On Integrability of a Special Class of Two-Component (2+1)-Dimensional Hydrodynamic-Type Systems*

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Abstract. The particular case of the integrable two component (2+1)-dimensional hydrodynamical type systems, which generalises the so-called Hamiltonian subcase, is considered. The associated system in involution is integrated in a parametric form. A dispersionless Lax formulation is found.

Key words: hydrodynamic-type system; dispersionless Lax representation

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1 Introduction

Quasilinear (2+1)-dimensional systems of the first order,

$$A^i_k(\mathbf{u})u^k_t + B^i_k(\mathbf{u})u^k_y + C^i_k(\mathbf{u})u^k_x = 0,$$

play an important role in the description of variety of physical phenomena. The method of the hydrodynamical reductions (see e.g. [1]) enables us to pick from this class the integrable systems which possess sufficiently many hydrodynamic reductions, and thus infinitely many particular solutions. Recently, a system in involution describing the integrable (2+1)-dimensional hydrodynamical type systems

$$\begin{pmatrix} v \\ w \end{pmatrix}_t = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_y + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_x, \tag{1}$$

where A_{ik} and B_{ik} are functions of v and w, was derived in [2] using the method of hydrodynamic reductions. In a particular case

$$\begin{pmatrix} v \\ w \end{pmatrix}_t = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_y + \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_x, \tag{2}$$

where α and β are constants, the corresponding system in involution for two functions $r(v, w)$ and $q(v, w)$ simplifies to the form

$$q_{vv} = (qr)_w, \qquad q_{vw} = \frac{q_v q_w}{q} + q^2 r, \qquad q_{ww} = \frac{q_w^2}{q} + 2qq_v - \frac{q_w r_w}{r},$$

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$$r_{ww}=(qr)_v,\qquad r_{vw}=\frac{r_vr_w}{r}+r^2q,\qquad r_{vv}=\frac{r_v^2}{r}+2rr_w-\frac{q_vr_v}{q},\tag{3}$$

and a general solution therefore depends on 6 arbitrary constants. Two other functions $p(v,w)$ and $s(v,w)$ can be found in quadratures

$$dp=rqdv+q_vdw,\qquad ds=r_wdv+rqdw.\tag{4}$$

A general solution of system (3) was presented in [2] in a parametric form.

Under a simple linear transformation of independent variables (y,t) such that

$$\partial_t-\alpha\partial_y\rightarrow\partial_t,\qquad \partial_t-\beta\partial_y\rightarrow\partial_y,$$

system (2) reduces to the form

$$v_t=p(v,w)v_x+q(v,w)w_x,\qquad w_y=r(v,w)v_x+s(v,w)w_x.\tag{5}$$

Our first result is that the equations (5) reduces to the most compact form

$$v_t=\partial_xH_{ww},\qquad w_y=\partial_xH_{vv}\tag{6}$$

by introducing the potential function $H(v,w)$ such that $r=H_{vvv}$, $s=H_{vvw}$, $p=H_{vww}$, $q=H_{www}$. Then the equations (3) can be written in the form

$$\begin{aligned}
&H_{vvww}=H_{vvv}H_{www},\\
&H_{vvvvw}=\frac{H_{vvvv}H_{vvvw}}{H_{vvv}}+H_{vvv}^2H_{www},\\
&H_{vwwww}=\frac{H_{wwww}H_{vwww}}{H_{www}}+H_{www}^2H_{vvv},\\
&H_{vvvvv}=\frac{H_{vvvv}^2}{H_{vvv}}-\frac{H_{vvvv}H_{vwww}}{H_{www}}+2H_{vvv}H_{vvvw},\\
&H_{wwwww}=\frac{H_{wwww}^2}{H_{www}}-\frac{H_{wwww}H_{vvvw}}{H_{vvv}}+2H_{www}H_{vwww}.
\end{aligned}\tag{7}$$

Our second result is that we present a general solution of system in involution (7) in a new parametric form (33), which is more convenient for the investigation in the special cases. This approach is universal, i.e. any other (2+1)-dimensional quasilinear equations can be investigated effectively in the same fashion (see [6, 5]).

In the general case (1), a dispersionless Lax representation has the form (see [2])

$$\psi_t=a(\psi_x,v,w),\qquad \psi_y=c(\psi_x,v,w),\tag{8}$$

where $a(\mu,v,w)$ and $c(\mu,v,w)$ are some functions. The function ψ is called a pseudopotential. Equations (3) and (4) can be derived directly from the compatibility condition $(\psi_t)_y=(\psi_y)_t$.

