Integrable String Models in Terms of Chiral Invariants of $SU(n)$, $SO(n)$, $SP(n)$ Groups*

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Abstract. We considered two types of string models: on the Riemmann space of string coordinates with null torsion and on the Riemman–Cartan space of string coordinates with constant torsion. We used the hydrodynamic approach of Dubrovin, Novikov to integrable systems and Dubrovin solutions of WDVV associativity equation to construct new integrable string equations of hydrodynamic type on the torsionless Riemmann space of chiral currents in first case. We used the invariant local chiral currents of principal chiral models for $SU(n)$, $SO(n)$, $SP(n)$ groups to construct new integrable string equations of hydrodynamic type on the Riemmann space of the chiral primitive invariant currents and on the chiral non-primitive Casimir operators as Hamiltonians in second case. We also used Pohlmeyer tensor nonlocal currents to construct new nonlocal string equation.

Key words: string; integrable models; Poisson brackets; Casimir operators; chiral currents

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1 Introduction

String theory is a very promising candidate for a unified quantum theory of gravity and all the other forces of nature. For quantum description of string model we must have classical solutions of the string in the background fields. String theory in suitable space-time backgrounds can be considered as principal chiral model. The integrability of the classical principal chiral model is manifested through an infinite set of conserved charges, which can form non-Abelian algebra. Any charge from the commuting subset of charges and any Casimir operators of charge algebra can be considered as Hamiltonian in bi-Hamiltonian approach to integrable models. The bi-Hamiltonian approach to integrable systems was initiated by Magri [1]. Two Poisson brackets (PBs)

$$\{\phi^a(x), \phi^b(y)\}_0 = P_0^{ab}(x, y)(\phi), \qquad \{\phi^a(x), \phi^b(y)\}_1 = P_1^{ab}(x, y)(\phi)$$

are called compatible if any linear combination of these PBs

$$\{*, *\}_0 + \lambda\{*, *\}_1$$

is PB also for arbitrary constant λ. The functions $\phi^a(t, x)$, $a = 1, 2, \dots, n$ are local coordinates on a certain given smooth n-dimensional manifold M^n. The Hamiltonian operators $P_0^{ab}(x, y)(\phi)$, $P_1^{ab}(x, y)(\phi)$ are functions of local coordinates $\phi^a(x)$. It is possible to find such Hamiltonians H_0 and H_1 which satisfy bi-Hamiltonian condition [2]

$$\frac{d\phi^a(x)}{dt} = \{\phi^a(x), H_0\}_0 = \{\phi^a(x), H_1\}_1,$$

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where $H_M = \int_0^{2\pi} h_M(\phi(y))dy$, $M = 0, 1$. Two branches of hierarchies arise under two equations of motion under two different parameters of evolution t_{0M} and t_{M0} [2]

$$\frac{d\phi^a(x)}{dt_{01}} = \{\phi^a(x), H_0\}_1 = \int_0^{2\pi} P_1^{ab}(x,y)\frac{\partial h_0}{\partial\phi^b(y)}dy = \int_0^{2\pi} R_c^a(x,z)P_0^{cb}(z,y)\frac{\partial h_0}{\partial\phi^b(y)}dy,$$
$$\frac{d\phi^a(x)}{dt_{10}} = \{\phi^a(x), H_1\}_0 = \int_0^{2\pi} P_0^{ab}(x,y)\frac{\partial h_1}{\partial\phi^b(y)}dy = \int_0^{2\pi} (R^{-1})_c^a(x,z)P_0^{cb}(z,y)\frac{\partial h_0}{\partial\phi^b(y)}dy.$$

There $R_b^a(x,y)$ is a recursion operator and $(R^{-1})_b^a(x,y)$ is its inverse

$$R_c^a(x,y) = \int_0^{2\pi} P_1^{ab}(x,z)(P_0)^{-1}_{bc}(z,y)dz.$$

The first branch of the hierarchies of dynamical systems has the following form

$$\frac{d\phi^a(x)}{dt_{0N}} = \int_0^{2\pi}(R(x,y_1)\cdots R(y_{N-1}))_c^a P_0^{cb}(y_{N-1},y_N)\frac{\partial h_0}{\partial\phi^b(y_N)}dy_1\cdots dy_N, \quad N = 1,2,\dots,\infty.$$

The second branch of the hierarchies can be obtained by replacement $R \to R^{-1}$ and $t_{0N} \to t_{N0}$. We will consider only the first branch of the hierarchies.

The local PBs of hydrodynamic type were introduced by Dubrovin, Novikov [3, 4] for Hamiltonian description of equations of hydrodynamics. They were generalized by Ferapontov [5] and Mokhov, Ferapontov [6] to the non-local PBs of hydrodynamic type. The hydrodynamic type systems were considered by Tsarev [8], Maltsev [9], Ferapontov [10], Mokhov [12] (see also [7]), Pavlov [13] (see also [14]), Maltsev, Novikov [15]. The polynomials of local chiral currents were considered by Goldshmidt and Witten [16] (see also [17]). The local conserved chiral charges in principal chiral models were considered by Evans, Hassan, MacKay, Mountain [24]. The tensor nonlocal chiral charges were introduced by Pohlmeyer [26] (see also [27, 28]). The string models of hydrodynamic type were considered by author [18, 19]. In Section 3, the author applied hydrodynamic approach to integrable systems to obtain new integrable string equations. In Section 4, the author used the nonlocal Pohlmeyer charges to obtain a new string equation in terms of the nonlocal currents. In Section 5, the author applied the local invariant chiral currents to a simple Lie algebra to construct new integrable string equations.

2 String model of principal chiral model type

A string model is described by the Lagrangian

$$L = \frac{1}{2}\int_0^{2\pi} \eta^{\alpha\beta} g_{ab}(\phi(t,x))\frac{\partial\phi^a(t,x)}{\partial x^\alpha}\frac{\partial\phi^b(t,x)}{\partial x^\beta}dx \tag{1}$$

and by two first kind constraints

$$g_{ab}(\phi(x))\left[\frac{\partial\phi^a(x)}{\partial t}\frac{\partial\phi^b(x)}{\partial t} + \frac{\partial\phi^a(x)}{\partial x}\frac{\partial\phi^b(x)}{\partial x}\right] \approx 0, \qquad g_{ab}(\phi(x))\frac{\partial\phi^a(x)}{\partial t}\frac{\partial\phi^b(x)}{\partial x} \approx 0.$$

The target space local coordinates $\phi^a(x)$, $a = 1,\dots,n$ belong to certain given smooth n-dimensional manifold M^n with nondegenerate metric tensor

$$g_{ab}(\phi(x)) = \eta_{\mu\nu}e_a^\mu(\phi(x))e_b^\nu(\phi(x)),$$

where $\mu,\nu = 1,\dots,n$ are indices of tangent space to manifold M^n on some point $\phi^a(x)$. The veilbein $e_a^\mu(\phi)$ and its inverse $e_\mu^a(\phi)$ satisfy the conditions

$$e_a^\mu e_\mu^b = \delta_a^b, \qquad e_a^\mu e^{a\nu} = \eta^{\mu\nu}.$$

The coordinates x^α ($x^0 = t$, $x^1 = x$) belong to world sheet with metric tensor $g_{\alpha\beta}$ in conformal gauge. The string equations of motion have the form

$$\eta^{\alpha\beta}[\partial_{\alpha\beta}\phi^a + \Gamma^a_{bc}(\phi)\partial_\alpha\phi^b\partial_\beta\phi^c] = 0, \qquad \partial_\alpha = \frac{\partial}{\partial x^\alpha}, \qquad \alpha = 0, 1,$$

where

$$\Gamma^a_{bc}(\phi) = \frac{1}{2}e^a_\mu \left[\frac{\partial e^\mu_b}{\partial\phi^c} + \frac{\partial e^\mu_c}{\partial\phi^b}\right]$$

is the connection. In terms of canonical currents

$$J^\mu_\alpha(\phi) = e^\mu_a(\phi)\partial_\alpha\phi^a$$

the equations of motion have the form

$$\eta^{\alpha\beta}\partial_\alpha J^\mu_\beta(\phi(t,x)) = 0, \qquad \partial_\alpha J^\mu_\beta(\phi) - \partial_\beta J^\mu_\alpha(\phi) - C^\mu_{\nu\lambda}(\phi)J^\nu_\alpha(\phi)J^\lambda_\beta(\phi) = 0,$$

where

$$C^\mu_{\nu\lambda}(\phi) = e^a_\nu e^b_\lambda \left[\frac{\partial e^\mu_a}{\partial\phi^b} - \frac{\partial e^\mu_b}{\partial\phi^a}\right]$$

is the torsion. The Hamiltonian has the form

$$H = \frac{1}{2}\int_0^{2\pi}[\eta^{\mu\nu}J_{0\mu}J_{0\nu} + \eta_{\mu\nu}J^\mu_1 J^\nu_1]dx,$$

where $J_{0\mu}(\phi) = e^a_\mu(\phi)p_a$, $J^\mu_1(\phi) = e^\mu_a \frac{\partial}{\partial x}\phi^a$ and $p_a(t,x) = \eta_{\mu\nu}e^\mu_a e^\nu_b \frac{\partial}{\partial t}\phi^b$ is the canonical momentum. The canonical commutation relations of currents are as follows

$$\{J_{0\mu}(\phi(x)), J_{0\nu}(\phi(y))\} = C^\lambda_{\mu\nu}(\phi(x))J_{0\lambda}(\phi(x))\delta(x-y),$$
$$\{J_{0\mu}(\phi(x)), J^\nu_1(\phi(y))\} = C^\nu_{\mu\lambda}(\phi(x))J^\lambda_1(\phi(x))\delta(x-y) - \frac{1}{2}\delta^\nu_\mu\frac{\partial}{\partial x}\delta(x-y),$$
$$\{J^\mu_1(\phi(x)), J^\nu_1(\phi(y))\} = 0.$$

