On Affine Fusion and the Phase Model*

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Received August 01, 2012, in final form November 08, 2012; Published online November 15, 2012
http://dx.doi.org/10.3842/SIGMA.2012.086

Abstract. A brief review is given of the integrable realization of affine fusion discovered recently by Korff and Stroppel. They showed that the affine fusion of the $su(n)$ Wess–Zumino–Novikov–Witten (WZNW) conformal field theories appears in a simple integrable system known as the phase model. The Yang–Baxter equation leads to the construction of commuting operators as Schur polynomials, with noncommuting hopping operators as arguments. The algebraic Bethe ansatz diagonalizes them, revealing a connection to the modular S matrix and fusion of the $su(n)$ WZNW model. The noncommutative Schur polynomials play roles similar to those of the primary field operators in the corresponding WZNW model. In particular, their 3-point functions are the $su(n)$ fusion multiplicities. We show here how the new phase model realization of affine fusion makes obvious the existence of threshold levels, and how it accommodates higher-genus fusion.

Key words: affine fusion; phase model; integrable system; conformal field theory; noncommutative Schur polynomials; threshold level; higher-genus Verlinde dimensions

2010 Mathematics Subject Classification: 81T40; 81R10; 81R12; 17B37; 17B81; 05E05

1 Introduction

Affine fusion is a natural generalization of the tensor product of representations of simple Lie algebras. It is a simple truncation thereof controlled by a non-negative integer, the level. As such, it is a basic mathematical object, found in many different mathematical and physical contexts. The physical context preferred by this author is provided by conformal field theory, and the so-called Wess–Zumino–Novikov–Witten (WZNW) models (see [4], for example). WZNW models realize at a fixed non-negative integer level k a non-twisted affine Kac–Moody algebra $g^{(1)}$ based on a simple Lie algebra g, or g_k for short. Their primary fields furnish representations of g_k and their operator products are governed by the corresponding affine fusion algebra.

Recently, Korff and Stroppel [16] found a much simpler physical realization of affine fusion, for the $su(n)_k$ case. The phase model [2] is an integrable multi-particle model whose integrals of motion may be diagonalized by the algebraic Bethe ansatz [13, 17]. Its integrability is not only crucial to its realization of $su(n)_k$ fusion, but also explains certain properties. The integrable, or phase-model realization of affine fusion raises hope that a better understanding of affine fusion and its physical contexts will result from its study.

This paper is meant to be a gentle, non-rigorous introduction to the phase-model realization of affine fusion. We hope that others will share our interest in the topic and the mathematical tools involved, and perhaps help develop them further. Other reviews can be found in [14, 15].

Sections 2–4 constitute the introductory review. Section 5 contains some new results: threshold levels (and threshold multiplicities and polynomials) and higher-genus Verlinde dimensions are both treated in the phase model there for the first time. Section 6 is a short conclusion.

*This paper is a contribution to the Special Issue "Superintegrability, Exact Solvability, and Special Functions". The full collection is available at http://www.emis.de/journals/SIGMA/SESSF2012.html

2 Phase model: Hilbert space and operator algebras

The set of highest weights λ of integrable highest-weight representations $L(\lambda)$ of $su(n)$ is

$$P_+ := \left\{ (\lambda_1, \lambda_2, \dots, \lambda_{n-1}) := \sum_{a=1}^{n-1} \lambda_a \Lambda^a \,|\, \lambda_a \in \mathbb{Z}_{\geq 0} \right\},$$

where Λ^a is the a-th fundamental weight. Identifying this $su(n)$ as the horizontal subalgebra of the affine Kac–Moody algebra $su(n)_k$ at level k,

$$P_+^k := \left\{ \lambda = [\lambda_1, \dots, \lambda_{n-1}, \lambda_n] := \sum_{a=1}^{n} \lambda_a \Lambda^a \,|\, \lambda_a \in \mathbb{Z}_{\geq 0}, \sum_{a=1}^{n} \lambda_a = k \right\}$$

is the set of affine highest weights at level k.

The phase model has a Hilbert space $\mathcal{H}$ with basis labeled by affine highest weights: $|\lambda\rangle = |\lambda_1, \dots, \lambda_{n-1}, \lambda_n\rangle$. The Dynkin labels are interpreted as the numbers of particles at n sites on a circle, corresponding to the nodes of the affine Dynkin diagram. If N_a denotes the number operator for site a, then the level k is the total number of particles

$$N_a|\lambda\rangle = \lambda_a|\lambda\rangle \quad \Rightarrow \quad N|\lambda\rangle = k|\lambda\rangle, \qquad N := \sum_{a=1}^{n} N_a.$$

The basis of states is orthonormal: $\langle\lambda|\mu\rangle = \delta_{\lambda,\mu}$. Notice that this means states of different levels (numbers of particles) are orthogonal.

Define operators $\varphi_i^\dagger$ and φ_i that create and annihilate (respectively) particles at site i:

$$\varphi_i^\dagger|\lambda\rangle = |\dots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \dots\rangle,$$
$$\varphi_i|\lambda\rangle = \begin{cases} |\dots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \dots\rangle, & \lambda_i \geq 1, \\ 0, & \lambda_i = 0. \end{cases}$$

In the phase model, these operators obey the so-called phase algebra [2, 16], generated by $\varphi_i^\dagger$, φ_i and the number operators N_i, for $i \in \{1, \dots, n\}$, with relations

$$\begin{aligned} &[\varphi_i, \varphi_j] = [\varphi_i^\dagger, \varphi_j^\dagger] = [N_i, N_j] = 0, \qquad [N_i, \varphi_j^\dagger] = \delta_{i,j}\varphi_i^\dagger, \qquad [N_i, \varphi_j] = -\delta_{i,j}\varphi_i, \\ &N_i(1 - \varphi_i^\dagger\varphi_i) = 0 = (1 - \varphi_i^\dagger\varphi_i)N_i, \qquad [\varphi_i, \varphi_j^\dagger] = 0 \quad \text{if} \quad i \neq j, \quad \text{but} \quad \varphi_i\varphi_i^\dagger = 1. \end{aligned} \tag{1}$$

Notice that the commutator of φ_i and $\varphi_i^\dagger$ does not appear in the defining relations of this algebra. That's because the phase model is the crystal limit of the q-boson hopping model, as made precise in [14], so that a q-commutator reduces at $q = 0$ to the last relation of (1). The operator $\pi_i := (1 - \varphi_i^\dagger\varphi_i)$ projects to states with no particles at site i (so $\pi_i^2 = \pi_i$).

