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On Classical r-Matrix for the Kowalevski Gyrostat on $so(4)$

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Abstract. We present the trigonometric Lax matrix and classical r-matrix for the Kowalevski gyrostat on $so(4)$ algebra by using the auxiliary matrix algebras $so(3,2)$ or $sp(4)$.

Key words: Kowalevski top; Lax matrices; classical r-matrix

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1 Introduction

The classical r-matrix structure is an important tool for investigating integrable systems. It encodes the Hamiltonian structure of the Lax equation, provides the involution of integrals of motion and gives a natural framework for quantizing integrable systems. The aim of this paper is severalfold. First, we present formulae for the classical r-matrices of the Kowalevski gyrostat on Lie algebra $so(4)$, derived in the framework of the Hamiltonian reduction. In the process we shall get new form of its 5×5 Lax matrix and discuss the properties of the r-matrices. Finally, we get the 4×4 Lax matrix on the auxiliary $sp(4)$ algebra.

Remind, the Kowalevski top is the third integrable case of motion of rigid body rotating in a constant homogeneous field [5]. This is an integrable system on the orbits of the Euclidean Lie algebra $e(3)$ with a quadratic and a quartic in angular momenta integrals of motion.

The Kowalevski top can be generalized in several directions. We can change either initial phase space or the form of the Hamilton function. In this paper we consider the Kowalevski gyrostat with the Hamiltonian

$$H = J_1^2 + J_2^2 + 2J_3^2 + 2\rho J_3 + 2y_1, \qquad \rho\in\mathbb{R}, \tag{1}$$

on a generic orbit of the $so(4)$ Lie algebra with the Poisson brackets

$$\{J_i, J_j\} = \varepsilon_{ijk}J_k, \qquad \{J_i, y_j\} = \varepsilon_{ijk}y_k, \qquad \{y_i, y_j\} = \varkappa^2\varepsilon_{ijk}J_k, \tag{2}$$

where ε_{ijk} is the totally skew-symmetric tensor and $\varkappa\in\mathbb{C}$ (see [6] for references). Fixing values a and b of the Casimir functions

$$A = \sum_{i=1}^{3} y_i^2 + \varkappa^2\sum_{i=1}^{3} J_i^2, \qquad B = \sum_{i=1}^{3} x_i J_i \tag{3}$$

one gets a four-dimensional orbit of $so(4)$

$$\mathcal{O}_{ab}: \quad \{y,\, J\, :\, A=a,\ B=b\},$$

which is a reduced phase space for the deformed Kowalevski top.

Because physical quantities y, J in (1) should be real, $\varkappa^2$ must be real too and algebra (2) is reduced to its two real forms $so(4,\mathbb{R})$ or $so(3,1,\mathbb{R})$ for positive and negative $\varkappa^2$ respectively and to $e(3)$ for $\varkappa = 0$.

The Hamilton function (1) is fixed up to canonical transformations. For instance, the brackets (2) are invariant with respect to scale transformation $y_i \to cy_i$ and $\varkappa \to c\varkappa$ that allows to include scaling parameter c into the Hamiltonian, i.e. to change y_1 by cy_1. Some other transformations are discussed in [6].

Below we identify Lie algebra $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ by using invariant inner product and notation $\mathfrak{g}^*$ is used both for the dual Lie algebra and for the corresponding Poisson manifold.

2 The Kowalevski gyrostat: some known results

The Lax matrices for the Kowalevski gyrostat was found in [9] and [6] at $\varkappa = 0$ and $\varkappa \neq 0$ respectively. The corresponding classical r-matrices have been constructed in [7] and [10]. In these papers different definitions of the classical r-matrix [8, 2] were used, which we briefly discuss below.