Let us introduce chiral currents

$$U^\mu = \eta^{\mu\nu}J_{0\nu} + J^\mu_1, \qquad V^\mu = \eta^{\mu\nu}J_{0\nu} - J^\mu_1$$

The commutation relations of chiral currents are the following

$$\{U^\mu(\phi(x)), U^\nu(\phi(y))\} = C^{\mu\nu}_\lambda(\phi(x))\left[\frac{3}{2}U^\lambda(\phi(x)) - \frac{1}{2}V^\lambda(\phi(x))\right]\delta(x-y) - \eta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y),$$
$$\{U^\mu(\phi(x)), V^\nu(\phi(y))\} = C^{\mu\nu}_\lambda(\phi(x))[U^\lambda(\phi(x)) + V^\lambda(\phi(x))]\delta(x-y),$$
$$\{V^\mu(\phi(x)), V^\nu(\phi(y))\} = C^{\mu\nu}_\lambda(\phi(x))\left[\frac{3}{2}V^\lambda(\phi(x)) - \frac{1}{2}U^\lambda(\phi(x))\right]\delta(x-y) + \eta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y).$$

Equations of motion in light-cone coordinates

$$x^\pm = \frac{1}{2}(t \pm x), \qquad \frac{\partial}{\partial x^\pm} = \frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}$$

have the form

$$\partial_- U^\mu = C^\mu_{\nu\lambda}(\phi(x))U^\nu V^\lambda, \qquad \partial_- V^\mu = C^\mu_{\nu\lambda}(\phi(x))V^\nu U^\lambda.$$

In the case of the null torsion

$$C^\mu_{\nu\lambda} = 0, \qquad e^\mu_a(\phi) = \frac{\partial e^\mu}{\partial\phi^a}, \qquad \Gamma^a_{bc}(\phi) = e^a_\mu\frac{\partial^2 e^\mu}{\partial\phi^b\partial\phi^c}, \qquad R^\mu_{\nu\lambda\rho}(\phi) = 0$$

the string model is integrable.The Hamiltonian equations of motion under Hamiltonian (1) are described by two independent left and right movers: $U^\mu(t+x)$ and $V^\mu(t-x)$.

3 Integrable string models of hydrodynamic type with null torsion

We want to construct new integrable string models with Hamiltonians as polynomials of the initial chiral currents $U^\mu(\phi(x))$. The PB of chiral currents $U^\mu(x)$ coincides with the flat PB of Dubrovin, Novikov

$$\{U^\mu(x), U^\nu(y)\}_0 = -\eta^{\mu\nu}\frac{\partial}{\partial x}\delta(x-y).$$

Let us introduce a local Dubrovin, Novikov PB [3, 4]. It has the form

$$\{U^\mu(x), U^\nu(y)\}_1 = g^{\mu\nu}(U(x))\frac{\partial}{\partial x}\delta(x-y) - \Gamma^{\mu\nu}_\lambda(U(x))\frac{\partial U^\lambda(x)}{\partial x}\delta(x-y).$$

This PB is skew-symmetric if $g^{\mu\nu}(U) = g^{\nu\mu}(U)$ and it satisfies Jacobi identity if $\Gamma^a_{bc}(U) = \Gamma^a_{cb}(U)$, $C^a_{bc}(U) = 0$, $R^a_{bcd}(U) = 0$. In the case of non-zero curvature tensor we must include Weingarten operators into right side of the PB with the step-function $\mathrm{sgn}\,(x-y){=}(\frac{d}{dx})^{-1}\delta(x-y){=}\nu(x-y)$ [5, 6]. The PBs $\{*,*\}_0$ and $\{*,*\}_1$ are compatible by Magri [1] if the pencil $\{*,*\}_0 + \lambda\{*,*\}_1$ is also PB . As a result, Mokhov [12, 11] obtained the compatible pair of PBs

$$P_{0\mu\nu}(U)(x,y) = -\eta_{\mu\nu}\frac{\partial}{\partial x}\delta(x-y),$$
$$P_{1\mu\nu}(U)(x,y) = 2\frac{\partial^2 F(U)}{\partial U^\mu \partial U^\nu}\frac{\partial}{\partial x}\delta(x-y) + \frac{\partial^3 F(U)}{\partial U^\mu \partial U^\nu \partial U^\lambda}\frac{\partial U^\lambda}{\partial x}\delta(x-y).$$

The function $F(U)$ satisfies the equation

$$\frac{\partial^3 F(U)}{\partial U^\mu \partial U^\rho \partial U^\lambda}\eta^{\lambda\rho}\frac{\partial^3 F(U)}{\partial U^\nu \partial U^\sigma \partial U^\rho} = \frac{\partial^3 F(U)}{\partial U^\nu \partial U^\rho \partial U^\lambda}\eta^{\lambda\rho}\frac{\partial^3 F(U)}{\partial U^\mu \partial U^\sigma \partial U^\rho}.$$

This equation is WDVV [20, 21] associativity equation and it was obtained in 2D topological field theory. Dubrovin [22, 23] obtained a lot of solutions of WDVV equation. He showed that local fields $U^\mu(x)$ must belong Frobenius manifolds to solve the WDVV equation and gave examples of Frobenius structures. Associative Frobenius algebra may be written in the following form

$$\frac{\partial}{\partial U^\mu} * \frac{\partial}{\partial U^\nu} := d^\lambda_{\mu\nu}(U)\frac{\partial}{\partial U^\lambda}.$$

Totally symmetric structure function has the form

$$d_{\mu\nu\lambda}(U) = \frac{\partial F(U)}{\partial U^\mu \partial U^\nu \partial U^\lambda}, \qquad \mu,\nu,\lambda = 1,\dots,n$$

and associativity condition

$$\left(\frac{\partial}{\partial U^\mu} * \frac{\partial}{\partial U^\nu}\right) * \frac{\partial}{\partial U^\lambda} = \frac{\partial}{\partial U^\mu} * \left(\frac{\partial}{\partial U^\nu} * \frac{\partial}{\partial U^\lambda}\right)$$

leads to the WDVV equation. Function $F(U)$ is quasihomogeneous function of its variables

$$\left(d_\mu U^\mu \frac{\partial}{\partial U^\mu}\right) F(U) = d_F F(U) + A_{\mu\nu}U^\mu U^\nu + B_\mu U^\mu + C,$$

here numbers d_μ, d_F, $A_{\mu\nu}$, B_μ, C depend on the type of polynomial function $F(U)$. Here are some Dubrovin examples of solutions of the WDVV equation

$$n = 1, \qquad F(U) = U_1^3;$$

$$n=2, \qquad F(U)=\frac{1}{2}U_1^2U_2+e^{U_2}, \qquad d_1=1, \qquad d_2=2, \qquad d_F=2, \qquad A_{11}=1 \tag{2}$$

and quasihomogeneity condition for $n=2$ has the form

$$\left(d_1U_1\frac{\partial}{\partial U_1}+d_2\frac{\partial}{\partial U_2}\right)F(U)=d_FF(U)+A_{11}U_1^2.$$

We used local fields U_μ with low indices here for convenience. One of the Dubrovin polynomial solutions is

$$F(U)=\frac{1}{2}(U_1^2U_3+U_1U_2^2)+\frac{1}{4}U_2^2U_3^2+\frac{1}{60}U_3^5, \tag{3}$$

here $d_1=1$, $d_2=\frac{3}{2}$, $d_3=2$, $d_F=4$ and the polynomial function

$$f(U_2,U_3)=\frac{1}{4}U_2^2U_3^2+\frac{1}{60}U_3^5$$

is a solution of the additional PDE.

