdot m = n\#m$ (Corollary 2.13 in [21]). Thus the mapping $\mathfrak{Q}$ is again a monomorphism and its range is contained in $\mathcal{ZM}$. A density argument with respect to the strict topology implies that every $\mathfrak{Q}(\varphi)$ commutes with all the elements of $\mathcal{M}(\mathfrak{A})$, thus $\mathfrak{Q}[\mathcal{C}(T)] \subset \mathcal{ZM}(\mathfrak{A})$as required.

Now we only need to show nondegeneracy, i.e. the fact that $\mathfrak{Q}[\mathcal{C}(T)] \cdot \mathfrak{A}$ is dense in $\mathfrak{A}$. We show the even stronger assertion that $\mathcal{Q}[\mathcal{C}(T)] \cdot \mathcal{A}^\infty = \mathfrak{Q}[\mathcal{C}(T)] \cdot \mathfrak{A}^\infty$ is dense in $\mathfrak{A}$. This would follow if we knew that $\mathcal{Q}[\mathcal{C}(T)] \cdot \mathcal{A}^\infty$ is dense in $\mathfrak{A}^\infty$ with respect to its Fréchet topology given by the semi-norms (2); then we use denseness of $\mathfrak{A}^\infty$ in the weaker C^*-norm topology of $\mathfrak{A}$.

We recall from Section 2 that $\mathcal{A}^\infty$ and $\mathfrak{A}^\infty$ coincide even as Fréchet spaces. Therefore one is reduced to showing that $\mathcal{Q}[\mathcal{C}(T)] \cdot \mathcal{A}^\infty$ is dense in $\mathcal{A}^\infty$ for its Fréchet topology. Taking $\mathcal{I} = \mathcal{C}(T)$ in Lemma 4.2, we find out that $\mathcal{Q}[\mathcal{C}(T)] \cdot \mathcal{A}^\infty$ is dense in $\big(\overline{\mathcal{Q}[\mathcal{C}(T)] \cdot \mathcal{A}}\big) \cap \mathcal{A}^\infty$, which equals $\mathcal{A}^\infty$ since $\mathcal{Q}$ has been assumed nondegenerate. This finishes the proof. ∎

If $\mathcal{A}$ is a covariant $\mathcal{C}(T)$-algebra, then $\mathcal{I}(t) := \overline{\mathcal{Q}\left[\mathcal{C}_t(T)\right] \cdot \mathcal{A}}$ is an invariant ideal of $\mathcal{A}$. We can apply Rieffel quantization to $\mathcal{I}(t)$, to $\mathcal{A}(t) := \mathcal{A}/\mathcal{I}(t)$ (with the obvious actions of Ξ) and to the projection $\mathcal{P}(t) : \mathcal{A} \to \mathcal{A}(t)$. One gets C^*-algebras $\mathfrak{I}_t$, $\mathfrak{A}_t$ as well as the morphism $\mathfrak{P}_t : \mathfrak{A} \to \mathfrak{A}_t$. By Theorem 7.7 at [21] the kernel of $\mathfrak{P}_t$ is $\mathfrak{I}_t$, so $\mathfrak{A}_t$ can be identified to the quotient $\mathfrak{A}/\mathfrak{I}_t$.

On the other hand, by using the $\mathcal{C}(T)$-structure of the C^*-algebra $\mathfrak{A}$ given by Theorem 4.3, we have ideals $\mathfrak{I}(t) := \overline{\mathfrak{Q}\left[\mathcal{C}_t(T)\right] \cdot \mathfrak{A}}$ in $\mathfrak{A}$ as well as quotients $\mathfrak{A}(t) := \mathfrak{A}/\mathfrak{I}(t)$ to which we associates projections $\mathfrak{A} \xrightarrow{\mathfrak{P}(t)} \mathfrak{A}(t)$. However, one gets