As already mentioned, the level becomes the total particle number here. Therefore, to realize the fusion of a WZNW model, which has a fixed level k, we must restrict to a fixed total particle number. Hopping operators

$$a_i := \varphi_i^\dagger\varphi_{i-1}, \qquad i \in \{1, \dots, n\} \tag{2}$$

are then important[1]. Here the indices are defined mod n, so that $a_1 = \varphi_1^\dagger\varphi_n$.[2] The action of a_i is

$$a_i|\lambda\rangle = \begin{cases} |\dots, \lambda_{i-2}, \lambda_{i-1} - 1, \lambda_i + 1, \lambda_{i+2}, \dots\rangle = |\lambda - \Lambda^{i-1} + \Lambda^i\rangle, & \lambda_{i-1} \geq 1, \\ 0, & \lambda_{i-1} = 0. \end{cases} \tag{3}$$

[1]When their action is non-trivial, the operators $\varphi_{i-1}^\dagger\varphi_i = a_i^\dagger$ hop particles in the opposite direction around the sites of the affine $su(n)$ Dynkin diagram. We will focus on the a_i.

[2]For simplicity, we put the "magnetic flux parameter" z of [16] to 1.

The algebra of the hopping operators $\mathcal{A} = \langle a_1, a_2, \ldots, a_n \rangle$ is defined by the relations

$$\begin{aligned} \mathcal{A}: \quad & [a_i, a_j] = 0 \qquad \text{if} \quad i \neq j \pm 1 \mod n, \\ & a_i a_j^2 = a_j a_i a_j \quad \& \quad a_i^2 a_j = a_i a_j a_i \qquad \text{if} \quad i = j+1 \mod n, \end{aligned} \tag{4}$$

easily verified from the phase algebra (1), in view of (2). This algebra $\mathcal{A}$ is called the affine local plactic algebra in [16]. The first line of (4) is the locality condition. Plactic algebras were first defined by Lascoux and Schützenberger, and named by them because of their relation to tableaux (see [7]).

For $n = 3$, the relations are

$$\begin{aligned} & a_3^2 a_2 = a_3 a_2 a_3, \qquad a_3 a_2^2 = a_2 a_3 a_2, \qquad a_2^2 a_1 = a_2 a_1 a_2, \qquad a_2 a_1^2 = a_1 a_2 a_1, \\ & a_1^2 a_3 = a_1 a_3 a_1, \qquad a_1 a_3^2 = a_3 a_1 a_3. \end{aligned} \tag{5}$$

Notice that there are no locality relations for this case – each node on the Dynkin diagram is a nearest neighbour of the other 2.

When the indices of the relations defining $\mathcal{A}$ are not identified mod n, the algebra becomes the local plactic algebra $\bar{\mathcal{A}} = \langle a_1, a_2, \ldots, a_n \rangle$:[3]

$$\begin{aligned} \bar{\mathcal{A}}: \quad & [a_i, a_j] = 0 \qquad \text{if} \quad i \neq j \pm 1, \\ & a_{i+1} a_i^2 = a_i a_{i+1} a_i \quad \& \quad a_{i+1}^2 a_i = a_{i+1} a_i a_{i+1}, \qquad 1 \leq i \leq n-1. \end{aligned} \tag{6}$$

This algebra is relevant to Young tableaux and the Littlewood–Richardson algorithm that computes $su(n)$ tensor product decompositions.

The algebra $\bar{\mathcal{A}}$ can also be realized in terms of creation operators $\varphi_i^\dagger$ and annihilation operators φ_i obeying a phase algebra. More sites are needed, so that $i \in \{0, 1, 2, \ldots, n\}$, but $i = 0$ and $i = n$ are not identified. Then the generators are again constructed as $a_i = \varphi_i^\dagger \varphi_{i-1}$, for $i \in \{1, 2, \ldots, n\}$. Identifying $\varphi_0^\dagger$ and φ_0 with $\varphi_n^\dagger$ and φ_n (respectively) then transforms the construction for $\bar{\mathcal{A}}$ to that for $\mathcal{A}$.

The relations defining $\bar{\mathcal{A}}$ for $n = 3$ are the same as (5), except that the third line is replaced by $[a_1, a_3] = 0$. That is,

$$\bar{\mathcal{A}} \Rightarrow \mathcal{A}: \quad [a_1, a_n] = 0 \ \mapsto \ [a_1, a_1 a_n] = [a_1 a_n, a_n] = 0 \tag{7}$$

summarizes the difference between the affine local plactic algebra and local plactic algebra for $n = 3$. Looking at (4) and (6), we see that (7) applies for all $n \geq 3$: in $\mathcal{A}$ a_1 and a_n do not commute, but the product $a_1 a_n$ commutes with both a_1 and a_n.

To see that plactic algebras are connected to Young tableaux, notice that the hopping operator a_i is associated with the weight $\Lambda^i - \Lambda^{i-1}$. These affine weights have horizontal parts equal to the weights of the basic $su(n)$ irreducible representation $L(\Lambda^1)$. The horizontal weight for a_i is the weight of the Young tableau $\boxed{i}$ that labels a vector of $L(\Lambda^1)$ [7].

Now the states (vectors) of an irreducible $su(n)$ representation of highest weight μ are in 1-1 correspondence with Young tableaux of shape μ and entries in $\{1, 2, \ldots, n\}$. Such a Young tableau is built starting with a Young diagram of shape μ, i.e. one with μ_1 columns of height 1, to the right of μ_2 columns of height 2, etc., up to μ_{n-1} columns of height $n-1$. Since columns of height n correspond to the trivial representation in $su(n)$, they may be omitted.

The Young tableaux (also known as semi-standard tableaux) are obtained by filling the Young diagram with entries from 1 to n, such that they increase going down columns, and do

[3] By abuse of notation, we use the same symbols for the generators of $\mathcal{A}$ and $\bar{\mathcal{A}}$. It is important to note that $\bar{\mathcal{A}}$ differs from the local plactic algebra $\mathrm{Pl}_{\mathrm{fin}}$ defined in [16].

not decrease going across rows [7]. As an example, we display the Young tableaux for the adjoint representation of $su(3)$, of highest weight $\Lambda^1+\Lambda^2$:

$$\begin{array}{ccccc}
 & \begin{array}{|c|c|}\hline 1&2\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} & & \begin{array}{|c|c|}\hline 1&1\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} & \\
\begin{array}{|c|c|}\hline 2&2\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} & & \begin{array}{|c|c|}\hline 1&3\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{array}{|c|c|}\hline 1&2\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} & & \begin{array}{|c|c|}\hline 1&1\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} \\
 & \begin{array}{|c|c|}\hline 2&3\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} & & \begin{array}{|c|c|}\hline 1&3\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} &
\end{array} \tag{8}$$

The arrangement is meant to remind the reader of the corresponding weight diagram. The weights of $su(n)$ Young tableaux are determined by their entries: if there are $\#_i$ occurrences of $\boxed{i}$, $i=1,\ldots,n$, its weight is $\sum_i \#_i(\Lambda^i-\Lambda^{i-1})$.

