The Universal Askey–Wilson Algebra*

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Abstract. Let $\mathbb{F}$ denote a field, and fix a nonzero $q \in \mathbb{F}$ such that $q^4 \neq 1$. We define an associative $\mathbb{F}$-algebra $\Delta = \Delta_q$ by generators and relations in the following way. The generators are A, B, C. The relations assert that each of

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \qquad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}, \qquad C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}$$

is central in Δ. We call Δ the *universal Askey–Wilson algebra*. We discuss how Δ is related to the original Askey–Wilson algebra AW(3) introduced by A. Zhedanov. Multiply each of the above central elements by $q + q^{-1}$ to obtain α, β, γ. We give an alternate presentation for Δ by generators and relations; the generators are A, B, γ. We give a faithful action of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ on Δ as a group of automorphisms; one generator sends $(A, B, C) \mapsto (B, C, A)$ and another generator sends $(A, B, \gamma) \mapsto (B, A, \gamma)$. We show that $\{A^i B^j C^k \alpha^r \beta^s \gamma^t | i, j, k, r, s, t \geq 0\}$ is a basis for the $\mathbb{F}$-vector space Δ. We show that the center $Z(\Delta)$ contains the element

$$\Omega = qABC + q^2A^2 + q^{-2}B^2 + q^2C^2 - qA\alpha - q^{-1}B\beta - qC\gamma.$$

Under the assumption that q is not a root of unity, we show that $Z(\Delta)$ is generated by Ω, α, β, γ and that $Z(\Delta)$ is isomorphic to a polynomial algebra in 4 variables. Using the alternate presentation we relate Δ to the q-Onsager algebra. We describe the 2-sided ideal $\Delta[\Delta, \Delta]\Delta$ from several points of view. Our main result here is that $\Delta[\Delta, \Delta]\Delta + \mathbb{F}1$ is equal to the intersection of (i) the subalgebra of Δ generated by A, B; (ii) the subalgebra of Δ generated by B, C; (iii) the subalgebra of Δ generated by C, A.

Key words: Askey–Wilson relations; Leonard pair; modular group; q-Onsager algebra

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1 Introduction

In [65] A. Zhedanov introduced the Askey–Wilson algebra AW = AW(3) and used it to describe the Askey–Wilson polynomials [3]. Since then, AW has become one of the main objects in the theory of the Askey scheme of orthogonal polynomials [25, 26, 27, 37, 38, 39, 62, 63]. It is particularly useful in the theory of Leonard pairs [44, 54, 56, 57, 59, 60] and Leonard triples [19, 20, 40]. The algebra AW is related to the algebra $U_q(\mathfrak{sl}_2)$ [24, 26, 50, 51, 64] and the algebra $U_q(\mathrm{su}_2)$ [4, 5, 6]. There is a connection to the double affine Hecke algebra of type $(C_1^\vee, C_1)$ [32, 38, 39]. The $\mathbb{Z}_3$-symmetric quantum algebra $O'_q(\mathrm{so}_3)$ [18, Remark 6.11], [22], [23, Section 3], [28, 29, 35, 48] is a special case of AW, and the recently introduced Calabi–Yau algebras [21] give a generalization of AW. The algebra AW plays a role in integrable systems [2, 7, 8, 9, 10, 11, 12, 13, 14, 15, 41, 42, 43, 61] and quantum mechanics [46, 47], as well as the

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theory of quadratic algebras [35, 36, 49]. There is a classical version of AW that has a Poisson algebra structure [25], [40, equation (2.9)], [45, equations (26)–(28)], [66].

In this paper we introduce a central extension of AW called the *universal Askey–Wilson algebra*. This central extension, which we denote by Δ, is related to AW in the following way. There is a reduced $\mathbb{Z}_3$-symmetric presentation of AW that involves three scalar parameters besides q [40, equation (6.1)]. Up to normalization, the algebra Δ is what one gets from this presentation by reinterpreting the three scalar parameters as central elements in the algebra. By construction Δ has no scalar parameters besides q, and there exists a surjective algebra homomorphism $\Delta \to \mathrm{AW}$. One advantage of Δ over AW is that Δ has a larger automorphism group. Our definition of Δ was inspired by [32, Section 3], which in turn was motivated by [30].

Let us now bring in more detail, and recall the definition of AW. There are at least three presentations in the literature; the original one involving three generators [65, equations (1.1a)–(1.1c)], one involving two generators [38, equations (2.1), (2.2)], [57, Theorem 1.5], and a $\mathbb{Z}_3$-symmetric presentation involving three generators [40, equation (6.1)], [49, p. 101], [52], [64, Section 4.3]. We will use the presentation in [40, equation (6.1)], although we adjust the normalization and replace q by q^2 in order to illuminate the underlying symmetry.

Our conventions for the paper are as follows. An algebra is meant to be associative and have a 1. A subalgebra has the same 1 as the parent algebra. We fix a field $\mathbb{F}$ and a nonzero $q \in \mathbb{F}$ such that $q^4 \neq 1$.

Definition 1.1 ([40, equation (6.1)])**.** Let a, b, c denote scalars in $\mathbb{F}$. Define the $\mathbb{F}$-algebra $\mathrm{AW} = \mathrm{AW}_q(a,b,c)$ by generators and relations in the following way. The generators are A, B, C. The relations assert that

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \qquad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}, \qquad C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}$$

are equal to $a/(q+q^{-1})$, $b/(q+q^{-1})$, $c/(q+q^{-1})$ respectively. We call AW the *Askey–Wilson algebra* that corresponds to a, b, c.

We now introduce the algebra Δ.

Definition 1.2. Define an $\mathbb{F}$-algebra $\Delta = \Delta_q$ by generators and relations in the following way. The generators are A, B, C. The relations assert that each of

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \qquad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}, \qquad C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} \tag{1.1}$$

is central in Δ. We call Δ the *universal Askey–Wilson algebra*.

Definition 1.3. For the three central elements in (1.1), multiply each by $q+q^{-1}$ to get α, β, γ. Thus

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} = \frac{\alpha}{q+q^{-1}}, \tag{1.2}$$

$$B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} = \frac{\beta}{q+q^{-1}}, \tag{1.3}$$

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = \frac{\gamma}{q+q^{-1}}. \tag{1.4}$$

Note that each of α, β, γ is central in Δ. (The purpose of the factor $q+q^{-1}$ is to make the upcoming formula (1.5) more attractive.)

From the construction we obtain the following result.

Lemma 1.4. *Let a, b, c denote scalars in $\mathbb{F}$ and let AW denote the corresponding Askey–Wilson algebra. Then there exists a surjective $\mathbb{F}$-algebra homomorphism $\Delta \to \mathrm{AW}$ that sends*

$$A \mapsto A, \qquad B \mapsto B, \qquad C \mapsto C, \qquad \alpha \mapsto a, \qquad \beta \mapsto b, \qquad \gamma \mapsto c.$$

In this paper we begin a comprehensive study of the algebra Δ. For now we consider the ring-theoretic aspects, and leave the representation theory for some future paper. Our main results are summarized as follows. We give an alternate presentation for Δ by generators and relations; the generators are A, B, γ. Following [19, Lemma 5.2], [32, Theorem 5.1], [40], [45, Section 1.2] we give a faithful action of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ on Δ as a group of automorphisms; one generator sends $(A, B, C) \mapsto (B, C, A)$ and another generator sends $(A, B, \gamma) \mapsto (B, A, \gamma)$. Following [29, Theorem 1], [35, Proposition 6.6(i)] we show that

$$A^i B^j C^k \alpha^r \beta^s \gamma^t, \qquad i, j, k, r, s, t \geq 0$$

is a basis for the $\mathbb{F}$-vector space Δ. Following [29, Lemma 1], [40, Proposition 3], [65, equation (1.3)] we show that the center $Z(\Delta)$ contains a Casimir element

$$\Omega = qABC + q^2A^2 + q^{-2}B^2 + q^2C^2 - qA\alpha - q^{-1}B\beta - qC\gamma. \tag{1.5}$$

Under the assumption that q is not a root of unity, we show that $Z(\Delta)$ is generated by Ω, α, β, γ and that $Z(\Delta)$ is isomorphic to a polynomial algebra in 4 variables. Using the alternate presentation we relate Δ to the q-Onsager algebra [15, Section 4], [34], [55, Definition 3.9]. We describe the 2-sided ideal $\Delta[\Delta, \Delta]\Delta$ from several points of view. Our main result here is that $\Delta[\Delta, \Delta]\Delta + \mathbb{F}1$ is equal to the intersection of (i) the subalgebra of Δ generated by A, B; (ii) the subalgebra of Δ generated by B, C; (iii) the subalgebra of Δ generated by C, A. At the end of the paper we list some open problems that are intended to motivate further research.

2 Another presentation of Δ

A bit later in the paper we will discuss automorphisms of Δ. To facilitate this discussion we give another presentation for Δ by generators and relations.

Lemma 2.1. *The $\mathbb{F}$-algebra Δ is generated by A, B, γ. Moreover*

$$C = \frac{\gamma}{q + q^{-1}} - \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}, \tag{2.1}$$

$$\alpha = \frac{B^2A - (q^2 + q^{-2})BAB + AB^2 + (q^2 - q^{-2})^2A + (q - q^{-1})^2B\gamma}{(q - q^{-1})(q^2 - q^{-2})}, \tag{2.2}$$

$$\beta = \frac{A^2B - (q^2 + q^{-2})ABA + BA^2 + (q^2 - q^{-2})^2B + (q - q^{-1})^2A\gamma}{(q - q^{-1})(q^2 - q^{-2})}. \tag{2.3}$$

Proof. Line (2.1) is from (1.4). To get (2.2), (2.3) eliminate C in (1.2), (1.3) using (2.1). ■

Recall the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad n = 0, 1, 2, \ldots.$$

Theorem 2.2. *The $\mathbb{F}$-algebra Δ has a presentation by generators A, B, γ and relations*

$$A^3B - [3]_qA^2BA + [3]_qABA^2 - BA^3 = -\big(q^2 - q^{-2}\big)^2(AB - BA),$$

$$B^3A - [3]_qB^2AB + [3]_qBAB^2 - AB^3 = -\big(q^2 - q^{-2}\big)^2(BA - AB),$$

$$A^2B^2 - B^2A^2 + \big(q^2 + q^{-2}\big)(BABA - ABAB) = -\big(q - q^{-1}\big)^2(AB - BA)\gamma,$$

$$\gamma A = A\gamma, \qquad \gamma B = B\gamma.$$

Proof. Use Lemma 2.1 to express the defining relations for Δ in terms of A, B, γ. ■

Note 2.3. The first two equations in Theorem 2.2 are known as the *tridiagonal relations* [55, Definition 3.9]. These relations have appeared in algebraic combinatorics [53, Lemma 5.4], the theory of tridiagonal pairs [31, 33, 54, 55, 56], and integrable systems [7, 8, 9, 10, 11, 12, 13, 14, 15].