Proposition 4.4. *With notation as above, for each $t \in T$ we have $\mathfrak{I}(t) = \mathfrak{I}_t$. In particular, the fibers $\mathfrak{A}(t) = \mathfrak{A}/\mathfrak{I}(t)$ of the $\mathcal{C}(T)$-algebra $\mathfrak{A}$ are isomorphic to the Rieffel quantization $\mathfrak{A}_t$ of the fibers $\mathcal{A}(t) = \mathcal{A}/\mathcal{I}(t)$ of $\mathcal{A}$ and for each $f \in \mathfrak{A}$ the mapping $t \mapsto \|\mathfrak{P}(t)f\|_{\mathfrak{A}(t)} = \|\mathfrak{P}_t f\|_{\mathfrak{A}_t}$ is upper-semi-continuous.*

Proof. We recall that $\mathcal{I}(t)^\infty$ and $\mathfrak{I}(t)^\infty$ coincide as Fréchet spaces. By Lemma 4.2, $\mathfrak{Q}\left[\mathcal{C}_t(T)\right] \cdot \mathfrak{A}^\infty$ is dense in $\mathfrak{I}(t)^\infty$, thus in $\mathfrak{I}(t)$, and $\mathcal{Q}\left[\mathcal{C}_t(T)\right] \cdot \mathcal{A}^\infty$ is dense in $\mathcal{I}(t)^\infty = \mathfrak{I}(t)^\infty$, thus also dense in $\mathfrak{I}_t$.

By construction one has $\mathfrak{Q}\left[\mathcal{C}_t(T)\right] \cdot \mathfrak{A}^\infty = \mathcal{Q}\left[\mathcal{C}_t(T)\right] \cdot \mathcal{A}^\infty$; consequently $\mathfrak{I}(t) = \mathfrak{I}_t$ for every $t \in T$ and the proof is finished. ∎

Remark 4.5. We are going to say that $\left\{\mathcal{A} \xrightarrow{\mathcal{P}(t)} \mathcal{A}(t) \mid t \in T\right\}$ and $\left\{\mathfrak{A} \xrightarrow{\mathfrak{P}(t)} \mathfrak{A}(t) \mid t \in T\right\}$ are *covariant upper-semi-continuous fields of C^*-algebras*. The intrinsic definition, in the first case for instance, would be the following: $\left\{\mathcal{A} \xrightarrow{\mathcal{P}(t)} \mathcal{A}(t) \mid t \in T\right\}$ is required to be an upper semi-continuous field of C^*-algebras and we also ask the action Θ to leave invariant all the ideals $\mathcal{I}(t) = \ker[\mathcal{P}(t)]$. It is easily seen that this is equivalent to requiring the covariance of the associated $\mathcal{C}(T)$-structure. This makes the connection with Definition 3.1 in [20].

For section C^*-algebras of an upper-semi-continuous field it is known [24] that each irreducible representation factorizes through one of the fibers. Therefore we get

Corollary 4.6. *Let $(\mathcal{A}, \Theta, \Xi, [\![\cdot,\cdot]\!])$ be a classical data and assume that $\mathcal{A}$ is a Θ-covariant $\mathcal{C}(T)$-algebra with respect to a Hausdorff locally compact space T, with fibers $\{\mathcal{A}(t) \mid t \in T\}$. Denote, respectively, by $\mathfrak{A}$ and $\mathfrak{A}(t)$ the corresponding quantized C^*-algebras. Then any irreducible representation of $\mathfrak{A}$ factorizes through one of the algebras $\mathfrak{A}(t)$.*

The $\mathcal{C}(T)$-structure $\mathfrak{Q}$ of $\mathfrak{A}$, given by Theorem 4.3, defines canonically the map $\mathfrak{q}:\text{Prim}(\mathfrak{A})\to T$, as explained at the end of Section 3. If $\pi : \mathfrak{A} \to \mathbb{B}(\mathcal{H})$ is an irreducible Hilbert space representation of $\mathfrak{A}$, then the point t in Corollary 4.6 is $\mathfrak{q}[\ker(\pi)]$.

