On Transitive Systems of Subspaces in a Hilbert Space

Yuliya P. MOSKALEVA † *and Yurii S. SAMOĬLENKO* ‡

† *Taurida National University, 4 Vernads'kyi Str., Simferopol, 95007 Ukraine*
E-mail: *YulMosk@mail.ru*

‡ *Institute of Mathematics, National Academy of Sciences of Ukraine,*
3 Tereshchenkivs'ka Str., Kyiv-4, 01601 Ukraine
E-mail: *yurii_sam@imath.kiev.ua*

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Abstract. Methods of $*$-representations in Hilbert space are applied to study of systems of n subspaces in a linear space. It is proved that the problem of description of n-transitive subspaces in a finite-dimensional linear space is $*$-wild for $n \geq 5$.

Key words: algebras generated by projections; irreducible inequivalent representations; transitive nonisomorphic systems of subspaces

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1 Introduction

Systems of n subspaces $H_1, H_2, \dots, H_n$ of a Hilbert space H, denoted in the sequel by $S = (H; H_1, H_2, \dots, H_n)$, is a mathematical object that traditionally draws an interest both by itself [1, 4, 5, 6] and in connection with the discussion on whether there exists a deeper connection between this object and the famous H. Weyl problem, the Coxeter groups, singularity theory, and physical applications.

Systems of subspaces that can be regarded as candidates for being the simplest building blocks for arbitrary systems of subspaces are those that are indecomposable or transitive [4, 5, 6]. A description of transitive and indecomposable systems is carried out up to an isomorphism of the systems of subspaces. For a description of transitive and indecomposable systems of two subspaces of a Hilbert space, as well as for transitive and indecomposable triples of a finite dimensional linear space, see, e.g., [6]. For an infinite dimensional space, not only the problem of description but even the problem of existence of transitive and indecomposable triples of subspaces is an unsolved problem [2]. For a finite dimensional linear space, transitive quadruples of subspaces are described in [3], and [4, 5] give indecomposable quadruples. Examples of nonisomorphic transitive and indecomposable systems of four subspaces in an infinite dimensional space can be found, e.g., in [6].

In [6] the authors make a conjecture that there is a connection between systems of n subspaces and representations of $*$-algebras that are generated by the projections, — "There seems to be interesting relations of systems of n-subspaces with the study of representations of $*$-algebras generated by idempotents by S. Kruglyak, V. Ostrovskyi, V. Rabanovich, Yu. Samoĭlenko and other. But we do not know the exact implication …". The present article deals with this implication.

Let us consider systems of subspaces of the form $S_\pi = (H; P_1H, P_2H, \dots, P_nH)$, where the orthogonal projections $P_1, P_2, \dots, P_n$ make a $*$-representation π of the $*$-algebra generated by the projections, and H is the representation space. For the $*$-algebras $\mathcal{P}_{4,\mathrm{com}} = \mathbb{C}\langle p_1, p_2, p_3, p_4 \,|\, p_k^2 = p_k^* = p_k, \big[\sum_{k=1}^{4} p_k, p_i\big] = 0, \forall\, i = 1,2,3,4\rangle$, it was proved in [11] that irreducible inequivalent $*$-representations π of the $*$-algebra $\mathcal{P}_{4,\mathrm{com}}$ make a complete list of nonisomorphic transitive quadruples of subspaces S_π of a finite dimensional linear space.

In this paper, we make an analysis of complexity of the description problem for transitive systems of subspaces $S = (H; H_1, H_2, \ldots, H_n)$ for $n \geq 5$. In Section 3, we prove that it is an extremely difficult problem to describe nonisomorphic transitive quintuples of subspaces $S = (H; P_1H, P_2H, \ldots, P_5H)$ even under the assumption that the sum of the corresponding five projections equals $2I$; in other words, the problem of describing inequivalent $*$-representations of the $*$-algebras that give rise to nonisomorphic transitive systems, is $*$-wild.

Since the problem of describing the system of n subspaces up to an isomorphism is complicated, it seems natural to describe transitive systems that correspond to $*$-representations of various algebras generated by projections (Sections 4 and 5).

In Section 4, we consider transitive systems S_π of n subspaces, where $\pi \in \operatorname{Rep} \mathcal{P}_{n,\alpha}$, $\mathcal{P}_{n,\alpha} = \mathbb{C}\langle p_1, p_2, \ldots, p_n \,|\, p_1 + p_2 + \cdots + p_n = \alpha e, p_j^2 = p_j, p_j^* = p_j, \forall\, j = 1, \ldots, n\rangle$, and α takes values in a fixed set. In Section 5, using nonisomorphic transitive systems S_π of n subspaces, where π belongs to $\operatorname{Rep} \mathcal{P}_{n,\alpha}$, we construct nonisomorphic transitive systems $S_{\hat\pi}$ of $n+1$ subspaces, where $\hat\pi$ is in $\operatorname{Rep} \mathcal{P}_{n,\mathrm{abo},\tau}$, $\mathcal{P}_{n,\mathrm{abo},\tau} = \mathbb{C}\langle q_1, q_2, \ldots, q_n, p \,|\, q_1 + q_2 + \cdots + q_n = e, q_j p q_j = \tau q_j$, $q_j^2 = q_j$, $q_j^* = q_j, \forall\, j = 1, \ldots, n, p^2 = p, p^* = p\rangle$.

2 Definitions and main properties

In this section we make necessary definitions and recall known facts; the proofs can be found in [6, 9]. Let H be a Hilbert space and $H_1, H_2, \ldots, H_n$ be n subspaces of H. Denote by $S = (H; H_1, H_2, \ldots, H_n)$ the system of n subspaces of the space H. Let $S = (H; H_1, H_2, \ldots, H_n)$ be a system of n subspaces of a Hilbert space H and $\tilde S = (\tilde H; \tilde H_1, \tilde H_2, \ldots, \tilde H_n)$ a system of n subspaces of a Hilbert space $\tilde H$. A linear map $R : H \to \tilde H$ from the space H to the space $\tilde H$ is called a homomorphism of the system S into the system $\tilde S$ and denoted by $R : S \to \tilde S$, if $R(H_i) \subset \tilde H_i$, $i = 1, \ldots, n$. A homomorphism $R : S \to \tilde S$ of a system S into a system $\tilde S$ is called an isomorphism, $R : S \to \tilde S$, if the mapping $R : H \to \tilde H$ is a bijection and $R(H_i) = \tilde H_i$, $\forall\, i = 1, \ldots, n$. Systems S and $\tilde S$ will be called isomorphic, denoted by $S \cong \tilde S$, if there exists an isomorphism $R : S \to \tilde S$.

Denote by $\operatorname{Hom}(S, \tilde S)$ the set of homomorphisms of a system S into a system $\tilde S$ and by $\operatorname{End}(S) := \operatorname{Hom}(S, S)$ the algebra of endomorphisms of S into S, that is,

$$\operatorname{End}(S) = \{R \in B(H) \,|\, R(H_i) \subset H_i, i = 1, \ldots, n\}.$$

A system $S = (H; H_1, H_2, \ldots, H_n)$ of n subspaces of a Hilbert space H is called transitive, if $\operatorname{End}(S) = \mathbb{C} I_H$.

Denote

$$\operatorname{Idem}(S) = \{R \in B(H) \,|\, R(H_i) \subset H_i, i = 1, \ldots, n, R^2 = R\}.$$

A system $S = (H; H_1, H_2, \ldots, H_n)$ of n subspaces of a space H is called indecomposable, if $\operatorname{Idem}(S) = \{0, I_H\}$.

Isomorphic systems are either simultaneously transitive or intransitive, decomposable or indecomposable. We say that $S \cong \tilde S$ up to permutation of subspaces, if there exists a permutation $\sigma \in S_n$ such that the systems $\sigma(S)$ and $\tilde S$ are isomorphic, where $\sigma(S) = (H; H_{\sigma(1)}, H_{\sigma(2)}, \ldots, H_{\sigma(n)})$, so that there exists an invertible operator $R : H \to \tilde H$ such that $R(H_{\sigma(i)}) = \tilde H_i$, $\forall\, i = 1, \ldots, n$.

Let us now recall the notion of unitary equivalence for systems and collections of orthogonal projections. Systems S and $\tilde{S}$ are called unitary equivalent, or simply equivalent, if $S \cong \tilde{S}$ and it is possible to choose the isomorphism $R : S \to \tilde{S}$ to be a unitary operator.

To every system $S = (H; H_1, H_2, \ldots, H_n)$ of n subspaces of a Hilbert space H, one can naturally associate a system of orthogonal projections $P_1, P_2, \ldots, P_n$, where P_i is the orthogonal projection operator onto the space H_i, $i = 1, \ldots, n$. A system of projections $P_1, P_2, \ldots, P_n$ on a Hilbert space H such that $\operatorname{Im} P_i = H_i$ for $i = 1, \ldots, n$ is called a system of orthogonal projections associated to the system of subspaces, $S = (H; H_1, H_2, \ldots, H_n)$. Conversely, to each system of projections there naturally corresponds a system of subspaces. A system $S = (H; P_1H, P_2H, \ldots, P_nH)$ is called a system corresponding to the system of projections $P_1, P_2, \ldots, P_n$.

A system of orthogonal projections $P_1, P_2, \ldots, P_n$ on a Hilbert space H is called unitary equivalent to a system $\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_n$ on a Hilbert space $\tilde{H}$, if there exists a unitary operator $R : H \to \tilde{H}$ such that $RP_i = \tilde{P}_i R$, $i = 1, \ldots, n$. Systems S and $\tilde{S}$ are unitary equivalent if and only if the corresponding systems of orthogonal projections are unitary equivalent.

A system of orthogonal projections $P_1, P_2, \ldots, P_n$ on a Hilbert space H is called irreducible if zero and H are the only invariant subspaces. Unitary equivalent systems of orthogonal projections are both either reducible or irreducible.

If systems S and $\tilde{S}$ are unitary equivalent, then $S \cong \tilde{S}$. The converse is not true.

Example 1. Let $S = (\mathbb{C}^2; \mathbb{C}(1, 0), \mathbb{C}(\cos\theta, \sin\theta))$, $\theta \in (0, \pi/2)$, and $\tilde{S} = (\mathbb{C}^2; \mathbb{C}(1, 0), \mathbb{C}(0, 1))$. The decomposable system S that corresponds to an irreducible pair of orthogonal projections, is isomorphic but not unitary equivalent to the decomposable system $\tilde{S}$ that corresponds to a reducible pair of orthogonal projections.

Finally, let us mention the relationship between the notions of transitivity, indecomposability, and irreducibility. If a system of subspaces is transitive, then it is indecomposable, but not vice versa. Indecomposability of a system of subspaces implies irreducibility of the corresponding system of orthogonal projections, but not conversely.

3 On $*$-wildness of the description problem for transitive systems of n subspaces for $n \geq 5$

3.1 On $*$-wildness of the description problem for transitive systems that correspond to orthogonal projections

A description of transitive quadruples of subspaces of a finite dimensional linear space is given in [3]. We will show that such a problem for n subspaces, $n \geq 5$, is extremely complicated ($*$-wild).