3 An action of $\mathrm{PSL}_2(\mathbb{Z})$ on Δ

We now consider some automorphisms of Δ. Recall that the modular group $\mathrm{PSL}_2(\mathbb{Z})$ has a presentation by generators ρ, σ and relations $\rho^3 = 1$, $\sigma^2 = 1$. See for example [1].

Theorem 3.1. *The group* $\mathrm{PSL}_2(\mathbb{Z})$ *acts on* Δ *as a group of automorphisms in the following way:*

u	A	B	C	α	β	γ
$\rho(u)$	B	C	A	β	γ	α
$\sigma(u)$	B	A	$C + \frac{AB-BA}{q-q^{-1}}$	β	α	γ

Proof. By Definition 1.2 there exists an automorphism P of Δ that sends

$$A \mapsto B, \qquad B \mapsto C, \qquad C \mapsto A.$$

Observe $P^3 = 1$. By (1.2)–(1.4) the map P sends

$$\alpha \mapsto \beta, \qquad \beta \mapsto \gamma, \qquad \gamma \mapsto \alpha.$$

By Theorem 2.2 there exists an automorphism S of Δ that sends

$$A \mapsto B, \qquad B \mapsto A, \qquad \gamma \mapsto \gamma.$$

Observe $S^2 = 1$. By Lemma 2.1 the map S sends

$$\alpha \mapsto \beta, \qquad \beta \mapsto \alpha, \qquad C \mapsto C + \frac{AB - BA}{q - q^{-1}}.$$

The result follows. ■

In Theorem 3.1 we gave an action of $\mathrm{PSL}_2(\mathbb{Z})$ on Δ. Our next goal is to show that this action is faithful.

Let λ denote an indeterminate. Let $\mathbb{F}[\lambda, \lambda^{-1}]$ denote the $\mathbb{F}$-algebra consisting of the Laurent polynomials in λ that have all coefficients in $\mathbb{F}$. We will be discussing the $\mathbb{F}$-algebra

$$\Lambda = \mathrm{Mat}_2(\mathbb{F}) \otimes_{\mathbb{F}} \mathbb{F}\big[\lambda, \lambda^{-1}\big].$$

We view elements of Λ as 2×2 matrices that have entries in $\mathbb{F}[\lambda, \lambda^{-1}]$. From this point of view the product operation for Λ is ordinary matrix multiplication, and the multiplicative identity in Λ is the identity matrix I. For notational convenience define $\mu = \lambda + \lambda^{-1}$.

For later use we now describe the center $Z(\Lambda)$.

Lemma 3.2. *For all* $\eta \in \Lambda$ *the following (i), (ii) are equivalent:*

(i) $\eta \in Z(\Lambda)$.

(ii) There exists $\theta \in \mathbb{F}[\lambda, \lambda^{-1}]$ *such that* $\eta = \theta I$.

Proof. Routine. ■

Definition 3.3. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ denote the following elements of Λ:

$$\mathcal{A} = \begin{pmatrix} \lambda & 1-\lambda^{-1} \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} \lambda^{-1} & 0 \\ \lambda - 1 & \lambda \end{pmatrix}, \qquad \mathcal{C} = \begin{pmatrix} 1 & 1-\lambda \\ \lambda^{-1} - 1 & \lambda + \lambda^{-1} - 1 \end{pmatrix}.$$

Lemma 3.4. *We have*

$$\mathcal{ABC} = I, \qquad \mathcal{A} + \mathcal{A}^{-1} = \mu I, \qquad \mathcal{B} + \mathcal{B}^{-1} = \mu I, \qquad \mathcal{C} + \mathcal{C}^{-1} = \mu I.$$

Proof. Use Definition 3.3. ■

Lemma 3.5. *In the algebra Λ the elements $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ multiply as follows:*

	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$
$\mathcal{A}$	$\mu\mathcal{A} - I$	$\mu I - \mathcal{C}$	$\mu\mathcal{A} + \mathcal{B} + \mu\mathcal{C} - \mu^2 I$
$\mathcal{B}$	$\mu\mathcal{B} + \mathcal{C} + \mu\mathcal{A} - \mu^2 I$	$\mu\mathcal{B} - I$	$\mu I - \mathcal{A}$
$\mathcal{C}$	$\mu I - \mathcal{B}$	$\mu\mathcal{C} + \mathcal{A} + \mu\mathcal{B} - \mu^2 I$	$\mu\mathcal{C} - I$

Proof. Use Lemma 3.4. ■

The algebra Λ is not generated by $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$. However we do have the following result.

Lemma 3.6. *Suppose $\eta \in \Lambda$ commutes with at least two of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$. Then $\eta \in Z(\Lambda)$.*

Proof. For $1 \le i, j \le 2$ let η_{ij} denote the (i, j)-entry of η. The matrix η commutes with each of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ since $\mathcal{ABC} = I$. In the equation $\eta\mathcal{A} = \mathcal{A}\eta$, evaluate $\mathcal{A}$ using Definition 3.3, and simplify the result to get $\eta_{21} = 0$. Similarly using $\eta\mathcal{B} = \mathcal{B}\eta$ we find $\eta_{12} = 0$ and $\eta_{11} = \eta_{22}$. Therefore $\eta = \eta_{11} I \in Z(\Lambda)$. ■

Next we describe an action of $\mathrm{PSL}_2(\mathbb{Z})$ on Λ as a group of automorphisms.

Definition 3.7. Let p and s denote the following elements of Λ:

$$p = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \qquad s = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}.$$

Lemma 3.8. *The following (i)–(iv) hold.*

(*i*) $\det(p) = 1$ *and* $\det(s) = -\lambda$.

(*ii*) $p^3 = -I$ *and* $s^2 = \lambda I$.

(*iii*) $p\mathcal{A}p^{-1} = \mathcal{B}$, $p\mathcal{B}p^{-1} = \mathcal{C}$, $p\mathcal{C}p^{-1} = \mathcal{A}$.

(*iv*) $s\mathcal{A}s^{-1} = \mathcal{B}$ *and* $s\mathcal{B}s^{-1} = \mathcal{A}$.

Proof. (*i*), (*ii*) Use Definition 3.7. (*iii*), (*iv*) Use Definition 3.3 and Definition 3.7. ■

Lemma 3.9. *The group $\mathrm{PSL}_2(\mathbb{Z})$ acts on Λ as a group of automorphisms such that $\rho(\eta) = p\eta p^{-1}$ and $\sigma(\eta) = s\eta s^{-1}$ for all $\eta \in \Lambda$.*

Proof. By Lemma 3.8(*ii*) the elements p^3, s^2 are in $Z(\Lambda)$. ■

Lemma 3.10. *The action of $\mathrm{PSL}_2(\mathbb{Z})$ on Λ is faithful.*

Proof. Pick an integer $n \geq 1$. Consider an element $\eta \in \Lambda$ of the form $\eta = \eta_1\eta_2\cdots\eta_n$ such that for $1 \leq i \leq n$, $\eta_i = s$ for one parity of i and $\eta_i \in \{p, p^{-1}\}$ for the other parity of i. We show that $\eta \notin Z(\Lambda)$. To this end we assume $\eta \in Z(\Lambda)$ and get a contradiction. Let ℓ denote the number of times s occurs among $\{\eta_i\}_{i=1}^n$. Assume for the moment $\ell = 0$. Then $n = 1$ so $\eta = \eta_1 \in \{p, p^{-1}\}$. The elements p, p^{-1} are not in $Z(\Lambda)$, for a contradiction. Therefore $\ell \neq 0$. From the nature of the matrices p, s in Definition 3.7, we may view η as a polynomial in λ that has coefficients in $\mathrm{Mat}_2(\mathbb{F})$ and degree at most ℓ. Call this polynomial f. We claim that the degree of f is exactly ℓ. To prove the claim, write

$$s = s_0 + s_1\lambda, \qquad s_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad s_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let $m \in \mathrm{Mat}_2(\mathbb{F})$ denote the coefficient of λ^ℓ in f. The matrix m is obtained from $\eta_1\eta_2\cdots\eta_n$ by replacing each occurrence of s by s_1. Using $s_1ps_1 = -s_1$ and $s_1p^{-1}s_1 = s_1$ we find $m \in \{\pm p^i s_1 p^j | -1 \leq i, j \leq 1\}$. The matrix p is invertible and $s_1 \neq 0$ so $p^i s_1 p^j \neq 0$ for $-1 \leq i, j \leq 1$. Therefore $m \neq 0$ and the claim is proved. Let κ denote the $(1,1)$-entry of the matrix η. Then $\eta = \kappa I$ since $\eta \in Z(\Lambda)$. In the equation $\eta_1\eta_2\cdots\eta_n = \kappa I$ take the determinant of each side and use Lemma 3.8(i) to get $(-\lambda)^\ell = \kappa^2$. Therefore ℓ is even and $\kappa = \pm\lambda^{\ell/2}$. Now $\eta = \pm\lambda^{\ell/2}I$, so the above polynomial f has degree $\ell/2$. But $\ell \neq 0$ so $\ell > \ell/2$ for a contradiction. the result follows. ∎

We now display an algebra homomorphism $\Delta \to \Lambda$.

Lemma 3.11. *There exists a unique $\mathbb{F}$-algebra homomorphism $\pi : \Delta \to \Lambda$ that sends*

$$A \mapsto q\mathcal{A} + q^{-1}\mathcal{A}^{-1}, \qquad B \mapsto q\mathcal{B} + q^{-1}\mathcal{B}^{-1}, \qquad C \mapsto q\mathcal{C} + q^{-1}\mathcal{C}^{-1}.$$

The homomorphism π sends

$$\alpha \mapsto \nu I, \qquad \beta \mapsto \nu I, \qquad \gamma \mapsto \nu I, \tag{3.1}$$

where $\nu = (q^2 + q^{-2})\mu + \mu^2$.