2.1 The Lax matrices

By definition the Lax matrices L and M satisfy the Lax equation

$$\frac{d}{dt}L(\lambda) \equiv \big\{H, L(\lambda)\big\} = [M(\lambda), L(\lambda)] \tag{4}$$

with respect to evolution determined by Hamiltonian H. Usually the matrices L and M take values in some auxiliary algebra $\mathfrak{g}$ (or in its representation), whereas entries of L and M are functions on the phase space of a given integrable system depending on spectral parameter λ.

For $\varkappa = 0$ the Lax matrices for the Kowalevski gyrostat on $e(3)$ algebra were found by Reyman and Semenov-Tian-Shansky [9]

$$L_0(\lambda) = \begin{pmatrix} 0 & J_3 & -J_2 & \lambda - \dfrac{y_1}{\lambda} & 0 \\ -J_3 & 0 & J_1 & -\dfrac{y_2}{\lambda} & \lambda \\ J_2 & -J_1 & 0 & -\dfrac{y_3}{\lambda} & 0 \\ \lambda - \dfrac{y_1}{\lambda} & -\dfrac{y_2}{\lambda} & -\dfrac{y_3}{\lambda} & 0 & -J_3 - \rho \\ 0 & \lambda & 0 & J_3 + \rho & 0 \end{pmatrix} \tag{5}$$

and

$$M_0(\lambda) = 2\begin{pmatrix} 0 & -2J_3 & J_2 & -\lambda & 0 \\ 2J_3 & 0 & -J_1 & 0 & -\lambda \\ -J_2 & J_1 & 0 & 0 & 0 \\ -\lambda & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

These matrices belong to the twisted loop algebra $\mathfrak{g}_\lambda$ based on the auxiliary Lie algebra $\mathfrak{g} = so(3,2)$ in fundamental representation. We have to underline that the phase space of the Kowalevski gyrostat and the auxiliary space of these Lax matrices are essentially *different*.

Remind the auxiliary Lie algebra $so(3,2)$ may be defined by all the 5×5 matrices satisfying

$$X^T = -JXJ,$$

where $J = \mathrm{diag}\,(1,1,1,-1,-1)$, and T stands for matrix transposition. The Cartan involution on $\mathfrak{g} = so(3,2)$ is given by

$$\sigma X = -X^T$$

and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the corresponding Cartan decomposition where $\mathfrak{k} = so(3)\oplus so(2)$ is the maximal compact subalgebra of $so(3,2)$. The pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ is given by invariant inner product

$$(X,Y) = -\frac{1}{2}\,\mathrm{tr}\,XY \tag{7}$$

that is positively definite on $\mathfrak{k}$.

We extend the involution σ to the loop algebra $\mathfrak{g}_\lambda$ by setting $(\sigma X)(\lambda) = \sigma(X(-\lambda))$. By definition, the twisted loop algebra $\mathfrak{g}_\lambda$ consists of matrices $X(\lambda)$ such that

$$X(\lambda) = -X^T(-\lambda). \tag{8}$$

The pairing between $\mathfrak{g}_\lambda$ and $\mathfrak{g}^*_\lambda$ is given by

$$\langle X,Y\rangle = \mathrm{Res}\;\lambda^{-1}(X,Y). \tag{9}$$

At $\varkappa\neq 0$ the Lax matrices for the Kowalevski gyrostat on $so(4)$ were originally found in [6] as a deformation of the matrices $L_0(\lambda)$ and $M_0(\lambda)$

$$L = Y_c\cdot L_0,\qquad M = M_0\cdot Y_c^{-1},\qquad Y_c = \mathrm{diag}\left(1,1,1,\frac{\lambda^2}{\lambda^2-\varkappa^2},1\right). \tag{10}$$

Algebraic nature of the matrix $L(\lambda)$ (10) is appeared to be mysterious, because the diagonal matrix Y_c does not belong to the fundamental representation of the auxiliary $so(3,2)$ algebra, hence matrices (10) do not belong to the Reyman–Semenov-Tian-Shansky scheme [8].