Consider a system of five subspaces, which corresponds to the five orthogonal projections

$$P_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \quad P_3 = \frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix},$$
$$P_4 = \frac{1}{2}\begin{pmatrix} I & U \\ U^* & I \end{pmatrix}, \quad P_5 = \frac{1}{2}\begin{pmatrix} I & V \\ V^* & I \end{pmatrix}$$

that act on the space $\mathcal{H} = H \oplus H$, where H is a Hilbert space and U and V are unitary operators. Denote this system of subspaces by $S_{U,V}$. So, $S_{U,V} = (\mathcal{H}; P_1\mathcal{H}, P_2\mathcal{H}, P_3\mathcal{H}, P_4\mathcal{H}, P_5\mathcal{H})$. Consider the system $S_{\tilde{U},\tilde{V}} = (\tilde{\mathcal{H}}; \tilde{P}_1\tilde{\mathcal{H}}, \tilde{P}_2\tilde{\mathcal{H}}, \tilde{P}_3\tilde{\mathcal{H}}, \tilde{P}_4\tilde{\mathcal{H}}, \tilde{P}_5\tilde{\mathcal{H}})$ that corresponds to the collection of orthogonal projections $\tilde{P}_1$, $\tilde{P}_2$, $\tilde{P}_3$, $\tilde{P}_4$, $\tilde{P}_5$ that have the above type and act on the space $\tilde{\mathcal{H}} = \tilde{H} \oplus \tilde{H}$; here $\tilde{H}$ is a Hilbert space and $\tilde{U}$, $\tilde{V}$ is a pair of unitary operators.

Theorem 1. *The system $S_{U,V}$ is transitive if and only if the unitary operators U, V are irreducible. Also, $S_{U,V} \cong S_{\tilde{U},\tilde{V}}$ if and only if the pair of unitary operators U, V is unitary equivalent to the pair of unitary operators $\tilde{U}$, $\tilde{V}$.*

Proof. Denote $H_i = P_i\mathcal{H}$, $i = 1, \ldots, 5$. For H_1 and H_2, we have

$$H_1 = H \oplus 0, \qquad H_2 = 0 \oplus H.$$

For H_3, H_4, and H_5, respectively,

$$H_3 = \{(x, x) \,|\, x \in H\}, \quad H_4 = \{(Ux, x) \,|\, x \in H\}, \qquad H_5 = \{(V x, x)|x \in H\}.$$

Let us prove an auxiliary identity

$$\begin{aligned} &\{\mathcal{R} \in B(\mathcal{H}, \tilde{\mathcal{H}}) \,|\, \mathcal{R}(H_i) \subset \tilde{H}_i, i = 1, \ldots, 5\} \\ &\quad = \{R \oplus R \in B(\mathcal{H}, \tilde{\mathcal{H}}) \,|\, R \in B(H, \tilde{H}), RU = \tilde{U}R, RV = \tilde{V}R\}. \end{aligned} \tag{1}$$

The first three inclusions, $\mathcal{R}(H_i) \subset \tilde{H}_i$, $i = 1, 2, 3$, imply that any operator $\mathcal{R}$ in $B(\mathcal{H}, \tilde{\mathcal{H}})$ can be represented as $\mathcal{R} = R \oplus R$, where $R \in B(H, \tilde{H})$. The fourth inclusion, $\mathcal{R}(H_4) \subset \tilde{H}_4$, implies $RU = \tilde{U}R$, and the fifth one, $\mathcal{R}(H_5) \subset \tilde{H}_5$, gives $RV = \tilde{V}R$. The converse implications finish the proof of (1).

It directly follows from (1) that $S_{U,V} \cong S_{\tilde{U},\tilde{V}}$ if and only if the pair of unitary operators U, V is similar to the pair of unitary operators $\tilde{U}$, $\tilde{V}$. By [9], a pair of unitary operators U, V is similar to a pair of unitary operators $\tilde{U}$, $\tilde{V}$ if and only if the pair of unitary operators U, V is unitary equivalent to the pair of unitary operators $\tilde{U}$, $\tilde{V}$.

Now, setting $S_{\tilde{U},\tilde{V}} = S_{U,V}$, rewrite the identity (1) as follows:

$$\begin{aligned} \mathrm{End}(S_{U,V}) &= \{\mathcal{R} \in B(\mathcal{H}) \,|\, \mathcal{R}(H_i) \subset H_i, i = 1, \ldots, 5\} \\ &= \{R \oplus R \in B(\mathcal{H}) \,|\, R \in B(H), RU = UR, RV = V R\}. \end{aligned}$$

The latter identity immediately implies that the system $S_{U,V}$ is transitive if and only if the unitary operators U, V are irreducible. ■

Theorem 1 allows to identify the description problem for nonisomorphic transitive quintuples that correspond to five orthogonal projections of a special type with that for inequivalent irreducible pairs of unitary operators. The latter problem is ∗-wild in the theory of ∗-representations of ∗-algebras [8, 9].

3.2 On ∗-wildness of the description problem for transitive systems corresponding to orthogonal projections with an additional relation

Let P_1, P_2, P_3 be orthogonal projections on a Hilbert space H, and P_2, P_3 be mutually orthogonal. Introduce a system of five subspaces of the space H corresponding to the collection of orthogonal projections P_1, $P_1^{\perp}$,P_2, P_3,$(P_2 + P_3)^{\perp}$. Denote

$$S_{P_1,P_2\perp P_3} = (H; \mathrm{Im}\ P_1, \mathrm{Im}\ P_1^{\perp}, \mathrm{Im}\ P_2, \mathrm{Im}\ P_3, \mathrm{Im}\,(P_2 + P_3)^{\perp}).$$

Theorem 2. *Let P_1, P_2, P_3 be orthogonal projections on a Hilbert space H such that P_2 and P_3 are mutually orthogonal, and $\tilde{P}_1$, $\tilde{P}_2$, $\tilde{P}_3$ be orthogonal projections on a Hilbert space $\tilde{H}$ such that $\tilde{P}_2$ and $\tilde{P}_3$ are mutually orthogonal. Then the system $S_{P_1,P_2\perp P_3}$ is transitive if and only if the projections P_1, P_2, P_3 are irreducible. Also, $S_{P_1,P_2\perp P_3} \cong S_{\tilde{P}_1,\tilde{P}_2\perp\tilde{P}_3}$ if and only if the triple of the orthogonal projections P_1, P_2, P_3 is unitary equivalent to the triple of the orthogonal projections $\tilde{P}_1$, $\tilde{P}_2$, $\tilde{P}_3$.*

Proof. Denote $H_1 = \mathrm{Im}\ P_1$, $H_2 = \mathrm{Im}\ P_1^{\perp}$, $H_3 = \mathrm{Im}\ P_2$, $H_4 = \mathrm{Im}\ P_3$, $H_5 = \mathrm{Im}\,(P_2+P_3)^{\perp}$, and let $\tilde H_1 = \mathrm{Im}\ \tilde P_1$, $\tilde H_2 = \mathrm{Im}\ \tilde P_1^{\perp}$, $\tilde H_3 = \mathrm{Im}\ \tilde P_2$, $\tilde H_4 = \mathrm{Im}\ \tilde P_3$, $\tilde H_5 = \mathrm{Im}\,(\tilde P_2+\tilde P_3)^{\perp}$.

The proof of the theorem directly follows from the identity

$$\{R \in B(H,\tilde H) \,|\, R(H_i) \subset \tilde H_i, i = 1,\dots,5\} = \{R \in B(H,\tilde H) \,|\, RP_i = \tilde P_i R, i = 1,2,3\}. \qquad \blacksquare$$

Theorem 2 identifies the description problem for nonisomorphic transitive quintuples of subspaces corresponding to quintuples of orthogonal projections of a special type, the ones such that their sum equals $2I_H$, with that for inequivalent irreducible triples P_1, P_2, P_3 of orthogonal projections satisfying the condition $P_2 \perp P_3$. The latter problem is $*$-wild in the theory of $*$-representations of $*$-algebras [8, 9].

4 Transitive systems of subspaces corresponding to Rep $\mathcal{P}_{n,\mathrm{com}}$

4.1 On $*$-representations of the $*$-algebra $\mathcal{P}_{n,\mathrm{com}}$

Denote by Σ_n ($n \in \mathbb{N}$) the set $\alpha \in \mathbb{R}_+$ such that there exists at least one $*$-representation of the $*$-algebra $\mathcal{P}_{n,\alpha} = \mathbb{C}\langle p_1, p_2, \dots, p_n \,|\, p_k^2 = p_k^* = p_k, \sum_{k=1}^n p_k = \alpha e\rangle$, i.e., the set of real numbers α such that there exist n orthogonal projections $P_1, P_2, \dots, P_n$ on a Hilbert space H satisfying $\sum_{k=1}^n P_k = \alpha I_H$. It follows from the definition of the algebra $\mathcal{P}_{n,\mathrm{com}} = \mathbb{C}\langle p_1, p_2, \dots, p_n \,|\, p_k^2 = p_k^* = p_k, [\sum_{k=1}^n p_k, p_i] = 0, \forall\, i = 1,\dots,n\rangle$ that all irreducible $*$-representations of $\mathcal{P}_{n,\mathrm{com}}$ coincide with the union of irreducible $*$-representations of $\mathcal{P}_{n,\alpha}$ taken over all $\alpha \in \Sigma_n$.

A description of the set Σ_n for all $n \in \mathbb{N}$ is obtained by S.A. Kruglyak, V.I. Rabanovich, and Yu.S. Samoĭlenko in [7], and is given by

$$\Sigma_2 = \{0,1,2\}, \quad \Sigma_3 = \left\{0,1,\tfrac{3}{2},2,3\right\},$$
$$\Sigma_n = \left\{\Lambda_n^0, \Lambda_n^1, \left[\tfrac{n-\sqrt{n^2-4n}}{2}, \tfrac{n+\sqrt{n^2-4n}}{2}\right], n-\Lambda_n^1, n-\Lambda_n^0\right\} \ \text{for } n \ge 4,$$
$$\Lambda_n^0 = \left\{0, 1+\tfrac{1}{n-1}, 1+\tfrac{1}{(n-2)-\frac{1}{n-1}}, \dots, 1+\cfrac{1}{(n-2)-\cfrac{1}{(n-2)-\cfrac{1}{\ddots-\frac{1}{n-1}}}}, \dots\right\},$$
$$\Lambda_n^1 = \left\{1, 1+\tfrac{1}{n-2}, 1+\tfrac{1}{(n-2)-\frac{1}{n-2}}, \dots, 1+\cfrac{1}{(n-2)-\cfrac{1}{(n-2)-\cfrac{1}{\ddots-\frac{1}{n-2}}}}, \dots\right\}.$$

Here, the elements of the sets Λ_n^0, Λ_n^1, $n-\Lambda_n^1$, $n-\Lambda_n^0$, in what follows, will be called points of the discrete spectrum of the description problem for unitary representations of the algebra $\mathcal{P}_{n,\mathrm{com}}$, whereas the elements of the line segment $\left[\frac{n-\sqrt{n^2-4n}}{2}, \frac{n+\sqrt{n^2-4n}}{2}\right]$ are called point of the continuous spectrum. For each point α in the sets Λ_n^0, $n-\Lambda_n^0$ there exists, up to unitary equivalence, a unique irreducible $*$-representation of the $*$-algebra $\mathcal{P}_{n,\alpha}$ and, hence, that of $\mathcal{P}_{n,\mathrm{com}}$. For each point α in the sets Λ_n^1, $n-\Lambda_n^1$ there exist n inequivalent irreducible $*$-representations of the $*$-algebra $\mathcal{P}_{n,\alpha}$ and, hence, those of $\mathcal{P}_{n,\mathrm{com}}$.

An important instrument for describing the set Σ_n and representations of $\mathcal{P}_{n,\mathrm{com}}$ is use of Coxeter functors, constructed in [7], between the categories of $*$-representations of $\mathcal{P}_{n,\alpha}$ for different values of the parameters.

Define a functor $\mathcal{T} : \mathrm{Rep}\ \mathcal{P}_{n,\alpha} \to \mathrm{Rep}\ \mathcal{P}_{n,n-\alpha}$, see [7]. Let π be a representation of the algebra $\mathcal{P}_{n,\alpha}$, and $\pi(p_i) = P_i$, $i = 1,\dots,n$, be orthogonal projections on a representation space H.

