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Phase Space of Rolling Solutions of the Tippe Top*

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Abstract. Equations of motion of an axially symmetric sphere rolling and sliding on a plane are usually taken as model of the tippe top. We study these equations in the nonsliding regime both in the vector notation and in the Euler angle variables when they admit three integrals of motion that are linear and quadratic in momenta. In the Euler angle variables (θ, φ, ψ) these integrals give separation equations that have the same structure as the equations of the Lagrange top. It makes it possible to describe the whole space of solutions by representing them in the space of parameters (D, λ, E) being constant values of the integrals of motion.

Key words: nonholonomic dynamics; rigid body; rolling sphere; tippe top; integrals of motion

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1 Introduction

The tippe top (TT) has the shape of a truncated sphere with a knob. The tippe top is well known for its counterintuitive behavior that after being launched with sufficiently fast spin it turns upside down to spin on the knob until loss of energy makes it fall down again onto the spherical bottom.

It is well known now that the sliding friction is the only force producing a vertical component of the torque, which is needed for reducing the vertical component of the spin and for transferring the rotational energy into the potential energy by raising the center of mass (CM) and inverting the tippe top.

Equations describing rolling motions of an axially symmetric sphere are a limiting case of the TT equations and their solutions cannot display the TT rising phenomenon since the sliding friction is absent. They do, however indicate how solutions of the TT equations with weak friction behave in shorter time periods. According to experiments and numerical simulations [4, 5, 1] the rising of CM of TT has a wobbly character which means that the inclination angle of the symmetry axis $\theta(t)$ is rising in an oscillatory manner. The analysis of the purely rolling solutions presented here supports an understanding that rising of CM may be seen as a superposition of a generic nutational motion of a rolling sphere combined together with drift of the symmetry axis caused by the action of the frictional component of the torque.

The TT is modelled by a sphere of mass m and radius R having axially symmetric distribution of mass with the center of mass shifted along the symmetry axis by αR $(0 < \alpha < 1)$ w.r.t. the geometrical center O.

Full equations of the rolling and sliding TT admit an angular momentum type integral of motion $\lambda = -\boldsymbol{L}\boldsymbol{a}$ called the Jellett's integral, where $\boldsymbol{L}$ is the angular momentum w.r.t. CM and

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$\boldsymbol{a}$ is a vector pointing from CM toward the point A of contact with the supporting plane. The energy of the TT has, under assumption that the friction force $\boldsymbol{F}_f$ acts against the direction of the sliding velocity $\boldsymbol{v}_A$, negative time derivative $\dot{E} = \boldsymbol{v}_A \boldsymbol{F}_f(\boldsymbol{v}_A) < 0$ and is decreasing monotonously. These two features of the TT equations allow, under some additional assumptions about the reaction force, for complete description of all asymptotic motions of the TT and for analysis of their stability [7, 5, 12, 1]. The asymptotic solutions of the TT constitute an invariant manifold satisfying the conditions $\boldsymbol{v}_A = \boldsymbol{0}$ and $\dot{\boldsymbol{v}}_A = \boldsymbol{0}$ and it consists of vertically spinning motions and of tumbling solutions having constant inclination θ of the symmetry axis, so that the sphere is rolling along a circle around fixed center of mass.

The usual assumption about pure rolling of the TT sphere actually changes the model so that the reaction force is dynamically determined. Then the TT equations simplify, the energy E is conserved, and the equations admit a third integral of motion D already known to Routh [13]. The existence of three integrals of motion reduces the TT equations to three first order ODE's that can be solved by separation of variables in a similar way as the equations for the Lagrange top (LT) [10].

Separation of equations for a rolling axially symmetric sphere is a known fact [13, 2, 6] but detailed analysis of all possible rolling motions of the tippe top, as labeled here by integrals (D, λ, E), doesn't seem to be available in literature. A more general discussion of the Smale diagram for rolling solutions of an ellipsoid of revolution has been presented on [14]. A qualitative analysis of motion of a solid of revolution on an absolutely rough plane has been also performed in [11] by starting from canonical equations with nonholonomicity term and by referring to properties of the monodromy matrix. These results, when specialised to the case of sphere, lead to the same picture as presented here.

A different discussion (than presented here), of the main separation equation $\dot{\theta}^2 = f(\theta)$ and of the dependence of a modified effective potential on the Jellett's integral λ has been recently given in [6]. Authors of [6] also explain how the set of admissible (physical) trajectories depends on the nonsliding condition for the components of the reaction force.

In this paper, for completness of exposition, we discuss again integrals of motion formulated in a suitable way both in the vector notation as well as expressed through the Euler angles (θ, φ, ψ). We use them to reduce the equations of motion to the separated form $\dot{\theta}^2 = f(\theta, D, \lambda, E)$. The function $f(\theta, D, \lambda, E)$ is a complicated rational function of $z = \cos\theta$ and we study solutions of this equation by distinguishing special types of motions that are explained by revoking similarities with the LT.

All dynamical states of the rolling tippe top (rTT) are illustrated as a set in the space of parameters (D, λ, E). This set is bounded from below by the surface of minimal value of the energy function.

The advantage of representation of dynamical states as points in the space $(D, \lambda, E) \in \mathbb{R}^3$ is that the vector connecting two points can be used as a measure of distance between two states and its components can be given physical interpretation in terms of the energy difference and the angular momentum difference. This can be translated into the moments of force and the work needed for transferring the TT from one state to another.

2 Vector equations of the rolling and sliding TT

The tippe top (TT) is modelled by a sphere of mass m and radius R having axially symmetric distribution of mass. Its center of mass (CM) is shifted w.r.t. the geometric center O along the symmetry axis $\hat{\mathbf{3}}$ by αR, $0 < \alpha < 1$ as illustrated in Fig. 1. In describing the motion of TT we attach at the centre of mass (CM) a moving orthonormal reference frame $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ with the vector $\hat{\mathbf{3}}$ directed along the symmetry axis, with the vector $\hat{\mathbf{1}}$ (in the vertical plane of the picture) orthogonal to $\hat{\mathbf{3}}$ and the vector $\hat{\mathbf{2}} = \hat{\mathbf{3}} \times \hat{\mathbf{1}}$ that is always parallel to the plane of support and

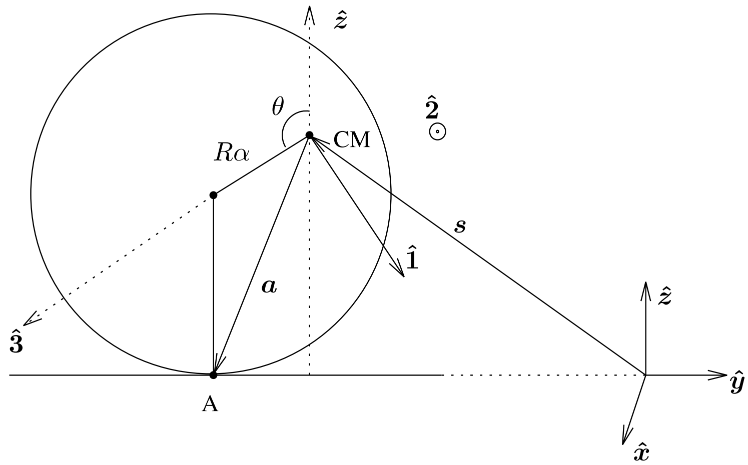

Figure 1. Model of TT.

is pointing out of the plane of the picture. The position of CM w.r.t. the inertial orthonormal reference frame $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$ is denoted by vector $\boldsymbol{s}$ and the vector connecting CM with the point A, of support by the horizontal plane, is denoted $\boldsymbol{a} = R\left(\alpha\hat{\mathbf{3}} - \hat{\boldsymbol{z}}\right)$ as follows from Fig. 1. The principal moments of inertia along axis $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ are denoted $I_1 = I_2$, I_3 and the inertia tensor has the form $\hat{\mathbb{I}} = I_1\left\{\mathbb{1} + \frac{I_3 - I_1}{I_1}|\hat{\mathbf{3}}\rangle\langle\hat{\mathbf{3}}|\right\}$, $\hat{\mathbb{I}}^{-1} = \frac{1}{I_1}\left\{\mathbb{1} - \frac{I_3 - I_1}{I_3}|\hat{\mathbf{3}}\rangle\langle\hat{\mathbf{3}}|\right\}$.