Proof. Define

$$A^\vee = q\mathcal{A} + q^{-1}\mathcal{A}^{-1}, \qquad B^\vee = q\mathcal{B} + q^{-1}\mathcal{B}^{-1}, \qquad C^\vee = q\mathcal{C} + q^{-1}\mathcal{C}^{-1}.$$

By Lemma 3.4 and Lemma 3.5,

$$(q + q^{-1})A^\vee + \frac{qB^\vee C^\vee - q^{-1}C^\vee B^\vee}{q - q^{-1}} = \nu I, \tag{3.2}$$

$$(q + q^{-1})B^\vee + \frac{qC^\vee A^\vee - q^{-1}A^\vee C^\vee}{q - q^{-1}} = \nu I, \tag{3.3}$$

$$(q + q^{-1})C^\vee + \frac{qA^\vee B^\vee - q^{-1}B^\vee A^\vee}{q - q^{-1}} = \nu I. \tag{3.4}$$

By (3.2)–(3.4) and since νI is central, the elements $A^\vee$, $B^\vee$, $C^\vee$ satisfy the defining relations for Δ from Definition 1.2. Therefore the homomorphism π exists. The homomorphism π is unique since A, B, C generate Δ. Line (3.1) follows from Definition 1.3 and (3.2)–(3.4). ∎

Lemma 3.12. *For $g \in \mathrm{PSL}_2(\mathbb{Z})$ the following diagram commutes:*

$$\begin{array}{ccc} \Delta & \xrightarrow{\pi} & \Lambda \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ \Delta & \xrightarrow{\pi} & \Lambda \end{array}$$

Proof. The elements ρ, σ form a generating set for $\mathrm{PSL}_2(\mathbb{Z})$; without loss we may assume that g is contained in this set. By Theorem 3.1 the action of ρ on Δ cyclically permutes A, B, C. By Lemma 3.8(*iii*) the action of ρ on Λ cyclically permutes $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$. By Theorem 3.1 the action of σ on Δ swaps A, B and fixes γ. By Lemma 3.8(*iv*) and the construction, the action of σ on Λ swaps $\mathcal{A}$, $\mathcal{B}$ and fixes I. The diagram commutes by these comments and Lemma 3.11. ∎

Theorem 3.13. *The action of* $\mathrm{PSL}_2(\mathbb{Z})$ *on* Δ *is faithful.*

Proof. Let g denote an element of $\mathrm{PSL}_2(\mathbb{Z})$ that fixes everything in Δ. We show that $g = 1$. By Lemma 3.9 and since ρ, σ generate $\mathrm{PSL}_2(\mathbb{Z})$, there exists an invertible $\xi \in \Lambda$ such that $g(\eta) = \xi\eta\xi^{-1}$ for all $\eta \in \Lambda$. By assumption g fixes the element A of Δ. Under the homomomorphim $\pi : \Delta \to \Lambda$ the image of A is $q\mathcal{A} + q^{-1}\mathcal{A}^{-1}$, so g fixes $q\mathcal{A} + q^{-1}\mathcal{A}^{-1}$ in view of Lemma 3.12. Therefore ξ commutes with $q\mathcal{A} + q^{-1}\mathcal{A}^{-1}$. Recall $\mathcal{A} + \mathcal{A}^{-1} = \mu I$ by Lemma 3.4, so ξ commutes with $\mathcal{A} + \mathcal{A}^{-1}$. By these comments and $q^2 \neq 1$ we find ξ commutes with $\mathcal{A}$. By a similar argument ξ commutes with $\mathcal{B}$. Now $\xi \in Z(\Lambda)$ by Lemma 3.6. Consequently g fixes everything in Λ, so $g = 1$ by Lemma 3.10. ∎

4 A basis for Δ

In this section we display a basis for the $\mathbb{F}$-vector space Δ.

Theorem 4.1. *The following is a basis for the* $\mathbb{F}$*-vector space* Δ.

$$A^i B^j C^k \alpha^r \beta^s \gamma^t, \qquad i, j, k, r, s, t \geq 0. \tag{4.1}$$

Proof. We invoke Bergman's Diamond Lemma [16, Theorem 1.2]. Consider the symbols

$$A, \quad B, \quad C, \quad \alpha, \quad \beta, \quad \gamma. \tag{4.2}$$

For an integer $n \geq 0$, by a Δ*-word of length* n we mean a sequence $x_1 x_2 \cdots x_n$ such that x_i is listed in (4.2) for $1 \leq i \leq n$. We interpret the Δ-word of length zero to be the multiplicative identity in Δ. Consider a Δ-word $w = x_1 x_2 \cdots x_n$. By an *inversion* for w we mean an ordered pair of integers (i, j) such that $1 \leq i < j \leq n$ and x_i is strictly to the right of x_j in the list (4.2). For example $CABA$ has 4 inversions and CB^2A has 5 inversions. A Δ-word is called *reducible* whenever it has at least one inversion, and *irreducible* otherwise. The list (4.1) consists of the irreducible Δ-words. For each integer $n \geq 0$ let W_n denote the set of Δ-words that have length n. Let $W = \cup_{n=0}^{\infty} W_n$ denote the set of all Δ-words. We now define a partial order $<$ on W. The definition has two aspects. (*i*) For all integers $n > m \geq 0$, every word in W_m is less than every word in W_n, with respect to $<$. (*ii*) For an integer $n \geq 0$ the restriction of $<$ to W_n is described as follows. Pick $w, w' \in W_n$ and write $w = x_1 x_2 \cdots x_n$. We say that w *covers* w' whenever there exists an integer j $(2 \leq j \leq n)$ such that $(j-1, j)$ is an inversion for w, and w' is obtained from w by interchanging x_{j-1}, x_j. In this case w' has one fewer inversions than w. Therefore the transitive closure of the covering relation on W_n is a partial order on W_n, and this is the restriction of $<$ to W_n. We have now defined a partial order $<$ on W. By construction this partial order is a semi-group partial order [16, p. 181] and satisfies the descending chain condition [16, p. 179]. By Definition 1.2 and Definition 1.3 the defining relations for Δ can be expressed as follows:

$$\begin{aligned}
BA &= q^2 AB + q(q^2 - q^{-2})C - q(q - q^{-1})\gamma, \\
CB &= q^2 BC + q(q^2 - q^{-2})A - q(q - q^{-1})\alpha, \\
CA &= q^{-2} AC + q^{-1}(q^{-2} - q^2)B - q^{-1}(q^{-1} - q)\beta,
\end{aligned}$$

$$\begin{aligned} \alpha A &= A\alpha, & \alpha B &= B\alpha, & \alpha C &= C\alpha,\\ \beta A &= A\beta, & \beta B &= B\beta, & \beta C &= C\beta,\\ \gamma A &= A\gamma, & \gamma B &= B\gamma, & \gamma C &= C\gamma,\\ \beta\alpha &= \alpha\beta, & \gamma\beta &= \beta\gamma, & \gamma\alpha &= \alpha\gamma. \end{aligned}$$

The above equations give reduction rules for Δ-words, as we now explain. Let $w = x_1x_2\cdots x_n$ denote a reducible Δ-word. Then there exists an integer j $(2 \le j \le n)$ such that $(j-1, j)$ is an inversion for w. In the above list of equations, there exists an equation with $x_{j-1}x_j$ on the left-hand side; in w we eliminate $x_{j-1}x_j$ using this equation and thereby express w as a linear combination of Δ-words, each less than w with respect to $<$. Therefore the reduction rules are compatible with $<$ in the sense of Bergman [16, p. 181]. In order to employ the Diamond Lemma, we must show that the ambiguities are resolvable in the sense of Bergman [16, p. 181]. There are potentially two kinds of ambiguities; inclusion ambiguities and overlap ambiguities [16, p. 181]. For the present example there are no inclusion ambiguities. The only nontrivial overlap ambiguity involves the word CBA. This word can be reduced in two ways; we could evaluate CB first or we could evaluate BA first. Either way, after a three-step reduction we obtain the same result, which is

$$\begin{aligned} q^{-1}CBA = qABC &+ \big(q^2-q^{-2}\big)A^2 - \big(q^2-q^{-2}\big)B^2 + \big(q^2-q^{-2}\big)C^2\\ &- \big(q-q^{-1}\big)A\alpha + \big(q-q^{-1}\big)B\beta - \big(q-q^{-1}\big)C\gamma. \end{aligned}$$

Therefore the overlap ambiguity CBA is resolvable. We conclude that every ambiguity is resolvable, so by the Diamond Lemma [16, Theorem 1.2] the elements (4.1) form a basis for Δ. ∎

On occasion we wish to discuss the coefficients when an element of Δ is written as a linear combination of the elements (4.1). To facilitate this discussion we define a bilinear form $(\,,\,) : \Delta \times \Delta \to \mathbb{F}$ such that $(u, v) = \delta_{u,v}$ for all elements u, v in the basis (4.1). In other words the basis (4.1) is orthonormal with respect to $(\,,\,)$. Observe that $(\,,\,)$ is symmetric. For $u \in \Delta$,

$$u = \sum \big(u, A^iB^jC^k\alpha^r\beta^s\gamma^t\big)A^iB^jC^k\alpha^r\beta^s\gamma^t, \tag{4.3}$$

where the sum is over all elements $A^iB^jC^k\alpha^r\beta^s\gamma^t$ in the basis (4.1).

Definition 4.2. Let $u \in \Delta$. A given element $A^iB^jC^k\alpha^r\beta^s\gamma^t$ in the basis (4.1) is said to *contribute to* u whenever $(u, A^iB^jC^k\alpha^r\beta^s\gamma^t) \neq 0$.

5 A filtration of Δ

In this section we obtain a filtration of Δ which is related to the basis from Theorem 4.1. This filtration will be useful when we investigate the center $Z(\Delta)$ later in the paper.

We recall some notation. For subspaces H, K of Δ define $HK = \mathrm{Span}\{hk | h \in H,\ k \in K\}$.

Definition 5.1. We define subspaces $\{\Delta_n\}_{n=0}^{\infty}$ of Δ such that

$$\Delta_0 = \mathbb{F}1, \qquad \Delta_1 = \Delta_0 + \mathrm{Span}\{A, B, C, \alpha, \beta, \gamma\}, \qquad \Delta_n = \Delta_1\Delta_{n-1}, \qquad n = 1, 2, \ldots$$

Lemma 5.2. *The following (i)–(iii) hold.*

(i) $\Delta_{n-1} \subseteq \Delta_n$ *for* $n \geq 1$.

(ii) $\Delta = \cup_{n=0}^{\infty}\Delta_n$.

(iii) $\Delta_m\Delta_n = \Delta_{m+n}$ *for* $m, n \geq 0$.

Proof. (*i*) Since $\Delta_n = \Delta_1 \Delta_{n-1}$ and $1 \in \Delta_1$. (*ii*) Since A, B, C, α, β, γ generate Δ. (*iii*) Each side is equal to $(\Delta_1)^{m+n}$. ■

By Lemma 5.2 the sequence $\{\Delta_n\}_{n=0}^{\infty}$ is a filtration of Δ in the sense of [17, p. 202].

Lemma 5.3. *Each of the following is contained in* Δ_1*:*

$$qAB - q^{-1}BA, \qquad qBC - q^{-1}CB, \qquad qCA - q^{-1}AC.$$

Proof. Each of the three expressions is a linear combination of A, B, C, α, β, γ and these are contained in Δ_1. ■

Theorem 5.4. *For all integers* $n \geq 0$ *the following is a basis for the* $\mathbb{F}$*-vector space* Δ_n*:*

$$A^i B^j C^k \alpha^r \beta^s \gamma^t, \qquad i, j, k, r, s, t \geq 0, \qquad i + j + k + r + s + t \leq n. \tag{5.1}$$

Proof. The elements (5.1) are linearly independent by Theorem 4.1. We show that the elements (5.1) span Δ_n. We will use induction on n. Assume $n \geq 2$; otherwise the result holds by Definition 5.1. By Definition 5.1 Δ_n is spanned by the set of elements of the form $x_1 x_2 \cdots x_n$ where $x_i \in \{1, A, B, C, \alpha, \beta, \gamma\}$ for $1 \leq i \leq n$. Therefore Δ_n is spanned by the set of elements of the form $x_1 x_2 \cdots x_m$ where $0 \leq m \leq n$ and $x_i \in \{A, B, C, \alpha, \beta, \gamma\}$ for $1 \leq i \leq m$. Therefore Δ_n is spanned by Δ_{n-1} together with the set of elements of the form $x_1 x_2 \cdots x_n$ where $x_i \in \{A, B, C, \alpha, \beta, \gamma\}$ for $1 \leq i \leq n$. Consider such an element $x_1 x_2 \cdots x_n$. By Lemma 5.3 and since each of α, β, γ is central, we find that for $2 \leq j \leq n$,

$$x_1 \cdots x_{j-1} x_j \cdots x_n \in \mathbb{F} x_1 \cdots x_j x_{j-1} \cdots x_n + \Delta_{n-1}.$$

By the above comments Δ_n is spanned by Δ_{n-1} together with the set

$$A^i B^j C^k \alpha^r \beta^s \gamma^t, \qquad i, j, k, r, s, t \geq 0, \qquad i + j + k + r + s + t = n.$$

By this and induction Δ_n is spanned by the elements (5.1). We have shown that the elements (5.1) form a basis for the $\mathbb{F}$-vector space Δ_n. ■

Let V denote a vector space over $\mathbb{F}$ and let U denote a subspace of V. By a *complement of* U *in* V we mean a subspace U' of V such that $V = U + U'$ (direct sum).