Then the representation $\hat{\pi} = \mathcal{T}(\pi)$ in Rep $\mathcal{P}_{n,n-\alpha}$ is defined by the identities $\hat{\pi}(p_i) = (I - P_i)$ that give orthogonal projections on H. We leave out a description of the action of the functor $\mathcal{T}$ on morphisms of the category Rep $\mathcal{P}_{n,\alpha}$, since it is not used in the sequel. Let us now define a functor $\mathcal{S}$: Rep $\mathcal{P}_{n,\alpha} \to$ Rep $\mathcal{P}_{n,\frac{\alpha}{\alpha-1}}$, see [7]. Again, let π denote a representation in Rep $\mathcal{P}_{n,\alpha}$, and by $P_1, P_2, \ldots, P_n$ denote the corresponding orthogonal projections on the representation space H. Consider the subspaces $H_i = \mathrm{Im}\ P_i$ $(i = 1, \ldots, n)$. Let $\Gamma_i : H_i \to H$, $i = 1, \ldots, n$, be the natural isometries. Then

$$\Gamma_i^* \Gamma_i = I_{H_i}, \qquad \Gamma_i \Gamma_i^* = P_i, \qquad i = 1, \ldots, n. \tag{2}$$

Let an operator Γ be defined by the matrix $\Gamma = [\Gamma_1, \Gamma_2, \ldots, \Gamma_n] : \mathcal{H} = H_1 \oplus H_2 \oplus \cdots \oplus H_n \to H$. Then the natural isometry $\sqrt{\frac{\alpha-1}{\alpha}}\Delta^*$ that acts from the orthogonal complement $\hat{H}$ to the subspace Im Γ^* into the space $\mathcal{H}$ defines the isometries $\Delta_k = \Delta|_{\mathrm{Im}\ P_k} : H_k \to \hat{H}$, $k = 1, \ldots, n$. The orthogonal projections $Q_i = \Delta_i \Delta_i^*$, $i = 1, \ldots, n$, on the space $\hat{H}$ make the corresponding representation in $\mathcal{S}$(Rep $\mathcal{P}_{n,\alpha}$), i.e. the representation $\hat{\pi} = \mathcal{S}(\pi)$ in Rep $\mathcal{P}_{n,\frac{\alpha}{\alpha-1}}$ is given by the identities $\hat{\pi}(p_i) = Q_i$. Write down the relations satisfied by the operators $\{\Delta_i\}_i^n$,

$$\Delta_i^* \Delta_i = I_{H_i}, \qquad \Delta_i \Delta_i^* = Q_i, \qquad i = 1, \ldots, n. \tag{3}$$

We will not describe the action of the functor $\mathcal{S}$ on morphisms of the category Rep $\mathcal{P}_{n,\alpha}$, since we will not use it in the sequel.

Following [7], introduce a functor Φ^+ : Rep $\mathcal{P}_{n,\alpha} \to$ Rep $\mathcal{P}_{n,1+\frac{1}{n-1-\alpha}}$ defined by $\Phi^+(\pi) = \mathcal{S}(\mathcal{T}(\pi))$ for $\alpha < n-1$. Denote by π_k $(k = 0, 1, \ldots, n)$ the following representations in Rep $\mathcal{P}_{n,\alpha}$: $\pi_0(p_i) = 0$, $i = 1, \ldots, n$, where the space of representation is $\mathbb{C}$; $\pi_k(p_i) = 0$ if $i \neq k$ and $\pi_k(p_k) = 1$, $k = 1, \ldots, n$, with $\mathbb{C}$ as the representation space. For an arbitrary irreducible representation π of the algebra $\mathcal{P}_{n,\alpha}$ in the case of points of the discrete spectrum, one can assert that either π or $\mathcal{T}(\pi)$ is unitary equivalent to a representation of the form $\Phi^{+s}(\breve{\pi})$, where the representation $\breve{\pi}$ is one of the simplest representations π_k, $k = \overline{0, n}$, and s is a natural number.

4.2 Transitive systems of n subspaces corresponding to Rep $\mathcal{P}_{n,\alpha}$

The systems of subspaces, S_{π_k}, $k = 0, 1, \ldots, n$, are clearly nonisomorphic transitive systems of n subspaces of the space $\mathbb{C}$. I.M. Gel'fand and V.A. Ponamarev in [4], by using the functor technique, construct from the systems S_{π_k}, $k = 0, 1, \ldots, n$, infinite series of indecomposable systems, which turn out to be are transitive, of n subspaces. In this section we show that the Coxeter functors in [7], as the functors in [4], transform nonisomorphic transitive systems into nonisomorphic transitive systems and, consequently, all systems of the form $S_{\Phi^{+s}(\breve{\pi})}$ and $S^{\perp}_{\Phi^{+s}(\breve{\pi})}$, where the representation $\breve{\pi}$ is one of the simplest representations π_k, $k = 0, 1, \ldots, n$, and s is a natural number, will be nonisomorphic transitive systems. Hence, we have the following theorem.

Theorem 3. *Systems of n subspaces S_π constructed from irreducible inequivalent representations $\pi \in$ Rep $\mathcal{P}_{n,\alpha}$, for α in the discrete spectrum, are nonisomorphic and transitive.*

To prove the theorem, by using the Coxeter functors $\mathcal{T}$ and $\mathcal{S}$ in [7], we construct auxiliary functors $\mathcal{T}'$ and $\mathcal{S}'$. The action of the functors $\mathcal{T}'$: Rep $\mathcal{P}_{n,\alpha} \to$ Rep $\mathcal{P}_{n,n-\alpha}$ and $\mathcal{S}'$: Rep $\mathcal{P}_{n,\alpha} \to$ Rep $\mathcal{P}_{n,\frac{\alpha}{\alpha-1}}$ on the objects of the category is defined to coincide with the actions of $\mathcal{T}$ and $\mathcal{S}$, that is, $\mathcal{T}(\pi) = \mathcal{T}'(\pi)$ and $\mathcal{S}(\pi) = \mathcal{S}'(\pi)$ $\forall\, \pi \in$ Rep $\mathcal{P}_{n,\alpha}$. The morphisms of the category of representations are defined differently. Let $\pi \in$ Rep $(\mathcal{P}_{n,\alpha}, H)$ and $\tilde{\pi} \in$ Rep $(\mathcal{P}_{n,\alpha}, \tilde{H})$. A linear operator $C \in B(H, \tilde{H})$ is called a morphism of the category of representations, $C \in \mathrm{Mor}(\pi, \tilde{\pi})$, if $C\pi(p_i) = \tilde{\pi}(p_i)C\pi(p_i)$, $i = 1, \dots, n$, that is,

$$CP_i = \tilde{P}_i C P_i, \qquad i = 1, \dots, n. \tag{4}$$

The restrictions $C|_{H_i}$, $i = 1, \dots, n$, are denoted by C_i. Let us show that the operators C_i map H_i into $\tilde{H}_i$, that is,

$$C_i(H_i) \subset \tilde{H}_i, \qquad i = 1, \dots, n. \tag{5}$$

Indeed, for $x \in H_i$, we have $C_i x = Cx = CP_i x = \tilde{P}_i C P_i x$ and, consequently, $C_i x \in \tilde{H}_i$.

If $x \in H_i$, then (4) and (5) give

$$C\Gamma_i x = Cx = CP_i x = \tilde{P}_i C P_i x = \tilde{P}_i C_i x = C_i x = \tilde{\Gamma}_i C_i x,$$

so that

$$C\Gamma_i = \tilde{\Gamma}_i C_i, \qquad i = 1, \dots, n. \tag{6}$$

The identities (4) are equivalent to the inclusions $C(H_i) \subset \tilde{H}_i$, $i = 1, \dots, n$, which immediately gives the following relations:

$$C_i = \tilde{\Gamma}_i^* C \Gamma_i, \qquad i = 1, \dots, n. \tag{7}$$

Formula (6) allows to represent C as

$$C = \frac{1}{\alpha} \sum_{i=1}^{n} \tilde{\Gamma}_i C_i \Gamma_i^*. \tag{8}$$

Indeed, $\frac{1}{\alpha} \sum\limits_{i=1}^{n} \tilde{\Gamma}_i C_i \Gamma_i^* = \frac{1}{\alpha} \sum\limits_{i=1}^{n} C \Gamma_i \Gamma_i^* = C(\frac{1}{\alpha} \sum\limits_{i=1}^{n} P_i) = C$.

Consider an operator $\hat{C} : \hat{H} \to \hat{\tilde{H}}$ defined by

$$\hat{C} = \frac{\alpha - 1}{\alpha} \sum_{i=1}^{n} \tilde{\Delta}_i C_i \Delta_i^*. \tag{9}$$

Using the following properties of the operators [7] $\{\Gamma_i\}_{i=1}^n$, $\{\Gamma_i^*\}_{i=1}^n$, $\{\Delta_i\}_{i=1}^n$, $\{\Delta_i^*\}_{i=1}^n$:

$$\sum_{i=1}^{n} \Gamma_i \Delta_i^* = 0, \tag{10}$$

$$\Delta_i^* \Delta_j = -\frac{1}{\alpha - 1} \Gamma_i^* \Gamma_j, \quad i \neq j, \tag{11}$$

let us prove that

$$\tilde{\Delta}_k^* \hat{C} = C_k \Delta_k^*, \quad k = 1, \dots, n. \tag{12}$$

Indeed,

$$\begin{aligned}
\tilde{\Delta}_k^* \hat{C} &= \tilde{\Delta}_k^* \left(\frac{\alpha - 1}{\alpha} \sum_{i=1}^{n} \tilde{\Delta}_i C_i \Delta_i^* \right) = \frac{\alpha - 1}{\alpha} \sum_{i=1}^{n} (\tilde{\Delta}_k^* \tilde{\Delta}_i) C_i \Delta_i^* \\
&= \frac{\alpha - 1}{\alpha} (\tilde{\Delta}_k^* \tilde{\Delta}_k) C_k \Delta_k^* + \frac{\alpha - 1}{\alpha} \sum_{\substack{i=1 \\ i \neq k}}^{n} (-\frac{1}{\alpha - 1}) \tilde{\Gamma}_k^* (\tilde{\Gamma}_i C_i) \Delta_i^*
\end{aligned}$$

$$= \frac{\alpha-1}{\alpha} C_k \Delta_k^* - \frac{1}{\alpha} \sum_{\substack{i=1\\ i\neq k}}^{n} \tilde{\Gamma}_k^* (\tilde{\Gamma}_i C_i) \Delta_i^* = \frac{\alpha-1}{\alpha} C_k \Delta_k^* - \frac{1}{\alpha} \tilde{\Gamma}_k^* C \sum_{\substack{i=1\\ i\neq k}}^{n} \Gamma_i \Delta_i^*$$

$$= \frac{\alpha-1}{\alpha} C_k \Delta_k^* - \frac{1}{\alpha} \tilde{\Gamma}_k^* C \left(\sum_{i=1}^{n} \Gamma_i \Delta_i^* - \Gamma_k \Delta_k^* \right) = \frac{\alpha-1}{\alpha} C_k \Delta_k^* + \frac{1}{\alpha} (\tilde{\Gamma}_k^*) C \Gamma_k \Delta_k^* = C_k \Delta_k^*.$$