In the bi-Hamiltonian approach to an integrable string model we must construct the recursion operator to generate a hierarchy of PBs and a hierarchy of Hamiltonians

$$\begin{aligned} R^\mu_\nu(x,y)&=\int_0^{2\pi}P_1^{\mu\lambda}(x,z)(P_0^{-1}(z,y))_{\lambda\nu}dz \\ &=2\frac{\partial^2F(U(x))}{\partial U^\mu(x)\partial U^\nu(x)}\delta(x-y)+\frac{\partial^3F(U(x))}{\partial U^\mu(x)\partial U^\nu(x)\partial U^\lambda(x)}\frac{\partial U^\lambda(y)}{\partial y}\nu(x-y).\end{aligned}$$

The Hamiltonian equation of motion with Hamiltonian H_0 is the following

$$H_0=\int_0^{2\pi}\eta_{\mu\nu}U^\mu(x)U^\nu(x)dx, \qquad \frac{\partial U^\mu}{\partial t}=\frac{\partial U^\mu}{\partial x}.$$

First of the new equations of motion under the new time t_1 has the form [12]

$$\frac{\partial U^\mu}{\partial t_1}=\int_0^{2\pi}R^\mu_\nu(x,y)\frac{\partial U^\mu(y)}{\partial y}dy=\eta^{\mu\nu}\frac{d}{dx}\left(\frac{\partial F(x)}{\partial U^\nu}\right). \tag{4}$$

This equation of motion can be obtained as Hamiltonian equation with new Hamiltonian H_1

$$H_1=\int_0^{2\pi}\frac{\partial F(U(x))}{\partial U^\mu}U^\mu(x)dx,$$

where $F(U)$ is each of Dubrovin solutions WDVV associativity equation (2), (3). Any system of the following hierarchy [12]

$$\frac{\partial U^\mu}{\partial t_M}=\int_0^{2\pi}(R(x,y_1)\cdots R(y_{M-1},y_M))^\mu_\nu\frac{\partial U^\nu}{\partial y_M}dy_1\cdots dy_M$$

is an integrable system. As result we obtain chiral currents $U^\mu(\phi(t_M,x))=f^\mu(\phi(t_M,x)$, where $f^\mu(\phi)$ is a solution of the equation of motion. In the case of the Hamiltonian H_1 and of the equation of motion (4) we can introduce new currents

$$J_0^\mu(t_1,x)=U^\mu(t_1,x),\ \ J_1^\mu(t_1,x)=\eta^{\mu\nu}\frac{\partial F(U(t_1,x))}{\partial U^\nu}.$$

Consequently, we can introduce a new metric tensor and a new veilbein depending of the new time coordinate. The equation for the new metric tensor has the form

$$e^\mu_a(\phi(t_1,x))\frac{\partial\phi^a(t_1,x)}{\partial x}=\frac{de^\mu(\phi(t_1,x))}{dx}=\eta^{\mu\nu}\frac{\partial F(f(\phi(t_1,x)))}{\partial f^\nu(\phi(t_1,x))}.$$

4 New string equation in terms of Pohlmeyer tensor nonlocal currents

In the case of the flat space $C^{\mu}_{\nu\lambda} = 0$ there exist nonlocal totally symmetric tensor chiral currents called "Pohlmeyer" currents [26, 27, 28]

$$R^{(M)}(U(x)) \equiv R^{(\mu_1\mu_2\dots\mu_M)}(U(x))$$
$$= U^{(\mu_1}(x)\int_0^x U^{\mu_2}(x_1)dx_1\cdots\int_0^{x_{M-2}} U^{\mu_M)}(x_{M-1})dx_{M-1},$$

where round brackets the mean totally symmetric product of chiral currents $U^{\mu}(U)$. The new Hamiltonians may have the following forms

$$H^{(M)} = \frac{1}{2}\int_0^{2\pi} R^{(M)}(U(x))d_{2M}R^{(M)}(U(x))dx,$$

where $d_M \equiv d_{(\mu_1\mu_2\dots\mu_M)}$ is totally symmetric invariant constant tensor, which can be constructed from Kronecker deltas. For example

$$R^{(2)} \equiv R^{\mu\nu}(U(x)) = \frac{1}{2}[U^{\mu}(x)\int_0^x U^{\nu}(x_1)dx_1 + U^{\nu}(x)\int_0^x U^{\mu}(x_1)dx_1],$$

$$H^{(2)} = \frac{1}{2}\int_0^{2\pi}\Bigg[U^{\mu}(x)U^{\mu}(x)\int_0^x U^{\nu}(x_1)dx_1\int_0^x U^{\nu}(x_2)dx_2$$
$$+ U^{\mu}(x)U^{\nu}(x)\int_0^x U^{\mu}(x_1)dx_1\int_0^x U^{\nu}(x_2)dx_2\Bigg]dx.$$

The Hamiltonian $H^{(2)}$ commutes with the Hamiltonian $H^{(1)} = \frac{1}{2}\int_0^{2\pi} U^{\mu}(x)U^{\mu}(x)dx$ and it commutes with the Casimir $\int_0^{2\pi} U^{\mu}(x)dx$. The equation of motion under the Hamiltonian $H^{(2)}$ is as follows

$$\frac{\partial U^{\mu}(x)}{\partial t} = \frac{\partial}{\partial x}\Bigg[U^{\mu}(x)\int_0^x U^{\nu}(x_1)dx_1\int_0^x U^{\nu}(x_2)dx_2 + U^{\nu}(x)\int_0^x U^{\mu}(x_1)dx_1\int_0^x U^{\nu}(x_2)dx_2\Bigg]$$
$$- U^{\nu}(x)U^{\nu}(x)\int_0^x U^{\mu}(x_1)dx_1 - U^{\mu}(x)U^{\nu}(x)\int_0^x U^{\nu}(x_1)dx_1.$$

In the variables

$$S^{\mu}(x) = \int_0^x U^{\mu}(y)dy$$

the latter equation can be rewritten as follows

$$\frac{\partial S^{\mu}}{\partial t} = \frac{\partial}{\partial x}(S^{\mu}(S^{\nu}S^{\nu})) + \int_0^x S^{\mu}\left(S^{\nu}\frac{\partial^2 S^{\nu}}{\partial^2 y}\right)dy, \qquad \mu,\nu = 1,2,\dots,n.$$

5 Integrable string models with constant torsion

Let us go back to the commutation relations of chiral currents. Let the torsion $C^{\mu}_{\nu\lambda}(\phi(x)) \neq 0$ and $C_{\mu\nu\lambda} = f_{\mu\nu\lambda}$ be structure constant of s simple Lie algebra. We will consider a string model with the constant torsion in light-cone gauge in target space. This model coincides with the principal chiral model on compact simple Lie group. We cannot divide the motion on right and left mover because of chiral currents $\partial_- U^\mu = f^\mu_{\nu\lambda} U^\nu V^\lambda$, $\partial_- V^\mu = f^\mu_{\nu\lambda} V^\nu U^\lambda$ are not conserved. The correspondent charges are not Casimirs. The present paper was stimulated by paper [24]. Evans, Hassan, MacKay, Mountain (see [24] and references therein) constructed local invariant chiral currents as polynomials of the initial chiral currents of $SU(n)$, $SO(n)$, $SP(n)$ principal chiral models and they found such combination of them that the corresponding charges are Casimir operators of these dynamical systems. Their paper was based on the paper of de Azcarraga, Macfarlane, MacKay, Perez Bueno (see [25] and references therein) about invariant tensors for simple Lie algebras. Let t_μ be $n \otimes n$ traceless hermitian matrix representations of generators Lie algebra

$$[t_\mu, t_\nu] = 2i f_{\mu\nu\lambda} t_\lambda, \qquad \mathrm{Tr}(t_\mu t_\nu) = 2\delta_{\mu\nu}.$$

Here is an additional relation for $SU(n)$ algebra

$$\{t_\mu, t_\nu\} = \frac{4}{n}\delta_{\mu\nu} + 2d_{\mu\nu\lambda} t_\lambda, \qquad \mu = 1, \ldots, n^2 - 1.$$

De Azcarraga et al. gave some examples of invariant tensors of simple Lie algebras and they gave a general method to calculate them. Invariant tensors may be constructed as invariant symmetric polynomials on $SU(n)$

$$d^{(M)}_{(\mu_1 \ldots \mu_M)} = \frac{1}{M!}\mathrm{STr}(t_{\mu_1} \cdots t_{\mu_M}),$$

where STr means the completely symmetrized product of matrices and $d^{(M)}_{(\mu_1 \ldots \mu_M)}$ is the totally symmetric tensor and $M = 2, 3, \ldots, \infty$. Another family of invariant symmetric tensors [29, 30] (see also [25]) called D-family based on the product of the symmetric structure constant $d_{\mu\nu\lambda}$ of the $SU(n)$ algebra is as follows:

$$D^{(M)}_{(\mu_1 \ldots \mu_M)} = d^{k_1}_{(\mu_1 \mu_2} d^{k_1 k_2}_{\mu_3} \cdots d^{k_{M-2} k_{M-3}}_{\mu_{M-2}} d^{k_{M-3}}_{\mu_{M-1} \mu_M)},$$

where $D^{(2)}_{\mu\nu} = \delta_{\mu\nu}, D^{(3)}_{\mu\nu\lambda} = d_{\mu\nu\lambda}$ and $M = 4, 5, \ldots, \infty$.