Our third result is that the dispersionless Lax representation (8) for (2+1)-dimensional hydrodynamical type system (5) reduces to the most compact form

$$\psi_t=a(\psi_x,w),\qquad \psi_y=c(\psi_x,v),\tag{9}$$

i.e., each of the functions a and c depends on two arguments only.

Moreover, we were able to integrate the corresponding nonlinear system for the functions $a(\mu,w)$ and $c(\mu,v)$, i.e. we found the dispersionless Lax pair (9) for the (2+1)-dimensional quasilinear equations (6). In general, the compatibility condition $(\psi_t)_y=(\psi_y)_t$ for the system

$$\psi_t=a,\qquad \psi_y=c,$$

where $a = a(\mu, u^1, u^2, \ldots, u^N)$ and $c = c(\mu, u^1, u^2, \ldots, u^N)$ are some functions (cf. (9)), yields (2+1)-dimensional quasilinear systems of the nonlinear equations of the first order. Recently, a complete classification of pseudopotentials that satisfy the single equation $\psi_t = a(\psi_x, w)$ was given in [6].

Our fourth result is that we were able to extract the functions $a(\mu, w)$ and $c(\mu, v)$ such that (9) yields a system of the form (6).

Let us mention that the particular case, when $H_{vvv} = H_{www}$,

$$v_t = \partial_x h_v, \qquad w_y = \partial_x h_w. \tag{10}$$

was considered in [1] and a complete classification of all admissible functions $h(v, w)$ was presented in [4].

The paper is organized as follows. In Section 2, we prove that quasilinear system (6) admits the dispersionless Lax representation (9), which is a special case of (8), and conversely, a dispersionless Lax representation (9) yields an integrable (2+1)-dimensional quasilinear system (6). We also derive the associated system in involution (3) which in the case under study takes the form (7). The problem of computation of a single function $H(v, w)$ that solves (7) is reduced to quadratures. In Section 3, we present an effective method for direct integration of the system (3) and the corresponding equations are integrated in the parametric form. The last section contains the conclusion.

2 System in involution

A classification of integrable (2+1)-dimensional two-component Hamiltonian hydrodynamic-type systems (10) was obtained in [1] using the method of hydrodynamic reductions. In this paper, we use dispersionless Lax representations following the original papers [2] and [7] (see [5] for further details). It was proved in [2] that the pseudopotentials $\psi(x, y, t)$ for (2+1)-dimensional hydrodynamic-type systems (1) must satisfy the dispersionless Lax representation of the form (8). Also, it was proved in [4] that there exist the dispersionless Lax representations for (10) of the form (9). Moreover, in our case the dispersionless Lax representation (9) remains valid for more general case (5). Indeed, the following assertion holds.

Lemma 1. *The hydrodynamic-type system* (5) *can be obtained from the compatibility condition* $(\psi_t)_y = (\psi_y)_t$*, where the pseudopotential* ψ *satisfies dispersionless Lax representation* (9).

Proof. The compatibility condition for (8) implies

$$\partial_y a(\psi_x, v, w) = \partial_t c(\psi_x, v, w).$$

Substituting (5) into this equation yields

$$\begin{aligned} &a_\mu(c_v v_x + c_w w_x) + a_v v_y + a_w(r v_x + s w_x) \\ &\qquad = c_\mu(a_v v_x + a_w w_x) + c_v(p v_x + q w_x) + c_w w_t, \end{aligned}$$

where $\mu = \psi_x$. Since we are interested in a general solution, the coefficients at the t- and x-derivatives v and w on the left- and right-hand sides of the above equations must be identical, that is,

$$a_\mu c_v + r a_w = c_\mu a_v + p c_v, \qquad a_\mu c_w + s a_w = c_\mu a_w + q c_v,$$

and $a_v = 0, c_w = 0$. Thus, a is independent of v, and c is independent of w. This means that (8) reduces to (9). The lemma is proved. ■

Theorem 1. *The dispersionless Lax representation* (9) *uniquely determines a hydrodynamic-type system of the form* (5).