One version of the Littlewood–Richardson rule calculates the decomposition of the tensor-product $L(\lambda)\otimes L(\mu)$ as follows. Take the Young tableaux of shape λ and add them to the Young diagram of shape μ, column-by-column, from right to left, to obtain "mixed tableaux". When adding each column, simply adjoin each box $\boxed{i}$ to the i-th row of the mixed tableau. If after adding a column, the mixed tableau has an invalid shape, there is no contribution from the original Young tableau. If, on the other hand, the final mixed tableau survives, its shape ν indicates the appearance of $L(\nu)$ in the desired decomposition.

For example, suppose $\lambda=\Lambda^1+\Lambda^2$ and $\mu=\Lambda^1$, so that the appropriate Young diagram is $\boxed{}$ and the relevant Young tableaux are those of (8). Adding the rightmost column of $\begin{array}{|c|c|}\hline 1&3\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}$ to $\boxed{}$ yields:

$$\begin{array}{|c|c|}\hline 1&3\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} + \boxed{} \Rightarrow \begin{array}{c}\boxed{}\\ \\ \boxed{3}\end{array}$$

a mixed tableau with an invalid shape, so this Young tableau produces no contribution to the tensor-product decomposition. On the other hand, we find the sequence

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} + \boxed{} \Rightarrow \begin{array}{|c|}\hline \\\hline 2\\\hline\end{array}\ ,\ \begin{array}{|c|c|}\hline &1\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1} 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}$$

so the Young tableau $\begin{array}{|c|c|}\hline 1&2\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}$ reveals a representation $L(\Lambda^1)$ in the decomposition.

Notice that adding a single box $\boxed{i}$ to a Young diagram or mixed tableau of shape λ produces a mixed tableaux of shape $\lambda+\Lambda^i-\Lambda^{i-1}$ or vanishes, in precise correspondence with (3).

As an example of the Littlewood–Richardson rule, the Young tableaux of (8) may be added to the Young diagram $\begin{array}{|c|c|}\hline &\\\hline &\multicolumn{1}{c}{}\\\cline{1-1}\end{array}$ to verify the $su(n)$ tensor product decomposition

$$L\big(\Lambda^1+\Lambda^2\big)^{\otimes 2} \hookrightarrow L(0)\oplus 2L\big(\Lambda^1+\Lambda^2\big)\oplus L\big(3\Lambda^1\big)\oplus L\big(3\Lambda^2\big)\oplus L\big(2\Lambda^1+2\Lambda^2\big). \tag{9}$$

To understand the connection of the plactic algebra with the Littlewood–Richardson rule and Young tableaux, we must introduce words [7]. The (column) word of a Young tableau is obtained by listing its entries in the order from bottom to top in the left-most row, then from bottom to top in the next-to-left-most row, continuing until the top entry of the right-most row is listed. A plactic monomial is the result of substituting in the word the hopping operator a_i for the number i. For example, the plactic monomials of the Young tableaux in (8) are

$$\begin{array}{ccccc}
 & a_2a_1a_2 & & a_2a_1a_1 & \\
a_3a_2a_2 & & a_2a_1a_3\quad a_3a_1a_2 & & a_3a_1a_1 \\
 & a_3a_2a_3 & & a_3a_1a_3 &
\end{array}$$

Acting on the state $|\mu\rangle$ with the plactic monomials of Young tableaux of a fixed shape λ is equivalent to using the Littlewood–Richardson rule to calculate the decomposition of the tensor product $L(\lambda)\otimes L(\mu)$.

Now, if a triple tensor product $L(\kappa)\otimes L(\lambda)\otimes L(\mu)$ is considered, the procedure is not unique. Most straightforwardly, calculating $L(\lambda)\otimes L(\mu)$ first leads to a set of mixed tableaux, one for each irreducible highest-weight representation in the decomposition. If the mixed tableaux are replaced by Young diagrams of the same shape, then the result can be calculated with the rule already described, applied a second time: $L(\kappa)\otimes(L(\lambda)\otimes L(\mu))$.

On the other hand, one can also multiply the Young tableaux of shape κ and those of shape λ, to obtain a new set of Young tableaux. These product Young tableaux can then be added in the usual way to the Young diagram of shape μ, to obtain the desired decomposition $(\,L(\kappa)\otimes L(\lambda)\,)\otimes L(\mu)$. The required product $\bullet$ of Young tableaux is described by the "bumping" process [7]. Fundamental examples are

$$\boxed{j}\boxed{k}\bullet\boxed{i}=\begin{array}{|c|c|}\hline i & k\\\hline j & \\\cline{1-1}\end{array},\quad i<j\le k;\qquad \boxed{i}\boxed{k}\bullet\boxed{j}=\begin{array}{|c|c|}\hline i & j\\\hline k & \\\cline{1-1}\end{array},\quad i\le j<k. \tag{10}$$

If the Young tableaux are translated into words, these bumping identities are translated into relations for the corresponding plactic algebra. For example, with $j=k=i+1$, (10) yields $a_{i+1}^2a_i=a_{i+1}a_ia_{i+1}$; and with $j=k-1=i$, we get $a_ia_{i+1}a_i=a_{i+1}a_i^2$. The full relations of (6) guarantee that performing the Young tableaux calculations instead with the plactic monomials, in a fully algebraic way, will yield equivalent results.

3 Phase model: Yang–Baxter equation and Bethe ansatz

We now write the Yang–Baxter equation for the phase model and apply the algebraic Bethe ansatz in a standard way, following [16]. For an elementary introduction to the algebraic Bethe ansatz, see [17], and for a comprehensive treatment, consult [13], for example.

First, introduce an auxiliary space isomorphic to $\mathbb{C}^2$, and work in $\mathbb{C}^2\otimes\mathcal{H}$. Write

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}:=\begin{pmatrix}1&0\\0&0\end{pmatrix}\otimes\alpha+\begin{pmatrix}0&1\\0&0\end{pmatrix}\otimes\beta+\begin{pmatrix}0&0\\1&0\end{pmatrix}\otimes\gamma+\begin{pmatrix}0&0\\0&1\end{pmatrix}\otimes\delta,$$

for α, β, γ, δ operators acting on $\mathcal{H}$ (or endomorphisms of $\mathcal{H}$).