Corollary 5.5. *For all integers* $n \geq 1$ *the following is a basis for a complement of* Δ_{n-1} *in* Δ_n*:*

$$A^i B^j C^k \alpha^r \beta^s \gamma^t, \qquad i, j, k, r, s, t \geq 0, \qquad i + j + k + r + s + t = n.$$

Proof. Use Theorem 5.4. ■

6 The Casimir element Ω

We turn our attention to the center $Z(\Delta)$. In this section we discuss a certain element $\Omega \in Z(\Delta)$ called the Casimir element. The name is motivated by [65, equation (1.3)]. In Section 7 we will use Ω to describe $Z(\Delta)$. We acknowledge that the results of this section are extensions of [29, Lemma 1], [40, Section 6], [65, equation (1.3)].

Lemma 6.1. *The following elements of* Δ *coincide:*

$$qABC + q^2A^2 + q^{-2}B^2 + q^2C^2 - qA\alpha - q^{-1}B\beta - qC\gamma,$$
$$qBCA + q^2A^2 + q^2B^2 + q^{-2}C^2 - qA\alpha - qB\beta - q^{-1}C\gamma,$$

$$qCAB + q^{-2}A^2 + q^2B^2 + q^2C^2 - q^{-1}A\alpha - qB\beta - qC\gamma,$$
$$q^{-1}CBA + q^{-2}A^2 + q^2B^2 + q^{-2}C^2 - q^{-1}A\alpha - qB\beta - q^{-1}C\gamma,$$
$$q^{-1}ACB + q^{-2}A^2 + q^{-2}B^2 + q^2C^2 - q^{-1}A\alpha - q^{-1}B\beta - qC\gamma,$$
$$q^{-1}BAC + q^2A^2 + q^{-2}B^2 + q^{-2}C^2 - qA\alpha - q^{-1}B\beta - q^{-1}C\gamma.$$

We denote this common element by Ω.

Proof. Denote the displayed sequence of elements by Ω_B^+, Ω_C^+, Ω_A^+, Ω_B^-, Ω_C^-, Ω_A^-. The automorphism ρ cyclically permutes Ω_A^+, Ω_B^+, Ω_C^+ and cyclically permutes Ω_A^-, Ω_B^-, Ω_C^-. The element $\Omega_B^+ - \Omega_C^-$ is equal to $(q - q^{-1})A$ times

$$\big(q + q^{-1}\big)A + \frac{qBC - q^{-1}CB}{q - q^{-1}} - \alpha. \tag{6.1}$$

The element (6.1) is zero by Definition 1.3 so $\Omega_B^+ = \Omega_C^-$. In this equation we apply ρ twice to get $\Omega_C^+ = \Omega_A^-$ and $\Omega_A^+ = \Omega_B^-$. The element $\Omega_B^+ - \Omega_A^-$ is equal to

$$\big(q + q^{-1}\big)C + \frac{qAB - q^{-1}BA}{q - q^{-1}} - \gamma \tag{6.2}$$

times $(q - q^{-1})C$. The element (6.2) is zero by Definition 1.3 so $\Omega_B^+ = \Omega_A^-$. Applying ρ twice we get $\Omega_C^+ = \Omega_B^-$ and $\Omega_A^+ = \Omega_C^-$. By these comments Ω_B^+, Ω_C^+, Ω_A^+, Ω_B^-, Ω_C^-, Ω_A^- coincide. ∎

Theorem 6.2. *The element* Ω *from Lemma 6.1 is central in* Δ.

Proof. We first show $\Omega A = A\Omega$. We will work with the equations (1.3), (1.4) from Definition 1.3. Consider the equation which is qC times (1.3) plus (1.3) times $q^{-1}C$ minus γ times (1.3) plus β times (1.4) minus $q^{-1}B$ times (1.4) minus (1.4) times qB. After some cancellation this equation yields $\Omega_B^+A - A\Omega_C^+ = 0$, where Ω_B^+, Ω_C^+ are from the proof of Lemma 6.1. Therefore $\Omega A = A\Omega$. One similarly finds $\Omega B = B\Omega$ and $\Omega C = C\Omega$. The elements A, B, C generate Δ so Ω is central in Δ. ∎

Definition 6.3. We call Ω the *Casimir* element of Δ.

Theorem 6.4. *The Casimir element* Ω *is fixed by everything in* $\mathrm{PSL}_2(\mathbb{Z})$.

Proof. Since ρ, σ generate $\mathrm{PSL}_2(\mathbb{Z})$ it suffices to show that each of ρ, σ fixes Ω. We use the notation Ω_A^+, Ω_B^+ from the proof of Lemma 6.1. Observe that ρ fixes Ω since $\rho(\Omega_A^+) = \Omega_B^+$. To verify that σ fixes Ω we show that $\sigma(\Omega_B^+) = \Omega_A^+$. For notational convenience define

$$C' = C + \frac{AB - BA}{q - q^{-1}}.$$

By Theorem 3.1 and the definition Ω_B^+,

$$\sigma(\Omega_B^+) = qBAC' + q^2B^2 + q^{-2}A^2 + q^2C'^2 - qB\beta - q^{-1}A\alpha - qC'\gamma.$$

By this and the definition of Ω_A^+,

$$\sigma(\Omega_B^+) - \Omega_A^+ = \big(BA + q^{-1}C + qC' - \gamma\big)qC' - qC\big(AB + qC + q^{-1}C' - \gamma\big).$$

In the above equation each parenthetical expression is zero so $\sigma(\Omega_B^+) = \Omega_A^+$. Therefore σ fixes Ω. ∎

7 A basis for Δ that involves Ω

In Theorem 4.1 we displayed a basis for Δ. In this section we display a related basis for Δ that involves the Casimir element Ω. In the next section we will use the related basis to describe the center $Z(\Delta)$.

Recall the filtration $\{\Delta_n\}_{n=0}^{\infty}$ of Δ from Definition 5.1.

Lemma 7.1. *For all integers $\ell \geq 1$ the following hold:*

(i) $\Omega^\ell \in \Delta_{3\ell}$.

(ii) $\Omega^\ell - q^{\ell^2} A^\ell B^\ell C^\ell \in \Delta_{3\ell-1}$.

Proof. Consider the expression for Ω from the first displayed line of Lemma 6.1. The term $qABC$ is in Δ_3 and the remaining terms are in Δ_2. Therefore $\Omega \in \Delta_3$ and $\Omega - qABC \in \Delta_2$. By this and Lemma 5.2(*iii*) we find $\Omega^\ell \in \Delta_{3\ell}$ and $\Omega^\ell - q^\ell (ABC)^\ell \in \Delta_{3\ell-1}$. Using Lemma 5.3 we obtain $(ABC)^\ell - q^{\ell(\ell-1)} A^\ell B^\ell C^\ell \in \Delta_{3\ell-1}$. By these comments $\Omega^\ell - q^{\ell^2} A^\ell B^\ell C^\ell \in \Delta_{3\ell-1}$. ■

Lemma 7.2. *For all integers $n \geq 1$ the following is a basis for a complement of Δ_{n-1} in Δ_n:*

$$A^i B^j C^k \Omega^\ell \alpha^r \beta^s \gamma^t, \qquad i,j,k,\ell,r,s,t \geq 0, \qquad ijk = 0, \qquad i+j+k+3\ell+r+s+t = n.$$

Proof. Let $\mathbb{I}_n$ denote the set consisting of the 6-tuples (i,j,k,r,s,t) of nonnegative integers whose sum is n. By Corollary 5.5 the following is a basis for a complement of Δ_{n-1} in Δ_n:

$$A^i B^j C^k \alpha^r \beta^s \gamma^t, \qquad (i,j,k,r,s,t) \in \mathbb{I}_n.$$

Let $\mathbb{J}_n$ denote the set consisting of the 7-tuples (i,j,k,ℓ,r,s,t) of nonnegative integers such that $ijk = 0$ and $i+j+k+3\ell+r+s+t = n$. Observe that the map

$$\mathbb{J}_n \to \mathbb{I}_n, \qquad (i,j,k,\ell,r,s,t) \mapsto (i+\ell, j+\ell, k+\ell, r, s, t)$$

is a bijection. Suppose we are given $(i,j,k,\ell,r,s,t) \in \mathbb{J}_n$. By Lemma 7.1, Δ_{n-1} contains

$$A^i B^j C^k \Omega^\ell \alpha^r \beta^s \gamma^t - q^{\ell^2} A^i B^j C^k A^\ell B^\ell C^\ell \alpha^r \beta^s \gamma^t.$$

By Lemma 5.3, Δ_{n-1} contains

$$A^i B^j C^k A^\ell B^\ell C^\ell \alpha^r \beta^s \gamma^t - q^{2j\ell} A^{i+\ell} B^{j+\ell} C^{k+\ell} \alpha^r \beta^s \gamma^t.$$

Therefore Δ_{n-1} contains

$$A^i B^j C^k \Omega^\ell \alpha^r \beta^s \gamma^t - q^{\ell(2j+\ell)} A^{i+\ell} B^{j+\ell} C^{k+\ell} \alpha^r \beta^s \gamma^t.$$

By these comments the following is a basis for a complement of Δ_{n-1} in Δ_n:

$$A^i B^j C^k \Omega^\ell \alpha^r \beta^s \gamma^t, \qquad (i,j,k,\ell,r,s,t) \in \mathbb{J}_n.$$

The result follows. ■

Note 7.3. Pick an integer $n \geq 1$. In Corollary 5.5 and Lemma 7.2 we mentioned a complement of Δ_{n-1} in Δ_n. These complements are not the same in general.