Now, let us show that

$$C_k = \tilde{\Delta}_k^* \hat{C} \Delta_k, \qquad k = 1, \dots, n. \tag{13}$$

Using (2), (3), (7), (8), (9), and (11) we get

$$\tilde{\Delta}_k^* \hat{C} \Delta_k = \tilde{\Delta}_k^* \left(\frac{\alpha-1}{\alpha} \sum_{i=1}^{n} \tilde{\Delta}_i C_i \Delta_i^* \right) \Delta_k = \frac{\alpha-1}{\alpha} \sum_{i=1}^{n} \tilde{\Delta}_k^* \tilde{\Delta}_i C_i \Delta_i^* \Delta_k$$

$$= \frac{\alpha-1}{\alpha} \tilde{\Delta}_k^* \tilde{\Delta}_k C_k \Delta_k^* \Delta_k + \frac{\alpha-1}{\alpha} \sum_{\substack{i=1\\ i\neq k}}^{n} \tilde{\Delta}_k^* \tilde{\Delta}_i C_i \Delta_i^* \Delta_k$$

$$= \frac{\alpha-1}{\alpha} C_k + \frac{1}{\alpha(\alpha-1)} \sum_{\substack{i=1\\ i\neq k}}^{n} \tilde{\Gamma}_k^* \tilde{\Gamma}_i C_i \Gamma_i^* \Gamma_k$$

$$= \frac{\alpha-1}{\alpha} C_k + \frac{1}{\alpha-1} \tilde{\Gamma}_k^* \left(\frac{1}{\alpha} \sum_{\substack{i=1\\ i\neq k}}^{n} \tilde{\Gamma}_i C_i \Gamma_i^* \right) \Gamma_k$$

$$= \frac{\alpha-1}{\alpha} C_k + \frac{1}{\alpha-1} \tilde{\Gamma}_k^* C \Gamma_k - \frac{1}{\alpha(\alpha-1)} \tilde{\Gamma}_k^* \tilde{\Gamma}_k C_k \Gamma_k^* \Gamma_k = C_k.$$

Now, it follows from (12) and (13) that $\tilde{Q}_k \hat{C} = \tilde{\Delta}_k \tilde{\Delta}_k^* \hat{C} = \tilde{\Delta}_k C_k \Delta_k^* = \tilde{Q}_k \hat{C} Q_k$, that is, $\tilde{Q}_k \hat{C} = \tilde{Q}_k \hat{C} Q_k$, $k = 1, \dots, n$. Whence,

$$\hat{C}^* \tilde{Q}_k = Q_k \hat{C}^* \tilde{Q}_k, \qquad k = 1, \dots, n. \tag{14}$$

The latter identities mean that $\hat{C}^* \in \mathrm{Mor}(\mathcal{S}'(\tilde{\pi}), \mathcal{S}'(\pi))$. The action of the auxiliary functors $\mathcal{T}'$ and $\mathcal{S}'$ on morphisms of the category Rep $\mathcal{P}_{n,\alpha}$ are defined by $\mathcal{T}'(C) = C^*$ and $\mathcal{S}'(C) = \hat{C}^*$ for any $C \in \mathrm{Mor}(\pi, \tilde{\pi})$. This completes the construction of the auxiliary functors.

Lemma 1. *The functors $\mathcal{T}'$ and $\mathcal{S}'$ are category equivalences.*

Proof. It is easy to check by using the definition that the functor $\mathcal{T}'$ is univalent and complete. $\mathcal{T}^2 = \mathrm{Id}$ and $\mathcal{T}'(\pi) = \mathcal{T}(\pi)$ for any $\pi \in$ Rep $\mathcal{P}_{n,\alpha}$. Consequently, the functor $\mathcal{T}'$ is an equivalence between the categories Rep $\mathcal{P}_{n,\alpha}$ and Rep $\mathcal{P}_{n,n-\alpha}$.

Now, let us prove the lemma for the functor $\mathcal{S}'$. Let us show that the functor $\mathcal{S}'$ is univalent. Let $C, D \in \mathrm{Mor}(\pi, \tilde{\pi})$ and $C \neq D$, and show that $\mathcal{S}'(C) \neq \mathcal{S}'(D)$. Indeed, if $\mathcal{S}'(C) = \mathcal{S}'(D)$, then $\hat{C}^* = \hat{D}^*$ and $\hat{C} = \hat{D}$. By (13), we have

$$C_i = \tilde{\Delta}_i^* \hat{C} \Delta_i = \tilde{\Delta}_i^* \hat{D} \Delta_i = D_i, \qquad i = 1, \dots, n.$$

Using the decomposition (8) we get

$$C = \frac{1}{\alpha} \sum_{i=1}^{n} \tilde{\Gamma}_i C_i \Gamma_i^*, \qquad D = \frac{1}{\alpha} \sum_{i=1}^{n} \tilde{\Gamma}_i D_i \Gamma_i^*.$$

Then $C = D$ and, hence, the functor $\mathcal{S}'$ is univalent.

Let us now show that $\mathcal{S}'$ is complete. Let $R \in \operatorname{Mor}(\mathcal{S}'(\tilde{\pi}), \mathcal{S}'(\pi))$. To prove the completeness, construct a linear operator from the set $\operatorname{Mor}(\pi, \tilde{\pi})$ such that the functor takes on this morphism the value R. Since $R \in \operatorname{Mor}(\mathcal{S}'(\tilde{\pi}), \mathcal{S}'(\pi))$, the operator $R : \hat{\tilde{H}} \to \hat{H}$ satisfies

$$R\tilde{Q}_k = Q_k R \tilde{Q}_k, \qquad k = 1, \dots, n.$$

Consider an operator $\hat{r}$ in $B(\hat{H}, \hat{\tilde{H}})$ such that $\hat{r}^* = R$. Then the former identities can be written as

$$\hat{r}^* \tilde{Q}_k = Q_k \hat{r}^* \tilde{Q}_k, \qquad k = 1, \dots, n,$$

and, consequently,

$$\tilde{Q}_k \hat{r} = \tilde{Q}_k \hat{r} Q_k, \qquad k = 1, \dots, n. \tag{15}$$

Denote by r_k the operators $r_k = \tilde{\Delta}_k^* \hat{r} \Delta_k : H_k \to \tilde{H}_k$, $k = 1, \dots, n$, and show that $\hat{r}$ can be represented as

$$\hat{r} = \frac{\alpha - 1}{\alpha} \sum_{k=1}^{n} \tilde{\Delta}_k r_k \Delta_k^*. \tag{16}$$

Indeed,

$$\frac{\alpha - 1}{\alpha} \sum_{k=1}^{n} \tilde{\Delta}_k r_k \Delta_k^* = \frac{\alpha - 1}{\alpha} \sum_{k=1}^{n} \tilde{\Delta}_k \tilde{\Delta}_k^* \hat{r} \Delta_k \Delta_k^* = \frac{\alpha - 1}{\alpha} \sum_{k=1}^{n} \tilde{Q}_k \hat{r} Q_k = \left(\frac{\alpha - 1}{\alpha} \sum_{k=1}^{n} \tilde{Q}_k \right) \hat{r} = \hat{r}.$$

It follows from the definition of r_k and identities (3), (15) that

$$\begin{aligned} r_k \Delta_k^* &= (\tilde{\Delta}_k^* \hat{r} \Delta_k) \Delta_k^* = \tilde{\Delta}_k^* \hat{r} (\Delta_k \Delta_k^*) = \tilde{\Delta}_k^* \hat{r} Q_k = I_{\tilde{H}_k} \tilde{\Delta}_k^* \hat{r} Q_k = (\tilde{\Delta}_k^* \tilde{\Delta}_k) \tilde{\Delta}_k^* \hat{r} Q_k \\ &= \tilde{\Delta}_k^* (\tilde{\Delta}_k \tilde{\Delta}_k^*) \hat{r} Q_k = \tilde{\Delta}_k^* \tilde{Q}_k \hat{r} Q_k = \tilde{\Delta}_k^* \tilde{Q}_k \hat{r} = \tilde{\Delta}_k^* (\tilde{\Delta}_k \tilde{\Delta}_k^*) \hat{r} = (\tilde{\Delta}_k^* \tilde{\Delta}_k) \tilde{\Delta}_k^* \hat{r} = \tilde{\Delta}_k^* \hat{r}. \end{aligned}$$

Hence, we have

$$r_k \Delta_k^* = \tilde{\Delta}_k^* \hat{r}, \qquad k = 1, \dots, n. \tag{17}$$

Consider the operator

$$r = \frac{1}{\alpha} \sum_{i=1}^{n} \tilde{\Gamma}_i r_i \Gamma_i^*. \tag{18}$$

Using (2), (10), (11), (17) we get

$$r \Gamma_k = \tilde{\Gamma}_k r_k, \qquad k = 1, \dots, n, \tag{19}$$

$$r_k = \tilde{\Gamma}_k^* r \Gamma_k, \qquad k = 1, \dots, n. \tag{20}$$

Indeed,

$$\begin{aligned} r\Gamma_k &= \frac{1}{\alpha} \sum_{i=1}^{n} \tilde{\Gamma}_i r_i \Gamma_i^* \Gamma_k = \frac{1}{\alpha} \tilde{\Gamma}_k r_k + \frac{1}{\alpha} \sum_{\substack{i=1 \\ i \neq j}}^{n} \tilde{\Gamma}_i r_i (\Gamma_i^* \Gamma_k) = \frac{1}{\alpha} \tilde{\Gamma}_k r_k - \frac{\alpha - 1}{\alpha} \sum_{\substack{i=1 \\ i \neq j}}^{n} \tilde{\Gamma}_i (r_i \Delta_i^*) \Delta_k \\ &= \frac{1}{\alpha} \tilde{\Gamma}_k r_k - \frac{\alpha - 1}{\alpha} \sum_{\substack{i=1 \\ i \neq j}}^{n} \tilde{\Gamma}_i (\tilde{\Delta}_i^* \hat{r}) \Delta_k = \frac{1}{\alpha} \tilde{\Gamma}_k r_k + \frac{\alpha - 1}{\alpha} \tilde{\Gamma}_k \Delta_k^* \hat{r} \Delta_k = \tilde{\Gamma}_k r_k \end{aligned}$$

and

$$\tilde{\Gamma}_k^* r \Gamma_k = \frac{1}{\alpha}\tilde{\Gamma}_k^*\left(\sum_{i=1}^n \tilde{\Gamma}_i r_i \Gamma_i^*\right)\Gamma_k = \frac{1}{\alpha} r_k + \frac{1}{\alpha}\sum_{\substack{i=1\\ i\neq j}}^n \tilde{\Gamma}_k^*\tilde{\Gamma}_i r_i \Gamma_i^*\Gamma_k$$
$$= \frac{1}{\alpha} r_k + \frac{(\alpha-1)^2}{\alpha}\sum_{\substack{i=1\\ i\neq j}}^n \tilde{\Delta}_k^*\tilde{\Delta}_i r_i \Delta_i^*\Delta_k = \frac{1}{\alpha} r_k + (\alpha-1)\tilde{\Delta}_k^*\hat{r}\Delta_k - \frac{(\alpha-1)^2}{\alpha} r_k = r_k.$$

It follows from (19) and (20) that $rP_k = r\Gamma_k\Gamma_k^* = \tilde{\Gamma}_k r_k \Gamma_k^* = \tilde{\Gamma}_k\tilde{\Gamma}_k^* r_k \Gamma_k \Gamma_k^* = \tilde{P}_k r P_k$, which means that $r \in \mathrm{Mor}(\pi, \tilde{\pi})$.