The orientation of the moving reference frame $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ w.r.t. the inertial reference frame $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$ is described by two angles (θ, φ), as in Fig. 2, and the angular velocity of the moving frame can be read from Figs. 1 and 2

$$\boldsymbol{\omega}_{\text{ref}} = -\dot{\varphi}\sin\theta\hat{\mathbf{1}} + \dot{\theta}\hat{\mathbf{2}} + \dot{\varphi}\cos\theta\hat{\mathbf{3}},$$

where dots denote time derivatives of the angles (θ, φ). The angular velocity of TT is then

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\text{ref}} + \dot{\psi}\hat{\mathbf{3}} = -\dot{\varphi}\sin\theta\hat{\mathbf{1}} + \dot{\theta}\hat{\mathbf{2}} + \left(\dot{\psi} + \dot{\varphi}\cos\theta\right)\hat{\mathbf{3}}.$$

It contains an extra term $\dot{\psi}\hat{\mathbf{3}}$ that describes rotation of TT by the angle ψ around the symmetry axis $\hat{\mathbf{3}}$; we shall denote $\omega_3 = \dot{\psi} + \dot{\varphi}\cos\theta$. These definitions entail the following kinematic equations for rotation of the reference frame $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$

$$\dot{\hat{\mathbf{1}}} = \boldsymbol{\omega}_{\text{ref}} \times \hat{\mathbf{1}} = \dot{\varphi}\cos\theta\hat{\mathbf{2}} - \dot{\theta}\hat{\mathbf{3}},$$
$$\dot{\hat{\mathbf{2}}} = \boldsymbol{\omega}_{\text{ref}} \times \hat{\mathbf{2}} = -\dot{\varphi}\cos\theta\hat{\mathbf{1}} - \dot{\varphi}\sin\theta\hat{\mathbf{3}},$$
$$\dot{\hat{\mathbf{3}}} = \boldsymbol{\omega}_{\text{ref}} \times \hat{\mathbf{3}} = \boldsymbol{\omega} \times \hat{\mathbf{3}} = \dot{\theta}\hat{\mathbf{1}} + \dot{\varphi}\sin\theta\hat{\mathbf{2}}.$$

The dynamics of the rolling and sliding TT is described by two ordinary differential equations (ODE's), one for motion of CM and another one for rotation about CM

$$m\dot{\boldsymbol{v}}_{\text{CM}} = \boldsymbol{F}_R + \boldsymbol{F}_f(\boldsymbol{v}_A) - mg\hat{\boldsymbol{z}}, \tag{1a}$$
$$\dot{\boldsymbol{L}} = \boldsymbol{a} \times \left[\boldsymbol{F}_R + \boldsymbol{F}_f(\boldsymbol{v}_A)\right], \tag{1b}$$

where $\boldsymbol{v}_{\text{CM}} = \dot{\boldsymbol{s}}$, $\boldsymbol{L} = \hat{\mathbb{I}}\boldsymbol{\omega}$ is the angular momentum and $\boldsymbol{v}_A = \boldsymbol{v}_{\text{CM}} + \boldsymbol{\omega} \times \boldsymbol{a}$ is the velocity of the point of contact A. The gravity force $-mg\hat{\boldsymbol{z}}$ acts at the center of mass and the contact force $\boldsymbol{F} = \boldsymbol{F}_R + \boldsymbol{F}_f(\boldsymbol{v}_A)$ acts at the point A. It is a sum of the friction force $\boldsymbol{F}_f(\boldsymbol{v}_A)$ parallel to the supporting plane and of the reaction force $\boldsymbol{F}_R(\boldsymbol{v}_{\text{CM}}, \boldsymbol{L})$ that depends on the dynamical state of TT and does not have to be orthogonal to the supporting plane. About the friction

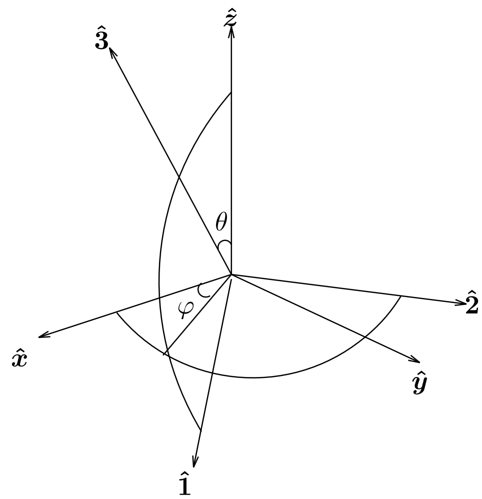

Figure 2. Axis orientation in TT.

force we assume that it vanishes at zero sliding velocity $\boldsymbol{F}_f(\boldsymbol{v}_A = \mathbf{0}) = \mathbf{0}$, but remarkably many qualitative aspects of the motion of TT are independent of the friction law that specifies how $\boldsymbol{F}_f(\boldsymbol{v}_A)$ depends on the contact velocity $\boldsymbol{v}_A$. This feature of equations (1a), (1b) is the reason why the popular toy models of TT persistently exhibit the inverting behavior for majority of supporting surfaces and for different materials that TT is made of.

In order to close the system of vector equations (1a), (1b) we need to add the equation

$$\dot{\hat{\mathbf{3}}} = \frac{1}{I_1}\boldsymbol{L} \times \hat{\mathbf{3}} \tag{1c}$$

that follows from $\dot{\hat{\mathbf{3}}} = \boldsymbol{\omega} \times \hat{\mathbf{3}} = \hat{\mathbb{I}}^{-1}\boldsymbol{L} \times \hat{\mathbf{3}} = \frac{1}{I_1}\left\{\boldsymbol{L} - \frac{I_3 - I_1}{I_3}\left(\boldsymbol{L}\hat{\mathbf{3}}\right)\hat{\mathbf{3}}\right\} \times \hat{\mathbf{3}} = \frac{1}{I_1}\boldsymbol{L} \times \hat{\mathbf{3}}$ by the axial symmetry of TT.

In this paper we are studying only solutions that stay in the supporting plane and, therefore, satisfy identically with respect to time t the algebraic condition $\hat{\boldsymbol{z}}\left[\boldsymbol{s}(t) + \boldsymbol{a}(t)\right] = 0$. This condition is compatible with the structure of equations (1) if all time derivatives of $\hat{\boldsymbol{z}}\left[\boldsymbol{s}(t) + \boldsymbol{a}(t)\right] = 0$ are also equal to zero. The requirement of vanishing first derivative $0 = \hat{\boldsymbol{z}}\big[\dot{\boldsymbol{s}} + \dot{\boldsymbol{a}}\big] = \hat{\boldsymbol{z}}\left[\dot{\boldsymbol{s}}(t) + \boldsymbol{\omega}(t) \times \boldsymbol{a}(t)\right] = \hat{\boldsymbol{z}}\boldsymbol{v}_A$ says that the contact velocity $\boldsymbol{v}_A$ has to stay in the supporting plane all time and the requirement of vanishing second derivative

$$\begin{aligned} 0 &= \hat{\boldsymbol{z}}\left[\ddot{\boldsymbol{s}}(t) + \frac{d}{dt}(\boldsymbol{\omega}(t) \times \boldsymbol{a}(t))\right] = \hat{\boldsymbol{z}}\left[\boldsymbol{F}_R + \boldsymbol{F}_f(\boldsymbol{v}_A) - mg\hat{\boldsymbol{z}} + m\frac{d}{dt}(\boldsymbol{\omega}(t) \times \boldsymbol{a}(t))\right] \\ &= \hat{\boldsymbol{z}}\left[\boldsymbol{F}_R - mg\hat{\boldsymbol{z}} + m\frac{d}{dt}(\boldsymbol{\omega}(t) \times \boldsymbol{a}(t))\right] \end{aligned}$$

determines the vertical component of the reaction force $\hat{\boldsymbol{z}}\boldsymbol{F}_R = mg - m\hat{\boldsymbol{z}}(\boldsymbol{\omega}(t) \times \boldsymbol{a}(t))\dot{}$. The planar component of $\boldsymbol{F}_R$ has to be defined as an external law of the reaction force or may be determined by some extra conditions for motions satisfying (1). For instance in the model of the rising tippe top [5, 12] there has been assumed that $\boldsymbol{F}_R = g_n(\boldsymbol{v}_{\text{CM}}, \boldsymbol{L}, \hat{\mathbf{3}})\hat{\boldsymbol{z}}$ is orthogonal to the plane when $\boldsymbol{F}_f(\boldsymbol{v}_A) = -\mu g_n \boldsymbol{v}_A$ vanishes.