Proposition 7.4. *For all integers $n \geq 0$ the following is a basis for the $\mathbb{F}$-vector space Δ_n:*

$$A^i B^j C^k \Omega^\ell \alpha^r \beta^s \gamma^t, \qquad i,j,k,\ell,r,s,t \geq 0, \qquad ijk = 0, \qquad i+j+k+3\ell+r+s+t \leq n.$$

Proof. By Lemma 7.2 and $\Delta_0 = \mathbb{F}1$. ■

Theorem 7.5. *The following is a basis for the $\mathbb{F}$-vector space Δ:*

$$A^i B^j C^k \Omega^\ell \alpha^r \beta^s \gamma^t, \qquad i,j,k,\ell,r,s,t \geq 0, \qquad ijk = 0.$$

Proof. Combine Lemma 5.2(*ii*) and Proposition 7.4. ■

8 The center $Z(\Delta)$

In this section we give a detailed description of the center $Z(\Delta)$, under the assumption that q is not a root of unity. For such q we show that $Z(\Delta)$ is generated by Ω, α, β, γ and isomorphic to a polynomial algebra in four variables.

Recall the commutator $[r,s]=rs-sr$.

Lemma 8.1. *Let i, j, k denote nonnegative integers. Then Δ_{i+j+k} contains each of the following:*

$$\big[A,A^iB^jC^k\big]-\big(1-q^{2j-2k}\big)A^{i+1}B^jC^k, \tag{8.1}$$
$$\big[B,A^iB^jC^k\big]-\big(q^{2i}-q^{2k}\big)A^iB^{j+1}C^k, \tag{8.2}$$
$$\big[C,A^iB^jC^k\big]-\big(q^{2j-2i}-1\big)A^iB^jC^{k+1}. \tag{8.3}$$

Proof. Concerning (8.1), observe

$$\big[A,A^iB^jC^k\big]=A^{i+1}B^jC^k-A^iB^jC^kA.$$

By Lemma 5.3 Δ_{i+j+k} contains

$$A^iB^jC^kA-q^{2j-2k}A^{i+1}B^jC^k.$$

By these comments Δ_{i+j+k} contains (8.1). One similarly finds that Δ_{i+j+k} contains (8.2), (8.3). ■

Theorem 8.2. *Assume that q is not a root of unity. Then the following is a basis for the $\mathbb{F}$-vector space $Z(\Delta)$.*

$$\Omega^\ell\alpha^r\beta^s\gamma^t, \qquad \ell,r,s,t\geq 0. \tag{8.4}$$

Proof. Abbreviate $Z=Z(\Delta)$. The elements (8.4) are linearly independent by Theorem 7.5, so it suffices to show that they span Z. Let Z' denote the subspace of Δ spanned by (8.4), and note that $Z'\subseteq Z$. To show $Z'=Z$, we assume that Z' is properly contained in Z and get a contradiction. Define the set $E=Z\backslash Z'$ and note $E\neq\varnothing$. We have $\Delta_0=\mathbb{F}1\subseteq Z'$ so $E\cap\Delta_0=\varnothing$. By this and Lemma 5.2(*i*),(*ii*) there exists a unique integer $n\geq 1$ such that $E\cap\Delta_n\neq\varnothing$ and $E\cap\Delta_{n-1}=\varnothing$. Fix $u\in E\cap\Delta_n$. Let $S=S(u)$ denote the set of 6-tuples (i,j,k,r,s,t) of nonnegative integers whose sum is n and $A^iB^jC^k\alpha^r\beta^s\gamma^t$ contributes to u in the sense of Definition 4.2. By (4.3) and Corollary 5.5, Δ_{n-1} contains

$$u-\sum_{(i,j,k,r,s,t)\in S}\big(u,A^iB^jC^k\alpha^r\beta^s\gamma^t\big)A^iB^jC^k\alpha^r\beta^s\gamma^t. \tag{8.5}$$

We are going to show that $i=j=k$ for all $(i,j,k,r,s,t)\in S$. To this end we first claim that $j=k$ for all $(i,j,k,r,s,t)\in S$. To prove the claim, take the commutator of A with (8.5) and evaluate the result using the following facts. By construction $u\in E\subseteq Z$ so $[A,u]=0$. By Lemma 5.2(*iii*) $A\Delta_{n-1}\subseteq\Delta_n$ and $\Delta_{n-1}A\subseteq\Delta_n$ so $[A,\Delta_{n-1}]\subseteq\Delta_n$. Moreover each of α, β, γ is central. The above evaluation shows that Δ_n contains

$$\sum_{(i,j,k,r,s,t)\in S}\big(u,A^iB^jC^k\alpha^r\beta^s\gamma^t\big)\big[A,A^iB^jC^k\big]\alpha^r\beta^s\gamma^t.$$

Pick $(i,j,k,r,s,t)\in S$. By Lemma 5.2(*iii*) and Lemma 8.1, Δ_n contains

$$\big[A,A^iB^jC^k\big]\alpha^r\beta^s\gamma^t-\big(1-q^{2j-2k}\big)A^{i+1}B^jC^k\alpha^r\beta^s\gamma^t.$$

By the above comments Δ_n contains

$$\sum_{(i,j,k,r,s,t)\in S} \big(u, A^i B^j C^k \alpha^r \beta^s \gamma^t\big)\big(1-q^{2j-2k}\big) A^{i+1} B^j C^k \alpha^r \beta^s \gamma^t.$$

For all $(i,j,k,r,s,t) \in S$ the element $A^{i+1}B^jC^k\alpha^r\beta^s\gamma^t$ is contained in the basis for the complement of Δ_n in Δ_{n+1} given in Corollary 5.5. Therefore

$$\big(u, A^i B^j C^k \alpha^r \beta^s \gamma^t\big)\big(1-q^{2j-2k}\big) = 0 \qquad \forall\ (i,j,k,r,s,t) \in S.$$

By the definition of S we have $(u, A^iB^jC^k\alpha^r\beta^s\gamma^t) \neq 0$ for all $(i,j,k,r,s,t) \in S$. Therefore $1 - q^{2j-2k} = 0$ for all $(i,j,k,r,s,t) \in S$. The scalar q is not a root of unity so $j = k$ for all $(i,j,k,r,s,t) \in S$. The claim is proved. We next claim that $i = j$ for all $(i,j,k,r,s,t) \in S$. This claim is proved like the previous one, except that as we begin the argument below (8.5), we use C instead of A in the commutator. By the two claims $i = j = k$ for all $(i,j,k,r,s,t) \in S$. In light of this we revisit the assertion above (8.5) and conclude that Δ_{n-1} contains

$$u - \sum_{(i,i,i,r,s,t)\in S} \big(u, A^i B^i C^i \alpha^r \beta^s \gamma^t\big) A^i B^i C^i \alpha^r \beta^s \gamma^t.$$

Pick $(i,i,i,r,s,t) \in S$. By Lemma 5.2(*iii*) and Lemma 7.1, Δ_{n-1} contains

$$A^i B^i C^i \alpha^r \beta^s \gamma^t - q^{-i^2} \Omega^i \alpha^r \beta^s \gamma^t.$$

By these comments Δ_{n-1} contains

$$u - \sum_{(i,i,i,r,s,t)\in S} q^{-i^2} \big(u, A^i B^i C^i \alpha^r \beta^s \gamma^t\big) \Omega^i \alpha^r \beta^s \gamma^t.$$

In the above expression let ψ denote the main sum, so that $u - \psi \in \Delta_{n-1}$. Observe $\psi \in Z' \subseteq Z$. Recall $u \in E = Z \backslash Z'$ so $u \in Z$ and $u \notin Z'$. By these comments $u - \psi \in Z$ and $u - \psi \notin Z'$. Therefore $u - \psi \in E$ so $u - \psi \in E \cap \Delta_{n-1}$. This contradicts $E \cap \Delta_{n-1} = \varnothing$ so $Z = Z'$. The result follows. ■

We mention two corollaries of Theorem 8.2.

Corollary 8.3. *Assume that q is not a root of unity. Then $Z(\Delta)$ is generated by Ω, α, β, γ.*

Let $\{\lambda_i\}_{i=1}^4$ denote mutually commuting indeterminates. Let $\mathbb{F}[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$ denote the $\mathbb{F}$-algebra consisting of the polynomials in $\{\lambda_i\}_{i=1}^4$ that have all coefficients in $\mathbb{F}$.

Corollary 8.4. *Assume that q is not a root of unity. Then there exists an $\mathbb{F}$-algebra isomorphism $Z(\Delta) \to \mathbb{F}[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$ that sends*

$$\Omega \mapsto \lambda_1, \qquad \alpha \mapsto \lambda_2, \qquad \beta \mapsto \lambda_3, \qquad \gamma \mapsto \lambda_4.$$

9 The q-Onsager algebra $\mathcal{O}$

In the theory of tridiagonal pairs there is an algebra known as the tridiagonal algebra [55, Definition 3.9]. This algebra is defined using several parameters, and for a certain value of these parameters the algebra is sometimes called the q-Onsager algebra $\mathcal{O}$ [15, Section 4]. Our next goal is to show how $\mathcal{O}$ and Δ are related. In this section we define $\mathcal{O}$ and discuss some of its properties. In the next section we will relate $\mathcal{O}$ and Δ.

Definition 9.1 ([55, Definition 3.9]). Let $\mathcal{O} = \mathcal{O}_q$ denote the $\mathbb{F}$-algebra defined by generators X, Y and relations

$$X^3Y - [3]_q X^2YX + [3]_q XYX^2 - YX^3 = -\big(q^2 - q^{-2}\big)^2(XY - YX), \tag{9.1}$$

$$Y^3X - [3]_q Y^2XY + [3]_q YXY^2 - XY^3 = -\big(q^2 - q^{-2}\big)^2(YX - XY). \tag{9.2}$$

We call $\mathcal{O}$ the *q-Onsager algebra.*

The following definition is motivated by Theorem 2.2.

Definition 9.2. Let ξ_1, ξ_2 denote the following elements of $\mathcal{O}$:

$$\xi_1 = XY - YX,$$
$$\xi_2 = X^2Y^2 - Y^2X^2 + \big(q^2 + q^{-2}\big)(YXYX - XYXY).$$

Referring to Definition 9.2, we are going to show that ξ_1, ξ_2 commute if and only if $q^6 \neq 1$. We will use the following results, which apply to any $\mathbb{F}$-algebra.

Lemma 9.3. *Let x, y denote elements in any $\mathbb{F}$-algebra, and consider the commutator*

$$\big[xy - yx, x^2y^2 - y^2x^2 + \big(q^2 + q^{-2}\big)(yxyx - xyxy)\big]. \tag{9.3}$$

(*i*) *The element (9.3) is equal to*

$$xyx^2y^2 - x^2y^2xy + yxy^2x^2 - y^2x^2yx + x^2y^3x - xy^3x^2 + y^2x^3y - yx^3y^2$$
$$- \big(q^2 + q^{-2}\big)\big(xyxy^2x - xy^2xyx + yxyx^2y - yx^2yxy\big).$$

(*ii*) *The element (9.3) times $[3]_q$ is equal to*

$$[y, [y, [x, [x, [x, y]\,]_q]_{q^{-1}}]_q]_{q^{-1}} + [x, [x, [y, [y, [y, x]\,]_q]_{q^{-1}}]_q]_{q^{-1}}, \tag{9.4}$$

where $[u, v]_\epsilon$ means $\epsilon uv - \epsilon^{-1}vu$.