5 Lower-semi-continuity under Rieffel quantization

We keep the previous setting and inquire now if lower-semi-continuity of the mappings $t \mapsto \|\mathcal{P}(t)f\|_{\mathcal{A}(t)}$ for all $f \in \mathcal{A}$ implies lower-semi-continuity of the mappings $t \mapsto \|\mathfrak{P}(t)f\|_{\mathfrak{A}(t)}$ for all $f \in \mathfrak{A}$. We start by noticing that $\mathrm{Prim}(\mathcal{A})$ and $\mathrm{Prim}(\mathfrak{A})$ are canonically endowed with continuous actions of the group Ξ; once again these actions will be denoted by Θ. By the discussion at the end of Section 3 we are left with proving

Proposition 5.1. *Suppose that $\mathcal{Q} : \mathcal{C}(T) \to \mathcal{ZM}(\mathcal{A})$ is a covariant $\mathcal{C}(T)$-algebra structure on $\mathcal{A}$ and that the associated function $q : \mathrm{Prim}(\mathcal{A}) \to T$ is open. Then the function $\mathfrak{q} : \mathrm{Prim}(\mathfrak{A}) \to T$ associated to $\mathfrak{Q} : \mathcal{C}(T) \to \mathcal{Z}(\mathfrak{A})$ is also open.*

Proof. We remark first that q is Θ-covariant (Lemma 8.1 in [24]), i.e. one has $q \circ \Theta_X = q$ for every $X \in \Xi$. Consequently, if $\mathcal{O} \subset \mathrm{Prim}(\mathcal{A})$ is an open set, then $\Theta_\Xi(\mathcal{O}) := \{\Theta_X(\mathcal{K}) \mid X \in \Xi, \mathcal{K} \in \mathcal{O}\}$ will also be an open set and $q(\mathcal{O}) = q\,[\Theta_\Xi(\mathcal{O}))]$. So q will be open iff it sends open *invariant* subsets of $\mathrm{Prim}(\mathcal{A})$ into open subsets of T. The same is true for $\mathfrak{q} : \mathrm{Prim}(\mathfrak{A}) \to T$. But the most general open subset of $\mathrm{Prim}(\mathcal{A})$ has the form

$$\mathcal{O}_{\mathcal{J}} := \{\mathcal{K} \in \mathrm{Prim}(\mathcal{A}) \mid \mathcal{J} \not\subset \mathcal{K}\} = h(\mathcal{J})^c$$

for some ideal $\mathcal{J}$ of $\mathcal{A}$, being the complement of the hull $h(\mathcal{J})$ of this ideal. In addition, $\mathcal{O}_{\mathcal{J}}$ is Θ-invariant iff $\mathcal{J}$ is an invariant ideal. We also recall that Rieffel quantization establishes a one-to-one correspondence between invariant ideals of $\mathcal{A}$ and invariant ideals of $\mathfrak{A}$.

So let $\mathcal{J}$ be an invariant ideal in $\mathcal{A}$ and $\mathfrak{J}$ its quantization (an invariant ideal in $\mathfrak{A}$). We would like to show that $q\,(\mathcal{O}_{\mathcal{J}}) = \mathfrak{q}\,(\mathcal{O}_{\mathfrak{J}})$; by the discussion above this would imply that q and $\mathfrak{q}$ are simultaneously open. Using the fact that $q(\mathcal{K}) = t$ if and only if $\mathcal{I}(t) \subseteq \mathcal{K}$ and similarly for $\mathfrak{q}$, one gets

$$q\,(\mathcal{O}_{\mathcal{J}}) = \{t \in T \mid \exists\, \mathcal{K} \in \mathrm{Prim}(\mathcal{A}),\ \mathcal{J} \not\subset \mathcal{K},\ \mathcal{I}(t) \subset \mathcal{K}\}$$

and

$$\mathfrak{q}\,(\mathcal{O}_{\mathfrak{J}}) = \{t \in T \mid \exists\, \mathfrak{K} \in \mathrm{Prim}(\mathfrak{A}),\ \mathfrak{J} \not\subset \mathfrak{K},\ \mathfrak{I}(t) \subset \mathfrak{K}\}.$$