In the case of pure rolling solutions it is the rolling condition that allows for determining the value of the total force $\boldsymbol{F} = \boldsymbol{F}_R$ (since $\boldsymbol{F}_f(\boldsymbol{v}_A = \mathbf{0}) = \mathbf{0}$) so that rolling without sliding takes place and $\boldsymbol{F}_R$ usually is not orthogonal to the supporting plane.

The pure rolling solutions of TT equations have to satisfy an additional algebraic condition $\boldsymbol{v}_A(t) = \boldsymbol{v}_{\text{CM}} + [\boldsymbol{\omega}(t) \times \boldsymbol{a}(t)] = \mathbf{0}$. For discussing the rolling solutions of the TT equations (1) we rewrite them as equations for the new unknowns $\boldsymbol{v}_A$, $\hat{\mathbf{3}}$ and $\boldsymbol{L}$ (or $\boldsymbol{\omega} = \hat{\mathbb{I}}^{-1}\boldsymbol{L}$)

$$m\big[\dot{\boldsymbol{v}}_A - (\boldsymbol{\omega}\times\boldsymbol{a})^{\dot{}}\big] = \boldsymbol{F}_R + \boldsymbol{F}_f(\boldsymbol{v}_A) - mg\hat{\boldsymbol{z}}, \tag{2a}$$

$$\dot{\boldsymbol{L}} = \boldsymbol{a}\times\big[mg\hat{\boldsymbol{z}} + m\dot{\boldsymbol{v}}_A - m(\boldsymbol{\omega}\times\boldsymbol{a})^{\dot{}}\big], \tag{2b}$$

$$\dot{\hat{\mathbf{3}}} = \boldsymbol{\omega}_{\text{ref}}\times\hat{\mathbf{3}} = \boldsymbol{\omega}\times\hat{\mathbf{3}} = \frac{1}{I_1}\boldsymbol{L}\times\hat{\mathbf{3}}. \tag{2c}$$

The requirement of vanishing $\boldsymbol{v}_A(t) = \boldsymbol{v}_{CM}(t) + [\boldsymbol{\omega}(t)\times\boldsymbol{a}(t)] = \mathbf{0}$ entails that $\dot{\boldsymbol{v}}_A = \dot{\boldsymbol{v}}_{CM} + (\boldsymbol{\omega}\times\boldsymbol{a})^{\dot{}} = \mathbf{0}$ vanishes as well. Then equations (2b), (2c) become an autonomous system of equations, with a polynomial vector field, for $\boldsymbol{L}$ (or $\boldsymbol{\omega}$) and $\hat{\mathbf{3}}$. They have solutions by the existence theorem for dynamical systems and $\boldsymbol{v}_{CM}$ is determined from $\boldsymbol{v}_{\text{CM}} = -[\boldsymbol{\omega}(t)\times\boldsymbol{a}(t)]$. The condition $\dot{\boldsymbol{v}}_{\text{CM}} = -(\boldsymbol{\omega}(t)\times\boldsymbol{a}(t))^{\dot{}}$ automatically follows and from the first equation (2a) the unknown total force $\boldsymbol{F} = \boldsymbol{F}_R + \boldsymbol{F}_f(\boldsymbol{v}_A) = m\big[-(\boldsymbol{\omega}\times\boldsymbol{a})^{\dot{}}\big] + mg\hat{\boldsymbol{z}}$, needed for maintaining the pure rolling motion, can be calculated. Thus we have shown.

Proposition 1. *The pure rolling constraint $\boldsymbol{v}_A = \boldsymbol{v}_{CM} + \boldsymbol{\omega}\times\boldsymbol{a}$ reduces equations (1) to the closed system of equations*

$$(\hat{\mathbb{I}}\boldsymbol{\omega})^{\dot{}} = \boldsymbol{a}\times\big[mg\hat{\boldsymbol{z}} - m(\boldsymbol{\omega}\times\boldsymbol{a})^{\dot{}}\big], \tag{3a}$$

$$\dot{\hat{\mathbf{3}}} = \boldsymbol{\omega}\times\hat{\mathbf{3}} \tag{3b}$$

for the unknowns $\boldsymbol{\omega}$ and $\hat{\mathbf{3}}$ where $\hat{\mathbb{I}} = I_1\big\{\mathbb{1} + \frac{I_3-I_1}{I_1}|\hat{\mathbf{3}}\rangle\langle\hat{\mathbf{3}}|\big\}$. It is consistent with equations (1) when we assume that the total force is dynamically determined as $\boldsymbol{F} = \boldsymbol{F}_R + \boldsymbol{F}_f = -m\big(\boldsymbol{\omega}\times\boldsymbol{a}\big)^{\dot{}} + mg\hat{\boldsymbol{z}}$.

Thus the pure rolling constraint changes the model of the TT by saying that the force $\boldsymbol{F}$ applied to the body at point A is dynamically determined. This means that general rolling solutions presented here usually do not satisfy the TT model with $\boldsymbol{F}_R = g_n\hat{\boldsymbol{z}}$, $\boldsymbol{F}_f = -\mu g_z\boldsymbol{v}_A$ except the vertical spinning solutions and the tumbling solutions with CM fixed in space [5, 12].

3 Coordinate form of the rolling TT (rTT) equations. Integrals of motion

The autonomous system of rTT equations (3a), (3b) can be expressed in the moving reference frame $\hat{\mathbf{1}}$, $\hat{\mathbf{2}}$, $\hat{\mathbf{3}}$. Recall that $\hat{\boldsymbol{z}} = -\sin\theta\hat{\mathbf{1}} + \cos\theta\hat{\mathbf{3}}$, $\boldsymbol{a} = R(\alpha\hat{\mathbf{3}} - \hat{\boldsymbol{z}}) = R\big(\sin\theta\hat{\mathbf{1}} + (\alpha-\cos\theta)\hat{\mathbf{3}}\big)$, $\boldsymbol{\omega} = \boldsymbol{\omega}_{\text{ref}} + \dot{\psi}\hat{\mathbf{3}} = -\dot{\varphi}\sin\theta\hat{\mathbf{1}} + \dot{\theta}\hat{\mathbf{2}} + (\dot{\psi} + \dot{\varphi}\cos\theta)\hat{\mathbf{3}} = -\dot{\varphi}\sin\theta\hat{\mathbf{1}} + \dot{\theta}\hat{\mathbf{2}} + \omega_3\hat{\mathbf{3}}$. By substituting these expressions into (3a) we get at $\hat{\mathbf{1}}$, $\hat{\mathbf{2}}$, $\hat{\mathbf{3}}$ the following system of equations for the Euler angles (θ, φ, ψ)

$$\begin{aligned} &I_3\omega_3\dot{\theta} - 2I_1\dot{\varphi}\dot{\theta}\cos\theta - I_1\ddot{\varphi}\sin\theta + mR^2(\alpha-\cos\theta)\big(-\ddot{\varphi}\sin\theta(\alpha-\cos\theta) \\ &\qquad - 2\dot{\varphi}\dot{\theta}\cos\theta(\alpha-\cos\theta) - \dot{\omega}_3\sin\theta - \omega_3\dot{\theta}\cos\theta\big) = 0, \end{aligned} \tag{4a}$$

$$\begin{aligned} &I_1\ddot{\theta} - I_1\dot{\varphi}^2\sin\theta\cos\theta + I_3\omega_3\dot{\varphi}\sin\theta + mR^2\sin\theta\big(\dot{\theta}^2\alpha + \omega_3\dot{\varphi}\sin^2\theta + \ddot{\theta}\sin\theta \\ &\qquad + \dot{\varphi}^2\sin^2\theta(\alpha-\cos\theta)\big) + mR^2(\alpha-\cos\theta)\big(\ddot{\theta}(\alpha-\cos\theta) - \dot{\varphi}\omega_3\sin\theta\cos\theta \\ &\qquad - \dot{\varphi}^2\sin\theta\cos\theta(\alpha-\cos\theta)\big) = -\alpha mgR\sin\theta, \end{aligned} \tag{4b}$$

$$I_3\dot{\omega}_3 + mR^2\sin\theta\big(2\dot{\varphi}\dot{\theta}\cos\theta(\alpha-\cos\theta) + \ddot{\varphi}\sin\theta(\alpha-\cos\theta) + \dot{\omega}_3\sin\theta + \omega_3\dot{\theta}\cos\theta\big) = 0. \tag{4c}$$