Proof. (*i*) Expand (9.3) and simplify the result. (*ii*) Expand (9.4) and compare it with the expression in (*i*) above. ∎

We return our attention to the elements ξ_1, ξ_2 in $\mathcal{O}$.

Proposition 9.4. *Referring to Definition 9.2, the elements ξ_1, ξ_2 commute if and only if $q^6 \neq 1$.*

Proof. First assume $q^6 \neq 1$, so that $[3]_q$ is nonzero. Applying Lemma 9.3(*ii*) to the elements $x = X$ and $y = Y$ in the algebra $\mathcal{O}$, we find that $[\xi_1, \xi_2]$ times $[3]_q$ is equal to

$$[Y, [Y, [X, [X, [X, Y]\,]_q]_{q^{-1}}]_q]_{q^{-1}} + [X, [X, [Y, [Y, [Y, X]\,]_q]_{q^{-1}}]_q]_{q^{-1}}. \tag{9.5}$$

We show that the element (9.5) is zero. Observe

$$[X, [X, [X, Y]\,]_q]_{q^{-1}} = X^3Y - [3]_q X^2YX + [3]_q XYX^2 - YX^3$$
$$= -\big(q^2 - q^{-2}\big)^2[X, Y] = \big(q^2 - q^{-2}\big)^2[Y, X].$$

Similarly

$$[Y, [Y, [Y, X]\,]_q]_{q^{-1}} = \big(q^2 - q^{-2}\big)^2[X, Y].$$

Therefore

$$[Y,[Y,[X,[X,[X,Y]\,]_q]_{q^{-1}}]_q]_{q^{-1}} = \big(q^2-q^{-2}\big)^2[Y,[Y,[Y,X]\,]_q]_{q^{-1}} = (q^2-q^{-2})^4[X,Y].$$

Similarly

$$[X,[X,[Y,[Y,[Y,X]\,]_q]_{q^{-1}}]_q]_{q^{-1}} = \big(q^2-q^{-2}\big)^4[Y,X].$$

By these comments the element (9.5) is zero, and therefore ξ_1, ξ_2 commute.

Next assume $q^6=1$, so that $[3]_q=0$ and $(q^2-q^{-2})^2=-3$. In this case the relations (9.1), (9.2) become

$$X^3Y - YX^3 = 3(XY-YX), \qquad Y^3X - XY^3 = 3(YX-XY).$$

In the above line the relation on the left (resp. right) asserts that X^3-3X (resp. Y^3-3Y) commutes with Y (resp. X) and is therefore central in $\mathcal{O}$. We show that ξ_1, ξ_2 do not commute by displaying an $\mathcal{O}$-module on which $[\xi_1,\xi_2]$ is nonzero. Define a group G to be the free product $H*K$ where each of H, K is a cylic group of order 3. Let h (resp. k) denote a generator for H (resp. K). Let $\mathbb{F}G$ denote the group $\mathbb{F}$-algebra. We give $\mathbb{F}G$ an $\mathcal{O}$-module structure. To do this we specify the action of X, Y on the basis G of $\mathbb{F}G$. For the identity $1\in G$ let $X.1=h$ and $Y.1=k$. For $1\neq g\in G$ write $g=g_1g_2\cdots g_n$ such that for $1\le i\le n$, $g_i\in\{h,h^{-1}\}$ for one parity of i and $g_i\in\{k,k^{-1}\}$ for the other parity of i. Let X, Y act on g as follows:

Case	$g_1=h$	$g_1=h^{-1}$	$g_1=k$	$g_1=k^{-1}$
$X.g$	hg	$3h^{-1}g$	hg	hg
$Y.g$	kg	kg	kg	$3k^{-1}g$

We have now specified the actions of X, Y on G. These actions give $\mathbb{F}G$ an $\mathcal{O}$-module structure on which $X^3=3X$ and $Y^3=3Y$. For the $\mathcal{O}$-module $\mathbb{F}G$ we now apply $[\xi_1,\xi_2]$ to the vector 1. By Lemma 9.3(i) and the construction, the element $[\xi_1,\xi_2]$ sends 1 to

$$\begin{aligned} &hkh^{-1}k^{-1} - h^{-1}k^{-1}hk + khk^{-1}h^{-1} - k^{-1}h^{-1}kh + 3h^{-1}kh - 3hkh^{-1} + 3k^{-1}hk - 3khk^{-1}\\ &\quad + hkhk^{-1}h - hk^{-1}hkh + khkh^{-1}k - kh^{-1}khk. \end{aligned}$$

The above element of $\mathbb{F}G$ is nonzero. Therefore $[\xi_1,\xi_2]\neq 0$ so ξ_1, ξ_2 do not commute. ∎

10 How $\mathcal{O}$ and Δ are related

Recall the q-Onsager algebra $\mathcal{O}$ from Definition 9.1. In this section we discuss how $\mathcal{O}$ and Δ are related.

Definition 10.1. Let λ denote an indeterminate that commutes with everything in $\mathcal{O}$. Let $\mathcal{O}[\lambda]$ denote the $\mathbb{F}$-algebra consisting of the polynomials in λ that have all coefficients in $\mathcal{O}$. We view $\mathcal{O}$ as an $\mathbb{F}$-subalgebra of $\mathcal{O}[\lambda]$.

Lemma 10.2. *There exists a unique* $\mathbb{F}$*-algebra homomorphism* $\varphi:\mathcal{O}[\lambda]\to\Delta$ *that sends*

$$X\to A, \qquad Y\to B, \qquad \lambda\to\gamma.$$

Moreover φ *is surjective.*

Proof. Compare Theorem 2.2 and Definition 9.1. ∎

Let J denote the 2-sided ideal of $\mathcal{O}[\lambda]$ generated by $(q-q^{-1})^2\xi_1\lambda+\xi_2$, where ξ_1, ξ_2 are from Definition 9.2. By Theorem 2.2, J is the kernel of φ. Therefore φ induces an isomorphism of $\mathbb{F}$-algebras $\mathcal{O}[\lambda]/J\to\Delta$.

Theorem 10.3. *The $\mathbb{F}$-algebra Δ is isomorphic to $\mathcal{O}[\lambda]/J$, where J is the 2-sided ideal of $\mathcal{O}[\lambda]$ generated by $(q-q^{-1})^2\xi_1\lambda+\xi_2$.*

We now adjust our point of view.

Definition 10.4. Let $\phi : \mathcal{O} \to \Delta$ denote the $\mathbb{F}$-algebra homomorphism that sends $X \mapsto A$ and $Y \mapsto B$. Observe that ϕ is the restriction of φ to $\mathcal{O}$.

We now describe the image and kernel of the homomorphism ϕ in Definition 10.4. We will use the following notation.

Definition 10.5. For any subset $\mathcal{S} \subseteq \Delta$ let $\langle \mathcal{S} \rangle$ denote the $\mathbb{F}$-subalgebra of Δ generated by $\mathcal{S}$.

Lemma 10.6. *For the homomorphism $\phi : \mathcal{O} \to \Delta$ from Definition 10.4, the image is $\langle A, B\rangle$. The kernel is $\mathcal{O} \cap J$, where J is defined above Theorem 10.3.*

Proof. Routine. ∎

We are going to show that ϕ is not injective. To do this we display some nonzero elements in the kernel $\mathcal{O} \cap J$.

Lemma 10.7. *$\mathcal{O} \cap J$ contains the elements*

$$\xi_1 z\xi_2 - \xi_2 z\xi_1, \qquad z \in \mathcal{O}.$$

Here ξ_1, ξ_2 are from Definition 9.2.

Proof. Let z be given. Each of z, ξ_1, ξ_2 is in $\mathcal{O}$ so $\xi_1 z\xi_2 - \xi_2 z\xi_1 \in \mathcal{O}$. The element $\xi_1 z\xi_2 - \xi_2 z\xi_1$ is equal to

$$\xi_1 z\big((q-q^{-1})^2\xi_1\lambda+\xi_2\big) - \big((q-q^{-1})^2\xi_1\lambda+\xi_2\big)z\xi_1$$

and is therefore contained in J. The result follows. ∎

We now display some elements z in $\mathcal{O}$ such that $\xi_1 z\xi_2 - \xi_2 z\xi_1$ is nonzero. For general q we cannot take $z = 1$ in view of Proposition 9.4, so we proceed to the next simplest case.

Lemma 10.8. *The following elements of $\mathcal{O}$ are nonzero:*

$$\xi_1 X\xi_2 - \xi_2 X\xi_1, \qquad \xi_1 Y\xi_2 - \xi_2 Y\xi_1.$$

Moreover $\mathcal{O} \cap J$ is nonzero.

Proof. To show $\xi_1 X\xi_2 - \xi_2 X\xi_1$ is nonzero we display an $\mathcal{O}$-module on which $\xi_1 X\xi_2 - \xi_2 X\xi_1$ is nonzero. This $\mathcal{O}$-module is a variation on an $\mathcal{O}$-module due to M. Vidar [58, Theorem 9.1]. Define $\theta_i = q^{2i}+q^{-2i}$ for $i = 0, 1, 2$. Define $\vartheta = (q^4-q^{-4})(q^2-q^{-2})(q-q^{-1})^2$. Adapting [58, Theorem 9.1] there exists a four-dimensional $\mathcal{O}$-module V with the following property: V has a basis with respect to which the matrices representing X, Y are

$$X : \begin{pmatrix} \theta_0 & 0 & 0 & 0 \\ 1 & \theta_1 & 0 & 0 \\ 0 & 0 & \theta_1 & 0 \\ 0 & 1 & 0 & \theta_2 \end{pmatrix}, \qquad Y : \begin{pmatrix} \theta_0 & \vartheta & q & 0 \\ 0 & \theta_1 & 0 & 0 \\ 0 & 0 & \theta_1 & 1 \\ 0 & 0 & 0 & \theta_2 \end{pmatrix}.$$

Consider the matrix that represents $\xi_1 X\xi_2 - \xi_2 X\xi_1$ with respect to the above basis. For this matrix the $(4,3)$-entry is $-q^2$. This entry is nonzero so $\xi_1 X\xi_2 - \xi_2 X\xi_1$ is nonzero. Interchanging the roles of X, Y in the above argument, we see that $\xi_1 Y\xi_2 - \xi_2 Y\xi_1$ is nonzero. The result follows. ∎

Theorem 10.9. *The homomorphism $\phi : \mathcal{O} \to \Delta$ from Definition 10.4 is not injective.*

Proof. Combine Lemma 10.6 and Lemma 10.8. ∎

11 The 2-sided ideal $\Delta[\Delta,\Delta]\Delta$

We will be discussing the following subspace of Δ:

$$[\Delta,\Delta] = \text{Span}\{[u,v] \mid u,v \in \Delta\}.$$

Observe that $\Delta[\Delta,\Delta]\Delta$ is the 2-sided ideal of Δ generated by $[\Delta,\Delta]$. In this section we describe this ideal from several points of view.