Using the hull application and the fact that both the hull and the kernel are decreasing, one can write

$$t \notin q\,(\mathcal{O}_{\mathcal{J}}) \iff h[\mathcal{I}(t)] \cap h[\mathcal{J}]^c = \varnothing \iff h[\mathcal{I}(t)] \subset h[\mathcal{J}] \iff \mathcal{I}(t) \supset \mathcal{J}$$

and

$$t \notin \mathfrak{q}\,(\mathcal{O}_{\mathfrak{J}}) \iff h[\mathfrak{I}(t)] \cap h[\mathfrak{J}]^c = \varnothing \iff h[\mathfrak{I}(t)] \subset h[\mathfrak{J}] \iff \mathfrak{I}(t) \supset \mathfrak{J}.$$

To finish the proof one only needs to notice that the Rieffel quantization of invariant ideals preserves inclusions. ■

Remark 5.2. The definition of *a covariant continuous field of C^*-algebras* is naturally obtained by adding the lower-semi-continuity condition to the definition of an upper-semi-continuous field of C^*-algebras contained in Remark 4.5. Using this notion, *Theorem 1.1 is now fully justified.* Another proof, based on the theory of twisted crossed products and on results from the deep work [20] can be found in [1], as we indicated in the introduction.

The C^*-dynamical system $(\mathcal{A}, \Theta, \Xi)$ being given, one could try one of the choices $T = \mathrm{Orb}[\mathrm{Prim}(\mathcal{A})]$ (*the orbit space*) or $T = \mathrm{Quorb}[\mathrm{Prim}(\mathcal{A})]$ (*the quasi-orbit space*), both associated to the natural action of Ξ on the space $\mathrm{Prim}(\mathcal{A})$. We recall that, by definition, *a quasi-orbit* is the closure of an orbit and we refer to [24] for all the fairly standard assertions we are going to make about these spaces. The two spaces are quotients of $\mathrm{Prim}(\mathcal{A})$ with respect to obvious equivalence relations. Endowed with the quotient topology they are locally compact, but they may fail to possess the Hausdorff property. But *the orbit map* $p : \mathrm{Prim}(\mathcal{A}) \to \mathrm{Orb}[\mathrm{Prim}(\mathcal{A})]$ and *the quasi-orbit map* $q : \mathrm{Prim}(\mathcal{A}) \to \mathrm{Quorb}[\mathrm{Prim}(\mathcal{A})]$ are continuous open surjections. So one can state:

Corollary 5.3. *If the quasi-orbit space associated to the dynamical system* $(\mathrm{Prim}(\mathcal{A}), \Theta, \Xi)$ *is Hausdorff, then the deformed C^*-algebra $\mathfrak{A}$ can be expressed as a continuous field of C^*-algebras over the base* $\mathrm{Quorb}[\mathrm{Prim}(\mathcal{A})]$*. A similar statement holds with "quasi-orbit" replaced by "orbit" and* $\mathrm{Quorb}[\mathrm{Prim}(\mathcal{A})]$ *replaced by* $\mathrm{Orb}[\mathrm{Prim}(\mathcal{A})]$*.*

Notice that, when $\mathrm{Orb}[\mathrm{Prim}(\mathcal{A})]$ happens to be Hausdorff, the orbits will be automatically closed (as inverse images by p of points); so one would actually have $\mathrm{Orb}[\mathrm{Prim}(\mathcal{A})] = \mathrm{Quorb}[\mathrm{Prim}(\mathcal{A})]$.

6 Some examples

The most important is the Abelian case, that has been introduced at the end of Section 2. We make a brief review of this case in conjunction to continuity matters, following the more detailed version of [1, Section 2]. This will illustrate the general theory and will also be a preparation for some of the examples we intend to present below.

We recall that an action Θ of Ξ by homeomorphisms of the locally compact space Σ is given. We also assume given a continuous surjection $q : \Sigma \to T$. Then we have the disjoint decomposition of Σ in closed subsets

$$\Sigma = \sqcup_{t \in T} \Sigma_t, \qquad \Sigma_t := q^{-1}(\{t\}).$$

Associated to the canonical injections $j_t : \Sigma_t \to \Sigma$, we have associated restriction epimorphisms

$$\mathcal{R}(t) : \ \mathcal{C}(\Sigma) \to \mathcal{C}(\Sigma_t), \qquad \mathcal{R}(t) f := f|_{\Sigma_t} = f \circ j_t, \quad \forall\, t \in T.$$

We say that the continuous surjection q *is Θ-covariant* if each Σ_t is Θ-invariant.