Here are $n-1$ primitive invariant tensors on $SU(n)$. The invariant tensors for $M \geq n$ are functions of primitive tensors. The Casimir operators on $SU(n)$ algebra have the form

$$C^{(M)}(t) = d^M_{(\mu_1 \ldots \mu_M)} t_{\mu_1} \cdots t_{\mu_M}.$$

Evans et al. introduced local chiral currents based on the invariant symmetric polynomials on simple Lie groups

$$J^{(M)}(U) = \mathrm{STr}(U \cdots U) \equiv \mathrm{STr}\, U^M = d^{(M)}_{\mu_1 \ldots \mu_M} U^{\mu_1} \cdots U^{\mu_M}, \tag{5}$$

where $U = t_\mu U^\mu$ and $\mu = 1, \ldots, n^2 - 1$. It is possible to decompose the invariant symmetric chiral currents $J^{(M)}(U)$ into product of the basic invariant chiral currents $D^{(M)}(U)$

$$D^{(2)}(U) = d^{(2)}_{\mu\nu} U^\mu U^\nu = \eta_{\mu\nu} U^\mu U^\nu, \qquad D(3)(U) = d_{\mu\nu\lambda} U^\mu U^\nu U^\lambda,$$
$$D^{(M)}(U(x)) = d^{k_1}_{\mu_1 \mu_2} d^{k_1 k_2}_{\mu_3} \cdots d^{k_{M-2} k_{M-3}}_{\mu_{M-2}} d^{k_{M-3}}_{\mu_{M-1} \mu_M} U_{\mu_1} U_{\mu_2} \cdots U_{\mu_M},$$

where $M = 4, 5, \ldots, \infty$. The author obtained the following expressions for local invariant chiral currents $J^{(M)}(U)$

$$J^{(2)} = 2D^{(2)}, \qquad J^{(3)} = 2D^{(3)}, \qquad J^{(4)} = 2D^{(4)} + \frac{4}{n} D^{(2)2},$$

$$J^{(5)} = 2D^{(5)} + \frac{8}{n}D^{(2)}D^{(3)}, \qquad J^{(6)} = 2D^{(6)} + \frac{4}{n}D^{(3)2} + \frac{8}{n}D^{(2)}D^{(4)} + \frac{8}{n^2}D^{(2)3},$$
$$J^{(7)} = 2D^{(7)} + \frac{8}{n}D^{(3)}D^{(4)} + \frac{8}{n}D^{(2)}D^{(5)} + \frac{24}{n^2}D^{(2)2}D^{(3)},$$
$$J^{(8)} = 2D^{(8)} + \frac{4}{n}D^{(4)2} + \frac{8}{n}D^{(3)}D^{(5)} + \frac{8}{n}D^{(2)}D^{(6)} + \frac{24}{n^2}D^{(2)}D^{(3)2} + \frac{24}{n^2}D^{(2)2}D^{(4)} + \frac{16}{n^3}D^{(2)4},$$
$$J^{(9)} = 2D^{(9)} + \frac{8}{n}D^{(4)}D^{(5)} + \frac{8}{n}D^{(3)}D^{(6)} + \frac{8}{n}D^{(2)}D^{(7)} + \frac{8}{n^2}D^{(3)3} + \frac{48}{n^2}D^{(2)}D^{(3)}D^{(4)} + \frac{24}{n^2}D^{(2)2}D^{(5)} + \frac{64}{n^3}D^{(2)3}D^{(3)}.$$

Both families of invariant chiral currents $J^{(M)}(U(x))$ and $D^{(M)}(U(x))$ satisfy the conservation equations $\partial_- J^{(M)}(U(x)) = 0$, $\partial_- D^{(M)}(U(x)) = 0$.

The commutation relations of invariant chiral currents $J^{(M)}(U(x))$ show that these currents are not densities of dynamical Casimir operators for $SU(n)$ group. Therefore, we will not consider these currents in the following.

We considered abasic family of invariant chiral currents $D^{(M)}(U)$ and we proved that the invariant chiral currents $D^{(M)}(U)$ form closed algebra under canonical PB and corresponding charges are dynamical Casimir operators. The commutation relations of invariant chiral currents $D^{(M)}(U(x))$ and $D^{(N)}(U(y))$ for $M, N = 2, 3, 4$ and for $M = 2$, $N = 2, 3, \ldots, \infty$ are as follows

$$\{D^{(M)}(x), D^{(N)}(y)\} = -MND^{(M+N-2)}(x)\frac{\partial}{\partial x}\delta(x-y) - \frac{MN(N-1)}{M+N-2}\frac{\partial D^{(M+N-2)}(x)}{\partial x}\delta(x-y).$$

The commutation relations for $M \geq 5$, $N \geq 3$ are as follows

$$\{D^{(5)}(x), D^{(3)}(y)\} = -[12D^{(6)}(x) + 3D^{(6,1)}(x)]\frac{\partial}{\partial x}\delta(x-y) - \frac{1}{3}\frac{\partial}{\partial x}[12D^{(6)}(x) + 3D^{(6,1)}(x)]\delta(x-y),$$
$$\{D^{(5)}(x), D^{(4)}(y)\} = -[16D^{(7)}(x) + 4D^{(7,1)}(x)]\frac{\partial}{\partial x}\delta(x-y) - \frac{3}{7}\frac{\partial}{\partial x}[16D^{(7)}(x) + 4D^{(7,1)}(x)]\delta(x-y),$$
$$\{D^{(6)}(x), D^{(3)}(y)\} = -[12D^{(7)}(x) + 6D^{(7,1)}(x)]\frac{\partial}{\partial x}\delta(x-y) - \frac{2}{7}\frac{\partial}{\partial x}[12D^{(7)}(x) + 6D^{(7,1)}(x)]\delta(x-y),$$
$$\{D^{(5)}(x), D^{(5)}(y)\} = -[16D^{(8)}(x) + 8D^{(8,1)}(x) + D^{(8,2)}(x)]\frac{\partial}{\partial x}\delta(x-y) - \frac{1}{2}\frac{\partial}{\partial x}[16D^{(8)}(x) + 8D^{(8,1)}(x) + D^{(8,2)}(x)]\delta(x-y),$$
$$\{D^{(6)}(x), D^{(4)}(y)\} = -[16D^{(8)}(x) + 8D^{(8,3)}(x)]\frac{\partial}{\partial x}\delta(x-y) - \frac{3}{8}\frac{\partial}{\partial x}[16D^{(8)}(x) + 8D^{(8,3)}(x)]\delta(x-y),$$
$$\{D^{(7)}(x), D^{(3)}(y)\} = -[12D^{(8)}(x) + 6D^{(8,1)}(x) + 3D^{(8,3)}(x)]\frac{\partial}{\partial x}\delta(x-y) - \frac{1}{4}\frac{\partial}{\partial x}[12D^{(8)}(x) + 6D^{(8,1)}(x) + 3D^{(8,3)}(x)]\delta(x-y),$$
$$\{D^{(8)}(x), D^{(3)}(y)\} = -[12D^{(9)}(x) + 6D^{(9,1)}(x) + 6D^{(9,2)}(x)]\frac{\partial}{\partial x}\delta(x-y)$$