It is easy to check that $\mathcal{S}'(r) = R$ and, consequently, the functor $\mathcal{S}'$ is complete. So the univalence and completeness properties of the functor $\mathcal{S}'$ are checked, $\mathcal{S}^2 = \mathrm{Id}$ and $\mathcal{S}'(\pi) = \mathcal{S}(\pi)$ for any $\pi \in \mathrm{Rep}\ \mathcal{P}_{n,\alpha}$. Consequently, the functor $\mathcal{S}'$ is an equivalence between the categories $\mathrm{Rep}\ \mathcal{P}_{n,\alpha}$ and $\mathrm{Rep}\ \mathcal{P}_{n,\frac{\alpha}{\alpha-1}}$. ∎

Lemma 2. *If a system S_π, $\pi \in \mathrm{Rep}\ \mathcal{P}_{n,\mathrm{com}}$, of subspaces is transitive, then the system $S_{\Phi^+(\pi)}$ of subspaces is transitive. Here, $S_\pi \cong S_{\tilde{\pi}}$ if and only if $S_{\Phi^+(\pi)} \cong S_{\Phi^+(\tilde{\pi})}$.*

Proof. For the functors $\mathcal{T}$ and $\mathcal{S}$, we have $\mathcal{T}(\pi) = \mathcal{T}'(\pi)$ and $\mathcal{S}(\pi) = \mathcal{S}'(\pi)$ for any $\pi \in \mathrm{Rep}\ \mathcal{P}_{n,\alpha}$. Consequently, $S_{\mathcal{T}(\pi)} = S_{\mathcal{T}'(\pi)}$ and $S_{\mathcal{S}(\pi)} = S_{\mathcal{S}'(\pi)}$. By Lemma 1, $\mathcal{T}'$ is an equivalence of the categories that shows that if a system S_π, $\pi \in \mathrm{Rep}\ \mathcal{P}_{n,\mathrm{com}}$, of subspaces is transitive, then the system $S_{\mathcal{T}(\pi)}$ of subspaces is transitive. We also have that $S_\pi \cong S_{\tilde{\pi}}$ if and only if $S_{\mathcal{T}(\pi)} \cong S_{\mathcal{T}(\tilde{\pi})}$.

Let us now consider the systems $S_{\mathcal{S}(\pi)}$, $\pi \in \mathrm{Rep}\ \mathcal{P}_{n,\alpha}$ of subspaces. Let $\pi, \tilde{\pi} \in \mathrm{Rep}\ \mathcal{P}_{n,\alpha}$. Consider the systems of subspaces $S_\pi = (H; H_1, H_2, \dots, H_n)$ and $S_{\tilde{\pi}} = (\tilde{H}; \tilde{H}_1, \tilde{H}_2, \dots, \tilde{H}_n)$, that, respectively, correspond to the representations π and $\tilde{\pi}$. Let the systems of subspaces be isomorphic, that is, $S_\pi \cong S_{\tilde{\pi}}$. By the definition of isomorphic systems, there exists a linear operator $T \in B(H, \tilde{H})$ such that $T^{-1} \in B(\tilde{H}, H)$ and $T(H_i) = \tilde{H}_i$, $i = 1, \dots, n$. It follows from $T(H_i) = \tilde{H}_i$, $i = 1, \dots, n$, that $T(H_i) \subset \tilde{H}_i$, $i = 1, \dots, n$, and, consequently, we get the relations $TP_i = \tilde{P}_i T P_i$, $i = 1, \dots, n$. The latter relations mean that $T \in \mathrm{Mor}(\pi, \tilde{\pi})$ if $\hat{T}^* \in \mathrm{Mor}(\mathcal{S}'(\tilde{\pi}), \mathcal{S}'(\pi))$, and

$$\hat{T}^*(\mathrm{Im}\ \tilde{Q}_i) \subset (\mathrm{Im}\ Q_i), \qquad i = 1, \dots, n. \tag{21}$$

Again, using $T(H_i) = \tilde{H}_i$, $i = 1, \dots, n$, we get $T(H_i) \supset \tilde{H}_i$, $i = 1, \dots, n$, so that $T^{-1}(\tilde{H}_i) \subset H_i$, $i = 1, \dots, n$, and, respectively, $T^{-1}\tilde{P}_i = P_i T^{-1}\tilde{P}_i$, $i = 1, \dots, n$. This means that $T^{-1} \in \mathrm{Mor}(\tilde{\pi}, \pi)$, hence, $\widehat{T^{-1}}^* \in \mathrm{Mor}(\mathcal{S}'(\pi), \mathcal{S}'(\tilde{\pi}))$, and using $\widehat{T^{-1}}^* = (\hat{T}^{-1})^* = (\hat{T}^*)^{-1}$ we get

$$\mathrm{Im}\ \tilde{Q}_i \supset (\hat{T}^*)^{-1}(\mathrm{Im}\ Q_i), \qquad i = 1, \dots, n,$$

so that

$$\hat{T}^*(\mathrm{Im}\ \tilde{Q}_i) \supset \mathrm{Im}\ Q_i, \qquad i = 1, \dots, n. \tag{22}$$

It follows from (21) and (22) that

$$\hat{T}^*(\mathrm{Im}\ \tilde{Q}_i) = \mathrm{Im}\ Q_i, \qquad i = 1, \dots, n,$$

i.e., it is an isomorphism of the systems corresponding to the representations $\mathcal{S}'(\pi)$ and $\mathcal{S}'(\tilde{\pi})$ and, since the functors $\mathcal{S}'$ and $\mathcal{S}$ coincide on the objects of the categories, it is an isomorphism of the systems corresponding to the representations $\mathcal{S}(\pi)$ and $\mathcal{S}(\tilde{\pi})$.

Since $\mathcal{S}'$ is complete, using similar reasonings it is easy to show that the functor $\mathcal{S}'$ and, hence, $\mathcal{S}$ takes the representations corresponding to nonisomorphic systems to representations that also correspond to nonisomorphic systems.

Let again π be a representation of the algebra $\mathcal{P}_{n,\alpha}$, and $\pi(p_i) = P_i$, $i = 1, \dots, n$, be orthogonal projections on a representation space H. Assume that the system of projections $P_1, P_2, \dots, P_n$ gives rise to a transitive system of subspaces $S_\pi = (H; H_1, H_2, \dots, H_n)$, where $H_i = P_i H$, $i = 1, \dots, n$, that is,

$$\mathrm{End}(S_\pi) = \{r \in B(H) \,|\, r(H_i) \subset H_i, i = 1, \dots, n\} = \mathrm{Mor}(\pi, \pi) = \mathbb{C}I.$$

Consider $\mathcal{S}'(\pi) = \hat{\pi}$, where $\hat{\pi}(q_i) = Q_i$, $i = 1, \dots, n$, and the corresponding system of subspaces $S_{\hat{\pi}}$. Let now $R \in \mathrm{End}(S_{\hat{\pi}})$. Since $\mathrm{End}(S_{\hat{\pi}}) = \mathrm{Mor}(\mathcal{S}'(\pi), \mathcal{S}'(\pi))$ and the functor $\mathcal{S}'$ is complete, we see that $\mathcal{S}'(r) = R$, where $r \in \mathrm{Mor}(\pi, \pi)$ is constructed from the operator $R^* = \frac{\alpha-1}{\alpha} \sum_{k=1}^{n} \tilde{\Delta}_k r_k \Delta_k^*$, $r_i = \tilde{\Delta}_i^* R^* \Delta_i : H_i \to \tilde{H}_i$, $i = 1, \dots, n$, as follows:

$$r = \frac{1}{\alpha} \sum_{i=1}^{n} \Gamma_i r_i \Gamma_i^*. \tag{23}$$

By using $R \in \mathrm{Mor}(\mathcal{S}'(\pi), \mathcal{S}'(\pi))$, we obtain, similarly to (20), that

$$r_i = \Gamma_i^* r \Gamma_i, \qquad i = 1, \dots, n. \tag{24}$$

Since the system S_π is transitive, the operator r is a scalar, that is, $r = \lambda I_H$. Using that $\Gamma_i^* \Gamma_i = I_{H_i}$, $i = 1, \dots, n$, and (24) we get

$$r_i = \lambda I_{H_i}, \qquad i = 1, \dots, n.$$

Then R^* is a scalar operator and, consequently, R is also a scalar operator that means that the system $S_{\mathcal{S}'(\pi)}$ is transitive and such is $S_{\mathcal{S}(\pi)}$. ■

The statement of Theorem 3 follows directly from Lemma 2.

5 Transitive systems of subspaces corresponding to Rep $\mathcal{P}_{n,\mathrm{abo},\tau}$

5.1 Equivalence of the categories Rep $\mathcal{P}_{n,\alpha}$ and Rep $\mathcal{P}_{n,\mathrm{abo},\tau}$

Let us examine the equivalence $\mathcal{F}$, constructed in [10], between the categories of $*$-representations $\mathcal{P}_{n,\alpha}$ and $\mathcal{P}_{n,\mathrm{abo},\frac{1}{\alpha}}$, $\alpha \neq 0$. Theorem 3 allows to consider nonisomorphic transitive systems of n subspaces of the form S_π, constructed from representations of the algebras $\mathcal{P}_{n,\alpha}$ for α lying in the discrete spectrum. The equivalence $\mathcal{F}$, in its turn, allows to construct nonisomorphic transitive systems $S_{\mathcal{F}(\pi)}$ of $n+1$ subspaces starting with nonisomorphic transitive systems S_π, $\pi \in \mathcal{P}_{n,\alpha}$, of n subspaces.

Let us describe the equivalence $\mathcal{F}$. Let π be a representation of the algebra $\mathcal{P}_{n,\alpha}$, and $\pi(p_i) = P_i$, $i = 1, \dots, n$, be orthogonal projections on a representation space H. As it was done in Section 4, let us introduce the spaces $H_i = \mathrm{Im}\, P_i$ and the natural isometries $\Gamma_i : H_i \to H$. Let $\mathcal{H} = H_1 \oplus H_2 \oplus \cdots \oplus H_n$. Define a linear operator $\Gamma : H_1 \oplus H_2 \oplus \cdots \oplus H_n \to H$ in terms of the matrix $\Gamma = (\Gamma_1\ \Gamma_2\ \dots \Gamma_n)$ of the dimension $n \times 1$. Let Q_i denote n orthogonal projections, $Q_i = \mathrm{diag}(0, \dots, 0, I_{H_i}, 0, \dots, 0)$, $i = 1, \dots, n$, and $P : \mathcal{H} \to \mathcal{H}$ an orthogonal projection defined by $P = \frac{1}{\alpha} \Gamma^* \Gamma$ with the block matrix $P = \frac{1}{\alpha} ||\Gamma_i^* \Gamma_j||_{i,j=1}^n$ on the space $H_1 \oplus H_2 \oplus \cdots \oplus H_n$.

Let a functor $\mathcal{F} : \mathrm{Rep}\, \mathcal{P}_{n,\alpha} \to \mathrm{Rep}\, \mathcal{P}_{n,\mathrm{abo},\frac{1}{\alpha}}$, $\alpha \neq 0$, be defined on objects of the category of representations as follows: $\mathcal{F}(\pi) = \hat{\pi}$, where $\hat{\pi}(q_i) = Q_i$, $i = 1, \dots, n$, and $\hat{\pi}(p) = P$. The identities $\sum\limits_{i=1}^{n} Q_i = I$ and $Q_i P Q_i = \frac{1}{\alpha} Q_i$, $i = 1, \dots, n$, are easily checked. We do not describe the action of the functor $\mathcal{F}$ on morphisms of the category Rep $\mathcal{P}_{n,\alpha}$, since we will not use it.