Proof. The compatibility condition $(\psi_t)_y = (\psi_y)_t$ implies

$$a_\mu c_v v_x + a_w w_y = c_\mu a_w w_x + c_v v_t, \tag{11}$$

where the dispersionless Lax representation (9) is re-written in the form

$$\mu_t = \partial_x a(\mu, w), \qquad \mu_y = \partial_x c(\mu, v) \tag{12}$$

Since $a_w \neq 0$, differentiating the equation (11)

$$\frac{a_\mu c_v}{a_w} v_x + w_y = c_\mu w_x + \frac{c_v}{a_w} v_t$$

with respect to μ yields

$$\left(\frac{a_\mu c_v}{a_w}\right)_\mu v_x = c_{\mu\mu} w_x + \left(\frac{c_v}{a_w}\right)_\mu v_t.$$

Suppose that $(c_v/a_w)_\mu \neq 0$, then the equation

$$v_t = \frac{\left(\frac{a_\mu c_v}{a_w}\right)_\mu}{\left(\frac{c_v}{a_w}\right)_\mu} v_x - \frac{c_{\mu\mu}}{\left(\frac{c_v}{a_w}\right)_\mu} w_x \tag{13}$$

cannot depend on μ. This means that (see (5)) we have

$$\frac{\left(\frac{a_\mu c_v}{a_w}\right)_\mu}{\left(\frac{c_v}{a_w}\right)_\mu} = p(v, w), \qquad \frac{c_{\mu\mu}}{\left(\frac{c_v}{a_w}\right)_\mu} = -q(v, w)$$

for some functions $p(v, w)$ and $q(v, w)$ that do not depend on μ. Then the above equation (13) reduces to the form (cf. (5))

$$v_t = p v_x + q w_x.$$

Since $c_v \neq 0$, two other relations of the above type,

$$\frac{a_{\mu\mu}}{\left(\frac{a_w}{c_v}\right)_\mu} = -r(v, w), \qquad \frac{\left(\frac{c_\mu a_w}{c_v}\right)_\mu}{\left(\frac{a_w}{c_v}\right)_\mu} = s(v, w),$$

where $r(v, w)$ and $s(v, w)$ are some other functions independent of μ, can be derived in analogy with the above, along with the second equation of the hydrodynamic-type system (5). Thus, the hydrodynamic-type system (5) is indeed uniquely determined by the compatibility condition $(\psi_t)_y = (\psi_y)_t$. The theorem is proved. ■

Substituting (5) into (11) implies the relations

$$a_w = \frac{q c_v}{s - c_\mu}, \qquad rq = (s - c_\mu)(p - a_\mu). \tag{14}$$

among the first derivatives of the functions $a(\mu, w)$ and $c(\mu, v)$.

The compatibility conditions $(a_w)_\mu = (a_\mu)_w$, $(a_w)_v = 0$ and $(a_\mu)_v = 0$ yield

$$\begin{aligned} c_{vv} &= \frac{p_v}{rq} c_\mu c_v + \frac{qr_v - sp_v}{rq} c_v, \\ c_{\mu v} &= \frac{p_v}{rq} c_\mu^2 + \left(\frac{q_v}{q} + \frac{r_v}{r} - 2\frac{sp_v}{rq} \right) c_\mu + \frac{s^2 p_v}{rq} - \left(\frac{q_v}{q} + \frac{r_v}{r} \right) s + s_v, \\ c_{\mu\mu} &= \frac{1}{qc_v} \left[\frac{p_v}{r}(c_\mu - s)^3 + \left(p_w + q_v + \frac{qr_v}{r} \right)(c_\mu - s)^2 + (rq_w + qr_w + qs_v)(c_\mu - s) + rqs_w \right]. \end{aligned} \tag{15}$$

Similar formulas can be obtained from the other compatibility conditions, namely, $(c_v)_\mu = (c_\mu)_v$, $(c_v)_w = 0$ and $(c_\mu)_w = 0$. The compatibility conditions $(c_{\mu\mu})_v = (c_{\mu v})_\mu$, $(c_{\mu v})_v = (c_{vv})_\mu$, $(a_{\mu\mu})_w = (a_{\mu w})_\mu$, $(a_{\mu w})_w = (a_{ww})_\mu$ yield the system