$$
\begin{aligned}
&\quad - \frac{2}{9}\frac{\partial}{\partial x}[12D^{(9)}(x) + 6D^{(9,1)}(x) + 6D^{(9,2)}(x)]\delta(x-y),\\
\{D^{(7)}(x), D^{(4)}(y)\} &= -[16D^{(9)}(x) + 8D^{(9,2)}(x) + 4D^{(9,3)}(x)]\frac{\partial}{\partial x}\delta(x-y)\\
&\quad - \frac{1}{3}\frac{\partial}{\partial x}[16D^{(9)}(x) + 8D^{(9,2)}(x) + 4D^{(9,3)}(x)]\delta(x-y),\\
\{D^{(6)}(x), D^{(5)(y)}\} &= -[16D^{(9)}(x) + 4D^{(9,1)}(x) + 8D^{(9,2)}(x) + 2D^{(9,4)}(x)]\frac{\partial}{\partial x}\delta(x-y)\\
&\quad - \frac{4}{9}\frac{\partial}{\partial x}[16D^{(9)}(x) + 4D^{(9,1)}(x) + 8D^{(9,2)}(x) + 2D^{(9,4)}(x)]\delta(x-y),\\
\{D^{(9)}(x), D^{(3)}(y)\} &= -[12D^{(10)}(x) + 6D^{(10,1)}(x) + 6D^{(10,2)}(x) + 3D^{(10,3)}(x)]\frac{\partial}{\partial x}\delta(x-y)\\
&\quad - \frac{1}{5}\frac{\partial}{\partial x}[12D^{(10)}(x) + 6D^{(10,1)}(x) + 6D^{(10,2)}(x) + 3D^{(10,3)}(x)]\delta(x-y),\\
\{D^{(8)}(x), D^{(4)}(y)\} &= -[16D^{(10)}(x) + 8D^{(10,2)}(x) + 8D^{(10,4)}(x)]\frac{\partial}{\partial x}\delta(x-y)\\
&\quad - \frac{3}{10}\frac{\partial}{\partial x}[16D^{(10)}(x) + 8D^{(10,2)}(x) + 8D^{(10,4)}(x)]\delta(x-y),\\
\{D^{(7)}(x), D^{(5)}(y)\} &= -[16D^{(10)}(x) + 8D^{(10,3)}(x) + 4D^{(10,1)}(x)\\
&\quad + 4D^{(10,4)}(x) + 2D^{(10,5)}(x) + D^{(10,6)}(x)]\frac{\partial}{\partial x}\delta(x-y)\\
&\quad - \frac{2}{5}\frac{\partial}{\partial x}[16D^{(10)}(x) + 8D^{(10,3)}(x) + 4D^{(10,1)}(x) + 4D^{(10,4)}(x)\\
&\quad + 2D^{(10,5)}(x) + D^{(10,6)}(x)]\delta(x-y),\\
\{D^{(6)}(x), D^{(6)}(y)\} &= -[16D^{(10)}(x) + 16D^{(10,2)}(x) + 4D^{(10,7)}(x)]\frac{\partial}{\partial x}\delta(x-y)\\
&\quad - \frac{1}{2}\frac{\partial}{\partial x}[16D^{(10)}(x) + 16D^{(10,2)}(x) + 4D^{(10,7)}(x)]\delta(x-y).
\end{aligned}
$$

The new dependent invariant chiral currents $D^{(6,1)}$, $D^{(7,1)}$, $D^{(8,1)} - D^{(8,3)}$, $D^{(9,1)} - D^{(9,4)}$, $D^{(10,1)} - D^{(10,7)}$ (see Appendix A) have the form

$$
\begin{aligned}
D^{(6,1)} &= d^k_{\mu\nu} d^l_{\lambda\rho} d^n_{\sigma\varphi} d^{kln} U^\mu U^\nu U^\lambda U^\rho U^\sigma U^\varphi,\\
D^{(7,1)} &= d^k_{\mu\nu} d^l_{\lambda\rho} d^n_{\sigma\varphi} d^{nm}_{\tau} d^{klm} U^\mu U^\nu U^\lambda U^\rho U^\sigma U^\varphi U^\tau,\\
D^{(8,1)} &= [d^k_{\mu\nu} d^{kl}_{\lambda} d^{ln}_{\rho}][d^m_{\sigma\varphi}][d^p_{\tau\theta}] d^{nmp} U^\mu U^\nu U^\lambda U^\rho U^\sigma U^\varphi U^\tau U^\theta,\\
D^{(8,2)} &= [d^k_{\mu\nu}][d^l_{\lambda\rho}][d^n_{\sigma\varphi}][d^m_{\tau\theta}] d^{klp} d^{nmp} U^\mu U^\nu U^\lambda U^\rho U^\sigma U^\varphi U^\tau U^\theta,\\
D^{(8,3)} &= [d^k_{\mu\nu} d^{kl}_{\lambda}][d^n_{\rho\sigma} d^{nm}_{\varphi}][d^p_{\tau\theta}] d^{lmp} U^\mu U^\nu U^\lambda U^\rho U^\sigma U^\varphi U^\tau U^\theta,\\
D^{(9,1)} &= [d^k_{\mu\nu} d^{kl}_{\lambda} d^{ln}_{\rho} d^{nm}_{\sigma}][d^p_{\varphi\tau}][d^r_{\theta\omega}] d^{mpr} U^\mu U^\nu U^\lambda U^\rho U^\sigma U^\varphi U^\tau U^\theta U^\omega,\\
D^{(9,2)} &= [d^k_{\mu\nu} d^{kl}_{\lambda} d^{ln}_{\rho}][d^m_{\sigma\varphi} d^{mp}_{\tau}][d^r_{\theta\omega}] d^{npr} U^\mu U^\nu U^\lambda U^\rho U^\sigma U^\varphi U^\tau U^\theta U^\omega,\\
D^{(9,3)} &= [d^k_{\mu\nu} d^{kl}_{\lambda}][d^n_{\rho\sigma} d^{nm}_{\varphi}][d^p_{\tau\theta} d^{pr}_{\omega}] d^{lmr} U^\mu U^\nu U^\lambda U^\rho U^\sigma U^\varphi U^\tau U^\theta U^\omega,\\
D^{(9,4)} &= [d^k_{\mu\nu} d^{kl}_{\lambda}][d^n_{\rho\sigma}][d^m_{\varphi\tau}][d^p_{\theta\omega}] d^{lnr} d^{mpr} U^\mu U^\nu U^\lambda U^\rho U^\sigma U^\varphi U^\tau U^\theta U^\omega,\\
D^{(10,1)} &= [d^k_{\mu\nu} d^{kl}_{\lambda} d^{ln}_{\rho} d^{nm}_{\sigma} d^{mp}_{\varphi}][d^r_{\tau\theta}][d^s_{\omega\beta}] d^{prs} U^\mu U^\nu U^\lambda U^\rho U^\sigma U^\varphi U^\tau U^\theta U^\omega U^\beta,\\
D^{(10,2)} &= [d^k_{\mu\nu} d^{kl}_{\lambda} d^{ln}_{\rho} d^{nm}_{\sigma}][d^p_{\varphi\tau} d^{pr}_{\theta}][d^s_{\omega\beta}] d^{mrs} U^\mu U^\nu U^\lambda U^\rho U^\sigma U^\varphi U^\tau U^\theta U^\omega U^\beta,\\
D^{(10,3)} &= [d^k_{\mu\nu} d^{kl}_{\lambda} d^{ln}_{\rho}][d^m_{\sigma\varphi} d^{mp}_{\varphi} d^{pr}_{\tau}][d^s_{\omega\beta}] d^{nrs} U^\mu U^\nu U^\lambda U^\rho U^\sigma U^\varphi U^\tau U^\theta U^\omega U^\beta,\\
D^{(10,4)} &= [d^k_{\mu\nu} d^{kl}_{\lambda} d^{ln}_{\rho}][d^m_{\sigma\varphi} d^{mp}_{\tau}][d^r_{\theta\omega} d^{rs}_{\beta}] d^{nps} U^\mu U^\nu U^\lambda U^\rho U^\sigma U^\varphi U^\tau U^\theta U^\omega U^\beta,\\
D^{(10,5)} &= [d^k_{\mu\nu} d^{kl}_{\lambda} d^{ln}_{\rho}][d^m_{\sigma\varphi}][d^p_{\tau\theta}][d^r_{\omega\beta}] d^{nms} d^{prs} U^\mu U^\nu U^\lambda U^\rho U^\sigma U^\varphi U^\tau U^\theta U^\omega U^\beta,
\end{aligned}
$$

$$D^{(10,6)}=[d^k_{\mu\nu}d^{kl}_{\lambda}][d^n_{\rho\sigma}d^{nm}_{\varphi}][d^p_{\tau\theta}][d^r_{\omega\beta}]d^{lms}d^{prs}U^\mu U^\nu U^\lambda U^\rho U^\sigma U^\varphi U^\tau U^\theta U^\omega U^\beta,$$
$$D^{(10,7)}=[d^k_{\mu\nu}d^{kl}_{\lambda}][d^n_{\rho\sigma}]d^m_{\varphi}][d^{mp}_{\tau\theta}][d^r_{\omega\beta}]d^{lns}d^{prs}U^\mu U^\nu U^\lambda U^\rho U^\sigma U^\varphi U^\tau U^\theta U^\omega U^\beta.$$

Let us apply the hydrodynamic approach to integrable string models with constant torsion. In this case we must consider the conserved primitive chiral currents $D^{(M)}(U(x))$, $(M=2,3,\ldots,n-1)$ as local fields of the Riemmann manifold. The non-primitive local charges of invariant chiral currents with $M\geq n$ form the hierarchy of new Hamiltonians in the bi-Hamiltonian approach to integrable systems. The commutation relations of invariant chiral currents are local PBs of hydrodynamic type.