Use the creation and annihilation operators of the phase model to define a Lax matrix, or L-operator on $\mathbb{C}^2\otimes\mathcal{H}$

$$L_i(u):=\begin{pmatrix}1 & u\varphi_i^\dagger\\ \varphi_i & u\end{pmatrix}, \tag{11}$$

where u is the spectral parameter. The monodromy matrix is then

$$M(u)=L_n(u)L_{n-1}(u)\cdots L_1(u)=\begin{pmatrix}A(u) & B(u)\\ C(u) & D(u)\end{pmatrix},$$

where the last equality just establishes the standard notation. For the simple L-operator of (11), one finds $B(u)=D(u)\varphi_n^\dagger$ and $C(u)=\varphi_nA(u)$. For $n=3$ we find

$$A(u)=1+u(\varphi_2^\dagger\varphi_1+\varphi_3^\dagger\varphi_2)+u^2\varphi_3^\dagger\varphi_1,\qquad D(u)=u^3+u^2(\varphi_3\varphi_2^\dagger+\varphi_2\varphi_1^\dagger)+u\varphi_3\varphi_1^\dagger. \tag{12}$$

The monodromy matrix satisfies the fundamental relation

$$R_{12}(u/v)M_1(u)M_2(v)=M_2(v)M_1(u)R_{12}(u/v), \tag{13}$$

with R-matrix given by

$$R(x)=\begin{pmatrix}\frac{x}{x-1} & 0 & 0 & 0\\ 0 & 0 & \frac{x}{x-1} & 0\\ 0 & \frac{1}{x-1} & 1 & 0\\ 0 & 0 & 0 & \frac{x}{x-1}\end{pmatrix}.$$

This relation works on $\mathcal{H}$ extended by two copies of the auxiliary $\mathbb{C}^2$, and the indices 1 and 2 indicate which of them the operators implicate.

The R-matrix satisfies the quantum Yang–Baxter equation

$$R_{12}(u/v)R_{13}(u)R_{23}(v)=R_{23}(v)R_{13}(u)R_{12}(u/v). \tag{14}$$

On the other hand, (13) defines the so-called quantum Yang–Baxter algebra, satisfied by the entries $A(u)$, $B(u)$, $C(u)$, $D(u)$ of the monodromy matrix. For example, we find

$$[B(u),B(v)]=0, \tag{15}$$

important here. The commuting $B(u)$ will be used as creation operators for a basis of states in the phase-model Hilbert space.

To demonstrate integrability, define the transfer matrix

$$T(u):=\operatorname{tr}M(u)=A(u)+D(u), \tag{16}$$

where the trace is over the auxiliary space. Equation (13) guarantees that

$$[T(u),T(v)]=0, \tag{17}$$

so that $T(u)=\sum_r u^r T_r$ is the generating function of integrals of motion: $[T_r,T_s]=0$. The Hamiltonian of the phase model is recovered as

$$H=-\frac{1}{2}(T_1+T_{n-1})=-\frac{1}{2}\sum_{i=1}^{n}\left(\varphi_i\varphi_{i+1}^\dagger+\varphi_i^\dagger\varphi_{i+1}\right).$$

By (12), the transfer matrix is

$$\begin{aligned}T(u)&=1+u\left(\varphi_2^\dagger\varphi_1+\varphi_3^\dagger\varphi_2+\varphi_1^\dagger\varphi_3\right)+u^2\left(\varphi_3^\dagger\varphi_1+\varphi_2^\dagger\varphi_3+\varphi_1^\dagger\varphi_2\right)+u^3\\ &=1+u(a_2+a_3+a_1)+u^2(a_3a_2+a_2a_1+a_1a_3)+u^3.\end{aligned}$$

The general result [16] is

$$T(u)=\sum_{r=0}^{n}u^r e_r(\mathcal{A}), \tag{18}$$

where $e_r(\mathcal{A})$ indicates the r-th cyclic elementary symmetric polynomial, the sum of all cyclically ordered products of r distinct hopping operators a_i

$$e_r(\mathcal{A})=\sum_{|I|=r}\prod_{i\in I}^{\circlearrowleft}a_i. \tag{19}$$

In a monomial $a_{i_1}a_{i_2}\cdots a_{i_r}$, the relative order of 2 operators a_{i_j}, a_{i_k} only matters if i_j and i_k differ by $1 \bmod n$, because of (4). Suppose the nodes of the circular affine $su(n)$ Dynkin diagram are numbered from 1 to n in clockwise fashion. Then anticlockwise cyclic ordering $\circlearrowleft$ specifies that if $i_k = i_j+1 \bmod n$, then a_{i_j} occurs to the right of a_{i_k}. Thus, for $n = 3$, a_3a_2 is anticlockwise circularly ordered, while a_2a_3 is not – the latter is connected to the "long way" anticlockwise around the circular Dynkin diagram.

By (17), we know that

$$[e_r(\mathcal{A}), e_{r'}(\mathcal{A})] = 0. \tag{20}$$

For $n = 3$, the only non-trivial relation is

$$[e_1(\mathcal{A}), e_2(\mathcal{A})] = [a_1 + a_2 + a_3, a_3a_2 + a_2a_1 + a_1a_3] = 0.$$

Rewriting the defining relations (5) as

$$[a_3, a_3a_2] = [a_3, a_1a_3] = [a_2, a_2a_1] = [a_2, a_3a_2] = [a_1, a_1a_3] = [a_1, a_2a_1] = 0$$

makes obvious that it is satisfied.

The integrals of motion $e_r(\mathcal{A})$ are related to Schur polynomials. By the substitution $a_i \to x_i$, one recovers the elementary symmetric polynomials $e_r(x) = s_{\Lambda^r}(x)$, the Schur polynomials for the fundamental $su(n)$ representations. The definition (19) therefore produces noncommutative Schur polynomials for the fundamental representations of $su(n)$. The integrability result (20) allows us to define noncommutative Schur polynomials for all $su(n)$ representations. Since the $e_r(\mathcal{A})$ commute, the Jacobi–Trudy formula

$$s_\lambda(\mathcal{A}) = \det\left(e_{\lambda_i^t - i + j}(\mathcal{A})\right) \tag{21}$$

makes sense. Here λ_i^t is the i-th integer of λ^t, the transpose of the partition specifying λ.