Let $\overline{A}$, $\overline{B}$, $\overline{C}$ denote mutually commuting indeterminates. Let $\mathbb{F}[\overline{A},\overline{B},\overline{C}]$ denote the $\mathbb{F}$-algebra consisting of the polynomials in $\overline{A}$, $\overline{B}$, $\overline{C}$ that have all coefficients in $\mathbb{F}$.

Lemma 11.1. *There exists a unique $\mathbb{F}$-algebra homomorphism $\Delta \to \mathbb{F}[\overline{A},\overline{B},\overline{C}]$ that sends*

$$A \mapsto \overline{A}, \qquad B \mapsto \overline{B}, \qquad C \mapsto \overline{C}.$$

This homomorphism is surjective.

Proof. The algebra $\mathbb{F}[\overline{A},\overline{B},\overline{C}]$ is commutative, so each of

$$\overline{A} + \frac{q\overline{B}\,\overline{C} - q^{-1}\overline{C}\,\overline{B}}{q^2-q^{-2}}, \qquad \overline{B} + \frac{q\overline{C}\,\overline{A} - q^{-1}\overline{A}\,\overline{C}}{q^2-q^{-2}}, \qquad \overline{C} + \frac{q\overline{A}\,\overline{B} - q^{-1}\overline{B}\,\overline{A}}{q^2-q^{-2}}$$

is central in $\mathbb{F}[\overline{A},\overline{B},\overline{C}]$. Consequently $\overline{A}$, $\overline{B}$, $\overline{C}$ satisfy the defining relations for Δ from Definition 1.2. Therefore the homomorphism exists. The homomorphism is unique since A, B, C generate Δ. The homomorphism is surjective since $\overline{A}$, $\overline{B}$, $\overline{C}$ generate $\mathbb{F}[\overline{A},\overline{B},\overline{C}]$. ■

Definition 11.2. Referring to the map $\Delta \to \mathbb{F}[\overline{A},\overline{B},\overline{C}]$ from Lemma 11.1, for all $u \in \Delta$ let $\overline{u}$ denote the image of u.

Lemma 11.3. *We have*

$$\overline{\alpha} = \big(q+q^{-1}\big)\overline{A} + \overline{B}\,\overline{C}, \qquad \overline{\beta} = \big(q+q^{-1}\big)\overline{B} + \overline{C}\,\overline{A},$$
$$\overline{\gamma} = \big(q+q^{-1}\big)\overline{C} + \overline{A}\,\overline{B}, \qquad \overline{\Omega} = -\big(q+q^{-1}\big)\overline{A}\,\overline{B}\,\overline{C} - \overline{A}^2 - \overline{B}^2 - \overline{C}^2.$$

Proof. The assertions about α, β, γ follow from Definition 1.3. The assertion about Ω follows from its definition in Lemma 6.1. ■

Proposition 11.4. *The following coincide:*

(*i*) *The 2-sided ideal $\Delta[\Delta,\Delta]\Delta$;*

(*ii*) *The kernel of the homomorphism $\Delta \to \mathbb{F}[\overline{A},\overline{B},\overline{C}]$ from Lemma 11.1.*

Proof. Let Γ denote the kernel of the homomorphism $\Delta \to \mathbb{F}[\overline{A},\overline{B},\overline{C}]$ and note that Γ is a 2-sided ideal of Δ. We have $[\Delta,\Delta] \subseteq \Gamma$ since $\mathbb{F}[\overline{A},\overline{B},\overline{C}]$ is commutative, and $\Delta[\Delta,\Delta]\Delta \subseteq \Gamma$ since Γ is a 2-sided ideal of Δ. The elements $\{\overline{A}^i\overline{B}^j\overline{C}^k | i,j,k \geq 0\}$ form a basis for the $\mathbb{F}$-vector space $\mathbb{F}[\overline{A},\overline{B},\overline{C}]$. Therefore the elements $\{A^iB^jC^k | i,j,k \geq 0\}$ form a basis for a complement of Γ in Δ. Denote this complement by M, so the sum $\Delta = M + \Gamma$ is direct. We now show $\Delta = M + \Delta[\Delta,\Delta]\Delta$. The $\mathbb{F}$-algebra Δ is generated by A, B, C. Therefore the $\mathbb{F}$-vector space Δ is spanned by elements of the form $x_1x_2\cdots x_n$ where $n \geq 0$ and $x_i \in \{A,B,C\}$ for $1 \leq i \leq n$. Let $x_1x_2\cdots x_n$ denote such an element. Then $\Delta[\Delta,\Delta]\Delta$ contains

$$x_1\cdots x_{i-1}x_i\cdots x_n - x_1\cdots x_ix_{i-1}\cdots x_n$$

for $2 \leq i \leq n$. Therefore $\Delta[\Delta,\Delta]\Delta$ contains

$$x_1x_2\cdots x_n - A^iB^jC^k,$$

where i, j, k denote the number of times A, B, C appear among $x_1,x_2,\dots,x_n$. Observe $A^iB^jC^k \in M$ so $x_1x_2\cdots x_n \in M + \Delta[\Delta,\Delta]\Delta$. Therefore $\Delta = M + \Delta[\Delta,\Delta]\Delta$. We already showed $\Delta[\Delta,\Delta]\Delta \subseteq \Gamma$ and the sum $\Delta = M+\Gamma$ is direct. By these comments $\Delta[\Delta,\Delta]\Delta = \Gamma$. ■

Recall the $\mathrm{PSL}_2(\mathbb{Z})$-action on Δ from Theorem 3.1. We now relate this action to the homomorphism $\Delta \to \mathbb{F}[\overline{A}, \overline{B}, \overline{C}]$ from Lemma 11.1.

Lemma 11.5. *The group* $\mathrm{PSL}_2(\mathbb{Z})$ *acts on* $\mathbb{F}[\overline{A}, \overline{B}, \overline{C}]$ *as a group of automorphisms in the following way:*

u	$\overline{A}$	$\overline{B}$	$\overline{C}$
$\rho(u)$	$\overline{B}$	$\overline{C}$	$\overline{A}$
$\sigma(u)$	$\overline{B}$	$\overline{A}$	$\overline{C}$

Proof. $\mathbb{F}[\overline{A}, \overline{B}, \overline{C}]$ has an automorphism of order 3 that sends $(\overline{A}, \overline{B}, \overline{C}) \to (\overline{B}, \overline{C}, \overline{A})$, and an automorphism of order 2 that sends $(\overline{A}, \overline{B}, \overline{C}) \to (\overline{B}, \overline{A}, \overline{C})$. ■

Lemma 11.6. *For* $g \in \mathrm{PSL}_2(\mathbb{Z})$ *the following diagram commutes:*

$$\begin{array}{ccc} \Delta & \xrightarrow{u \mapsto \overline{u}} & \mathbb{F}[\overline{A}, \overline{B}, \overline{C}] \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ \Delta & \xrightarrow[u \mapsto \overline{u}]{} & \mathbb{F}[\overline{A}, \overline{B}, \overline{C}] \end{array}$$

Proof. Without loss we may assume that g is one of ρ, σ. By Theorem 3.1 the action of ρ on Δ cyclically permutes A, B, C. By Lemma 11.5 the action of ρ on $\mathbb{F}[\overline{A}, \overline{B}, \overline{C}]$ cyclically permutes $\overline{A}$, $\overline{B}$, $\overline{C}$. By Theorem 3.1 the action of σ on Δ swaps A, B and fixes γ. By Lemma 11.3 and Lemma 11.5, the action of σ on $\mathbb{F}[\overline{A}, \overline{B}, \overline{C}]$ swaps $\overline{A}$, $\overline{B}$ and fixes $\overline{\gamma}$. By these comments the diagram commutes. ■

Definition 11.7. By Lemma 11.5 each element of $\mathrm{PSL}_2(\mathbb{Z})$ permutes $\overline{A}$, $\overline{B}$, $\overline{C}$. This induces a group homomorphism from $\mathrm{PSL}_2(\mathbb{Z})$ onto the symmetric group S_3. Let $\mathbb{P}$ denote the kernel of this homomorphism. Thus $\mathbb{P}$ is a normal subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ and the quotient group $\mathrm{PSL}_2(\mathbb{Z})/\mathbb{P}$ is isomorphic to S_3.

Our last main goal is to show that

$$\Delta[\Delta, \Delta]\Delta + \mathbb{F}1 = \langle A, B\rangle \cap \langle B, C\rangle \cap \langle A, C\rangle. \tag{11.1}$$

Note that in (11.1) the sum on the left is direct; otherwise the ideal $\Delta[\Delta, \Delta]\Delta$ contains 1 and is therefore equal to Δ, contradicting Proposition 11.4.

Definition 11.8. For notational convenience abbreviate $\mathbb{O} = \langle A, B\rangle$. Thus $\mathbb{O}$ is the image of $\mathcal{O}$ under the homomoprhism ϕ from Definition 10.4.

Lemma 11.9. $\Delta = \sum_{n=0}^{\infty} \mathbb{O}\gamma^n$.

Proof. The algebra Δ is generated by $\mathbb{O}$, γ. Moreover γ is central in Δ. ■

Note 11.10. The sum in Lemma 11.9 is not direct, by the third displayed equation in Theorem 2.2.

Note that $\mathbb{O}[A, B]\mathbb{O}$ is the 2-sided ideal of $\mathbb{O}$ generated by $[A, B]$.

Lemma 11.11. *The following (i)–(iii) hold.*

(i) $[\mathbb{O}, \mathbb{O}] \subseteq \mathbb{O}[A, B]\mathbb{O}$.

(ii) $[A, B]\gamma \in [\mathbb{O}, \mathbb{O}]$.

(iii) $\mathbb{O}[A, B]\mathbb{O}\gamma \subseteq \mathbb{O}[A, B]\mathbb{O}$.

Proof. (i) Abbreviate $R = \mathbb{O}[A,B]\mathbb{O}$ and consider the quotient algebra $\mathbb{O}/R$. The elements A, B generate $\mathbb{O}$, and these generators satisfy $[A,B] \in R$. Therefore $A+R$, $B+R$ generate $\mathbb{O}/R$, and these generators commute. This shows that $\mathbb{O}/R$ is commutative. Consider the canonical map $\mathbb{O} \to \mathbb{O}/R$. This map has kernel R. The map sends $[\mathbb{O},\mathbb{O}] \mapsto 0$ since $\mathbb{O}/R$ is commutative. Therefore $[\mathbb{O},\mathbb{O}] \subseteq R$.

(ii) In the third displayed equation of Theorem 2.2, the expression on the right is a nonzero scalar multiple of $[A,B]\gamma$. The expression on the left is equal to $[A^2,B^2] + (q^2+q^{-2})[B,ABA]$ and is therefore in $[\mathbb{O},\mathbb{O}]$. The result follows.