$$\begin{aligned} & p_{vv} = \frac{p_v}{rq}(rp_w + qr_v), \qquad p_{vw} = \frac{p_v}{rq}(rq_w + qr_w), \qquad p_{ww} = \frac{1}{rq}(rp_w q_w + qp_v s_w), \\ & s_{ww} = \frac{s_w}{rq}(rq_w + qs_v), \qquad s_{vw} = \frac{s_w}{rq}(rq_v + qr_v), \qquad s_{vv} = \frac{1}{rq}(qr_v s_v + rp_v s_w), \\ & rq_{vv} + qr_{vv} = \frac{1}{rq}(r^2 p_v q_w + q^2 r_v^2) + 2s_v p_v + r_w p_v - r_v q_v, \\ & rq_{ww} + qr_{ww} = \frac{1}{rq}(q^2 s_w r_v + r^2 q_w^2) + 2p_w s_w + q_v s_w - q_w r_w, \\ & rq_{vw} + qr_{vw} = \frac{1}{rq}(r^2 q_v q_w + q^2 r_v r_w) + 2p_v s_w. \end{aligned} \tag{16}$$

The last equation can be replaced by the pair of equations

$$r_{vw} = \frac{1}{rq}(rp_v s_w + qr_v r_w), \qquad q_{vw} = \frac{1}{rq}(rq_v q_w + qp_v s_w),$$

which can be obtained from the compatibility conditions $(p_{vw})_v = (p_{vv})_w$ and $(s_{vw})_w = (s_{ww})_v$. The second and fifth equations of (16) can be integrated once to yield

$$p_v = rq\varphi_1(v), \qquad s_w = rq\varphi_2(w), \tag{17}$$

where $\varphi_1(v)$ and $\varphi_2(w)$ are arbitrary functions. However, without loss of generality these functions can be set equal to 1, because these functions can be eliminated from all of the above equations upon using the scaling $\int \varphi_1(v)dv \to v$, $\int \varphi_2(w)dw \to w$, $\varphi_1(v)p/\varphi_2(w) \to q$, $\varphi_2(w)r/\varphi_1(v) \to r$. Thus, substituting $p_v = rq$, $s_w = rq$ into (16), we finally obtain the system in involution (3) together with (4) (cf. [2]).

Thus, integrable (2+1)-dimensional system (5) can be written in the most compact form (6), and the reduction to the Hamiltonian case (10) is given by the symmetric constraint $r = q$. The function H can be reconstructed via the complete differentials,

$$\begin{aligned} & dH_{vv} = rdv + sdw, \qquad dH_{vw} = sdv + pdw, \qquad dH_{ww} = pdv + qdw, \\ & dH_v = H_{vv}dv + H_{vw}dw, \qquad dH_w = H_{vw}dv + H_{ww}dw, \qquad dH = H_v dv + H_w dw. \end{aligned}$$

Remark 1. The above choice (17) uniquely fixes system (5) in conservative form (6). This means that the integrable (2+1)-dimensional hydrodynamic-type system (5) is reducible to (6) under appropriate choice of functions $\varphi_1(v)$ and $\varphi_2(w)$ in (17).

3 Dispersionless Lax representation

In this section, we obtain a general solution of the system in involution (7) and simultaneously reconstruct the functions $a(\mu, w)$ and $c(\mu, v)$ (see (9)).

Rewrite system (15) for the function $c(\mu, v)$ in the form

$$\begin{aligned} c_{vv} &= A_1 c_\mu c_v + A_2 c_v, \\ c_{\mu v} &= B_1 c_\mu^2 + B_2 c_\mu + B_3, \\ c_v c_{\mu\mu} &= D_1 c_\mu^3 + D_2 c_\mu^2 + D_3 c_\mu + D_4, \end{aligned} \tag{18}$$

where all the coefficients A_k, B_n, D_m depend on v alone and are to be determined. The compatibility conditions $(c_{vv})_\mu = (c_{\mu v})_v$, $(c_{\mu\mu})_v = (c_{\mu v})_\mu$ give rise to the following system for the coefficients A_k, B_n, D_m:

$$\begin{aligned} A_1 &= B_1 = D_1, \\ B_1' &= A_2 B_1 - 2B_1 B_2 + B_1 D_2, \\ B_2' &= A_2 B_2 - B_1 B_3 + B_1 D_3 - B_2^2, \\ B_3' &= A_2 B_3 + B_1 D_4 - B_2 B_3, \\ D_2' &= A_2 D_2 - 3B_1 B_3 + 2B_1 D_3 - B_2 D_2, \\ D_3' &= A_2 D_3 + 3B_1 D_4 - 2B_3 D_2, \\ D_4' &= A_2 D_4 + B_2 D_4 - B_3 D_3, \end{aligned} \tag{19}$$

where the prime denotes the derivative with respect to v.