The invariant chiral currents $D^{(M)}$ with $M\geq 3$ for the $SU(3)$ group can be obtained from the following relation

$$d_{kln}d_{kmp}+d_{klm}d_{knp}+d_{klp}d_{knm}=\frac{1}{3}(\delta_{ln}\delta_{mp}+\delta_{lm}\delta_{np}+\delta_{lp}\delta_{nm}).$$

The corresponding invariant chiral currents for $SU(3)$ group have the form

$$D^{(2N)}=\frac{1}{3^{N-1}}(\eta_{\mu\nu}U^\mu U^\nu)^N=\frac{1}{3^{N-1}}D^{(2)N},$$
$$D^{(2N+1)}=\frac{1}{3^{N-1}}(\eta_{\mu\nu}U^\mu U^\nu)^{N-1}d_{kln}U^kU^lU^n=\frac{1}{3^{N-1}}D^{(2)N-1}D^{(3)}.$$

The invariant chiral currents $D^{(2)}$, $D^{(3)}$ are local coordinates of the Riemmann manifold M^2. The local charges $D^{(2N)}$, $N\geq 2$ form a hierarchy of Hamiltonians. The new nonlinear equations of motion for chiral currents are as follows

$$\frac{\partial D^{(k)}(U(x))}{\partial t_N}=\left\{D^{(k)}(U(x)),\int_0^{2\pi}D^{(2)N}(U(y))dy\right\},\qquad k=2,3,\qquad N=2,\ldots,\infty.$$
$$\frac{\partial D^{(2)}(U(x))}{\partial t_N}=-2(2N-1)\frac{\partial D^{(2)N}(U(x))}{\partial x},$$
$$\frac{\partial D^{(3)}(U(x))}{\partial t_N}=-6ND^{(3)}(U(x))\frac{\partial D^{(2)N-1}(U(x))}{\partial x}-2ND^{(2)N-1}(U(x))\frac{\partial D^{(3)}(U(x))}{\partial x}.$$

The construction of integrable equations with $SU(n)$ symmetries for $n\geq 4$ has difficulties in reduction of non-primitive invariant currents to primitive currents.

The similar method of construction of chiral currents for $SO(2l+1)=B_l$, $SP(2l)=C_l$ groups was used by Evans et al. [24] on the base of symmetric invariant tensors of de Azcarraga et al. [25]. In the defining representation these group generators corresponding to algebras t_μ satisfy the rules

$$[t_\mu,t_\nu]=2if^\lambda_{\mu\nu}t_\lambda,\qquad \mathrm{Tr}(t_\mu t_\nu)=2\delta_{\mu\nu},\qquad t_\mu\eta=-\eta t^t_\mu,$$

where η is a Euclidean or symplectic structure.

The symmetric tensor structure constants for these groups were introduced through completely symmetrized product of three generators of corresponding algebras

$$t_{(\mu}t_\nu t_{\lambda)}=v^\rho_{\mu\nu\lambda}t_\rho,$$

where $v_{\mu\nu\lambda\rho}$ is a totally symmetric tensor. The basic invariant symmetric tensors have the form [25]

$$V^{(2)}_{\mu\nu}=\delta_{\mu\nu},\ V^{(2N)}_{(\mu_1\mu_2\ldots\mu_{2N-1}\mu_{2N})}=v^{\nu_1}_{(\mu_1\mu_2\mu_3}v^{\nu_1\nu_2}_{\mu_4\mu_5}\cdots v^{\nu_{2N-3}}_{\mu_{2N-2}\mu_{2N-1}\mu_{2N})},\qquad N=2,\ldots,\infty.$$

The invariant chiral currents $J^{(2N)}$ (5) coincide with the basis invariant chiral currents $V^{(2N)}$

$$J^{(2N)} = 2V^{(2N)}_{\mu 1 \dots \mu_{2N}} U^{\mu_1} \cdots U^{\mu_{2N}}.$$

The commutation relations of invariant chiral currents are PBs of hydrodynamic type

$$\{J^{(M)}(x), J^{(N)}(y)\} = -MNJ^{(M+N-2)}(x)\frac{\partial}{\partial x}\delta(x-y) - \frac{MN(N-1)}{M+N-2}\frac{\partial J^{(M+N-2)}(x)}{\partial x}\delta(x-y). \tag{6}$$

The commuting charges of these invariant chiral currents are dynamical Casimir operators on $SO(2l+1)$, $SP(2l)$. The metric tensor of Riemmann space of invariant chiral currents is as follows

$$g_{MN}(J(x)) = -MN(M+N-2)J^{(M+N-2)}(x).$$

The commutation relations (6) coincide with commutation relations, which was obtained by Evans at al. [24].

We used relations for new symmetric invariant tensors $V^{(2N,1)}_{(\mu_1 \dots \mu_{2N})}$ (see Appendix B), which we obtained during calculation PB (6)

$$v^k_{(\mu_1\mu_2\mu_3}v^l_{\mu_4\mu_5\mu_6}v^n_{\mu_7\mu_8\mu_9}v^{kln}_{\mu_{10})} = V^{(10)}_{(\mu_1\dots\mu_{10})},$$
$$v^k_{(\mu_1\mu_2\mu_3}v^l_{\mu_4\mu_5\mu_6}v^n_{\mu_7\mu_8\mu_9}v^m_{\mu_{10}\mu_{11}\mu_{12})}v^{klnm} = V^{(12)}_{(\mu_1\dots\mu_{12})},$$
$$v^k_{(\mu_1\mu_2\mu_3}v^l_{\mu_4\mu_5\mu_6}v^n_{\mu_7\mu_8\mu_9}v^m_{\mu_{10}\mu_{11}\mu_{12}}v^{klp}_{\mu_{13}}v^{nmp}_{\mu_{14})} = V^{(14)}_{(\mu_1\dots\mu_{14})}.$$

Appendix A

The new dependent invariant chiral currents and the new dependent totally symmetric invariant tensors for $SU(N)$ group can be obtained under different order of calculation of trace of the product of the generators of $SU(n)$ algebra. Let us mark the matrix product of two generators t_μ, t_ν in round brackets

$$(t_\mu t_\nu) = \frac{2}{n}\delta_{\mu\nu} + (d^k_{\mu\nu} + if^k_{\mu\nu})t_k. \tag{7}$$

The expression of invariant chiral currents $J_M(U)$ depends on the position of the matrix product of two generators in the general list of generators. For example

$$J^{(6)} = \mathrm{Tr}[t(tt)(tt)t] = 2D^{(6)} + \frac{4}{n}D^{(3)2} + \frac{8}{n}D^{(2)}D^{(4)} + \frac{8}{n^2}D^{(2)3},$$
$$J^{(6)} = \mathrm{Tr}[(tt)(tt)(tt)] = 2D^{(6,1)} + \frac{12}{n}D^{(2)}D^{(4)} + \frac{8}{n^2}D^{(2)3},$$
$$J^{(7)} = \mathrm{Tr}[t(tt)t(tt)t] = 2D^{(7)} + \frac{8}{n}D^{(3)}D^{(4)} + \frac{8}{n^2}D^{(2)}D^{(5)} + \frac{24}{n^2}D^{(2)2}D^{(3)},$$
$$J^{(7)} = \mathrm{Tr}[(tt)(tt)(tt)t] = 2D^{(7,1)} + \frac{4}{n}D^{(3)}D^{(4)} + \frac{12}{n^2}D^{(2)}D^{(5)} + \frac{24}{n^2}D^{(2)2}D^{(3)},$$
$$J^{(8)} = \mathrm{Tr}[t(tt)tt(tt)t] = 2D^{(8)} + \frac{4}{n}D^{(4)2} + \frac{8}{n}D^{(3)}D^{(5)} + \frac{8}{n}D^{(2)}D^{(6)} + \frac{24}{n^2}D^{(2)}D^{(3)2} + \frac{24}{n^2}D^{(2)2}D^{(4)} + \frac{16}{n^3}D^{(2)4},$$