The Rieffel-quantized C^*-algebras $\mathfrak{C}(\Sigma)$ and $\mathfrak{C}(\Sigma_t)$ as well as the epimorphisms $\mathfrak{R}(t) : \mathfrak{C}(\Sigma) \to \mathfrak{C}(\Sigma_t)$ were introduced above. Applying now the results obtained in Sections 4 and 5, one gets as in [1, Section 2]

Corollary 6.1. *Assume that the mapping $q : \Sigma \to T$ is a Θ-covariant continuous surjection. Then $\left\{ \mathfrak{C}(\Sigma) \xrightarrow{\mathfrak{R}(t)} \mathfrak{C}(\Sigma_t) \mid t \in T \right\}$ is a covariant upper-semi-continuous field of noncommutative C^*-algebras. If q is also open, then the field is continuous.*

Let us assume now that the orbit space $\mathrm{Orb}(\Sigma)$ is Hausdorff. Any orbit, being the inverse image of a point in $\mathrm{Orb}(\Sigma)$, will be closed in Σ and invariant; it will also be homeomorphic to the quotient of Ξ by the corresponding stability group. As a precise particular case of Corollary 5.3 one can state:

Corollary 6.2. *If the orbit space of the dynamical system (Σ, Θ, Ξ) is Hausdorff, then the deformed C^*-algebra $\mathfrak{C}(\Sigma)$ can be expressed as a continuous field of C^*-algebras over the base space $\mathrm{Orb}(\Sigma)$. The fiber over $\mathcal{O} \in \mathrm{Orb}(\Sigma)$ is the deformation of the Abelian algebra $\mathcal{C}(\mathcal{O}) \cong \mathcal{C}(\Xi/\Xi_{\mathcal{O}})$.*

Remark 6.3. It is known that the orbit space is Hausdorff if the action Θ is proper; this happens for instance if Σ is a Hausdorff locally compact group on which the closed subgroup Ξ acts by left translations. More generally, assume that the action Θ factorizes through a compact group $\widehat{\Xi}$, i.e. the kernel of Θ contains a closed co-compact subgroup Z of Ξ (with $\widehat{\Xi} = \Xi/Z$). Then the orbit space under the initial action is the same as the orbit space of the action of the compact quotient. But the action of a compact group is proper and Corollary 6.2 applies.

Example 6.4. Let $\mathscr{A}$ be a C^*-algebra and T a locally compact space. On

$$\mathcal{A} \equiv \mathcal{C}(T;\mathscr{A}) := \{f : T \to \mathscr{A} \mid f \text{ is continuous and small at infinity}\}$$

we consider the natural structure of C^*-algebra. It clearly defines a continuous field of C^*-algebras

$$\left\{\mathcal{C}(T;\mathscr{A}) \xrightarrow{\delta(t)} \mathscr{A} \mid t \in T\right\}, \qquad \delta(t)f := f(t).$$

The associated $\mathcal{C}(T)$-structure is given by $[\mathcal{Q}(\varphi)f](t) := \varphi(t)f(t)$ for $\varphi \in \mathcal{C}(T)$, $f \in \mathcal{A}$, $t \in T$. For each $t \in T$ an action θ^t of Ξ on $\mathscr{A}$ is given; we require that each $f \in \mathcal{A}$ verifies

$$\sup_{t\in T} \|\theta^t_X[f(t)] - f(t)\|_{\mathscr{A}} \underset{X\to 0}{\longrightarrow} 0.$$