In the next section we prove that the Lax matrix $L(\lambda)$ at $\varkappa\neq 0$ is a trigonometric deformation of the rational Lax matrix $L_0(\lambda)$ on the same auxiliary space.

2.2 Classical r-matrix: operator notations

The classical r-matrix is a linear operator $\mathbf{r}\in\mathrm{End}\,\mathfrak{g}$ that determines second Lie bracket on $\mathfrak{g}$ by the rule

$$[X,Y]_r = [\mathbf{r}X,Y] + [X,\mathbf{r}Y].$$

The operator $\mathbf{r}$ is a classical r-matrix for a given integrable system, if the corresponding equations of motion with respect to the r-brackets have the Lax form (4) and the second Lax matrix M is given by

$$M = \frac{1}{2}\,\mathbf{r}\big(dH\big).$$

In the most common cases $\mathbf{r}$ is a skew-symmetric operator such that

$$\mathbf{r} = P_+ - P_-, \tag{11}$$

where $P_\pm$ are projection operators onto complementary subalgebras $\mathfrak{g}_\pm$ of $\mathfrak{g}$. In this case there exists a complete classification theory. All details may be found in the book [8] and references therein.

Marchall [7] has shown that the Lax matrices (5) for the Kowalevski gyrostat on $e(3)$ may be obtained by direct application of this r-matrix approach. Let us introduce the standard decomposition of any element $X \in \mathfrak{g}_\lambda$

$$X(\lambda) = X_+(\lambda) + X_0 + X_-(\lambda), \tag{12}$$

where $X_+(\lambda)$ is a Taylor series in λ, X_0 is an independent of λ and $X_-(\lambda)$ is a series in λ^{-1}. If $P_\pm$ and P_0 are the projection operators onto $\mathfrak{g}_\lambda$ parallel to the complementary subalgebras (12), the operator

$$\mathbf{r} = P_- + \varrho \circ P_0 - P_+. \tag{13}$$

defines the second Lie structure on $\mathfrak{g}_\lambda$. According to [7] the r-matrix (13) is the classical r-matrix for the Kowalevski gyrostat. In the standard case (11) operator ϱ is identity, however for the Kowalevski gyrostat ϱ is a difference of projectors in the base $\mathfrak{g} = so(3,2)$ (see details in [7]).

2.3 Classical r-matrix: tensor notations

Another definition of the classical r-matrix is more familiar in the inverse scattering method [8, 1, 2]. According to [1], the commutativity of the spectral invariant of the matrix $L(\lambda)$ is equivalent to existence of a classical r-matrix $r_{12}(\lambda,\mu)$ such that the Poisson brackets between the entries of $L(\lambda)$ may be rewritten in the following commutator form

$$\{\overset{1}{L}(\lambda), \overset{2}{L}(\mu)\} = [r_{12}(\lambda,\mu), \overset{1}{L}(\lambda)] - [r_{21}(\lambda,\mu), \overset{2}{L}(\mu)]. \tag{14}$$

Here

$$\overset{1}{L}(\lambda) = L(\lambda)\otimes 1, \qquad \overset{2}{L}(\mu) = 1\otimes L(\mu), \qquad r_{21}(\lambda,\mu) = \Pi\, r_{12}(\mu,\lambda)\,\Pi,$$

and Π is a permutation operator $\Pi X\otimes Y = Y\otimes X\Pi$ for any numerical matrices X, Y.

For a given Lax matrix $L(\lambda)$, r-matrices are far from being uniquely defined. The possible ambiguities are discussed in [8, 1, 2].

If the Lax matrix takes values in some Lie algebra $\mathfrak{g}$ (or in its representation), the r-matrix takes values in $\mathfrak{g}\times\mathfrak{g}$ or its corresponding representation. The matrices r_{12}, r_{21} may be identified with kernels of the operators $\mathbf{r}\in \mathrm{End}\,\mathfrak{g}$ and $\mathbf{r}^*\in\mathrm{End}\,\mathfrak{g}^*$ respectively, using pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ (see discussion in [8]).