The derivative c_v can be expressed from the last equation of (18). Plugging this expression into the l.h.s. of the second equation of (18) yields

$$\frac{c_{\mu\mu\mu}}{c_{\mu\mu}^2} = \frac{(3D_1 - B_1)c_\mu^2 + (2D_2 - B_2)c_\mu + D_3 - B_3}{D_1 c_\mu^3 + D_2 c_\mu^2 + D_3 c_\mu + D_4}, \tag{20}$$

which can be expanded into simple fractions (see [6] for the general case)

$$\frac{c_{\mu\mu\mu}}{c_{\mu\mu}^2} = \frac{k_1}{c_\mu - b_1} + \frac{k_2}{c_\mu - b_2} + \frac{k_3}{c_\mu - b_3}, \tag{21}$$

where $k_i(v)$ are some functions such that $k_1 + k_2 + k_3 = 2$, and the functions $b_k(v)$ are roots of the cubic polynomial in the last equation of (18):

$$c_v c_{\mu\mu} = D_1(c_\mu - b_1)(c_\mu - b_2)(c_\mu - b_3).$$

Conversely, upon comparing (20) with (21) the above coefficients A_k, B_n, D_m can be expressed via the functions $b_k(v)$ in the symmetric form

$$-B_2/D_1 = k_1 b_1 + k_2 b_2 + k_3 b_3, \qquad B_3/D_1 = (1 - k_1)b_2 b_3 + (1 - k_2)b_1 b_3 + (1 - k_3)b_1 b_2,$$
$$-D_2/D_1 = b_1 + b_2 + b_3, \qquad D_3/D_1 = b_1 b_2 + b_1 b_3 + b_2 b_3, \qquad -D_4/D_1 = b_1 b_2 b_3.$$

Lemma 2. *k_i are constants.*

Proof. Integrating (21) yields the following equation:

$$c_{\mu\mu} = b(c_\mu - b_1)^{k_1}(c_\mu - b_2)^{k_2}(c_\mu - b_3)^{k_3},$$

where $b(v)$ is a function of v alone. Then (see the last equation in (18)) we have

$$c_v = \frac{D_1}{b}(c_\mu - b_1)^{1-k_1}(c_\mu - b_2)^{1-k_2}(c_\mu - b_3)^{1-k_3}. \tag{22}$$

The compatibility condition $(c_v)_{\mu\mu} = (c_{\mu\mu})_v$ is satisfied only if k_i are constants. The lemma is proved. ■

Theorem 2. *Under the above substitutions, the system (19) reduces to the form*

$$b_i' = D_1(1-k_i)\prod_{j\neq i}(b_i - b_j), \qquad i = 1, 2, 3, \tag{23}$$

In this case we have

$$A_2 = D_1'/D_1 + D_1(b_1 + b_2 + b_3) - 2D_1(k_1b_1 + k_2b_2 + k_3b_3). \tag{24}$$

Proof. The coefficient A_2 in the form (24) can be expressed from the second equation of (19). The last three equations of (19) are linear with respect to the first derivatives b_k'. Solving this linear system with respect to b_k immediately yields (23). Moreover, the third and fourth equations of (19) are then automatically satisfied. The theorem is proved. ■

Let us choose $A_1 = B_1 = D_1 = 1$ in this formulation (see formulae (18), (19), (23), (24)) in agreement with the normalization (17). This means that a solution of the system (see (23), where $D_1 = 1$)

$$b_i' = (1-k_i)\prod_{j\neq i}(b_i - b_j), \qquad i = 1, 2, 3 \tag{25}$$

determines the coefficients p, q, r, s of (2+1)-dimensional integrable hydrodynamic-type system (5) written in the form (6).