(iii) By (i), (ii) above and since γ is central. ■

Lemma 11.12. *$\mathbb{O}[A,B]\mathbb{O}$ is a 2-sided ideal of Δ.*

Proof. Abbreviate $R = \mathbb{O}[A,B]\mathbb{O}$. We show $\Delta R \subseteq R$ and $R\Delta \subseteq R$. Recall that Δ is generated by $\mathbb{O}$, γ. By construction $\mathbb{O}R \subseteq R$ and $R\mathbb{O} \subseteq R$. By Lemma 11.11(iii) and since γ is central we have $\gamma R \subseteq R$ and $R\gamma \subseteq R$. By these comments $\Delta R \subseteq R$ and $R\Delta \subseteq R$. ■

Lemma 11.13. *We have*

$$\mathbb{O}[A,B]\mathbb{O} = \Delta[\Delta,\Delta]\Delta.$$

Proof. We have $\mathbb{O} \subseteq \Delta$ and $[A,B] \in [\Delta,\Delta]$ so $\mathbb{O}[A,B]\mathbb{O} \subseteq \Delta[\Delta,\Delta]\Delta$. We now show the reverse inclusion. To this end we analyze $[\Delta,\Delta]$ using Lemma 11.9. For integers $m,n \geq 0$,

$$[\mathbb{O}\gamma^m, \mathbb{O}\gamma^n] = [\mathbb{O},\mathbb{O}]\gamma^{m+n} \overset{\text{by Lemma 11.11}(i)}{\subseteq} \mathbb{O}[A,B]\mathbb{O}\gamma^{m+n} \overset{\text{by Lemma 11.12}}{\subseteq} \mathbb{O}[A,B]\mathbb{O}.$$

By this and Lemma 11.9 we obtain $[\Delta,\Delta] \subseteq \mathbb{O}[A,B]\mathbb{O}$. Now using Lemma 11.12 we obtain $\Delta[\Delta,\Delta]\Delta \subseteq \mathbb{O}[A,B]\mathbb{O}$. The result follows. ■

In the algebra $\mathbb{F}[\overline{A},\overline{B},\overline{C}]$ let $\mathbb{F}[\overline{A},\overline{B}]$ (resp. $\mathbb{F}[\overline{B},\overline{C}]$) (resp. $\mathbb{F}[\overline{A},\overline{C}]$) denote the subalgebra generated by $\overline{A}$, $\overline{B}$ (resp. $\overline{B}$, $\overline{C}$) (resp. $\overline{A}$, $\overline{C}$).

Proposition 11.14. *Referring to the homomorphism $\Delta \to \mathbb{F}[\overline{A},\overline{B},\overline{C}]$ from Lemma 11.1,*

- (i) *$\langle A,B\rangle$ is the preimage of $\mathbb{F}[\overline{A},\overline{B}]$;*
- (ii) *$\langle B,C\rangle$ is the preimage of $\mathbb{F}[\overline{B},\overline{C}]$;*
- (iii) *$\langle A,C\rangle$ is the preimage of $\mathbb{F}[\overline{A},\overline{C}]$.*

Proof. (i) Recall $\mathbb{O} = \langle A,B\rangle$. For the homomorphism in the proposition statement the image of $\mathbb{O}$ is $\mathbb{F}[\overline{A},\overline{B}]$. Therefore the preimage of $\mathbb{F}[\overline{A},\overline{B}]$ is $\mathbb{O}$ plus the kernel. The kernel is $\Delta[\Delta,\Delta]\Delta$ by Proposition 11.4, and this is contained in $\mathbb{O}$ by Lemma 11.13. Therefore $\mathbb{O}$ is the preimage of $\mathbb{F}[\overline{A},\overline{B}]$.

(ii), (iii) Apply ρ twice to everything in part (i) above. ■

Theorem 11.15. $\Delta[\Delta,\Delta]\Delta + \mathbb{F}1 = \langle A,B\rangle \cap \langle B,C\rangle \cap \langle A,C\rangle$.

Proof. By Proposition 11.4, Proposition 11.14, and since $\mathbb{F}[\overline{A},\overline{B}] \cap \mathbb{F}[\overline{B},\overline{C}] \cap \mathbb{F}[\overline{A},\overline{C}] = \mathbb{F}1$. ■

We finish with some comments related to Proposition 11.14 and Theorem 11.15.

Proposition 11.16. *In the table below, each space U is a subalgebra of Δ that contains $\Delta[\Delta,\Delta]\Delta$. The elements to the right of U form a basis for a complement of $\Delta[\Delta,\Delta]\Delta$ in U.*

U	basis for a complement of $\Delta[\Delta,\Delta]\Delta$ in U	
Δ	$A^iB^jC^k$	$i,j,k\geq 0$
$\langle A,B\rangle$	A^iB^j	$i,j\geq 0$
$\langle B,C\rangle$	B^jC^k	$j,k\geq 0$
$\langle A,C\rangle$	A^iC^k	$i,k\geq 0$
$\langle A,B\rangle\cap\langle A,C\rangle$	A^i	$i\geq 0$
$\langle A,B\rangle\cap\langle B,C\rangle$	B^j	$j\geq 0$
$\langle A,C\rangle\cap\langle B,C\rangle$	C^k	$k\geq 0$
$\langle A,B\rangle\cap\langle B,C\rangle\cap\langle A,C\rangle$	1	

Proof. By Proposition 11.4 and Proposition 11.14. ■

Proposition 11.17. *The automorphisms ρ and σ permute the subalgebras*

$$\langle A,B\rangle,\qquad \langle B,C\rangle,\qquad \langle A,C\rangle \tag{11.2}$$

in the following way:

U	$\langle A,B\rangle$	$\langle B,C\rangle$	$\langle A,C\rangle$
$\rho(U)$	$\langle B,C\rangle$	$\langle A,C\rangle$	$\langle A,B\rangle$
$\sigma(U)$	$\langle A,B\rangle$	$\langle A,C\rangle$	$\langle B,C\rangle$

Proof. By Lemmas 11.5, 11.6 and Proposition 11.14. ■

By Proposition 11.17 the action of $\mathrm{PSL}_2(\mathbb{Z})$ on Δ induces an action of $\mathrm{PSL}_2(\mathbb{Z})$ on the 3-element set (11.2). The kernel of this action is the group $\mathbb{P}$ from Definition 11.7.

Corollary 11.18. *Each of the subalgebras*

$$\langle A,B\rangle,\qquad \langle B,C\rangle,\qquad \langle A,C\rangle$$

is invariant under the group $\mathbb{P}$ from Definition 11.7.

12 Directions for further research

In this section we give some suggestions for further research. Recall the algebra Δ from Definition 1.2.

Problem 12.1. Recall from the Introduction that Δ was originally motivated by the Askey–Wilson polynomials. These polynomials are the most general family in a master class of orthogonal polynomials called the Askey scheme [37]. For each polynomial family in the Askey scheme, there should be an analog of Δ obtained from the appropriate version of AW(3) by interpreting parameters as central elements. Investigate these other algebras along the lines of the present paper.

Problem 12.2. For this problem assume the characteristic of $\mathbb{F}$ is not 2. By Theorem 3.1 and Theorem 3.13 the group $\mathrm{PSL}_2(\mathbb{Z})$ acts faithfully on Δ as a group of automorphisms. This action induces an injection of groups $\mathrm{PSL}_2(\mathbb{Z})\to\mathrm{Aut}(\Delta)$. This injection is not an isomorphism for the following reason. Given any element of Δ among A, B, C there exists a unique automorphism of Δ that fixes that element and changes the sign of the other two elements. This automorphism is not contained in the image of the above injection, because its induced action on $\mathbb{F}[\overline{A},\overline{B},\overline{C}]$ does not match the description given in Definition 11.7. The above three automorphisms are the nonidentity elements in a subgroup $\mathbb{K}\subseteq\mathrm{Aut}(\Delta)$ that is isomorphic to the Klein 4-group $\mathbb{Z}_2\times\mathbb{Z}_2$. Do $\mathbb{K}$ and $\mathrm{PSL}_2(\mathbb{Z})$ together generate $\mathrm{Aut}(\Delta)$?

Problem 12.3. View the $\mathbb{F}$-vector space Δ as a $\mathrm{PSL}_2(\mathbb{Z})$-module. Describe the irreducible $\mathrm{PSL}_2(\mathbb{Z})$-submodules of Δ. Is Δ a direct sum of irreducible $\mathrm{PSL}_2(\mathbb{Z})$-submodules?

Problem 12.4. Find all the 2-sided ideals of Δ. Which of these are $\mathrm{PSL}_2(\mathbb{Z})$-invariant?

Problem 12.5. Find all the $\mathrm{PSL}_2(\mathbb{Z})$-invariant subalgebras of Δ.

Problem 12.6. Describe the subalgebra of Δ consisting of the elements in Δ that are fixed by everything in the group $\mathbb{P}$ from Definition 11.7. This subalgebra contains $\langle \Omega, \alpha, \beta, \gamma \rangle$. Is this containment proper?

Problem 12.7. Describe the $\mathrm{PSL}_2(\mathbb{Z})$-submodule of Δ that is generated by $\langle A \rangle$. Also, describe the $\mathbb{P}$-submodule of Δ that is generated by $\langle A \rangle$.

Problem 12.8. Consider the basis for Δ given in Theorem 4.1 or Theorem 7.5. Find the matrices that represent ρ and σ with respect to this basis. Find the matrices that represent left-multiplication by A, B, C with respect to this basis. Hopefully the entries in the above matrices are attractive in some way. If not, then find a basis for Δ with respect to which the above matrix entries are attractive.

Problem 12.9. Find a basis for the center $Z(\Delta)$ under the assumption q is a root of unity.

Problem 12.10. Give a basis for the $\mathbb{F}$-vector space $\Delta[\Delta, \Delta]\Delta$.

Problem 12.11. Recall the homomorhism $\Delta \to \mathbb{F}[\overline{A}, \overline{B}, \overline{C}]$ from Lemma 11.1. Restrict this homomorphism to $Z(\Delta)$ or $\langle A, Z(\Delta) \rangle$. In each case find a basis for the kernel and image.

Problem 12.12. Each of the following is a commutative subalgebra of Δ; for each one give a basis and also a presentation by generators and relations.

(*i*) The intersection of $\langle A, B \rangle$ and $Z(\Delta)$.

(*ii*) The intersection of $\langle A, B \rangle$ and $\langle A, Z(\Delta) \rangle$.

(*iii*) The intersection of $\langle A, B \rangle$ and $\langle C, Z(\Delta) \rangle$.

Problem 12.13. Find a Hopf algebra structure for Δ. See [15, Proposition 4.1] and [18, Theorem 6.10] for some results in this direction.

Motivated by Theorem 3.1, let us view Δ as a Lie algebra with Lie bracket $[u, v] = uv - vu$ for all $u, v \in \Delta$.

Problem 12.14. Let L denote the Lie subalgebra of Δ generated by A, B, C. Show that

$$L \subseteq \mathbb{F}A + \mathbb{F}B + \mathbb{F}C + \Delta[\Delta, \Delta]\Delta.$$

Show that L is $\mathrm{PSL}_2(\mathbb{Z})$-invariant. Find a basis for the $\mathbb{F}$-vector space L. Describe $L \cap Z(\Delta)$. Give a presentation for L by generators and relations.

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