Theorem 4. *Systems of $n+1$ subspaces, $S_{\mathcal{F}(\pi)}$, constructed from irreducible inequivalent representations $\pi \in \operatorname{Rep} \mathcal{P}_{n,\alpha}$, where α is in the discrete spectrum, are nonisomorphic and transitive.*

To prove the theorem, construct an auxiliary functor $\mathcal{F}' : \operatorname{Rep} \mathcal{P}_{n,\alpha} \to \operatorname{Rep} \mathcal{P}_{n,\mathrm{abo},\frac{1}{\alpha}}$, $\alpha \neq 0$, the action of which on objects coincides with the action of $\mathcal{F}$, that is, $\mathcal{F}'(\pi) = \mathcal{F}(\pi)$ for all $\pi \in \operatorname{Rep} \mathcal{P}_{n,\alpha}$. Morphisms are defined as in Section 4. Let $\pi \in \operatorname{Rep}(\mathcal{P}_{n,\alpha}, H)$ and $\tilde{\pi} \in \operatorname{Rep}(\mathcal{P}_{n,\alpha}, \tilde{H})$. A linear operator $C \in B(H, \tilde{H})$ will be called a morphism of the category of representations, written $C \in \operatorname{Mor}(\pi, \tilde{\pi})$, if $C\pi(p_i) = \tilde{\pi}(p_i)C\pi(p_i)$, that is,

$$CP_i = \tilde{P}_i C P_i, \qquad i = 1, \dots, n. \tag{25}$$

As it was for the functors in Section 4, denote the restrictions $C|_{H_i}$, $i = 1, \dots, n$, by C_i. Then, as in Section 4, the operators C_i map H_i into $\tilde{H}_i$, that is,

$$C_i(H_i) \subset \tilde{H}_i, \qquad i = 1, \dots, n. \tag{26}$$

It follows from (25) and (26) that

$$C\Gamma_i = \tilde{\Gamma}_i C_i, \qquad i = 1, \dots, n. \tag{27}$$

The identities (25) are equivalent to the inclusions $C(H_i) \subset \tilde{H}_i$, $i = 1, \dots, n$, whence it follows that

$$C_i = \tilde{\Gamma}_i^* C \Gamma_i, \qquad i = 1, \dots, n. \tag{28}$$

Similarly to Section 4, identities (27) allow to represent C as

$$C = \frac{1}{\alpha} \sum_{i=1}^{n} \tilde{\Gamma}_i C_i \Gamma_i^*. \tag{29}$$

The above presents all the similarities with the calculations performed in Section 4; the operator $\hat{C}$ is now defined differently. For the operator $\hat{C} = \operatorname{diag}(C_1, C_2, \dots, C_n) : \mathcal{H} \to \tilde{\mathcal{H}}$, it is easy to check that $\hat{C}Q_i = \tilde{Q}_i\hat{C}$, $i = 1, \dots, n$. Then $Q_i\hat{C}^* = \hat{C}^*\tilde{Q}_i$, $i = 1, \dots, n$. The latter allows to conclude that $\hat{C}^*(\operatorname{Im} \tilde{Q}_i) \subset \operatorname{Im} Q_i$ and, consequently,

$$\hat{C}^*\tilde{Q}_i = Q_i\hat{C}^*\tilde{Q}_i, \qquad i = 1, \dots, n. \tag{30}$$

Denote by $(\tilde{P}\hat{C}P)_{ij}$ the elements of the block matrix of the operator $\tilde{P}\hat{C}P : H_1 \oplus H_2 \oplus \cdots \oplus H_n \to \tilde{H}_1 \oplus \tilde{H}_2 \oplus \cdots \oplus \tilde{H}_n$. Then $(\tilde{P}\hat{C}P)_{ij} = \frac{1}{\alpha^2} \sum\limits_{k=1}^{n} \tilde{\Gamma}_i^*\tilde{\Gamma}_k C_k \Gamma_k^* \Gamma_j = \frac{1}{\alpha^2}\tilde{\Gamma}_i^*(\sum\limits_{k=1}^{n} \tilde{\Gamma}_k C_k \Gamma_k^*)\Gamma_j = \frac{1}{\alpha}\tilde{\Gamma}_i^* C \Gamma_j = \frac{1}{\alpha}\tilde{\Gamma}_i^*\tilde{\Gamma}_j C_j = (\tilde{P}\hat{C})_{ij}$, that is, $\tilde{P}\hat{C}P = \tilde{P}\hat{C}$ and, consequently,

$$\hat{C}^*\tilde{P} = P\hat{C}^*\tilde{P}. \tag{31}$$

Identities (30) and (31) mean that $\hat{C}^* \in \operatorname{Mor}(\mathcal{F}'(\tilde{\pi}), \mathcal{F}'(\pi))$. Define $\mathcal{F}'(C) = \hat{C}^*$, and this finishes the construction of the functor $\mathcal{F}'$.

Lemma 3. *The functor $\mathcal{F}'$ is an equivalence between the categories.*

Proof. Let us show that the functor is univalent. Let $C, D \in \operatorname{Mor}(\pi, \tilde{\pi})$ and $C \neq D$, and show that $\mathcal{F}'(C) \neq \mathcal{F}'(D)$. Indeed, if $\mathcal{F}'(C) = \mathcal{F}'(D)$, i.e., $\hat{C}^* = \hat{D}^*$, then $C_i = D_i$, $\forall\, i = 1, \dots, n$. Let us use (29),

$$C = \frac{1}{\alpha} \sum_{i=1}^{n} \tilde{\Gamma}_i C_i \Gamma_i^*, \qquad D = \frac{1}{\alpha} \sum_{i=1}^{n} \tilde{\Gamma}_i D_i \Gamma_i^*.$$

It follows from $C_i = D_i$, $i = 1, \dots, n$, and the form of the representation operators C and D that $C = D$ and, hence, the functor $\mathcal{F}'$ is univalent.

Let us show that $\mathcal{F}'$ is complete. Let $R \in \mathrm{Mor}(\mathcal{F}'(\tilde{\pi}), \mathcal{F}'(\pi))$ and construct a linear operator in the set $\mathrm{Mor}(\pi, \tilde{\pi})$ such that the value of this functor on the morphism is R. It follows from $R \in \mathrm{Mor}(\mathcal{F}'(\tilde{\pi}), \mathcal{F}'(\pi))$ that the operator $R : \tilde{\mathcal{H}} \to \mathcal{H}$ satisfies

$$Q_i R \tilde{Q}_i = R \tilde{Q}_i, \qquad i = 1, \dots, n, \qquad P R \tilde{P} = R \tilde{P}.$$

Denote by $\hat{r}$ an operator in $B(\mathcal{H}, \tilde{\mathcal{H}})$ such that $\hat{r}^* = R$. Then the latter identities can be rewritten as follows:

$$Q_i \hat{r}^* \tilde{Q}_i = \hat{r}^* \tilde{Q}_i, \qquad i = 1, \dots, n, \qquad P \hat{r}^* \tilde{P} = \hat{r}^* \tilde{P},$$

and, consequently,

$$\tilde{Q}_i \hat{r} Q_i = \tilde{Q}_i \hat{r}, \qquad i = 1, \dots, n \tag{32}$$

and

$$\tilde{P} \hat{r} P = \tilde{P} \hat{r}. \tag{33}$$

Let now r_{ij} be elements of the block matrix of the operator $\hat{r}$ from $H_1 \oplus H_2 \oplus \cdots \oplus H_n$ into $\tilde{H}_1 \oplus \tilde{H}_2 \oplus \cdots \oplus \tilde{H}_n$. Identities (32) imply that if $i \neq j$, then $r_{ij} = 0$. Denote $r_i = r_{ii}$, $i = 1, \dots, n$. Then $r_i : H_i \to \tilde{H}_i$, $i = 1, \dots, n$, and $\hat{r} = \mathrm{diag}(r_1, r_2, \dots, r_n)$. Consider $r : H \to \tilde{H}$ defined by

$$r = \frac{1}{\alpha} \tilde{\Gamma} \hat{r} \Gamma^*. \tag{34}$$

Identity (33) and definition (34) imply that $\frac{1}{\alpha} \tilde{\Gamma}^* r \Gamma = \tilde{P} \hat{r} P = \tilde{P} \hat{r}$, then comparing the elements on the main diagonal of the corresponding block matrices gives

$$r_i = \tilde{\Gamma}_i^* r \Gamma_i, \qquad i = 1, \dots, n. \tag{35}$$

Using the relation $\frac{1}{\alpha} \tilde{\Gamma} \tilde{\Gamma}^* = I_{\tilde{H}}$ we get $r\Gamma = I_{\tilde{H}} r \Gamma = (\frac{1}{\alpha} \tilde{\Gamma} \tilde{\Gamma}^*) r \Gamma = \tilde{\Gamma} (\frac{1}{\alpha} \tilde{\Gamma}^* r \Gamma) = \tilde{\Gamma} (\tilde{P} \hat{r} P) = \tilde{\Gamma} (\tilde{P} \hat{r}) = \tilde{\Gamma} (\frac{1}{\alpha} \tilde{\Gamma}^* \tilde{\Gamma}) \hat{r} = (\frac{1}{\alpha} \tilde{\Gamma} \tilde{\Gamma}^*) \tilde{\Gamma} \hat{r} = I_{\tilde{H}} \tilde{\Gamma} \hat{r} = \tilde{\Gamma} \hat{r}$. Rewrite the identity $r\Gamma = \tilde{\Gamma} \hat{r}$ in the matrix form,

$$(r\Gamma_1 \, r\Gamma_2 \, \dots r\Gamma_n) = (\tilde{\Gamma}_1 r_1 \, \tilde{\Gamma}_2 r_2 \, \dots \, \tilde{\Gamma}_n r_n)$$

that gives

$$r \Gamma_i = \tilde{\Gamma}_i r_i, \qquad i = 1, \dots, n. \tag{36}$$

Using identities (35) and (36) we get

$$r P_i = \tilde{P}_i r P_i, \qquad i = 1, \dots, n. \tag{37}$$

Indeed, $rP_i = r\Gamma_i \Gamma_i^* = \tilde{\Gamma}_i r_i \Gamma_i^* = \tilde{\Gamma}_i \tilde{\Gamma}_i^* r \Gamma_i \Gamma_i^* = \tilde{P}_i r P_i$. Identities (37) mean that $r \in \mathrm{Mor}(\pi, \tilde{\pi})$. Let us check that $\mathcal{F}'(r) = \hat{r}^* = R$. Denote by C the constructed morphism r and find $\mathcal{F}'(C)$. Since $\mathcal{F}'(C) = \hat{C}^*$, where $\hat{C} = \mathrm{diag}(C_1, C_2, \dots, C_n) : \mathcal{H} \to \tilde{\mathcal{H}}$, let us find $C_i = C|_{H_i} = r|_{H_i}$, $i = 1, \dots, n$. Since $C \in \mathrm{Mor}(\pi, \tilde{\pi})$, it follows from (28) and (35) that $C_i = \tilde{\Gamma}_i^* C \Gamma_i = \tilde{\Gamma}_i^* r \Gamma_i = r_i$. Then $\hat{C} = \hat{r}$, $\hat{C}^* = \hat{r}^*$ and $\mathcal{F}'(r) = \mathcal{F}'(C) = \hat{C}^* = \hat{r}^* = R$. This proves that the functor $\mathcal{F}'$ is complete.