Introducing the "intermediate" independent variable $V(v)$ such that

$$V' = \xi V^{k_1}(1-V)^{k_3}, \tag{26}$$

where ξ is an arbitrary constant we obtain the following theorem:

Theorem 3. *General solution of system (25) can be written in the form*

$$b_2 = b_1 + \xi V^{k_1-1}(1-V)^{k_3}, \qquad b_3 = b_1 + \xi V^{k_1-1}(1-V)^{k_3-1},$$
$$b_1 = (1-k_1)\xi\int V^{k_1-2}(1-V)^{k_3-1}dV.$$

Proof. Introduce the auxiliary functions $b_{12} = b_2 - b_1$ and $b_{13} = b_3 - b_1$. Then the system (25) reduces to the form

$$b_1' = (1-k_1)b_{12}b_{13}, \qquad b_{12}' = b_{12}[(1-k_2)b_{12} - k_3b_{13}],$$
$$b_{13}' = b_{13}[(1-k_3)b_{13} - k_2b_{12}]. \tag{27}$$

The ratio of the last two equations

$$\frac{d\ln b_{12}}{d\ln b_{13}} = \frac{(1-k_2)b_{12} - k_3b_{13}}{(1-k_3)b_{13} - k_2b_{12}}$$

is nothing but a first-order ODE. Substituting the intermediate function $V = 1 - b_{12}/b_{13}$ into this ODE reduces to the following quadrature:

$$d\ln b_{13} = (k_3 - 1)d\ln(1-V) + (k_1 - 1)d\ln V.$$

Taking into account that $b_{12} = (1-V)b_{13}$, one can obtain the equality $b_{12} = \xi V^{k_1-1}(1-V)^{k_3}$, where the above quadrature is integrated in the parametric form $b_{13} = \xi V^{k_1-1}(1-V)^{k_3-1}$. Substituting these expressions for b_{12} and b_{13} into the first equation of (27), $b_1' = (1-k_1)b_{12}b_{13}$, yields the quadrature

$$db_1 = (1-k_1)\xi V^{k_1-2}(1-V)^{k_3-1}dV.$$

The remaining equations in (27) yield (26). The theorem is proved. ■

In turn, comparing the expressions for $c_{\mu\mu}$, $c_{\mu v}$, and c_{vv} from (15) with their counterparts (18) and equating the coefficients at the powers of c_μ and c_v gives rise the following quadratures:

$$dp = q[rdv + (D_2 - B_2 + s)dw], \tag{28}$$

$$d\ln q = \frac{2s^2 + (2D_2 - B_2)s + D_3 - B_3}{r}dw + (B_2 + 2s)dv - d\ln r, \tag{29}$$

where

$$r = \frac{s^3 + D_2 s^2 + D_3 s + D_4}{s_w}, \tag{30}$$

and the function $s(v, w)$ satisfies the Riccati equation

$$s_v = s^2 + B_2 s + B_3. \tag{31}$$

This equation can be reduced to the linear ODE

$$f_v = [k_3(b_3 - b_1) + k_2(b_2 - b_1)]f - 1$$

by the substitution $s = b_1 + 1/f(v, w)$ with the solution given by

$$f = \frac{1 - VW}{b_2 - b_1},$$

where $W(w)$ is an integration "constant". With this in mind, the function r can be found from (30). In turn, the function $W(w)$ cannot be found from the compatibility conditions $\partial_v(\partial_w \ln q) = \partial_w(\partial_v \ln q)$ (see (29)) or $\partial_v(p_w) = \partial_w(p_v)$ (see (28)). A substitution of $q = s_w/r$ (see (4)) into (29) yields an equation

$$W' = \bar{\xi} W^{k_2} (1 - W)^{k_3} \tag{32}$$

which is similar to (26). Here $\bar{\xi}$ is an arbitrary constant. Then the two quadratures (28) and (29) can be performed explicitly.

Thus, the functions p, q, r, s are given by the following expressions:

$$\begin{aligned}
s &= \xi \frac{V^{k_1-1}(1 - V)^{k_3}}{1 - VW} + (1 - k_1)\xi \int V^{k_1-2}(1 - V)^{k_3-1} dV, \\
r &= -\frac{\xi^2}{\bar{\xi}} \frac{W^{1-k_2}(1 - W)^{1-k_3}}{1 - VW} V^{2k_1-1}(1 - V)^{2k_3-1}, \\
q &= -\frac{\bar{\xi}^2}{\xi} W^{2k_2-1}(1 - W)^{2k_3-1} \frac{V^{1-k_1}(1 - V)^{1-k_3}}{1 - VW}, \\
p &= \bar{\xi} \frac{W^{k_2-1}(1 - W)^{k_3}}{1 - VW} + (1 - k_2)\bar{\xi} \int W^{k_2-2}(1 - W)^{k_3-1} dW.
\end{aligned} \tag{33}$$