After resolving w.r.t. $(\ddot{\theta}, \ddot{\varphi}, \dot{\omega}_3)$ we obtain

$$\ddot{\theta} = \frac{\sin\theta}{I_1 + mR^2\big((\alpha-\cos\theta)^2 + \sin^2\theta\big)}\Big[\dot{\varphi}^2\big(-mR^2(\alpha-\cos\theta)(1-\alpha\cos\theta) + I_1\cos\theta\big)$$

$$+\,\omega_3\dot\varphi\left(mR^2(\alpha\cos\theta-1)-I_3\right)-mR^2\dot\theta^2\alpha-mR\alpha g\Big], \tag{5a}$$

$$\ddot\varphi=\frac{\omega_3\dot\theta}{\sin\theta}\left[\frac{I_3^2+mR^2I_3(1-\alpha\cos\theta)}{I_1I_3+mR^2I_1\sin^2\theta+mR^2I_3(\alpha-\cos\theta)^2}\right]-\frac{2\dot\varphi\dot\theta\cos\theta}{\sin\theta}, \tag{5b}$$

$$\dot\omega_3=-\omega_3\dot\theta\sin\theta\left[\frac{mR^2I_3(\alpha-\cos\theta)+mR^2I_1\cos\theta}{mR^2I_3(\alpha-\cos\theta)^2+I_1I_3+mR^2I_1\sin^2\theta}\right]. \tag{5c}$$

Since φ and ψ are cyclic coordinates, that do not appear in the right hand side of equations (5), it is effectively a fourth order dynamical system for the variables θ, $\dot\theta$, $\dot\varphi$ and $\omega_3=\dot\psi+\dot\varphi\cos\theta$. It admits three functionally independent integrals of motion [13].

Equation (5c) can be integrated directly to the Routh integral

$$D=I_3\omega_3\left[\gamma+\beta(\alpha-\cos\theta)^2+\beta\gamma\sin^2\theta\right]^{\frac12}=:I_3\omega_3\sqrt{d(\theta)},$$

where $\beta=\frac{mR^2}{I_3}$, $\gamma=\frac{I_1}{I_3}$ and $d(\theta)=\gamma+\beta\big(\alpha-\cos\theta\big)^2+\beta\gamma\sin^2\theta$.

The Jellett's integral

$$\lambda=I_1\dot\varphi\sin^2\theta-(\alpha-\cos\theta)\,I_3\omega_3$$

follows from equations (5b) and (5c) and the energy integral

$$E=\frac12 mR^2\left[\dot\theta^2(\alpha-\cos\theta)^2+\sin^2\theta\dot\varphi^2(\alpha-\cos\theta)^2+2\sin^2\theta\dot\varphi\omega_3(\alpha-\cos\theta)\right.$$
$$\left.+\sin^2\theta\omega_3^2+\dot\theta^2\sin^2\theta\right]+\frac12\left[I_1\dot\varphi^2\sin^2\theta+I_1\dot\theta^2+I_3\omega_3^2\right]+mgR(1-\alpha\cos\theta)$$

is a consequence of all three equations (5) as can be checked by direct differentiation w.r.t. time.

Proposition 2. *The rTT equations of motion* (3) *admit three time independent integrals of motion*

$$D=(\boldsymbol\omega\hat{\mathbf 3})\left[I_1I_3+mR^2I_3\left(\alpha-(\hat{\boldsymbol z}\hat{\mathbf 3})\right)^2+mR^2I_1\left(1-(\hat{\boldsymbol z}\hat{\mathbf 3})^2\right)\right]^{\frac12}$$
$$=\omega_3\left[I_1I_3+mR^2I_3(\alpha-\cos\theta)^2+mR^2I_1\sin^2\theta\right]^{\frac12}, \tag{6a}$$

$$\lambda=-\boldsymbol L\boldsymbol a=I_1\dot\varphi\sin^2\theta-(\alpha-\cos\theta)\,I_3\omega_3, \tag{6b}$$

$$E=\frac12 m\boldsymbol v_{CM}^2+\frac12\boldsymbol\omega\boldsymbol L+mg\boldsymbol s\hat{\boldsymbol z}=\frac12 mR^2\left[\dot\theta^2(\alpha-\cos\theta)^2+\sin^2\theta\omega_3^2\right.$$
$$\left.+\sin^2\theta\dot\varphi^2(\alpha-\cos\theta)^2+2\sin^2\theta\dot\varphi\omega_3(\alpha-\cos\theta)+\dot\theta^2\sin^2\theta\right]$$
$$+\frac12\left[I_1\dot\varphi^2\sin^2\theta+I_1\dot\theta^2+I_3\omega_3^2\right]+mgR(1-\alpha\cos\theta). \tag{6c}$$

Proof. As we have seen, the coordinate form of the integrals of motion follows easily from the coordinate equations (5). It is instructive also to see how the vector form of the integrals of motion is related to the vector form of equations (3).

We see that the Jellett's integral $\lambda=-\boldsymbol L\boldsymbol a$ is a scalar product of the angular momentum $\boldsymbol L$ and a vector vector $\boldsymbol a$. When deriving

$$-\dot\lambda=\big(\boldsymbol L\boldsymbol a\big)^{\cdot}=\dot{\boldsymbol L}\boldsymbol a+\boldsymbol L\dot{\boldsymbol a}=(\boldsymbol a\times\boldsymbol F)\,\boldsymbol a+\boldsymbol L\big(\alpha\dot{\hat{\mathbf 3}}-R\dot{\hat{\boldsymbol z}}\big)$$
$$=\alpha\boldsymbol L\dot{\hat{\mathbf 3}}=\alpha\frac{1}{I_1}\boldsymbol L\left(\boldsymbol L\times\hat{\mathbf 3}\right)=0$$

we see that each term disappears on its own. The first term disappears because the vector $\boldsymbol{a}$ is the same as in the dynamical equation (3a). The second term disappears since $\dot{\hat{\mathbf{3}}} = \boldsymbol{\omega} \times \hat{\mathbf{3}} = \frac{1}{I_1}\left(\boldsymbol{L} \times \hat{\mathbf{3}}\right)$.

In calculating time derivative of the energy integral we use the equality $\dot{\boldsymbol{\omega}}\boldsymbol{L} = \boldsymbol{\omega}\dot{\boldsymbol{L}}$ that follows by taking the derivative of $\boldsymbol{\omega} = \hat{\mathbb{I}}^{-1}\boldsymbol{L} = \frac{1}{I_1}\left\{\mathbb{1} + \frac{I_3 - I_1}{I_3}|\hat{\mathbf{3}}\rangle\langle\hat{\mathbf{3}}|\right\}\boldsymbol{L}$. Then for the general equations of TT with sliding (1) we get

$$\begin{aligned}\dot{E} &= m\boldsymbol{v}_{CM}\dot{\boldsymbol{v}}_{CM} + \boldsymbol{\omega}\dot{\boldsymbol{L}} + mg\dot{\boldsymbol{s}}\hat{\boldsymbol{z}} \\ &= \boldsymbol{F}\boldsymbol{v}_A - (\boldsymbol{\omega} \times \boldsymbol{a})\,\boldsymbol{F} - \boldsymbol{F}\,(\boldsymbol{\omega} \times \boldsymbol{a}) + mg\,(\dot{\boldsymbol{s}} + \boldsymbol{\omega} \times \boldsymbol{a})\,\hat{\boldsymbol{z}} - mg\boldsymbol{v}_A\hat{\boldsymbol{z}} = \boldsymbol{F}\boldsymbol{v}_A.\end{aligned}$$

For rTT $\boldsymbol{v}_A = \mathbf{0}$ and the energy is conserved since the force $\boldsymbol{F}$ doesn't perform any work.