For example, with $n = 3$ and $\lambda = \Lambda^1 + \Lambda^2$, we find

$$\begin{aligned} s_{\Lambda^1+\Lambda^2}(\mathcal{A}) &= \det\begin{pmatrix} e_2(\mathcal{A}) & e_3(\mathcal{A}) \\ e_0(\mathcal{A}) & e_1(\mathcal{A}) \end{pmatrix} = \det\begin{pmatrix} a_3a_2 + a_2a_1 + a_1a_3 & 1 \\ 1 & a_1 + a_2 + a_3 \end{pmatrix} \\ &= a_2a_1^2 + a_1a_3a_1 + a_3a_2^2 + a_2a_1a_2 + a_3a_2a_3 + a_1a_3^2 \\ &\quad + (a_3a_2a_1 + a_1a_3a_2 + a_2a_1a_3 - 1). \end{aligned} \tag{22}$$

The terms of vanishing weight are enclosed in brackets.

Furthermore, the integrability (20) implies that the noncommutative Schur polynomials commute among themselves[4]

$$[s_\lambda(\mathcal{A}), s_\mu(\mathcal{A})] = 0. \tag{23}$$

One can therefore hope to find a basis diagonal in all these operators.

For that to be possible, the so-called Bethe ansatz equations must be satisfied [13, 17]. In more detail, the Bethe state (or vector) uses the commuting operators $B(u)$ as creation operators to construct basis elements from the vacuum

$$|b(x)\rangle := B\big(x_1^{-1}\big)B\big(x_2^{-1}\big)\cdots B\big(x_k^{-1}\big)|0\rangle, \tag{24}$$

that depend on invertible indeterminates $x = (x_1, \dots, x_k)$. Since $[N, B(u)] = B(u)$, each operator $B(x_i^{-1})$ injects a particle, and the states (24) have level k. Recalling (15), we see that $|b(x)\rangle$

[4]Terminology aside, it may be surprising that the noncommutative Schur polynomials commute. It was shown in [6], however, that the case studied here is but one of a more general class of such noncommutative Schur polynomials, that commute among themselves. The noncommutative arguments need only satisfy relations that are implied by those in (4) but do not themselves imply (4).

is completely symmetric in the variables $x_1^{-1},\dots,x_k^{-1}$. This can be made completely explicit using level-k symmetric polynomials [16]

$$|b(x)\rangle = \sum_{\lambda\in P_+^k} s_{\lambda^t}\big(x_1^{-1},\dots,x_k^{-1}\big)|\lambda\rangle. \tag{25}$$

Now the Bethe vector $|b(x)\rangle$ can be shown to be an eigenvector of the transfer matrix (16)

$$T(u)|b(x)\rangle = \left\{\big[1+(-1)^k e_k(x)u^{n+k}\big]\prod_{i=1}^{k}\frac{1}{1-ux_i}\right\}|b(x)\rangle, \tag{26}$$

using the Yang–Baxter algebra (which follows from the fundamental relation (13)) and properties of the 0-particle vacuum $|0\rangle$ [16][5]. But this works only if x obeys the Bethe ansatz equations

$$x_1^{n+k} = \cdots = x_k^{n+k} = (-1)^{k-1}x_1x_2\cdots x_k. \tag{27}$$

Remarkably, the solutions to (27) are in 1-1 correspondence with weights in P_+^k. To see roughly how this works, consider the variables $y_i = x_i^{-1}x_{i+1}$, with indices defined cyclically $\bmod\, k$. Think of a pie that can be divided into $n+k$ equal portions of angles $2\pi/(k+n)$ [1]. Each y_i is an $(n+k)$-th root of unity, and so determines a slice with a number of portions, the slice size. Since $y_1y_2\cdots y_k = 1$, each $x=(x_1,\dots,x_k)$ determines a slicing of the pie into k slices, or a k-slicing. Furthermore, there is an n-slicing complementary to each k-slicing: where the pie is cut and where it is not cut are interchanged. The slice sizes in the n-slicing give the Dynkin labels of shifted weights $\sigma+\rho$ and thus the weights $\sigma\in P_+^k$. The solutions to the Bethe ansatz equations can therefore be labelled by these $\sigma\in P_+^k$: $x=x_\sigma$.

For complete detail, see [16]. The result, valid for all n and k, is that the solutions to the Bethe ansatz equations, or Bethe roots, are in 1-1 correspondence with weights in P_+^k.

4 Affine fusion

With the Bethe ansatz equations satisfied at $x=x_\sigma$, so is equation (26). Then $|b(x_\sigma)\rangle$ is an eigenvector of the transfer matrix, and an eigenvector of all the $e_r(\mathcal{A})$, in view of (18). The eigenvalues can be determined from (26), and one finds

$$e_r(\mathcal{A})|b(x_\sigma)\rangle = h_r(x_\sigma)|b(x_\sigma)\rangle,$$

where $h_r(x)$ is the r-th complete symmetric polynomial. The noncommutative Jacobi–Trudy formula (21) then implies

$$s_\lambda(\mathcal{A})|b(x_\sigma)\rangle = \det\Big(h_{\lambda_i^t-i+j}(x_\sigma)\Big)|b(x_\sigma)\rangle = s_{\lambda^t}(x_\sigma)|b(x_\sigma)\rangle. \tag{28}$$

The last equality follows from a well-known identity for symmetric polynomials, an alternative, dual Jacobi–Trudy formula.

The connection with affine fusion now becomes clear, because

$$s_{\lambda^t}(x_\sigma) = \frac{S_{\lambda,\sigma}}{S_{k\Lambda^n,\sigma}}. \tag{29}$$

Here $S_{\lambda,\sigma}$ denotes an element of the unitary modular S-matrix [11] for $su(n)_k$. Since

$${}^{(k)}N_{\lambda,\mu}^{\nu} = \sum_{\kappa\in P_+^k}\frac{S_{\lambda,\kappa}S_{\mu,\kappa}S_{\nu^*,\kappa}}{S_{k\Lambda^n,\kappa}} \tag{30}$$

[5] In (26), $e_k(x) = x_1\cdots x_k$ is the k-th elementary symmetric polynomial.

by the Verlinde formula [18], the fusion eigenvalues $S_{\lambda,\sigma}/S_{k\Lambda^n,\sigma}$ obey

$$\left(\frac{S_{\lambda,\sigma}}{S_{k\Lambda^n,\sigma}}\right)\left(\frac{S_{\mu,\sigma}}{S_{k\Lambda^n,\sigma}}\right) = \sum_{\nu\in P_+^k} {}^{(k)}N_{\lambda,\mu}^{\nu}\left(\frac{S_{\nu,\sigma}}{S_{k\Lambda^n,\sigma}}\right).$$

Therefore, (28) and (29) combine into

$$s_\lambda(\mathcal{A})|b(x_\sigma)\rangle = \frac{S_{\lambda,\sigma}}{S_{k\Lambda^n,\sigma}}|b(x_\sigma)\rangle,$$

so that

$$s_\lambda(\mathcal{A})s_\mu(\mathcal{A})|b(x_\sigma)\rangle = \sum_{\nu\in P_+^k} {}^{(k)}N_{\lambda,\mu}^{\nu}\left(\frac{S_{\nu,\sigma}}{S_{k\Lambda^n,\sigma}}\right)|b(x_\sigma)\rangle = \sum_{\nu\in P_+^k} {}^{(k)}N_{\lambda,\mu}^{\nu} s_\nu(\mathcal{A})|b(x_\sigma)\rangle.$$

The fusion algebra is commutative, ${}^{(k)}N_{\lambda,\mu}^{\nu} = {}^{(k)}N_{\mu,\lambda}^{\nu}$. It is significant that the commutativity is guaranteed here by integrability: the noncommutative Schur polynomials commute by (23) because they are integrals of motion, existing due to the Yang–Baxter equation (14).