This means that the function $H(v, w)$ (see (7)) is determined via its third derivatives (see the end of Section 2). Thus, we proved that a single function $c(\mu, v)$ completely determines a (2+1)-dimensional quasilinear system (6), and the second function $a(\mu, w)$ is determined via the first derivatives of c:

$$da = \frac{qc_v}{s - c_\mu}dw + \left(p - \frac{rq}{s - c_\mu}\right)d\mu. \tag{34}$$

In order to compute the function $c(\mu, v)$ we integrate the second equation in (18). Indeed, the equation in question (recall that we have $B_1 = 1$ in this normalization)

$$\partial_v c_\mu = c_\mu^2 + B_2 c_\mu + B_3$$

coincides with (31) up to the replacement $c_\mu \leftrightarrow s$. Since $D_1 = 1$, the compatibility condition $(c_v)_{\mu\mu} = (c_{\mu\mu})_v$ implies $b(v) = 1$ in (22). Thus, the derivative c_v can also be found. Finally, substituting (33) and just obtained expressions for c_μ, c_v into (34) yields the corresponding dispersionless Lax representation (see (12), (14), (18)) which is now determined by means of the formulas

$$c_\mu = \xi\frac{V^{k_1-1}(1-V)^{k_3}}{1-\epsilon V} + (1-k_1)\xi\int V^{k_1-2}(1-V)^{k_3-1}dV,$$
$$c_V = -\frac{\xi}{\tilde{\xi}}\epsilon^{1-k_2}(1-\epsilon)^{1-k_3}\frac{V^{k_1-1}(1-V)^{k_3-1}}{1-\epsilon V},$$
$$a_W = -\frac{\tilde{\xi}}{\xi}\epsilon^{1-k_1}(1-\epsilon)^{1-k_3}\frac{W^{k_2-1}(1-W)^{k_3-1}}{1-\epsilon W},$$
$$a_\mu = \tilde{\xi}\frac{W^{k_2-1}(1-W)^{k_3}}{1-\epsilon W} + (1-k_2)\tilde{\xi}\int W^{k_2-2}(1-W)^{k_3-1}dW,$$

where the auxiliary variable $\epsilon(\mu)$ is determined by the formula (cf. (32))

$$\epsilon'(\mu) = \xi_0\epsilon^{k_2}(1-\epsilon)^{k_3}.$$

4 Conclusion

As it was mentioned in the Introduction, the integrable (2+1)-dimensional quasilinear systems of the nonlinear first-order equations (see [5] for details) are determined by the compatibility condition $(\psi_t)_y = (\psi_y)_t$, where in general $\psi_t = a(\mu, u^1, u^2, \dots, u^N)$ and $\psi_y = c(\mu, u^1, u^2, \dots, u^N)$, cf. (8) and (9). An open problem is whether it is possible to construct hydrodynamic chains (see, for instance, [3]) associated with such (2+1)-dimensional quasilinear systems. The theory of integrable hydrodynamic chains is much simpler than the theory of integrable (2+1)-dimensional quasilinear equations, because the former still is a theory of integrable (1+1)-dimensional hydrodynamic-type systems with just one nontrivial extension – allowing for infinitely many components. Thus, an integrable hydrodynamic chain possesses the properties that are well-known in the theory of finite-component systems (dispersive or dispersionless), such as infinite series of conservation laws, infinite series of commuting flows, and infinite series of Hamiltonian structures.

At least, we can answer the above question regarding the construction of the associated hydrodynamic chain for the case of (9), but this will be the subject of a separate paper. Moreover, if we fix the first equation in (9), then an associated integrable hierarchy can be found from the dispersionless Lax representations (cf. (8) and (9))

$$\psi_t = a(\psi_x, w), \qquad \psi_{y^N} = c\big(\psi_x, v^1, v^2, \dots, v^N\big),$$

where the first member of this hierarchy is given by (6) and uniquely determined by the dispersionless Lax representation (9). This means that infinitely many commuting flows (numbered by the "times" y^k) will be determined. It would be interesting to find the associated hydrodynamic chains for the case when, instead of the above dispersionless Lax representation, one has a more general ansatz

$$\psi_t = a\big(\psi_x, w^1, w^2, \dots, w^M\big), \qquad \psi_{y^N} = c\big(\psi_x, v^1, v^2, \dots, v^N\big),$$

where M and N are arbitrary positive integers.

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