The Routh integral follows remarkably simple from the coordinate equation (5c) but is considerably more difficult to see in the vector notation. The time derivative

$$\dot{D} = I_3\left[\frac{2\left(\dot{\boldsymbol{\omega}}\hat{\mathbf{3}}\right)d\left(\hat{\boldsymbol{z}}\hat{\mathbf{3}}\right) + \left(\boldsymbol{\omega}\hat{\mathbf{3}}\right)d'\left(\hat{\boldsymbol{z}}\hat{\mathbf{3}}\right)}{2\sqrt{d\left(\hat{\boldsymbol{z}}\hat{\mathbf{3}}\right)}}\right]$$

contains the term $\left(\dot{\boldsymbol{\omega}}\hat{\mathbf{3}}\right)$ that has to be eliminated with the use of equations (3). To do this we express $\dot{\boldsymbol{L}}$ through $\dot{\boldsymbol{\omega}}$ by differentiating $\boldsymbol{L} = \hat{\mathbb{I}}\boldsymbol{\omega}$ and calculate the $\hat{\mathbf{1}}$ and $\hat{\mathbf{3}}$ components that in the vector notation correspond to (4a) and (4c). We obtain two linear algebraic equations for the unknowns $\left(\dot{\boldsymbol{\omega}}\hat{\mathbf{3}}\right)$ and $(\dot{\boldsymbol{\omega}}\hat{\boldsymbol{z}})$. One by multiplying with $\hat{\mathbf{3}}$

$$\left(\dot{\boldsymbol{\omega}}\hat{\mathbf{3}}\right) = \frac{1}{I_3}(\hat{3}\dot{\boldsymbol{L}}) = \beta\big[\left(\dot{\boldsymbol{\omega}}\hat{\mathbf{3}}\right)\left(\left(\hat{\boldsymbol{z}}\hat{\mathbf{3}}\right)\alpha - 1\right) - \left(\dot{\boldsymbol{\omega}}\hat{\boldsymbol{z}}\right)\left(\alpha - \left(\hat{\boldsymbol{z}}\hat{\mathbf{3}}\right)\right) + \alpha\left(\boldsymbol{\omega}\hat{\mathbf{3}}\right)\left(\left(\hat{\mathbf{3}} \times \hat{\boldsymbol{z}}\right)\boldsymbol{\omega}\right)\big] \tag{7}$$

and a second equation by multiplying with $\hat{\mathbf{1}}$ and substituting $\hat{\mathbf{1}} = \frac{\hat{3}(\hat{\boldsymbol{z}}\hat{\mathbf{3}}) - \hat{z}}{\sqrt{1 - (\hat{\boldsymbol{z}}\hat{\mathbf{3}})^2}}$. This equation is more complicated but also involves $\left(\dot{\boldsymbol{\omega}}\hat{\mathbf{3}}\right)$ and $(\dot{\boldsymbol{\omega}}\hat{\boldsymbol{z}})$ like equation (7). From this linear system we determine $\left(\dot{\boldsymbol{\omega}}\hat{\mathbf{3}}\right)$ and substitute into the $\dot{D}$ expression. One then obtains

$$\begin{aligned}\dot{D} &= I_3\left[\frac{2\left(\dot{\boldsymbol{\omega}}\hat{\mathbf{3}}\right)d\left(\hat{\boldsymbol{z}}\hat{\mathbf{3}}\right) + \left(\boldsymbol{\omega}\hat{\mathbf{3}}\right)d'\left(\hat{\boldsymbol{z}}\hat{\mathbf{3}}\right)}{2\sqrt{d\left(\hat{\boldsymbol{z}}\hat{\mathbf{3}}\right)}}\right] \\ &= \frac{2I_3\left(\boldsymbol{\omega}\hat{\mathbf{3}}\right)\left(\left(\hat{\mathbf{3}} \times \hat{\boldsymbol{z}}\right)\boldsymbol{\omega}\right)\beta}{2\sqrt{d\left(\hat{\boldsymbol{z}}\hat{\mathbf{3}}\right)}}\left[\begin{array}{l}\text{rational expression depending only}\\ \text{on }\left(\hat{\boldsymbol{z}}\hat{\mathbf{3}}\right)\text{ that vanishes identically}\end{array}\right] = 0.\end{aligned}$$ ■

In [13] and [6] a quadratic integral of motion D^2 is taken because D^2 enters the expression (9) for the effective potential $V(\theta; D, \lambda)$. It seems natural, however, to speak about linear integral $D = I_3\omega_3\sqrt{d(\theta)}$, since it is well defined due to $d(\theta) > 0$. Any axially symmetric rolling rigid body is integrable [2, 9, 14] and it admits, beside energy E, a 2-parameter family of integrals of motion depending linearly on ω. These integrals are defined through transcendental functions satisfying a certain linear 2nd order ODE with variable coefficients that depend on the convex shape of the rigid body. The special feature of the rolling sphere integrals is that they are expressed explicitly through elementary functions.

In [6] authors also discuss the limit $R \to 0$ of the Jellett's and Routh integrals and they find they are equivalent to conservation of the vertical $\boldsymbol{L}\hat{\boldsymbol{z}}$ and the axial $\boldsymbol{L}\hat{\mathbf{3}}$ components of the angular momentum as for symmetric top with a fixed point tip. A slightly different way of finding the integral D has been given in [9]. The approach of [9] captures also the situation when the sphere is rolling along another sphere.

4 Separation equations for rTT

Equations (5) can be considered as a fourth order system for θ, $\dot\theta$, $\dot\varphi$, $\omega_3 = \dot\psi + \dot\varphi\cos\theta$ since φ, ψ are cyclic variables, but it is totaly a sixth order system of equations for the Euler angles (θ, φ, ψ). The existence of three integrals of motion reduces the differential order by three and we obtain the system of three equations

$$D = \omega_3\left[I_1 I_3 + mR^2 I_3\left(\alpha - \cos\theta\right)^2 + mR^2 I_1 \sin^2\theta\right]^{\frac{1}{2}} = I_3\omega_3\sqrt{d(\theta)}, \tag{8a}$$

$$\lambda = I_1\dot\varphi\sin^2\theta - \left(\alpha - \cos\theta\right) I_3\omega_3, \tag{8b}$$

$$\begin{aligned} E = {}& \frac{1}{2}mR^2\left[\dot\theta^2\left(\alpha - \cos\theta\right)^2 + \sin^2\theta\dot\varphi^2\left(\alpha - \cos\theta\right)^2 + 2\sin^2\theta\dot\varphi\omega_3\left(\alpha - \cos\theta\right)\right. \\ & \left. + \sin^2\theta\omega_3^2 + \dot\theta^2\sin^2\theta\right] + \frac{1}{2}\left[I_1\dot\varphi^2\sin^2\theta + I_1\dot\theta^2 + I_3\omega_3^2\right] + mgR\left(1 - \alpha\cos\theta\right). \end{aligned} \tag{8c}$$

It is separable when we resolve (8b) for $\dot\varphi = \frac{\lambda + (\alpha - \cos\theta) I_3\omega_3}{I_1\sin^2\theta}$, substitute $\dot\varphi$ into the energy integral (8c) and finally express ω_3 in (8c) through D and θ, using the Routh integral (8a). Then we obtain a separable equation of the form

$$E = g(\cos\theta)\dot\theta^2 + V(\cos\theta, D, \lambda)$$

with

$$g(\cos\theta) = \frac{1}{2}I_3\big(\beta\big(\left(\alpha - \cos\theta\right)^2 + 1 - \cos^2\theta\big) + \gamma\big)$$

$$\begin{aligned} V(z, D, \lambda) = {}& mgR\left(1 - \alpha z\right) + \frac{1}{2I_3 d(z)}\left[\frac{\left(\lambda\sqrt{d(z)} + \left(\alpha - z\right)D\right)^2\left(\beta\left(\alpha - z\right)^2 + \gamma\right)}{\gamma^2\left(1 - z^2\right)}\right. \\ & \left. + D^2\left(\beta\left(1 - z^2\right) + 1\right) + \frac{2D\beta\left(\alpha - z\right)\left(\lambda\sqrt{d(z)} + \left(\alpha - z\right)D\right)}{\gamma}\right]. \end{aligned} \tag{9}$$

We denote $z = \cos\theta$, $\gamma = \frac{I_1}{I_3}$, $\beta = \frac{mR^2}{I_3}$ and $d(z) = \gamma + \beta\left(\alpha - z\right)^2 + \gamma\beta\left(1 - z^2\right)$.

Notice that both $g(z) > 0$ and $d(z) > 0$ are strictly positive functions for $z \in [-1, 1]$ ($\theta \in [0, \pi]$). The potential $V(z = \cos\theta, D, \lambda)$ is well defined for all values of $\theta \in]0, \pi[$ and $V(\cos\theta, D, \lambda) \to +\infty$, $\theta \to 0, \pi$ if the quotient of integrals of motions $\frac{D}{\lambda} \neq \frac{-1}{(\alpha\pm1)}\left(\gamma + \beta\left(\alpha \pm 1\right)^2\right)^{\frac{1}{2}}$.