By the Bethe ansatz equations, the Bethe vectors $|b(x_\sigma)\rangle$ for $\sigma \in P_+^k$ form a complete orthogonal (but not normalized) basis of the Hilbert space at level k. Going back to (25), we can relate them to the standard basis

$$|b(x_\sigma)\rangle = \sum_{\lambda\in P_+^k} s_{\lambda^t}\left(x_\sigma^{-1}\right)|\lambda\rangle = \sum_{\lambda\in P_+^k} \frac{S_{\lambda,\sigma}^*}{S_{k\Lambda^n,\sigma}}|\lambda\rangle.$$

The unitarity of the modular S-matrix then yields

$$\sum_{\sigma\in P_+^k} S_{k\Lambda^n,\sigma} S_{\sigma,\mu}|b(x_\sigma)\rangle = |\mu\rangle,$$

and then applying $s_\lambda(\mathcal{A})$ leads to

$$s_\lambda(\mathcal{A})|\mu\rangle = \sum_{\nu\in P_+^k} {}^{(k)}N_{\lambda,\mu}^{\nu}|\nu\rangle, \tag{31}$$

taking the Verlinde formula into account.

Since ${}^{(k)}N_{\lambda,k\Lambda^n}^{\nu} = \delta_\lambda^\nu$ (the highest weight $k\Lambda^n$ labels the identity field), we find

$$s_\lambda(\mathcal{A})|k\Lambda^n\rangle = |\lambda\rangle.$$

This is highly reminiscent of the state-field correspondence in conformal field theory (see [4]), hinting that the operators $s_\lambda(\mathcal{A})$ play the role in the phase model of the primary fields in the corresponding WZNW model.

This becomes clear, however, when

$$\langle\nu|s_\lambda(\mathcal{A})|\mu\rangle = {}^{(k)}N_{\lambda,\mu}^{\nu} \qquad \forall\,\nu,\mu\in P_+^k \tag{32}$$

and

$${}^{(k)}N_{\lambda,\mu,\nu} = \langle k\Lambda^n|s_\lambda(\mathcal{A})s_\mu(\mathcal{A})s_\nu(\mathcal{A})|k\Lambda^n\rangle$$

are written. Indeed, the noncommutative Schur polynomials play the role of primary fields, for any number of them

$${}^{(k)}N_{\lambda_1,\lambda_2,\ldots,\lambda_N} = \langle\lambda_1^*|s_{\lambda_2}(\mathcal{A})\cdots s_{\lambda_{N-1}}(\mathcal{A})|\lambda_N\rangle = \langle k\Lambda^n|s_{\lambda_1}(\mathcal{A})\cdots s_{\lambda_N}(\mathcal{A})|k\Lambda^n\rangle.$$

Like the noncommutative Schur polynomials, the Bethe vectors have an affine-fusion-algebraic significance [15]. Define the (non-normalized) vectors

$$|b_\sigma\rangle = \frac{|b(x_\sigma)\rangle}{\langle b(x_\sigma)|b(x_\sigma)\rangle} = S_{k\Lambda^n,\sigma} \sum_{\lambda \in P_+^k} S^*_{\lambda,\sigma} |\lambda\rangle .$$

Then by Verlinde's formula (30), the formal fusion product $*$ yields

$$|\lambda\rangle * |\mu\rangle = \sum_{\nu \in P_+^k} {}^{(k)}N_{\lambda,\mu}^{\nu} |\nu\rangle \quad \Rightarrow \quad |b_\sigma\rangle * |b_\tau\rangle = \delta_{\sigma,\tau} |b_\sigma\rangle .$$

That is, rescaled Bethe vectors are the idempotents of the affine fusion algebra.

5 New perspective

Affine fusion appears in other integrable models – see [9], for early examples. The simple realization afforded by the phase model [16], however, provides a fresh, new perspective on the old subject. In this section we start to exploit it.

5.1 Threshold level

Perhaps the most striking property of the central result (31) is how the level-dependence of fusion is realized. The noncommutative Schur polynomial $s_\lambda(\mathcal{A})$ has no dependence on the level! At the price of noncommutativity, the same $s_\lambda(\mathcal{A})$ works for all levels k. In the expression $s_\lambda(\mathcal{A})\,|\mu\rangle$, all level-dependence lies in the state $|\mu\rangle$, a much simpler object.

Affine fusion has a simple dependence on the level, described well by the concept of a threshold level [3, 12]. Each highest weight representation in the decomposition of a fusion will appear at all levels greater than or equal to a minimum, non-negative integer value. This threshold level is best understood as a consequence of the Gepner–Witten depth rule [8], or a refinement thereof, conjectured in [12] and proved in [5].

All possible fusion decompositions can be given simply by treating the level as a variable, and writing multi-sets of threshold levels as subscripts. For example, we rewrite the $su(3)$ tensor product decomposition (9) as

$$L\big(\Lambda^1+\Lambda^2\big)^{\otimes 2} \hookrightarrow L(0)_2 \oplus 2L\big(\Lambda^1+\Lambda^2\big)_{2,3} \oplus L\big(3\Lambda^1\big)_3 \oplus L\big(3\Lambda^2\big)_3 \oplus L\big(2\Lambda^1+2\Lambda^2\big)_4 . \tag{33}$$

A multi-set of threshold levels can be replaced by a threshold polynomial $T(t)^{\nu}_{\lambda,\mu}$ with non-negative integer coefficients [10]; so we can also write

$$L\big(\Lambda^1+\Lambda^2\big)^{\otimes 2} \hookrightarrow t^2L(0) \oplus \big(t^2+t^3\big)L\big(\Lambda^1+\Lambda^2\big) \oplus t^3L\big(3\Lambda^1\big) \oplus t^3L\big(3\Lambda^2\big) \oplus t^4L\big(2\Lambda^1+2\Lambda^2\big).$$