$$
\begin{aligned}
J^{(8)} &= \mathrm{Tr}[(tt)(tt)t(tt)t] = 2D^{(8,1)} + \frac{4}{n}D^{(4)2} + \frac{4}{n}D^{(3)}D^{(5)} + \frac{24}{n^2}D^{(2)}D^{(3)2} \\
&\quad + \frac{12}{n}D^{(2)}D^{(6)} + \frac{24}{n^2}D^{(2)2}D^{(4)} + \frac{16}{n^3}D^{(2)4}, \\
J^{(8)} &= \mathrm{Tr}[(tt)(tt)(tt)(tt)] = 2D^{(8,2)} + \frac{4}{n}D^{(4)2} + \frac{16}{n}D^{(2)}D^{(6,1)} + \frac{32}{n^2}D^{(2)2}D^{(4)} + \frac{16}{n^3}D^{(2)4}, \\
J^{(8)} &= \mathrm{Tr}[t(tt)(tt)(tt)t] = 2D^{(8,3)} + \frac{12}{n}D^{(2)}D^{(6)} + \frac{8}{n}D^{(3)}D^{(5)} \\
&\quad + \frac{24}{n^2}D^{(2)2}D^{(4)} + \frac{24}{n^2}D^{(2)}D^{(3)2} + \frac{16}{n}D^{(2)4}, \\
J^{(9)} &= \mathrm{Tr}[t(tt)ttt(tt)t] = 2D^{(9)} + \frac{8}{n}D^{(4)}D^{(5)} + \frac{8}{n}D^{(3)}D^{(6)} + \frac{8}{n}D^{(2)}D^{(7)} + \frac{8}{n^2}D^{(3)3} \\
&\quad + \frac{48}{n^2}D^{(2)}D^{(3)}D^{(4)} + \frac{24}{n^2}D^{(2)2}D^{(5)} + \frac{64}{n^3}D^{(2)3}D^{(3)}, \\
J^{(9)} &= \mathrm{Tr}[t(tt)tt(tt)(tt)] \\
&= \begin{cases}
2D^{(9,1)} + \frac{4}{n}D^{(4)}D^{(5)} + \frac{4}{n}D^{(2)}D^{(7)} + \frac{4}{n}D^{(2)}D^{(7,1)} + \frac{8}{n}D^{(3)}D^{(6,1)} \\
\quad + \frac{32}{n^2}D^{(2)}D^{(3)}D^{(4)} + \frac{32}{n^2}D^{(2)2}D^{(5)} + \frac{64}{n^3}D^{(2)3}D^{(3)}, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
2D^{(9,4)} + \frac{4}{n}D^{(2)}D^{(7)} + \frac{4}{n}D^{(2)}D^{(7,1)} + \frac{12}{n}D^{(3)}D^{(6,1)} \\
\quad + \frac{32}{n^2}D^{(2)}D^{(3)}D^{(4)} + \frac{32}{n^2}D^{(2)2}D^{(5)} + \frac{64}{n^3}D^{(2)3}D^{(3)},
\end{cases} \\
J^{(9)} &= \mathrm{Tr}[t(tt)t(tt)t(tt)] \\
&= \begin{cases}
2D^{(9,2)} + \frac{4}{n}D^{(4)}D^{(5)} + \frac{8}{n}D^{(3)}D^{(6)} + \frac{8}{n}D^{(2)}D^{(7)} + \frac{4}{n}D^{(2)}D^{(7,1)} \\
\quad + \frac{8}{n^2}D^{(3)3} + \frac{40}{n^2}D^{(2)}D^{(3)}D^{(4)} + \frac{32}{n^2}D^{(2)2}D^{(5)} + \frac{64}{n^3}D^{(2)3}D^{(3)}, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
2D^{(9,3)} + \frac{8}{n}D^{(2)}D^{(7)} + \frac{4}{n}D^{(2)}D^{(71)} + \frac{12}{n}D^{(3)}D^{(6)} + \frac{8}{n}D^{(3)3} \\
\quad + \frac{40}{n^2}D^{(2)}D^{(3)}D^{(4)} + \frac{32}{n^2}D^{(2)2}D^{(5)} + \frac{64}{n^3}D^{(2)3}D^{(3)},
\end{cases}
\end{aligned}
$$

where $t = t_\mu U^\mu$ and two variants of two last expressions for $J^{(9)}(U)$ were obtained from two variants of expression for $J^{(6)}(U)$ during calculation $J^{(9)}(U)$. Because the result of calculation does not depend on the order of calculation, we can obtain relations between new invariant chiral currents and basic invariant currents $D^{(M)}(U)$

$$
\begin{aligned}
D^{(6,1)} &= D^{(6)} + \frac{2}{n}D^{(3)2} - \frac{2}{n}D^{(2)}D^{(4)}, \\
D^{(7,1)} &= D^{(7)} + \frac{4}{n}D^{(3)}D^{(4)} - \frac{4}{n}D^{(2)}D^{(5)}, \\
D^{(81)} &= D^{(8)} + \frac{2}{n}D^{(3)}D^{(5)} - \frac{2}{n}D^{(2)}D^{(6)}, \\
D^{(8,2)} &= D^{(8)} + \frac{4}{n}D^{(3)}D^{(5)} - \frac{4}{n}D^{(2)}D^{(6)} - \frac{4}{n^2}D^{(2)}D^{(3)2} + \frac{4}{n^2}D^{(2)2}D^{(4)}, \\
D^{(8,3)} &= D^{(8)} + \frac{2}{n}D^{(4)2} - \frac{2}{n}D^{(2)}D^{(6)}, \\
D^{(9,1)} &= D^{(9)} + \frac{2}{n}D^{(4)}D^{(5)} - \frac{4}{n^2}D^{(3)3} + \frac{8}{n^2}D^{(2)}D^{(3)}D^{(4)} + \frac{4}{n^2}D^{(2)2}D^{(5)},
\end{aligned}
$$

$$D^{(9,2)} = D^{(9)} + \frac{2}{n}D^{(4)}D^{(5)} - \frac{2}{n}D^{(2)}D^{(7)} - \frac{4}{n^2}D^{(2)}D^{(3)}D^{(4)} + \frac{4}{n^2}D^{(2)2}D^{(5)},$$
$$D^{(9,3)} = D^{(9)} + \frac{4}{n}D^{(4)}D^{(5)} - \frac{2}{n}D^{(2)}D^{(7)} - \frac{2}{n}D^{(3)}D^{(6)} - \frac{4}{n^2}D^{(2)}D^{(3)}D^{(4)} + \frac{4}{n^2}D^{(2)2}D^{(5)},$$
$$D^{(9,4)} = D^{(9)} + \frac{4}{n}D^{(4)}D^{(5)} - \frac{2}{n}D^{(3)}D^{(6)} - \frac{8}{n^2}D^{(3)3} + \frac{12}{n^2}D^{(2)}D^{(3)}D^{(4)} + \frac{4}{n^2}D^{(2)2}D^{(5)}.$$

Hence we can obtain the new relations for symmetric tensors

$$d^k_{(\mu\nu}d^l_{\lambda\rho}d^n_{\sigma\varphi)}d^{kln} = d^k_{(\mu\nu}d^{kl}_{\lambda}d^{ln}_{\rho}d^n_{\sigma\varphi)} + \frac{2}{n}d_{(\mu\nu\lambda}d_{\rho\sigma\varphi)} - \frac{2}{n}\delta_{(\mu\nu}d^k_{\lambda\rho}d^k_{\sigma\varphi)},$$
$$d^k_{(\mu\nu}d^l_{\lambda\rho}d^n_{\sigma\varphi}d^{nm}_{\tau)}d^{klm} = d^k_{(\mu\nu}d^{kl}_{\lambda}d^{ln}_{\rho}d^{nm}_{\sigma}d^m_{\varphi\tau)} + \frac{4}{n}d_{(\mu\nu\lambda}d^k_{\rho\sigma}d^k_{\varphi\tau)} - \frac{4}{n}\delta_{(\mu\nu}d^k_{\lambda\rho}d^{kl}_{\sigma}d^l_{\varphi\tau)},$$
$$d^k_{(\mu\nu}d^l_{\lambda\rho}d^n_{\sigma\varphi}d^{nm}_{\tau}d^{mp}_{\tau)}d^{klp} = d^k_{(\mu\nu}d^{kl}_{\lambda}d^{ln}_{\rho}d^{nm}_{\sigma}d^{mp}_{\varphi}d^p_{\tau\theta)} + \frac{4}{n}d_{(\mu\nu\lambda}d^k_{\rho\sigma}d^{kl}_{\varphi}d^l_{\tau\theta)} - \frac{2}{n}\delta_{(\mu\nu}d^k_{\lambda\rho}d^{kl}_{\sigma}d^{ln}_{\varphi}d^n_{\tau\theta)},$$
$$d^k_{(\mu\nu}d^l_{\lambda\rho}d^n_{\sigma\varphi}d^m_{\tau\theta)}d^{klp}d^{nmp} = d^k_{(\mu\nu}d^{kl}_{\lambda}d^{ln}_{\rho}d^{nm}_{\sigma}d^{mp}_{\varphi}d^p_{\tau\theta)} + \frac{4}{n}d_{(\mu\nu\lambda}d^k_{\rho\sigma}d^{kl}_{\varphi}d^l_{\tau\theta)}$$
$$- \frac{4}{n}\delta_{(\mu\nu}d^k_{\lambda\rho}d^{kl}_{\sigma}d^{ln}_{\varphi}d^n_{\tau\theta)} - \frac{4}{n^2}\delta_{(\mu\nu}d_{\lambda\rho\sigma}d_{\varphi\tau\theta)} + \frac{4}{n^2}\delta_{(\mu\nu}\delta_{\lambda\rho}d^k_{\sigma\varphi}d^k_{\tau\theta)}.$$