Generally speaking, the matrix $r_{12}(\lambda,\mu)$ is a function of dynamical variables [2, 4]. In the most extensively studied case of purely numeric r-matrices it satisfies the classical Yang–Baxter equation

$$[r_{12}(\lambda,\mu), r_{13}(\lambda,\nu) + r_{23}(\mu,\nu)] - [r_{13}(\lambda,\nu), r_{32}(\nu,\mu)] = 0, \tag{15}$$

which ensures the Jacobi identity for the Poisson brackets (14). If $r_{12}(\lambda,\mu)$ is a unitary numeric matrix depending on the difference of the spectral parameters $z=\lambda-\mu$, there exists a profound algebraic theory, which allows to classify r-matrices in various families [2, 8].

For the Kowalevski gyrostat the classical r-matrix $r_{12}(\lambda,\mu)$ entering (14) has been constructed in [10] by using the auxiliary Lie algebra $\mathfrak{g} = so(3,2)$ in fundamental representation. The generating set of this auxiliary space consists of one antisymmetric matrix

$$S_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and three symmetric matrices

$$Z_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Z_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which are the generators of the $so(3,2)$ algebra. Other generators are three symmetric matrices

$$H_i = [S_4, Z_i] \equiv S_4 Z_i - Z_i S_4, \qquad i = 1, 2, 3. \tag{16}$$

and three antisymmetric matrices

$$S_1 = [Z_2, Z_3], \qquad S_2 = [Z_3, Z_1], \qquad S_3 = [Z_1, Z_2]. \tag{17}$$

These matrices are orthogonal with respect to the form of trace (7). Four matrices S_k form maximal compact subalgebra $\mathfrak{f} = so(3) \oplus so(2)$ of $so(3,2)$ and their norm is 1, whereas six matrices Z_i and H_i belong to the complementary subspace $\mathfrak{p}$ in the Cartan decomposition $\mathfrak{g} = \mathfrak{f} + \mathfrak{p}$ and their norms are -1. Operators

$$P_{\mathfrak{f}} = \sum_{k=1}^{4} S_k \otimes S_k, \qquad \text{and} \qquad P_{\mathfrak{p}} = \sum_{i=1}^{3} \big(H_i \otimes H_i + Z_i \otimes Z_i\big)$$

are projectors onto the orthogonal subspaces $\mathfrak{f}$ and $\mathfrak{p}$ respectively.

In this basis the Lax matrix $L_0(\lambda)$ (5) for the Kowalevski gyrostat on $e(3)$ reads as

$$L_0 = \lambda(Z_1 + H_2) + \sum_{i=1}^{3} \big(J_i S_i - \lambda^{-1} x_i Z_i\big) + (J_3 + \rho) S_4.$$

According to [10] the corresponding r-matrix is equal to

$$\begin{aligned} r_{12}(\lambda,\mu) &= \frac{\lambda\mu}{\lambda^2-\mu^2} P_{\mathfrak{p}} - \frac{\mu^2}{\lambda^2-\mu^2} P_{\mathfrak{f}} + \big(S_3 - S_4\big) \otimes S_4 \\ &= \frac{\lambda\mu}{\lambda^2-\mu^2} \sum_{i=1}^{3} \big(H_i \otimes H_i + Z_i \otimes Z_i\big) - \frac{\mu^2}{\lambda^2-\mu^2} \sum_{k=1}^{4} S_k \otimes S_k + \big(S_3 - S_4\big) \otimes S_4. \end{aligned} \tag{18}$$

We can say that this matrix $r_{12}(\lambda,\mu)$ is a *specification* of the operator $\mathbf{r}$ (13) with respect to canonical pairing (7)–(9).

In applications to integrable models, see e.g. [8, 1, 2], the solutions of the Yang–Baxter equation (15) given by unitary numeric matrices, satisfying the relation

$$r_{12}(\lambda,\mu) = -r_{21}(\mu,\lambda)$$

and depending on the difference $\lambda - \mu$, had been studied most extensively.