In general, the threshold polynomials are

$$T(t)^{\nu}_{\lambda,\mu} = \sum_{t'}^{\infty} {}^{(t')}n^{\nu}_{\lambda,\mu} t^{t'} .$$

Here the threshold multiplicities ${}^{(t)}n^{\nu}_{\lambda,\mu}$ satisfy

$${}^{(k)}N^{\nu}_{\lambda,\mu} = \sum_{t}^{k} {}^{(t)}n^{\nu}_{\lambda,\mu},$$

so that $T(1)^{\nu}_{\lambda,\mu} = {}^{(\infty)}N^{\nu}_{\lambda,\mu} = T^{\nu}_{\lambda,\mu}$, the tensor-product multiplicities. We also find

$$ {}^{(k)}n^{\nu}_{\lambda,\mu} = {}^{(k)}N^{\nu}_{\lambda,\mu} - {}^{(k-1)}N^{\nu}_{\lambda,\mu}, \tag{34} $$

where we have put ${}^{(k-1)}N^{\nu}_{\lambda,\mu} = 0$ if any of λ, μ, ν are not in P^{k-1}_+.

In a similar way, the level-dependence can be incorporated into (31) simply by using $|\mu\rangle$ with variable level. The fusion decomposition (33) can be derived easily this way by applying (22) to $|\Lambda^1 + \Lambda^2 + (k-2)\Lambda^3\rangle$, for example.

More generally, write $\bar\mu = \mu_1\Lambda_1 + \cdots + \mu_{n-1}\Lambda^{n-1}$ and define $\bar\mu_k := \bar\mu + (k - \mu_1 - \mu_2 - \cdots - \mu_{n-1})\Lambda^n$. Then

$$ s_\lambda(\mathcal{A})|\bar\mu_k\rangle = \sum_{\bar\nu\in P^k_+}\sum_{t\le k} {}^{(t)}n^{\nu}_{\lambda,\mu}|\bar\nu_k\rangle. \tag{35} $$

In the limit of large k, the tensor product is recovered, and (35) becomes

$$ s_\lambda(\mathcal{A})|\bar\mu_\infty\rangle = \sum_{\bar\nu\in P^k_+}\sum_{t}^{\infty} {}^{(t)}n^{\nu}_{\lambda,\mu}|\bar\nu_\infty\rangle. \tag{36} $$

Since $s_\lambda(\mathcal{A})$ does not depend on the level of λ, so that $\lambda \to \bar\lambda_\infty$ doesn't change anything, this justifies our notation

$$ L(\bar\lambda)\otimes L(\bar\mu) \hookrightarrow \bigoplus_{\bar\nu\in P_+} T^{\bar\nu}_{\bar\lambda,\bar\mu}(t)L(\bar\nu). \tag{37} $$

Another advantage of the phase-model realization of affine fusion is that, unlike in the WZNW model, the level is not fixed – it is just the total particle number. Changes in level can be described in a simple, algebraic way by the operators $\varphi_i^\dagger$, φ_i of the phase algebra (1). In [16], recursion relations involving fusion multiplicities at levels k and $k+1$ were derived using this observation. Such relations are difficult to see in other ways[6].

Let us treat the threshold multiplicities (34) in similar spirit. Notice that $\varphi_n^\dagger|\bar\mu_{k-1}\rangle = |\bar\mu_k\rangle$. So we calculate

$$ [s_\lambda(\mathcal{A}),\varphi_n^\dagger]|\bar\mu_{k-1}\rangle = \sum_{\nu\in P^k_+} {}^{(k)}N^{\nu}_{\lambda,\mu}|\bar\nu_k\rangle - \sum_{\nu\in P^{k-1}_+} {}^{(k-1)}N^{\nu}_{\lambda,\mu}\varphi_n|\bar\nu_{k-1}\rangle. $$

So the phase-model version of (34) is

$$ \langle\bar\nu_k|[s_\lambda(\mathcal{A}),\varphi_n^\dagger]|\bar\mu_{k-1}\rangle = {}^{(k)}n^{\nu}_{\lambda,\mu}. \tag{38} $$

Once a particular noncommutative Schur polynomial $s_\lambda(\mathcal{A})$ is calculated, the interesting operator $[s_\lambda(\mathcal{A}),\varphi_n^\dagger]$ is easy to write down, since $[a_i,\varphi_n^\dagger] = \delta_{i,1}\varphi_1^\dagger\pi_n$.

5.2 Higher-genus Verlinde dimensions

As another new application of the phase-model realization of $su(n)_k$ affine fusion, we consider higher-genus fusion, i.e. higher-genus Verlinde dimensions [18].

In the WZNW model, the fusion multiplicity ${}^{(k)}N^{\nu}_{\lambda,\mu}$ is also the dimension of the space of conformal blocks for the corresponding 3-point function, its Verlinde dimension. The conformal blocks originate from correlation functions on the sphere with 3 points marked by the 3 primary

[6] See what were called "identities of the Feingold type" in [19], however, which relate fusion multiplicities at different levels.

fields, and the fusion multiplicity can be represented graphically by a 3-legged vertex that arises in a degenerate limit of the marked sphere.

A sphere with n marked points corresponds to a trivalent fusion graph with no loops. But higher-genus Riemann surfaces can also be considered, and so fusion graphs with loops are allowed. For such higher-genus Riemann surfaces with marked points, the trivalent graph that results is not unique. The conformal bootstrap, however, ensures that the Verlinde dimension calculated from any of the graphs is the same. So, all dimensions can be built from the genus-0 3-point ones, for example. By this reasoning, one can see that the Verlinde formula extends to [18]

$$ {}^{(k,g)}N_{\lambda_1,\lambda_2,\dots,\lambda_N} = \sum_{\sigma\in P_+^k} (S_{k\Lambda^n,\sigma})^{2(1-g)} \left(\frac{S_{\lambda_1,\sigma}}{S_{k\Lambda^n,\sigma}}\right)\cdots\left(\frac{S_{\lambda_N,\sigma}}{S_{k\Lambda^n,\sigma}}\right). $$

Here the left-hand side indicates the $su(n)_k$ Verlinde dimension for a genus-g Riemann sphere with N marked points.