It is possible to obtain similar relations for invariant symmetric tensors of ninth order. The commutation relations of chiral currents in terms of the basic invariant currents are as follows

$$\{D^{(5)}(x), D^{(3)}(y)\} = -\left[15D^{(6)}(x) + \frac{6}{n}D^{(3)2}(x) - \frac{6}{n}D^{(2)}(x)D^{(4)}(x)\right]\frac{\partial}{\partial x}\delta(x-y)$$
$$- \frac{1}{3}\frac{\partial}{\partial x}\left[15D^{(6)}(x) + \frac{6}{n}D^{(3)2}(x) - \frac{6}{n}D^{(2)}(x)D^{(4)}(x)\right]\delta(x-y),$$
$$\{D^{(5)}(x), D^{(4)}(y)\} = -\left[20D^{(7)}(x) + \frac{16}{n}D^{(3)}(x)D^{(4)}(x) - \frac{16}{n}D^{(2)}(x)D^{(5)}(x)\right]\frac{\partial}{\partial x}\delta(x-y)$$
$$- \frac{3}{7}\frac{\partial}{\partial x}\left[20D^{(7)}(x) + \frac{16}{n}D^{(3)}(x)D^{(4)}(x) - \frac{16}{n}D^{(2)}(x)D^{(5)}(x)\right]\delta(x-y),$$
$$\{D^{(5)}(x), D^{(5)}(y)\} = -\left[25D^{(8)}(x) + \frac{36}{n}D^{(3)}(x)D^{(5)}(x) - \frac{20}{n}D^{(2)}(x)D^{(6)}(x)\right.$$
$$\left. - \frac{4}{n}D^{(2)}(x)D^{(3)2}(x) + \frac{4}{n^2}D^{(2)2}(x)D^{(4)}(x)\right]\frac{\partial}{\partial x}\delta(x-y)$$
$$- \frac{1}{2}\frac{\partial}{\partial x}\left[25D^{(8)}(x) + \frac{36}{n}D^{(3)}(x)D^{(5)}(x) - \frac{20}{n}D^{(2)}(x)D^{(6)}(x)\right.$$
$$\left. - \frac{4}{n}D^{(2)}(x)D^{(3)2}(x) + \frac{4}{n^2}D^{(2)2}(x)D^{(4)}(x)\right],$$
$$\{D^{(6)}(x), D^{(4)}(y)\} = -\left[24D^{(8)} + \frac{12}{n}D^{(4)2} - \frac{12}{n}D^{(2)}D^{(6)}\right]\frac{\partial}{\partial x}\delta(x-y)$$
$$- \frac{3}{8}\frac{\partial}{\partial x}\left[24D^{(8)} + \frac{12}{n}D^{(4)2} - \frac{12}{n}D^{(2)}D^{(6)}\right]\delta(x-y),$$
$$\{D^{(7)}(x), D^{(3)}(y)\} = -\left[21D^{(8)} + \frac{6}{n}D^{(4)2} + \frac{12}{n}D^{(3)}D^{(5)} - \frac{18}{n}D^{(2)}D^{(6)}\right]\frac{\partial}{\partial x}\delta(x-y)$$
$$- \frac{1}{4}\frac{\partial}{\partial x}\left[21D^{(8)} + \frac{6}{n}D^{(4)2} + \frac{12}{n}D^{(3)}D^{(5)} - \frac{18}{n}D^{(2)}D^{(6)}\right]\delta(x-y),$$
$$\{D^{(8)}(x), D^{(3)}(y)\} = -\left[24D^{(9)} - \frac{12}{n}D^{(2)}D^{(7)} + \frac{24}{n}D^{(4)}D^{(5)} - \frac{24}{n^2}D^{(3)3} + \frac{24}{n^2}D^{(2)}D^{(3)}D^{(4)}\right.$$
$$\left. + \frac{48}{n^2}D^{(2)2}D^{(5)}\right]\frac{\partial}{\partial x}\delta(x-y) - \frac{2}{9}\frac{\partial}{\partial x}\left[24D^{(9)} - \frac{12}{n}D^{(2)}D^{(7)}\right.$$

$$
\begin{aligned}
&+\frac{24}{n}D^{(4)}D^{(5)}-\frac{24}{n^2}D^{(3)3}+\frac{24}{n^2}D^{(2)}D^{(3)}D^{(4)}+\frac{48}{n^2}D^{(2)2}D^{(5)}\Big]\delta(x-y),\\
\{D^{(7)}(x),D^{(4)}(y)\} = &-\Big[28D^{(9)}-\frac{8}{n}D^{(3)}D^{(6)}-\frac{24}{n}D^{(2)}D^{(7)}+\frac{32}{n}D^{(4)}D^{(5)}\\
&-\frac{48}{n^2}D^{(2)}D^{(3)}D^{(4)}+\frac{48}{n^2}D^{(2)2}D^{(5)}\Big]\frac{\partial}{\partial x}\delta(x-y)\\
&-\frac{1}{3}\frac{\partial}{\partial x}\Big[28D^{(9)}-\frac{8}{n}D^{(3)}D^{(6)}-\frac{24}{n}D^{(2)}D^{(7)}+\frac{32}{n}D^{(4)}D^{(5)}\\
&-\frac{48}{n^2}D^{(2)}D^{(3)}D^{(4)}+\frac{48}{n^2}D^{(2)2}D^{(5)}\Big]\delta(x-y),\\
\{D^{(6)}(x),D^{(5)}(y)\} = &-\Big[30D^{(9)}-\frac{4}{n}D^{(3)}D^{(6)}-\frac{12}{n}D^{(2)}D^{(7)}+\frac{32}{n}D^{(4)}D^{(5)}-\frac{32}{n^2}D^{(3)3}\\
&+\frac{24}{n^2}D^{(2)}D^{(3)}D^{(4)}+\frac{56}{n^2}D^{(2)2}D^{(5)}\Big]\frac{\partial}{\partial x}\delta(x-y)\\
&-\frac{4}{9}\frac{\partial}{\partial x}\Big[30D^{(9)}-\frac{4}{n}D^{(3)}D^{(6)}-\frac{12}{n}D^{(2)}D^{(7)}+\frac{32}{n}D^{(4)}D^{(5)}\\
&-\frac{32}{n^2}D^{(3)3}+\frac{24}{n^2}D^{(2)}D^{(3)}D^{(4)}+\frac{56}{n^2}D^{(2)2}D^{(5)}\Big]\delta(x-y).
\end{aligned}
$$

Appendix B

The invariant chiral currents $J^{(2N)}$ and $V^{(2N)}$ and the new dependent totally symmetric invariant tensors for $SO(2l+1)$, $SP(2l)$ groups can be obtained under different order of calculation of trace of the product of the generators of corresponding algebras. Let us mark the matrix product of three generators t_μ in round brackets

$$
(t_{(\mu}t_\nu t_{\lambda)}) = v_{\mu\nu\lambda\rho}t_\rho .
$$

A different position of this triplet inside of J^{2N} produces different expressions for V^{2N}

$$
\begin{aligned}
J^{(10)} &= \mathrm{Tr}[((t_1t_2t_3)t_4(t_5t_6t_7)(t_8t_9t_{10}))]U_1\cdots U_{10} = 2v^k_{123}v^{kl}_{45}v^{ln}_{67}v^n_{8910}U_1\cdots U_{10} = 2V^{(10)},\\
J^{(10)} &= \mathrm{Tr}[((t_1t_2t_3)(t_4t_5t_6)(t_7t_8t_9)t_{10})]U_1\cdots U_{10} = 2v^k_{123}v^l_{456}v^n_{789}v^{kln}_{10}U_1\cdots U_{10} = 2V^{(10,1)},\\
J^{(12)} &= \mathrm{Tr}[(t_1(t_2t_3t_4)t_5(t_6t_7t_8)t_9(t_{10}t_{11}t_{12}))]U_1\cdots U_{12}\\
&= 2v^k_{123}v^{kl}_{45}v^{ln}_{67}v^{nm}_{89}v^m_{101112}U_1\cdots U_{12} = 2V^{(12)},\\
J^{(12)} &= \mathrm{Tr}[((t_1t_2t_3)(t_4t_5t_6)(t_7t_8t_9)(t_{10}t_{11}t_{12}))]U_1\cdots U_{12}\\
&= 2v^k_{123}v^l_{456}v^n_{789}v^m_{101112}v^{klnm}U_1\cdots U_{12} = 2V^{(12,1)},\\
J^{(14)} &= \mathrm{Tr}[((t_1t_2t_3)t_4(t_5t_6t_7)t_8(t_9t_{10}t_{11})(t_{12}t_{13}t_{14}))]U_1\cdots U_{14}\\
&= 2v^k_{123}v^{kl}_{45}v^{ln}_{67}v^{nm}_{89}v^{mp}_{1011}v^p_{121314}U_1\cdots U_{14} = 2V^{(14)},\\
J^{(14)} &= \mathrm{Tr}[((t_1t_2t_3)(t_4t_5t_6)(t_7t_8t_9)(t_{10}t_{11}t_{12})t_{13}t_{14})]U_1\cdots U_{14}\\
&= 2v^k_{123}v^l_{456}v^n_{789}v^m_{101112}v^{klp}_{13}v^{nmp}_{14}U_1\cdots U_{14} = 2V^{(14,1)}.
\end{aligned}
$$

Here we introduced the short notation $t_{\mu_k}=t_k$, $U^{\mu_k}=U_k$ and $v^k_{\mu_l\mu_n\mu_m}$=v^k_{lnm}. New invariant chiral tensors do not lead to new invariant chiral currents.

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