In our case the matrix $r_{12}(\lambda,\mu)$ (18) is appeared to be purely numeric matrix, which depends on the ratio λ/μ only. It allows us to change the spectral parameters $\lambda=e^{u_1}$ and $\mu=e^{u_2}$ and rewrite this r-matrix in the following form

$$r_{12}(z)=\frac{1}{2\sinh(z)}P_{\mathfrak{p}}-\frac{1}{2\sinh z(\cosh z+\sinh z)}P_{\mathfrak{f}}+(S_3-S_4)\otimes S_4,$$

depending on one parameter $z=u_1-u_2$ via trigonometric functions. Therefore, the classical r-matrix for the Kowalevski gyrostat on $e(3)$ should be considered as *trigonometric* r-matrix according to generally accepted classification [2, 8].

At the same time it is natural to keep initial rational parameters λ, μ in the Lax matrix. Similar properties holds for the periodic Toda chain, for which $N\times N$ Lax matrix depends rationally on spectral parameters, while the corresponding r-matrix is trigonometric.

We have to underline that in contrast with usual cases this r-matrix is non-unitary. Moreover, it has a term $(S_3-S_4)\otimes S_4$, which is independent on spectral parameters and, therefore, the inequality $r_{12}(z)\neq -r_{12}(-z)$ takes place.

In order to understand the nature of these items we recall that the Lax matrix $L_0(\lambda)$ has been derived in the framework of the Hamiltonian reduction of the $so(3,2)$ top for which phase space coincides with the auxiliary space. The corresponding classical r-matrix, calculated in [10]

$$r_{12}^{\mathrm{so}(3,2)}(z)=\frac{1}{2\sinh(z)}\left(P_{\mathfrak{p}}-\frac{1}{\cosh z+\sinh z}P_{\mathfrak{f}}\right)$$

is a trigonometric r-matrix associated with the $so(3,2)$ Lie algebra [2]. So, the constant term $(S_3-S_4)\otimes S_4$ in (18) is an immediate result of the Hamilton reduction, which changes the phase space of our integrable system.

We recall that classical r-matrices $r_{12}(\lambda,\mu)$ are called regular solutions to the Yang–Baxter equation (15) if they pass through the unity at some λ and μ. In our case we have the following counterpart of this property of regularity

$$\operatorname{res} r_{12}(z)|_{z=0}=\frac{1}{2}\big(P_{\mathfrak{p}}-P_{\mathfrak{f}}\big).$$

3 Classical r-matrix for Kowalevski gyrostat on $so(4)$

Now let us consider Lax matrix $L(\lambda)$ (10) for the Kowalevski gyrostat on $so(4)$ algebra. After transformation $L(\lambda)\to\cos\phi\,Y_c^{-1/2}L(\lambda)Y_c^{1/2}$ of the Lax matrix $L(\lambda)$ (10) and change of the spectral parameter $\lambda=\varkappa/\sin\phi$ one gets a trigonometric Lax matrix on the auxiliary $so(3,2)$ algebra

$$L=\frac{\varkappa}{\sin\phi}\big(Z_1+\cos\phi H_2\big)+\sum_{i=1}^{3}\big(\cos\phi\,J_iS_i-\varkappa^{-1}\sin\phi\,y_iZ_i\big)+(J_3+\rho)S_4 \tag{19}$$

or

$$L=\begin{pmatrix} 0 & \cos\phi\,J_3 & -\cos\phi\,J_2 & \frac{\varkappa}{\sin\phi}-\frac{\sin\phi}{\varkappa}y_1 & 0\\ -\cos\phi\,J_3 & 0 & \cos\phi\,J_1 & -\frac{\sin\phi}{\varkappa}y_2 & \frac{\varkappa\cos\phi}{\sin\phi}\\ \cos\phi\,J_2 & -\cos\phi\,J_1 & 0 & -\frac{\sin\phi}{\varkappa}y_3 & 0\\ \frac{\varkappa}{\sin\phi}-\frac{\sin\phi}{\varkappa}y_1 & -\frac{\sin\phi}{\varkappa}y_2 & -\frac{\sin\phi}{\varkappa}y_3 & 0 & -J_3-\rho\\ 0 & \frac{\varkappa\cos\phi}{\sin\phi} & 0 & J_3+\rho & 0 \end{pmatrix}. \tag{20}$$