Since $V(\theta, D, \lambda)$ has no singularities in $\theta \in]0, \pi[$, the motion is confined to the interval determined by the equation $\dot\theta = 0$ which gives condition $E = V(\cos\theta, D, \lambda)$. For fixed values of λ, D and for the energy $E \geq V(z_{\min} = \cos\theta_{\min}, D, \lambda)$ the equation $E = V(\cos\theta, D, \lambda)$ has, at least, two solutions θ_1 and θ_2 and the one-dimensional θ-motion takes place between two turning points $0 \leq \theta_1, \theta_2 \leq \pi$. The motion of the symmetry axis $\hat{\mathbf{3}}$ on the unit sphere S^2 takes place between two latitudes and has nutational character similarly as the nutational motion of the Lagrange top (LT).

For $\frac{D}{\lambda} = \frac{-1}{(\alpha\pm1)}\left(\gamma + \beta\left(\alpha \pm 1\right)^2\right)^{\frac{1}{2}}$ the potential takes the form

$$\begin{aligned} V\left(z, D = \frac{-\lambda}{\left(\alpha \pm 1\right)}\left(\gamma + \beta\left(1 \pm \alpha\right)^2\right)^{\frac{1}{2}}, \lambda\right) = {}& mgR\left(1 - \alpha z\right) + \frac{\lambda^2}{2I_3 d(z)\left(\alpha \pm 1\right)^2} \\ \times \Bigg[\left(\gamma + \beta\left(\gamma \pm 1\right)^2\right)\left(\beta\left(1 - z^2\right) + 1\right) & + \frac{\left(\sqrt{d(z)} - \left(\alpha - z\right)\left(\gamma + \beta\left(\gamma \pm 1\right)^2\right)^{\frac{1}{2}}\right)^2}{\gamma^2\left(1 - z^2\right)} \\ - \frac{\left(\sqrt{d(z)}\left(\alpha \pm 1\right) - \left(\alpha - z\right)\left(\gamma + \beta\left(1 \pm \alpha\right)^2\right)^{\frac{1}{2}}\right)}{\gamma} & 2\beta\left(\alpha - z\right)\left(\gamma + \beta\left(1 \pm \alpha\right)^2\right)^{\frac{1}{2}}\Bigg]. \end{aligned}$$

In order to discuss the motion of the rTT we shall revoke similarities of equations (8) with the equations of the Lagrange top (LT) that follow from the Lagrangian [10]

$$\mathscr{L} = \frac{1}{2}I_1\big(\dot\theta^2 + \dot\varphi^2\sin^2\theta\big) + \frac{1}{2}I_3\big(\dot\psi + \dot\varphi\cos\theta\big)^2 - mgl\cos\theta,$$

where the angles θ, φ and ψ has the same meaning as for TT. The Lagrange equations of the LT admit the following three integrals of motion

$$L_3 = I_3\big(\dot\psi + \dot\varphi\cos\theta\big) = I_3\omega_3, \tag{10a}$$

$$L_z = I_1\dot\varphi\sin^2\theta + I_3\cos\theta\big(\dot\psi + \dot\varphi\cos\theta\big) = I_1\dot\varphi\sin^2\theta + I_3\omega_3\cos\theta, \tag{10b}$$

$$E = \frac{1}{2}I_1\big(\dot\theta^2 + \dot\varphi^2\sin^2\theta\big) + \frac{1}{2}I_3\big(\dot\psi + \dot\varphi\cos\theta\big)^2 + mgl\cos\theta, \tag{10c}$$

where L_3, L_z and E denote the constant values of integrals. There are transparent similarities between equations (8) and (10). The Routh integral D corresponds to $L_3 = I_3\omega_3$, the Jellett integral has a similar structure as L_z and in the energy E both $\dot\varphi$ and ω_3 can be eliminated to give a one-dimensional equation for $\theta(t)$. However in the case of rTT the effective potential $V(z, D, \lambda)$ becomes a more complicated function of z than the LT effective potential $V_{LT} = \frac{(L_z - L_3\cos\theta)^2}{I_1^2\sin^2\theta} + \frac{2mgl}{I_1}\cos\theta$. We shall analyze the character of motions of the rTT by studying some special types of solutions that are natural counterparts of special solutions to the LT.

In the case of LT one can distinguish several types of special solutions defined by an invariant algebraic conditions for dynamical variables and/or by fixing values of integrals of motion. Their trajectories are customarily represented by a curve on S^2 drawn by the symmetry axis $\hat{\mathbf{3}}$ of LT.

a) Vertical rotations are defined by the condition $\theta(t) \equiv 0$ or $\theta(t) \equiv \pi$, so that $L_z = \pm L_3 = I_3(\dot\psi \pm \dot\varphi)$. The vertical rotations are represented by the north and the south pole on S^2.

b) Planar pendulum motions have $L_3 = I_3\omega_3 = 0$, $L_z = I_1\dot\varphi\sin^2\theta + I_3\omega_3\cos\theta = 0$ so that either $\omega_3 = \dot\varphi = 0$ or $\omega_3 = 0$ and $\theta(t) \equiv 0, \pi$. They move along large circle arcs through the south pole of S^2.

c) Spherical pendulum motions have $L_3 = I_3\omega_3 = 0$, $L_z = I_1\dot\varphi\sin^2\theta + I_3\omega_3\cos\theta \neq 0$ so that $\omega_3 = 0$ and $L_z = I_1\dot\varphi\sin^2\theta$. The $\dot\varphi$ does not change the sign and the trajectory $(\varphi(t), \theta(t))$ of $\hat{\mathbf{3}}$ draws a wavelike curve between two latitudes $0 < \theta_1 < \theta_2 < \pi$ determined by the condition $E = \frac{1}{2}\frac{L_z^2}{I_1\sin^2\theta} + mgl\cos\theta$.

d) Precessional motions with $\dot\theta = 0$ which implies that $\theta(t) \equiv \theta_0$, $0 < \theta_0 < \pi$ and that $\dot\psi$ and $\dot\varphi$ are also constant. They are represented by latitude circles on S^2.

e) General nutational motions between two latitudes $0 \leq \theta_1 \leq \theta_2 \leq \pi$ given as solutions of the equation $E = \frac{1}{2I_1}\frac{(L_z - L_3\cos\theta)^2}{\sin^2\theta} + \frac{1}{2I_3}L_3^2 + mgl\cos\theta$. As is well known, the shape of the curve drawn by $\hat{\mathbf{3}}$ on the unit sphere depends on whether $\dot\varphi = \frac{L_z - L_3\cos\theta}{I_1\sin^2\theta}$ changes sign during motion or not. There are three cases:

   (i) $\frac{L_z}{L_3} \notin [\cos\theta_2, \cos\theta_1]$ so that $\dot\varphi$ has the same sign all the time and the trajectory $(\varphi(t), \theta(t))$ of $\hat{\mathbf{3}}$ on S^2 has a wavelike form.

   (ii) $\frac{L_z}{L_3} = \cos\theta_1$. In this case $\dot\varphi$ will be zero at θ_1 and the curve $(\varphi(t), \theta(t))$ has cusps at the latitude θ_1.

   (iii) $\frac{L_z}{L_3} \in]\cos\theta_2, \cos\theta_1[$ so that $\dot\varphi$ changes sign during motion and has different signs at the both turning angles θ_1 and θ_2. The axis $\hat{\mathbf{3}}$ draws a wave like curve with loops on S^2.