Since $\mathcal{F}'(\pi) = \mathcal{F}(\pi)$ for any $\pi \in \mathrm{Rep}\, \mathcal{P}_{n,\alpha}$ and $\mathcal{F}^2 = \mathrm{Id}$, we see that $\mathcal{F}'$ is an equivalence of the categories. ■

Lemma 4. *If a system S_π, $\pi \in \mathrm{Rep}\,\mathcal{P}_{n,\mathrm{com}}$, of n subspaces is transitive, then the system $S_{\mathcal{F}(\pi)}$ of $n+1$ subspaces is transitive. Also, $S_\pi \cong S_{\tilde\pi}$ if and only if $S_{\mathcal{F}(\pi)} \cong S_{\mathcal{F}(\tilde\pi)}$.*

Proof. Since the functors $\mathcal{F}$ and $\mathcal{F}'$ coincide on the objects of the categories, the representations constructed using the functors and the corresponding systems of subspaces will coincide, $S_{\mathcal{F}(\pi)} = S_{\mathcal{F}'(\pi)}$ for $\forall\, \pi \in \mathrm{Rep}\,\mathcal{P}_{n,\alpha}$. Let $\pi, \tilde\pi \in \mathrm{Rep}\,\mathcal{P}_{n,\alpha}$, $\alpha \neq 0$, and the systems of subspaces $S_\pi = (H; H_1, H_2, \dots, H_n)$ and $S_{\tilde\pi} = (\tilde H; \tilde H_1, \tilde H_2, \dots, \tilde H_n)$, which correspond to the representations π and $\tilde\pi$, be isomorphic, that is, $S_\pi \cong S_{\tilde\pi}$. By the definition of isomorphic systems, there exists a linear operator $T \in B(H, \tilde H)$ such that $T^{-1} \in B(\tilde H, H)$ and $T(H_i) = \tilde H_i$, $i = 1, \dots, n$. It follows from $T(H_i) = \tilde H_i$, $i = 1, \dots, n$, that $T(H_i) \subset \tilde H_i$, $i = 1, \dots, n$, and, consequently, $TP_i = \tilde P_i T P_i$, $i = 1, \dots, n$. The latter identities mean that $T \in \mathrm{Mor}(\pi, \tilde\pi)$. Then $\hat T^* \in \mathrm{Mor}(\mathcal{F}'(\tilde\pi), \mathcal{F}'(\pi))$ and

$$\hat T^*(\mathrm{Im}\,\tilde Q_i) \subset (\mathrm{Im}\,Q_i)\,(i = 1, \dots, n) \qquad \text{and} \qquad \hat T^*(\mathrm{Im}\,\tilde P) \subset (\mathrm{Im}\,P). \tag{38}$$

Again, considering the identities $T(H_i) = \tilde H_i$, $i = 1, \dots, n$, we conclude that $T(H_i) \supset \tilde H_i$, $i = 1, \dots, n$, that is, $T^{-1}(\tilde H_i) \subset H_i$, $i = 1, \dots, n$, and, respectively, $T^{-1}\tilde P_i = P_i T^{-1} \tilde P_i$, $i = 1, \dots, n$. These identities imply that $T^{-1} \in \mathrm{Mor}(\tilde\pi, \pi)$. Then $\widehat{T^{-1}}^* \in \mathrm{Mor}(\mathcal{F}'(\pi), \mathcal{F}'(\tilde\pi))$, whence using $\widehat{T^{-1}}^* = (\hat T^{-1})^* = (\hat T^*)^{-1}$ we have

$$\mathrm{Im}\,\tilde Q_i \supset (\hat T^*)^{-1}(\mathrm{Im}\,Q_i), \quad i = 1, \dots, n \qquad \text{and} \qquad \mathrm{Im}\,\tilde P \supset (\hat T^*)^{-1}(\mathrm{Im}\,P),$$

and, consequently,

$$\hat T^*(\mathrm{Im}\,\tilde Q_i) \supset \mathrm{Im}\,Q_i, \quad i = 1, \dots, n \qquad \text{and} \qquad \hat T^*(\mathrm{Im}\,\tilde P) \supset \mathrm{Im}\,P. \tag{39}$$

It follows from (38) and (39) that

$$\hat T^*(\mathrm{Im}\,\tilde Q_i) = \mathrm{Im}\,Q_i, \quad i = 1, \dots, n \qquad \text{and} \qquad \hat T^*(\mathrm{Im}\,\tilde P) = \mathrm{Im}\,P$$

that shows that it is an isomorphism of the systems that correspond to the representations $\mathcal{F}'(\pi)$ and $\mathcal{F}'(\tilde\pi)$ and, since the functors $\mathcal{F}'$ and $\mathcal{F}$ coincide on the objects of the category, it is an isomorphism of the systems corresponding to the representations $\mathcal{F}(\pi)$ and $\mathcal{F}(\tilde\pi)$.

Since the functor $\mathcal{F}'$ is complete, similar reasonings show that the representations that correspond to nonisomorphic systems are mapped by the functor $\mathcal{F}'$, and hence the functor $\mathcal{F}$, into representations that give rise to nonisomorphic systems.

Let us now prove the first part of the proposition. Let π be a representation of the algebra $\mathcal{P}_{n,\alpha}$ and $\pi(p_i) = P_i$, $i = 1, \dots, n$, be orthogonal projections on a representation space H. And let the system of orthogonal projections $P_1, P_2, \dots, P_n$ induce a transitive system of subspaces $S_\pi = (H; H_1, H_2, \dots, H_n)$, where $H_i = P_i H$, $i = 1, \dots, n$, that is,

$$\mathrm{End}(S_\pi) = \{r \in B(H) \,|\, r(H_i) \subset H_i, i = 1, \dots, n\} = \mathrm{Mor}(\pi, \pi) = \mathbb{C}I.$$

Consider $\mathcal{F}'(\pi) = \hat\pi$, where $\hat\pi(q_i) = Q_i$, $i = 1, \dots, n$, and $\hat\pi(p) = P$, and the corresponding system $S_{\hat\pi}$ of subspaces. Let now $R \in \mathrm{End}(S_{\hat\pi})$. Using $\mathrm{End}(S_{\hat\pi}) = \mathrm{Mor}(\mathcal{F}'(\pi), \mathcal{F}'(\pi))$ and since the functor $\mathcal{F}'$ is complete, we see that $\mathcal{F}'(r) = R$, where $r \in \mathrm{Mor}(\pi, \pi)$ is constructed from the diagonal operator $R^* = \mathrm{diag}(r_1, r_2, \dots, r_n)$ on the space $H_1 \oplus H_2 \oplus \dots \oplus H_n$ as follows:

$$r = \frac{1}{\alpha}\Gamma R^* \Gamma^*. \tag{40}$$

Using the inclusion $R \in \mathrm{Mor}(\mathcal{F}'(\pi), \mathcal{F}'(\pi))$, which is similar to identity (35), we get

$$r_i = \Gamma_i^* r \Gamma_i, \qquad i = 1, \dots, n. \tag{41}$$

Since the system S_π is transitive, the operator r is scalar, that is, $r = \lambda I_H$. Using $\Gamma_i^*\Gamma_i = I_{H_i}$, $i = 1, \dots, n$, and identities (41) we get

$$r_i = \lambda I_{H_i}, \qquad i = 1, \dots, n.$$

Then R^* is a scalar operator and, consequently, R is also a scalar operator that means that the system $S_{\mathcal{F}'(\pi)}$ is transitive and such is $S_{\mathcal{F}(\pi)}$. ■

The claim of Theorem 4 follows from Theorem 3 and Lemma 4.

5.2 Transitive quintuples of subspaces

By Theorem 4, the functor $\mathcal{F}$ maps known nonisomorphic transitive quadruples of subspaces of the form S_π, where $\pi \in \operatorname{Rep} \mathcal{P}_{4,\mathrm{com}}$, into nonisomorphic transitive quintuples $S_{\mathcal{F}(\pi)}$ [11, 12]. In this section, we give inequivalent irreducible $*$-representations of the $*$-algebras $\mathcal{P}_{4,\mathrm{abo},\tau}$, $\tau \in \tilde{\Sigma}_4$, where $\tilde{\Sigma}_4$ that is the set of $\tau \in \mathbb{R}_+$ such that there exists at least one $*$-representation of the $*$-algebra $\mathcal{P}_{4,\mathrm{abo},\tau}$, is related to Σ_4, the set $\alpha \in \mathbb{R}_+$ such that there exists at least one $*$-representation of the $*$-algebra $\mathcal{P}_{4,\alpha}$, via the following relation [10]:

$$\tilde{\Sigma}_4 = \{0\} \cup \left\{ \frac{1}{\alpha} \,|\, \alpha \neq 0, \alpha \in \Sigma_4 \right\}.$$

Here, by [7], $\Sigma_4 = \big\{0, 1, 2 - \frac{2}{2k+1}\,(k = 1, 2, \dots), 2 - \frac{1}{n}\,(n = 2, 3, \dots), 2, 2 + \frac{1}{n}\,(n = 2, 3, \dots), 2 + \frac{2}{2k+1}\,(k = 1, 2, \dots), 3, 4\big\}$. For these representations, the corresponding systems of subspaces are nonisomorphic and transitive.

Let $e_{i,j}^{r\times s}$ denote an $(r \times s)$-matrix that has 1 at the intersection of the ith row and the jth column, with other elements being zero.

1) The $*$-algebra $\mathcal{P}_{4,\mathrm{abo},0}$ has 4 irreducible inequivalent one-dimensional representations, $Q_1 = \dots = Q_{k-1} = Q_{k+1} = \dots = Q_4 = P = 0$, $Q_k = 1$.

2) The $*$-algebra $\mathcal{P}_{4,\mathrm{abo},1}$ has 4 irreducible inequivalent one-dimensional representations, $Q_1 = \dots = Q_{k-1} = Q_{k+1} = \dots = Q_4 = 0$, $P = Q_k = 1$.

3) The $*$-algebra $\mathcal{P}_{4,\mathrm{abo},\frac{1}{3}}$ has 4 irreducible inequivalent three-dimensional representations that are unitary equivalent, up to a permutation, to

$$Q_1 = 1 \oplus 0 \oplus 0, \qquad Q_3 = 0 \oplus 0 \oplus 1, \qquad P = \frac{1}{3}\sum_{i,j=1}^{3} e_{i,j}^{3\times 3},$$
$$Q_2 = 0 \oplus 1 \oplus 0, \qquad Q_4 = 0 \oplus 0 \oplus 0, \qquad \mathcal{H} = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}.$$

4) The $*$-algebra $\mathcal{P}_{4,\mathrm{abo},\frac{1}{4}}$ has a unique irreducible four dimensional representation,

$$Q_1 = 1 \oplus 0 \oplus 0 \oplus 0, \qquad Q_3 = 0 \oplus 0 \oplus 1 \oplus 0, \qquad P = \frac{1}{4}\sum_{i,j=1}^{4} e_{i,j}^{4\times 4},$$
$$Q_2 = 0 \oplus 1 \oplus 0 \oplus 0, \qquad Q_4 = 0 \oplus 0 \oplus 0 \oplus 1, \qquad \mathcal{H} = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}.$$

5) The $*$-algebra $\mathcal{P}_{4,\mathrm{abo},\frac{1}{2}}$ has 6 irreducible two-dimensional representations that are unitary equivalent, up to a permutation, to

$$Q_1 = 1 \oplus 0, \qquad Q_2 = 0 \oplus 1, \qquad Q_3 = 0 \oplus 0, \qquad Q_4 = 0 \oplus 0, \qquad P = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

where the representation space is $\mathcal{H} = \mathbb{C} \oplus \mathbb{C}$, and the following inequivalent four dimensional representations that depend on the points of the set $\Omega = \{(a, b, c) \in \mathbb{R} | a^2 + b^2 + c^2 = 1, a > 0, b > 0, c \in (-1, 1);$ or $a = 0, b^2 + c^2 = 1, b > 0, c > 0;$ or $b = 0, a^2 + c^2 = 1, a > 0, c > 0\}$:

$$Q_1 = 1 \oplus 0 \oplus 0 \oplus 0, \qquad Q_3 = 0 \oplus 0 \oplus 1 \oplus 0,$$
$$Q_2 = 0 \oplus 1 \oplus 0 \oplus 0, \qquad Q_4 = 0 \oplus 0 \oplus 0 \oplus 1,$$
$$P = \frac{1}{2}\begin{pmatrix} 1 & \frac{c(c-ib)}{\sqrt{1-a^2}} & \frac{b(b+ic)}{\sqrt{1-a^2}} & a \\ \frac{c(c+ib)}{\sqrt{1-a^2}} & 1 & -a & \frac{b(b-ic)}{\sqrt{1-a^2}} \\ \frac{b(b-ic)}{\sqrt{1-a^2}} & -a & 1 & \frac{c(c+ib)}{\sqrt{1-a^2}} \\ a & \frac{b(b+ic)}{\sqrt{1-a^2}} & \frac{c(c-ib)}{\sqrt{1-a^2}} & 1 \end{pmatrix},$$

where the representation space is $\mathcal{H} = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$.