Then obviously

$$\Theta : \ \Xi \to \mathrm{Aut}(\mathcal{A}), \qquad [\Theta_X(f)](t) := \theta^t_X[f(t)]$$

defines a continuous action of the vector group Ξ on $\mathcal{A}$. Each of the kernels

$$\mathcal{I}(t) := \ker[\delta(t)] = \{f \in \mathcal{C}(T;\mathscr{A}) \mid f(t) = 0\}$$

is Θ-invariant, so one actually has a covariant continuous field of C^*-algebras (see Remarks 4.5 and 5.2). It makes sense to apply Rieffel quantization, getting C^*-algebras (respectively) $\mathfrak{A} \equiv \mathfrak{C}(T;\mathscr{A})$ from the dynamical system $(\mathcal{A} \equiv \mathcal{C}(T;\mathscr{A}),\Theta)$ and $\mathfrak{A}(t)$ from the dynamical system $(\mathscr{A},\theta^t)$ for all $t \in T$. From the results above one concludes that $\left\{\mathfrak{A} \xrightarrow{\Delta(t)} \mathfrak{A}(t) \mid t \in T\right\}$ *is also a covariant field of C^*-algebras.* For each t we denoted by $\Delta(t)$ the Rieffel quantization of the morphism $\delta(t)$.

Example 6.5. A particular case, considered in [21, Chapter 8], consists in taking $T := \mathrm{End}(\Xi)$ the space of all linear maps $t : \Xi \to \Xi$; it is a locally compact (finite-dimensional vector) space with the obvious operator norm. If an initial action θ of Ξ on $\mathscr{A}$ is fixed, the choice $\theta^t_X := \theta_{tX}$ verify all the requirements above. Therefore one gets a covariant continuous field of C^*-algebras indexed by $\mathrm{End}(\Xi)$. This is basically [21, Theorem 8.3]; we think that our treatment gives a simpler and more unified proof of this result, especially concerning the lower-semi-continuous part. In particular, for any $f \in \mathcal{C}[\mathrm{End}(\Xi);\mathscr{A}]$, one has $\lim_{t\to 0} \|f(t)\|_{\mathfrak{A}(t)} = \|f(0)\|_{\mathscr{A}}$. An interesting particular case is obtained restricting the arguments to the compact subspace $T_0 := \{t = \sqrt{\hbar}\,\mathrm{id}_\Xi \mid \hbar \in [0,1]\} \subset T$. The number $\hbar$ corresponds to the Plank constant and, even for constant $f : [0,1] \to \mathscr{A}$, the relation $\lim_{\hbar\to 0} \|f\|_{\mathfrak{A}(\hbar)} = \|f\|_{\mathscr{A}}$ is nontrivial and has an important physical interpretation concerning the semiclassical behavior of the quantum mechanical formalism. We refer to [12, 21, 22] for much more on this topic.

Remark 6.6. A way to convert Example 6.4 into a more sophisticated one is as follows:

For every $t \in T$ pick $\mathcal{B}(t)$ to be a C^*-subalgebra of $\mathscr{A}$ which is invariant under the action θ^t. Construct the C^*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ defined as $\mathcal{B} := \{f \in \mathcal{C}(T;\mathscr{A}) \mid f(t) \in \mathcal{B}(t),\ \forall\, t \in T\}$, which is obviously invariant under the action Θ. One gets a covariant continuous field of C^*-algebras $\{\mathcal{B} \xrightarrow{\mathcal{P}(t)} \mathcal{B}(t) \mid t \in T\}$, where $\mathcal{P}(t)$ is a restriction of the epimorphism $\delta(t)$. The general theory developed in Sections 4 and 5 supplies another covariant continuous field of C^*-algebras $\{\mathfrak{B} \xrightarrow{\mathfrak{P}(t)} \mathfrak{B}(t) \mid t \in T\}$, where $\mathfrak{B}(t)$ is the quantization of $\mathcal{B}(t)$ and can be identified with an invariant C^*-subalgebra of $\mathfrak{A}(t)$.

Example 6.7. Crossed products associated to actions of $\mathscr{X} := \mathbb{R}^n$ on C^*-algebras can be obtained from Rieffel's quantization procedure, as it is explained in [21, Example 10.5]. From the results of the present article one could infer rather easily, as a particular case, that (informally) *the crossed product by a continuous field of C^*-algebras is a continuous field of crossed products.* Such results exist in a much greater generality, including (twisted) actions of amenable locally compact groups [15, 16, 20, 24], so we are not going to give details.