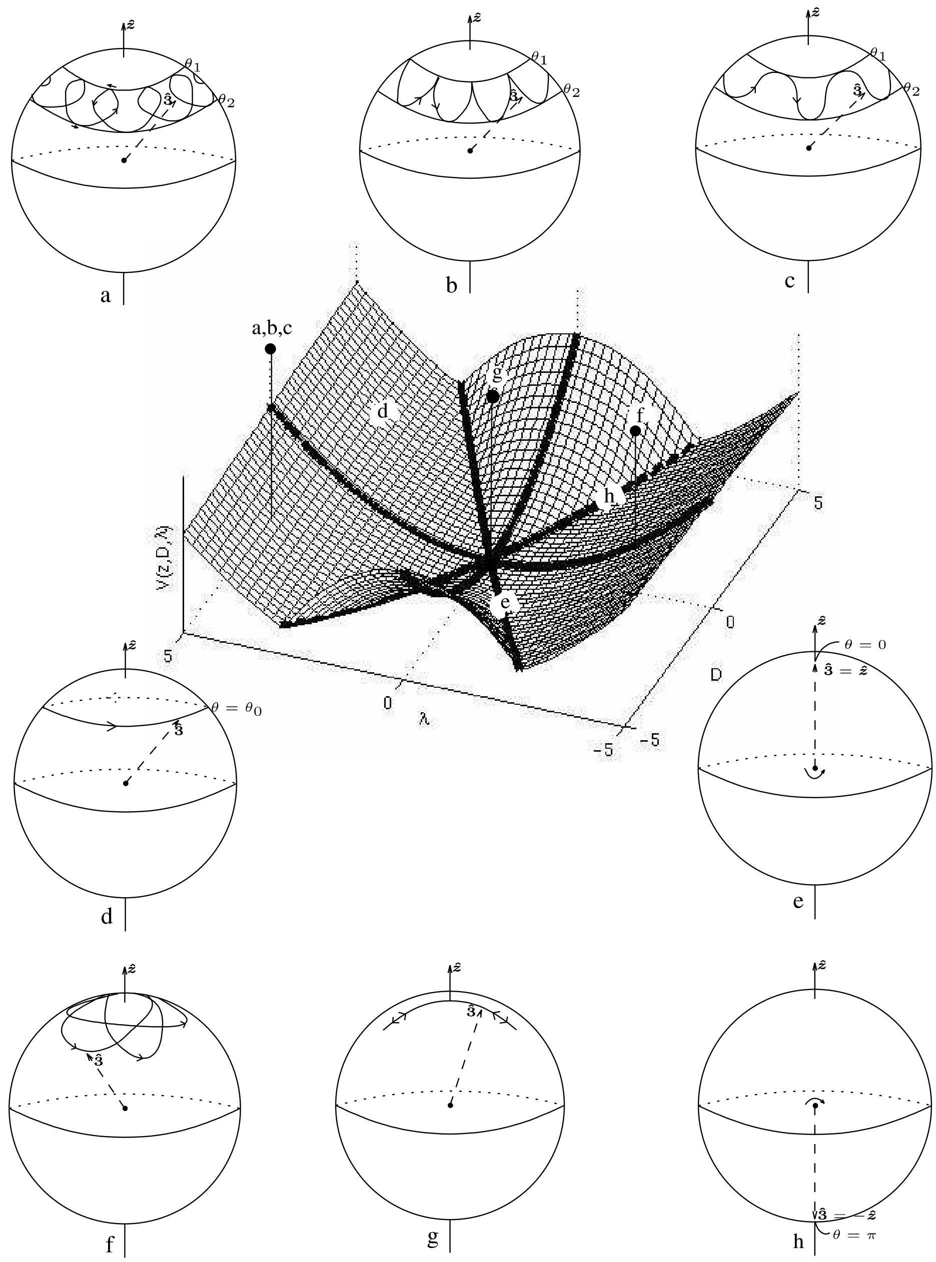

Figure 3. Illustration of the phase space picture of TT.

Due to similarity of separation equations (8) with (10) we can distinguish for rTT similar types of invariant solutions as for LT. They are easier to analyze than the general case $E = g(\cos\theta)\dot{\theta}^2 + V(\cos\theta, D, \lambda)$ and they well illustrate behavior of rTT for different initial conditions. They are again represented by curves drawn by the symmetry axis $\hat{\mathbf{3}}$ on S^2 (see Fig. 3).

a) Vertical rotations defined by the condition $\theta(t) \equiv 0$ or $\theta(t) \equiv \pi$. As for LT, the vertical rotations are represented by the north and the south pole on S^2. But here for vertical motions $\frac{D}{\lambda} = \frac{-1}{(\alpha \mp 1)}\left(\gamma + \beta\,(\alpha \mp 1)^2\right)^{\frac{1}{2}}$ for $\theta = 0, \pi$ respectively.

b) Analog of planar pendulum type of solutions: $D = I_3\omega_3\sqrt{d(\theta)} = 0$, $\lambda = I_1\dot{\varphi}\sin^2\theta - (\alpha - \cos\theta)\,I_3\omega_3 = 0$ so that either $\omega_3 = \dot{\varphi} = 0$ or $\omega_3 = 0$ and $\theta = 0, \pi$. The potential $V(z = \cos\theta, D = 0, \lambda = 0) = mgR(1 - \alpha\cos\theta)$ is a periodic bounded function, the coefficient $g(\cos\theta) = \frac{1}{2}\left[I_1 + (\alpha - \cos\theta)^2 + 1 - \cos^2\theta\right] > 0$ is also a positive periodic bounded function so that rTT admits pendulum type of solutions for low values of energy as well as the “rotational” type solutions (as in the mathematical pendulum) for higher values of energy.

c) Analog of the $L_3 \neq 0$, $L_z = 0$ case: $\lambda = I_1\dot{\varphi}\sin^2\theta - (\alpha - \cos\theta)\, I_3\omega_3 = 0$, $D \neq 0$ which means that $\boldsymbol{L}$ is always in the plane orthogonal to $\boldsymbol{a}$. Then

$$E = g(\cos\theta)\dot{\theta}^2 + V(z, D, \lambda = 0) = g(\cos\theta)\dot{\theta}^2 + \frac{D^2}{2I_3 d(z)}\Big[\beta\left(1 - z^2\right) + 1 + \frac{(\alpha - z)^2}{\gamma^2\left(1 - z^2\right)^2}\left(\beta\left(\alpha - z\right)^2 + \gamma\right) + \frac{2\beta\left(\alpha - z\right)^2}{\gamma}\Big] + mgR\left(1 - \alpha z\right)$$

and the potential $V(z = \cos\theta, D, \lambda = 0) \to +\infty$ as $\theta \to 0, \pi$. It has exactly one minimum for $\theta \in]0, \pi[$ because it is a convex function of z. To prove this we show that

$$\frac{d^2 V(z, D, \lambda = 0)}{dz^2} = \frac{-2D^2\alpha}{I_3\gamma^2\left(1 - z^2\right)}\left(z^3 - \frac{3}{2}\frac{1 + \alpha^2}{\alpha}z^2 + 3z - \frac{1 + \alpha^2}{2\alpha}\right)$$

is positive for all $-1 < z < 1$. The third order polynomial $p(z) = z^3 - \frac{3}{2}\frac{1+\alpha^2}{\alpha}z^2 + 3z - \frac{1+\alpha^2}{2\alpha}$ is negative for $z = 0$, and it is not changing sign because it does not have a zero in the interval $-1 < z < 1$ for $0 < \alpha < 1$. The derivative $p'(z) = 3\big(z^2 - \frac{1+\alpha^2}{\alpha}z + 1\big)$ has two zeros $z_1 = \alpha$, $z_2 = \frac{1}{\alpha}$ and $p(z_1 = \alpha) = -\frac{1}{2\alpha}(\alpha^2 - 1) < 0$. Since it is a local maximum, $p(z)$ can't change sign. The function $V(z = \cos\theta, D, 0)$ is convex and has only one minimum.

d) Analog of spherical pendulum solutions: $D = \omega_3\sqrt{d(\theta)} = 0$, $\lambda = I_1\dot{\varphi}\sin^2\theta \neq 0$ so that $\omega_3 = 0$ since $d(\theta) > 0$ for all $0 \leq \theta \leq \pi$. This means that during the rolling motion of rTT it's not performing any rotation around $\hat{\mathbf{3}}$-axis. Then $\dot{\varphi} = \frac{\lambda}{I_1\sin^2\theta}$ and

$$\begin{aligned} E &= g(\cos\theta)\dot{\theta}^2 + V(z = \cos\theta, D = 0, \lambda) \\ &= g(\cos\theta)\dot{\theta}^2 + \frac{\lambda^2}{2I_1\left(1 - z^2\right)\theta}\left(mR^2\left(\alpha - z\right)^2 + I_1\right) + mgR\left(1 - \alpha z\right). \end{aligned}$$