6) The $*$-algebras $\mathcal{P}_{4,\text{abo},\frac{1}{\alpha}}$, for $\alpha = 2 - \frac{2}{2k+1}$, $k = 1, 2, \ldots$, have unique irreducible representations

$$Q_1 = I \oplus 0 \oplus 0 \oplus 0, \qquad Q_3 = 0 \oplus 0 \oplus I \oplus 0,$$
$$Q_2 = 0 \oplus I \oplus 0 \oplus 0, \qquad Q_4 = 0 \oplus 0 \oplus 0 \oplus I,$$
$$P = \frac{1}{\alpha}\begin{pmatrix} A & B \\ B^t & C \end{pmatrix}, \qquad \text{where} \quad A = \begin{pmatrix} I & A_1 \\ A_1 & I \end{pmatrix}, \quad C = \begin{pmatrix} I & C_1 \\ C_1 & I \end{pmatrix}, \quad B = \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix},$$
$$A_1 = \frac{1}{2k+1}\sum_{i=1}^{k}(2k+3-4i)e_{i,i}^{k\times k}, \qquad C_1 = \frac{1}{2k+1}\sum_{i=1}^{k}(2k+1-4i)e_{i,i}^{k\times k},$$
$$B_{lm} = \frac{(-1)^\ell}{2k+1}\sum_{i=1}^{k}\sqrt{(2k-2i+1)(2i-1)}\, e_{i,i}^{k\times k} + \frac{(-1)^m}{2k+1}\sum_{i=1}^{k-1}\sqrt{(2k-2i)2i}\, e_{i+1,i}^{k\times k},$$

and the representation space is

$$\mathcal{H} = \mathbb{C}^k \oplus \mathbb{C}^k \oplus \mathbb{C}^k \oplus \mathbb{C}^k.$$

7) The $*$-algebras $\mathcal{P}_{4,\text{abo},\frac{1}{\alpha}}$, for $\alpha = 2 - \frac{1}{2k+1}$, $k = 1, 2, \ldots$, have unique irreducible representations

$$Q_1 = I \oplus 0 \oplus 0 \oplus 0, \qquad Q_3 = 0 \oplus 0 \oplus I \oplus 0,$$
$$Q_2 = 0 \oplus I \oplus 0 \oplus 0, \qquad Q_4 = 0 \oplus 0 \oplus 0 \oplus I,$$
$$P = \frac{1}{\alpha}\begin{pmatrix} A & B \\ B^t & C \end{pmatrix}, \qquad \text{where} \quad A = \begin{pmatrix} I & A_1 \\ A_1^t & I \end{pmatrix}, \quad C = \begin{pmatrix} I & C_1 \\ C_1 & I \end{pmatrix}, \quad B = \begin{pmatrix} \eta & \eta \\ B_{00} & B_{10} \\ B_{01} & B_{11} \end{pmatrix},$$
$$\eta = (\sqrt{\tfrac{k}{2k+1}}, \underbrace{0, 0, \ldots, 0}_{k-1}),$$
$$A_1 = -\frac{1}{2k+1}\sum_{i=1}^{k} 2ie_{i+1,i}^{(k+1)\times k}, \qquad C_1 = -\frac{1}{2k+1}\sum_{i=1}^{k}(2i-1)e_{i,i}^{k\times k},$$
$$B_{lm} = \frac{(-1)^\ell}{4k+2}\sum_{i=1}^{k}\sqrt{(2k-2i+1)(2k+2i)}\, e_{i,i}^{k\times k} + \frac{(-1)^m}{4k+2}\sum_{i=1}^{k-1}\sqrt{(2k-2i)(2k+2i+1)}\, e_{i,i+1}^{k\times k},$$

and the representation space is

$$\mathcal{H} = \mathbb{C}^{k+1} \oplus \mathbb{C}^k \oplus \mathbb{C}^k \oplus \mathbb{C}^k.$$

8) The $*$-algebras $\mathcal{P}_{4,\mathrm{abo},\frac{1}{\alpha}}$, for $\alpha = 2-\frac{1}{2k}$, $k = 1, 2, \ldots$, have unique irreducible representations

$$Q_1 = I \oplus 0 \oplus 0 \oplus 0, \qquad Q_3 = 0 \oplus 0 \oplus I \oplus 0,$$
$$Q_2 = 0 \oplus I \oplus 0 \oplus 0, \qquad Q_4 = 0 \oplus 0 \oplus 0 \oplus I,$$
$$P = \frac{1}{\alpha}\begin{pmatrix} A & B \\ B^t & C \end{pmatrix}, \qquad \text{where} \quad A = \begin{pmatrix} I & A_1 \\ A_1^t & I \end{pmatrix}, \quad C = \begin{pmatrix} I & C_1 \\ C_1 & I \end{pmatrix}, \quad B = \begin{pmatrix} B_{00} & B_{10} \\ \eta & \eta \\ B_{01} & B_{11} \end{pmatrix},$$
$$\eta = (\sqrt{\tfrac{2k-1}{4k}}, \underbrace{0, 0, \ldots, 0}_{k-1}),$$
$$A_1 = -\frac{1}{k}\sum_{i=1}^{k-1} i e_{i,i+1}^{(k-1)\times k}, \qquad C_1 = -\frac{1}{2k}\sum_{i=1}^{k}(2i-1)e_{i,i}^{k\times k},$$
$$B_{lm} = \frac{(-1)^\ell}{4k}\sum_{i=1}^{k-1}\sqrt{(2k-2i)(2k+2i-1)}\, e_{i,i}^{(k-1)\times k} + \frac{(-1)^m}{4k}\sum_{i=1}^{k-1}\sqrt{(2k-2i-1)(2k+2i)}\, e_{i,i+1}^{(k-1)\times k},$$

and the representation space is

$$\mathcal{H} = \mathbb{C}^{k-1} \oplus \mathbb{C}^k \oplus \mathbb{C}^k \oplus \mathbb{C}^k.$$

9) The $*$-algebras $\mathcal{P}_{4,\mathrm{abo},\frac{1}{\alpha}}$, for $\alpha = 2+\frac{1}{2k}$, $k = 1, 2, \ldots$, have unique irreducible representations

$$Q_1 = I \oplus 0 \oplus 0 \oplus 0, \qquad Q_3 = 0 \oplus 0 \oplus I \oplus 0,$$
$$Q_2 = 0 \oplus I \oplus 0 \oplus 0, \qquad Q_4 = 0 \oplus 0 \oplus 0 \oplus I,$$
$$P = \frac{1}{\alpha}\begin{pmatrix} A & B \\ B^t & C \end{pmatrix}, \qquad \text{where} \quad A = \begin{pmatrix} I & A_1 \\ A_1^t & I \end{pmatrix}, \quad C = \begin{pmatrix} I & C_1 \\ C_1 & I \end{pmatrix}, \quad B = \begin{pmatrix} \eta & \eta \\ B_{11} & B_{01} \\ B_{10} & B_{00} \end{pmatrix},$$
$$\eta = (\sqrt{\tfrac{2k+1}{4k}}, \underbrace{0, 0, \ldots, 0}_{k-1}),$$
$$A_1 = \frac{1}{k}\sum_{i=1}^{k} i e_{i+1,i}^{(k+1)\times k}, \qquad C_1 = \frac{1}{2k}\sum_{i=1}^{k}(2i-1)e_{i,i}^{k\times k},$$
$$B_{lm} = \frac{(-1)^\ell}{4k}\sum_{i=1}^{k}\sqrt{(2k+2i)(2k-2i+1)}\, e_{i,i}^{k\times k} + \frac{(-1)^m}{4k}\sum_{i=1}^{k-1}\sqrt{(2k+2i+1)(2k-2i)}\, e_{i,i+1}^{k\times k},$$

and the representation space is

$$\mathcal{H} = \mathbb{C}^{k+1} \oplus \mathbb{C}^k \oplus \mathbb{C}^k \oplus \mathbb{C}^k.$$

10) The $*$-algebras $\mathcal{P}_{4,\mathrm{abo},\frac{1}{\alpha}}$, for $\alpha = 2 + \frac{1}{2k+1}$, $k = 1, 2, \ldots$, have unique irreducible representations

$$Q_1 = I \oplus 0 \oplus 0 \oplus 0, \qquad Q_3 = 0 \oplus 0 \oplus I \oplus 0,$$
$$Q_2 = 0 \oplus I \oplus 0 \oplus 0, \qquad Q_4 = 0 \oplus 0 \oplus 0 \oplus I,$$

$$P = \frac{1}{\alpha}\begin{pmatrix} A & B \\ B^t & C \end{pmatrix}, \qquad \text{where} \quad A = \begin{pmatrix} I & A_1 \\ A_1^t & I \end{pmatrix}, \quad C = \begin{pmatrix} I & C_1 \\ C_1 & I \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{01} \\ \eta & \eta \\ B_{10} & B_{00} \end{pmatrix},$$

$$\eta = (\sqrt{\tfrac{k+1}{2k+1}}, \underbrace{0, 0, \ldots, 0}_{k}),$$

$$A_1 = \frac{1}{2k+1}\sum_{i=1}^{k} 2i e_{i,i+1}^{k\times(k+1)}, \qquad C_1 = \frac{1}{2k+1}\sum_{i=1}^{k+1}(2i-1)e_{i,i}^{(k+1)\times(k+1)},$$

$$B_{lm} = \frac{(-1)^\ell}{4k+2}\sum_{i=1}^{k}\sqrt{(2k-2i+2)(2k+2i+1)}\, e_{i,i}^{k\times(k+1)} + \frac{(-1)^m}{4k+2}\sum_{i=1}^{k}\sqrt{(2k+2i-1)(2k+2i+2)}\, e_{i,i+1}^{k\times(k+1)},$$

and the representation space is

$$\mathcal{H} = \mathbb{C}^k \oplus \mathbb{C}^{k+1} \oplus \mathbb{C}^{k+1} \oplus \mathbb{C}^{k+1}.$$

11) The $*$-algebras $\mathcal{P}_{4,\text{abo},\frac{1}{\alpha}}$, for $\alpha = 2 + \frac{2}{2k+1}$, $k = 1, 2, \ldots$, have unique irreducible representations

$$Q_1 = I \oplus 0 \oplus 0 \oplus 0, \qquad Q_3 = 0 \oplus 0 \oplus I \oplus 0,$$
$$Q_2 = 0 \oplus I \oplus 0 \oplus 0, \qquad Q_4 = 0 \oplus 0 \oplus 0 \oplus I,$$

$$P = \frac{1}{\alpha}\begin{pmatrix} A & B \\ B^t & C \end{pmatrix}, \qquad \text{where} \quad A = \begin{pmatrix} I & A_1 \\ A_1 & I \end{pmatrix}, \quad C = \begin{pmatrix} I & C_1 \\ C_1 & I \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{01} \\ B_{10} & B_{00} \end{pmatrix},$$

$$A_1 = -\frac{1}{2k+1}\sum_{i=1}^{k+1}(2k+3-4i)e_{i,i}^{(k+1)\times(k+1)},$$

$$C_1 = e_{1,1}^{(k+1)\times(k+1)} - \frac{1}{2k+1}\sum_{i=2}^{k+1}(2k+5-4i)e_{i,i}^{(k+1)\times(k+1)},$$

$$B_{lm} = \frac{1}{\sqrt{2k+1}}e_{1,1}^{(k+1)\times(k+1)} + \frac{(-1)^\ell}{2k+1}\sum_{i=2}^{k+1}\sqrt{(2k-2i+3)(2i-1)}\, e_{i,i}^{(k+1)\times(k+1)} + \frac{(-1)^m}{2k+1}\sum_{i=1}^{k}\sqrt{(2k-2i+2)2i}\, e_{i,i+1}^{(k+1)\times(k+1)},$$

and the representation space is

$$\mathcal{H} = \mathbb{C}^{k+1} \oplus \mathbb{C}^{k+1} \oplus \mathbb{C}^{k+1} \oplus \mathbb{C}^{k+1}.$$

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