Example 6.8. In [19] one constructs C^*-algebras which can be considered quantum versions of a certain class of compact connected Lie groups. We will have nothing to say about the extra structure making them quantum groups; we are only going to apply the results above to present these C^*-algebras as continuous fields.

Let Σ be a compact connected Lie group, containing *a toral subgroup*, i.e. a connected closed Abelian subgroup H. Such a toral group is isomorphic to an n-dimensional torus $\mathbb{T}^n$. Assume given a continuous group epimorphism $\eta : \mathbb{R}^n \to H$ (for example the exponential map defined on the Lie algebra $\mathfrak{H} \equiv \mathbb{R}^n$). We use η to define an action of $\Xi := \mathbb{R}^n \times \mathbb{R}^n$ on Σ by $\Theta_{(x,y)}(\sigma) := \eta(-x)\sigma\eta(y)$. Then, by applying Rieffel deformation to $\mathcal{A} := \mathcal{C}(\Sigma)$ using the action Θ (and a certain type of skew-symmetric operator on Ξ), one gets the C^*-algebra $\mathfrak{A} := \mathfrak{C}(\Sigma)$ which, endowed with suitable extra structure, is regarded as a quantum group corresponding to Σ.

It is obvious that the action factorizes through the compact group $H \times H$. Thus the orbit space $\mathrm{Orb}(\Sigma)$ is Hausdorff and Remark 6.3 and Corollary 6.2 serve to express $\mathfrak{C}(\Sigma)$ as a continuous field of C^*-algebras. For the stability group of any orbit $\mathcal{O}$ one can write $\Xi_{\mathcal{O}} \supset \ker(\Theta) \supset \ker(\eta) \times \ker(\eta)$, thus $\mathcal{O} \cong \Xi/\Xi_{\mathcal{O}}$ is a continuous image of $H \times H$.

Example 6.9. An interesting particular case, taken from [19], involves the construction of a quantum version of the compact Lie group $\Sigma := \mathbb{T} \times SU(2)$. Here $\mathbb{T}$ is the 1-torus, the group $SU(2)$ contains diagonally a second copy of $\mathbb{T}$ and can be parametrised by the 3-sphere $S^3 := \{(z, w) \in \mathbb{C}^2 \mid z^2 + w^2 = 1\}$, and so Σ contains a 2-torus. Initially $\Xi = \mathbb{R}^4$ acts on Σ in the given way, but it is shown in [19] (using results from [21]) that the same deformed algebra is obtained by the action

$$\Theta' : \ \Xi' := \mathbb{R}^2 \to \mathrm{Homeo}\left(\mathbb{T} \times S^3\right), \qquad \Theta'_{(x,y)}(\eta; z, w) := \left(e^{-2\pi i x}\eta; z, e^{4\pi i y} w\right).$$

The orbit space is homeomorphic with the closed unit disk $T := \{z \in \mathbb{C} \mid |z| \leq 1\}$. The orbits corresponding to $|z| < 1$ are 2-tori, while the orbits corresponding to $|z| = 1$ (implying $w = 0$) are 1-tori. If we set $\mathcal{A} := \mathcal{C}(\mathbb{T} \times SU(2))$, then the quantized C^*-algebra $\mathfrak{A} \cong \mathfrak{C}(\mathbb{T} \times SU(2))$ deserves to be called *a quantum* $\mathbb{T} \times SU(2)$. The deformation of the continuous functions on any of the 2-tori leads to a quantum torus. By multiplying the initial skew-symmetric form $[\![\cdot, \cdot]\!]$ with an irrational number β one can make this noncommutative torus $\mathfrak{C}_\beta(\mathbb{T}^2)$ irrational, which serves to show that the corresponding quantum $\mathbb{T} \times SU(2)$ (obtained for such a β) is not of type I. But applying the results obtained here one also gets the detailed information: *The algebra $\mathfrak{C}(\mathbb{T} \times SU(2))$ can be written over the closed unit disk T as a continuous field of noncommutative 2-tori and Abelian C^*-algebras (corresponding to the one-dimensional orbits).*

Many other particular cases can be worked out in detail. We propose to the reader the example $\Sigma := SU(2) \times SU(2)$.

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