The potential $V(z = \cos\theta, D = 0, \lambda) \to +\infty$ as $\theta \to 0, \pi$ and just as in the previous case it has exactly one minimum for $\theta \in]0, \pi[$ because it is a convex function of z. To prove this we show that for $0 < \alpha < 1$ and $\frac{\gamma}{\beta} > 0$, the derivative

$$\frac{d^2 V(z, 0, \lambda)}{dz^2} = -\frac{2\lambda^2 mR^2\alpha}{I_1^2\left(1 - z^2\right)^3}\left[z^3 - \frac{3\left(1 + \alpha^2 + \frac{\gamma}{\beta}\right)}{2\alpha}z^2 + 3z - \frac{1 + \alpha^2 + \frac{\gamma}{\beta}}{2\alpha}\right]$$

is positive for all $-1 < z < 1$. The third order polynomial $q(z) = z^3 - \frac{3\left(1+\alpha^2+\frac{\gamma}{\beta}\right)}{2\alpha}z^2 + 3z - \frac{1+\alpha^2+\frac{\gamma}{\beta}}{2\alpha}$ has three solutions. One can show that, for $0 < \alpha < 1$ and $\frac{\gamma}{\beta} > 0$, two of the solutions will be complex and the third solution will be greater than $z = 1$. Because the leading term in $q(z)$ is positive and the only real solution is greater than one, we conclude that $q(z)$ is negative for $-1 < z < 1$ and hence $\frac{d^2V(z,0,\lambda)}{dz^2}$ is positive for $-1 < z < 1$ and $V(z = \cos\theta, D = 0, \lambda)$ is therefore convex.
There are two types of orbits here:

(i) $E = V(\cos\theta_{\min}, D, \lambda)$ Then the motion is along the horizontal circle at $\theta(t) \equiv \theta_{\min}$.

(ii) $E > V(\cos\theta_{\min}, D, \lambda)$ Here the motion is confined between the circles $0 < \theta_1 \leq \theta_2 < \pi$ which are the solutions to $E = V(\cos\theta, \lambda, D = 0)$. Notice that $\dot{\varphi} = \frac{\lambda}{I_1\sin^2\theta}$ has the same sign during the motion.

e) Precessional motions: $\dot{\theta} = 0$ which implies that $\theta(t) \equiv \theta_0$, $0 < \theta_0 < \pi$ with ω_3 and $\dot{\varphi}$ equal to their initial values. The motion on S^2 is represented by the latitude circles $(\theta_0, \varphi(t))$.

f) General nutational motion: The trajectory on S^2 is confined between the two latitude circles $(\theta_1, \varphi(t))$ and $(\theta_2, \varphi(t))$. Where $0 \leq \theta_1 \leq \theta_2 \leq \pi$ are given as solutions of the equation $E = g(\cos\theta)\dot{\theta}^2 + V(z = \cos\theta, D, \lambda)$. The shape of the curve drawn by $\hat{\mathbf{3}}$ on the unit sphere depends on whether $\dot{\varphi} = \frac{D}{I_1 \sin^2\theta}\left[\frac{\lambda}{D} + \frac{\alpha - \cos\theta}{\sqrt{d(\theta)}}\right]$ changes sign during the motion or not. There are three qualitatively different cases because the function $h(z) = \frac{(\alpha - z)}{\sqrt{d(z)}}$ is monotone when, $-1 < z < 1$. To see that $h(z)$ is monotone observe that $\frac{dh(z)}{dz} = \frac{\gamma(1+\beta-\beta\alpha z)}{(d(z))^{\frac{3}{2}}} = 0$ implies $z = \frac{1}{\alpha} + \frac{1}{\alpha\beta} > 1$.

(i) $\frac{\lambda}{D} + \frac{(\alpha - \cos\theta)}{\sqrt{d(\theta)}} \neq 0$ for $\theta \in [\theta_1, \theta_2]$, so that $\dot{\varphi}$ has the same sign during the motion, and the trajectory $(\theta(t), \varphi(t))$ has a wavelike form touching tangentially the two boundary latitude circles θ_1 and θ_2.

(ii) $\frac{\lambda}{D} + \frac{(\alpha - \cos\theta)}{\sqrt{d(\theta)}} = 0$ for $\theta = \theta_1$. In this case $\dot{\varphi}$ is zero at θ_1, and both $\dot{\theta}$ and $\dot{\varphi}$ vanishes at $\theta(t) = \theta_1$ the motion momentarily stops, and the trajectory $(\theta(t), \varphi(t))$ will have cusps at the latitude θ_1.

(iii) $\frac{\lambda}{D} + \frac{(\alpha - \cos\theta)}{\sqrt{d(\theta)}} = 0$ for certain $\theta \in]\theta_1, \theta_2[$, so that $\dot{\varphi}$ changes sign during the motion, and is positive at the latitude θ_2 and negative at the latitude θ_1. The trajectory $(\theta(t), \varphi(t))$ of the $\hat{\mathbf{3}}$-axis on the unit sphere S^2 will have a wavelike form with loops when touching tangentially the latitude circle θ_1.

All these particular types of motion are illustrated in Fig. 3 presenting the minimal surface of the function $V(z = \cos\theta)$ in the space of parameters (D, λ, E). This surface is the lower boundary of the set of all admissible values of (D, λ, E). The values of parameters on the minimal surface corresponding to the special types of solutions are marked by distinguished black lines and the corresponding motions on the sphere S^2 are illustrated by the adjacent pictures marked with the same letter. All triples of parameters (D, λ, E) describing points on the minimal surface define precessional motions $\theta = \theta_0 = \text{const}$. They are denoted by the letter d. The degenerate forms of precessional motions are the vertical rotations $\theta_0 = 0$ and $\theta_0 = \pi$ that correspond to points on the lines e and h. All generic points above the minimal surface of $V(z)$ describe nutational motions denoted here by letters a, b, c. Two types of special motions corresponding to points above the minimal surface have been distinguished: the spherical pendulum type solutions that are marked by the letter f above the line $D = 0$ and the planar pendulum type solutions that are marked by the letter g above the point $D = 0$, $\lambda = 0$. The admissible set in the space of parameters (D, λ, E) represents all possible dynamical states of rTT modulo the choice of the initial angles φ_0, ψ_0 of the cyclic variables and of the initial value of the angle θ_0. The time dependence of $\theta(t)$, $\varphi(t)$, $\psi(t)$ is, up to these translations, fully determined by each admissible point in Fig. 3.

5 Final remarks

Separability of the rTT has been recognized many times in the literature [13, 2, 6] since the Routh observation that the system (1) admits three independent integrals of motion and is separable. There are several studies of stationary motions of the rTT [13, 2, 3, 8]. The separation equations, however, has been less used for analysis of the rTT and their apparent similarity with the equations of the Lagrange top, seemingly has not been discussed before.

In this paper we have discussed thoroughly the vector equations of the sliding tippe top and their reduction for pure rolling motion in the plane, the rTT equations (3a), (3b). The equations of motion (3a), (3b) admit three integrals of motion that are given here both in the vector form and also in the coordinate form, which provides the separation equations. Two integrals (D, λ), of angular momentum type, depend linearly on the momenta. Their existence can be understood as reflection of the fact that the dynamical equations (5) have two cyclic variables φ, ψ [14] although we are not speaking here about the Hamilton–Jacobi type separability. The third integral is the energy E that depends quadratically on momenta and gives rise to the main separation equation similarly as in the case of the Lagrange top.

According to the old result by Chaplygin [2] all axially symmetric rigid bodies rolling on the plane admit three integrals of motion – the energy and two integrals linear in momenta and are in principle separable. In general the linear integrals are however defined through two particular solutions of a second order linear differential equation with nonelementary solutions. For instance in the case of the rolling disc they are given by Legendre functions. These equations allow to formally express $\dot{\varphi}(\cos\theta)$ and $\omega_3(\cos\theta)$ as known functions of $\cos\theta$ to be substituted into the energy integral to give the θ-separation equation similarly as in this paper.

The special feature of the rTT equations is that the expressions for $\dot{\varphi}(\cos\theta)$ and $\omega_3(\cos\theta)$ are elementary functions of $\cos\theta$. When substituted explicitly into the energy integral they give an elementary expression for the effective potential $V(\cos\theta)$. This simplifies analysis of equations and enables representing all possible motions in the space of parameters (D, λ, E). Only two types of stationary motion describe stable trajectories with the reaction force orthogonal to the plane. They are the stable asymptotic motions of the genuine tippe top, of vertical spinning type and the tumbling solutions when the center of mass is fixed in the space and the sphere is rolling along a circle.

The approach presented here makes it possible to perform more general study of separability and of elementary separability (when $\dot{\varphi}(\cos\theta)$ and $\omega_3(\cos\theta)$ can be expressed by elementary functions) of axially symmetric rigid bodies. These questions are currently under study and the results will be presented in a subsequent paper.

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