In the phase-model realization, the argument above again applies, so that we can build all the required Verlinde dimensions from (32). So, for example,

$$ {}^{(k,1)}N_{\lambda_1,\lambda_2} = \sum_{\alpha,\beta\in P_+^k} {}^{(k,0)}N_{\alpha^*,\beta}^{\lambda_1^*}\, {}^{(k,0)}N_{\alpha,\lambda_2}^{\beta} = \sum_{\alpha,\beta\in P_+^k} \langle\lambda_1^*|s_{\alpha^*}|\beta\rangle\langle\beta|s_\alpha|\lambda_2\rangle = \langle\lambda_1^*|\sum_{\alpha\in P_+^k} s_{\alpha^*}s_\alpha|\lambda_2\rangle. $$

Here we have dropped the arguments from the noncommutative Schur polynomials, α^* indicates the weight charge-conjugate to α, e.g., and we have used the completeness of the standard basis states.

Using this genus-1, 2-point function, the general Verlinde dimension can be constructed, with the nice result

$$ \begin{aligned} {}^{(k,g)}N_{\lambda_1,\dots,\lambda_N} &= \langle\lambda_1^*|\left(\sum_{\alpha\in P_+^k} s_{\alpha^*}s_\alpha\right)^g s_{\lambda_2}\cdots s_{\lambda_{N-1}}|\lambda_N\rangle \\ &= \langle k\Lambda^n|\left(\sum_{\alpha\in P_+^k} s_{\alpha^*}s_\alpha\right)^g s_{\lambda_1}s_{\lambda_2}\cdots s_{\lambda_N}|k\Lambda^n\rangle. \end{aligned} \tag{39} $$

Recall that all the noncommutative Schur polynomials commute. Notice that $\sum_{\alpha\in P_+^k} s_{\alpha^*}s_\alpha$ can be interpreted as a genus-generating operator, or handle-creation operator.

6 Conclusion

Let us first point out the new results obtained. The existence of a threshold level for $su(n)_k$ affine fusion is made plain in the phase-model realization. The noncommutative Schur polynomials do not depend on the level; all dependence on k lies in the basis vectors $|\lambda\rangle$. The threshold-polynomial notation (37) was validated easily in the phase-model realization by (36). It was also shown in (38) how threshold multiplicities may be calculated using, in addition to the noncommutative Schur polynomials of the hopping (affine local plactic) algebra, the creation operator $\varphi_n^\dagger$ of the phase algebra.

The remarkable result (32) of [16] was generalized to the elegant formula (39) for arbitrary Verlinde dimensions, at any genus g and for any number N of marked points.

Most of this paper is not original, however. The bulk of it was devoted to a non-rigorous summary of the integrable, phase-model realization of affine $su(n)$ fusion discovered recently by Korff and Stroppel [16] (also reviewed in [14, 15]). The goal was to provide a brief, easily accessible treatment in the hope of interesting others in this nice work. I believe that the Korff–Stroppel integrable realization of affine fusion will help us understand better affine fusion, the WZNW models and perhaps more general rational conformal field theories.

Acknowledgements

I thank Elaine Beltaos, Terry Gannon, Ali Nassar and Andrew Urichuk for discussions and/or reading the manuscript. This research was supported in part by a Discovery Grant from the Natural Sciences and Engineering Research Council of Canada.

References

[1] Altschüler D., Bauer M., Itzykson C., The branching rules of conformal embeddings, *Comm. Math. Phys.* **132** (1990), 349–364.

[2] Bogoliubov N.M., Izergin A.G., Kitanine N.A., Correlation functions for a strongly correlated boson system, *Nuclear Phys. B* **516** (1998), 501–528, solv-int/9710002.

[3] Cummins C.J., Mathieu P., Walton M.A., Generating functions for WZNW fusion rules, *Phys. Lett. B* **254** (1991), 386–390.

[4] Di Francesco P., Mathieu P., Sénéchal D., Conformal field theory, *Graduate Texts in Contemporary Physics*, Springer-Verlag, New York, 1997.

[5] Feingold A.J., Fredenhagen S., A new perspective on the Frenkel–Zhu fusion rule theorem, *J. Algebra* **320** (2008), 2079–2100, arXiv:0710.1620.

[6] Fomin S., Greene C., Noncommutative Schur functions and their applications, *Discrete Math.* **193** (1998), 179–200.

[7] Fulton W., Young tableaux. With applications to representation theory and geometry, *London Mathematical Society Student Texts*, Vol. 35, Cambridge University Press, Cambridge, 1997.

[8] Gepner D., Witten E., String theory on group manifolds, *Nuclear Phys. B* **278** (1986), 493–549.

[9] Goodman F.M., Nakanishi T., Fusion algebras in integrable systems in two dimensions, *Phys. Lett. B* **262** (1991), 259–264.

[10] Irvine S.E., Walton M.A., Schubert calculus and threshold polynomials of affine fusion, *Nuclear Phys. B* **584** (2000), 795–809, hep-th/0004055.

[11] Kac V.G., Peterson D.H., Infinite-dimensional Lie algebras, theta functions and modular forms, *Adv. Math.* **53** (1984), 125–264.

[12] Kirillov A.N., Mathieu P., Sénéchal D., Walton M.A., Can fusion coefficients be calculated from the depth rule?, *Nuclear Phys. B* **391** (1993), 651–674, hep-th/9203004.

[13] Korepin V.E., Bogoliubov N.M., Izergin A.G., Quantum inverse scattering method and correlation functions, *Cambridge Monographs on Mathematical Physics*, Cambridge University Press, Cambridge, 1993.

[14] Korff C., Noncommutative Schur polynomials and the crystal limit of the $U_q\widehat{\mathfrak{sl}}(2)$-vertex model, *J. Phys. A: Math. Theor.* **43** (2010), 434021, 20 pages, arXiv:1006.4710.

[15] Korff C., The su(n) WZNW fusion ring as integrable model: a new algorithm to compute fusion coefficients, in Infinite Analysis 2010 – Developments in Quantum Integrable Systems, RIMS Kôkyûroku Bessatsu, B28, Res. Inst. Math. Sci. (RIMS), Kyoto, 2011, 121–153, arXiv:1106.5342.

[16] Korff C., Stroppel C., The $\widehat{\mathfrak{sl}}(n)_k$-WZNW fusion ring: a combinatorial construction and a realisation as quotient of quantum cohomology, *Adv. Math.* **225** (2010), 200–268, arXiv:0909.2347.

[17] Nepomechie R.I., A spin chain primer, *Internat. J. Modern Phys. B* **13** (1999), 2973–2985, hep-th/9810032.

[18] Verlinde E., Fusion rules and modular transformations in 2D conformal field theory, *Nuclear Phys. B* **300** (1988), 360–376.

[19] Walton M.A., Tensor products and fusion rules, *Canad. J. Phys.* **72** (1994), 527–536.