In order to consider the real forms $so(4,\mathbb{R})$ or $so(3,1,\mathbb{R})$ we have to use trigonometric or hyperbolic functions for positive and negative $\varkappa^2$, respectively.

If we put $\phi=\varkappa\lambda^{-1}$ and take the limit $\varkappa\to 0$ we find the rational Lax matrix $L_0(\lambda)$ (5) for the Kowalevski gyrostat on $e(3)$.

The Lax matrices $L(\phi)$ and $L_0(\lambda)$ are invariant with respect to the following involutions

$$L(\phi)\to -L^T(-\phi)\qquad\text{and}\qquad L_0(\lambda)\to -L_0^T(-\lambda), \tag{21}$$

that are compatible with the Cartan involution σ. This simple observation shows that for the Kowalevski $so(4)$ gyrostat the Reyman–Semenov-Tian-Shansky scheme [8] should be extended from rational to trigonometric case.

One can prove that the trigonometric Lax matrix $L(\phi)$ (19) satisfies relation

$$\bigl\{\overset{1}{L}(\phi),\overset{2}{L}(\theta)\bigr\}=\bigl[\,r_{12}(\phi,\theta),\overset{1}{L}(\phi)\,\bigr]-\bigl[r_{21}(\phi,\theta),\overset{2}{L}(\theta)\,\bigr].$$

with the following r-matrix

$$\begin{aligned} r_{12}(\phi,\theta)={}&\frac{\sin\phi\sin\theta}{\cos^2\phi-\cos^2\theta}\sum_{i=1}^{3}\bigl(\cos\theta\,H_i\otimes H_i+\cos\phi\,Z_i\otimes Z_i\bigr)\\ &-\frac{\sin^2\phi}{\cos^2\phi-\cos^2\theta}\sum_{k=1}^{4}\cos\theta\,S_k\otimes S_k+\left(S_3-\frac{\cos\phi\cos\theta+1}{\cos\phi+\cos\theta}\,S_4\right)\otimes S_4. \end{aligned}\tag{22}$$

If we put $\phi=\varkappa\lambda^{-1}$, $\theta=\varkappa\mu^{-1}$ and take the limit $\varkappa\to 0$ we get classical r-matrix for the Kowalevski gyrostat on $e(3)$ algebra (18). As above the matrix $r_{12}(\phi,\theta)$ satisfies the Yang–Baxter equation (15) and it has the same analog of the property of regularity

$$\operatorname{res} r_{12}(\phi,\theta)|_{\phi=\theta}\simeq\bigl(P_{\mathfrak{p}}-P_{\mathfrak{k}}\bigr).$$

In contrast with r-matrix (18) for the Kowalevski gyrostat on $e(3)$ we can not rewrite this r-matrix (22) as a function depending on the difference of the spectral parameters only. We suggest that it may be possible to present it in terms of elliptic functions of one spectral parameters after a proper similarity transformation and reparametrization.

The well known isomorphism between $so(3,2)$ and $sp(4)$ algebras allows us to consider 4×4 Lax matrix instead of 5×5 matrix (20). The generating set Z_1, Z_2, Z_3 and S_4 may be represented by different 4×4 real or complex matrices, for instance,

$$s_4=\frac{i}{2}\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix},$$

and

$$z_1=\frac{i}{2}\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix},\qquad z_2=\frac{1}{2}\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},\qquad z_3=\frac{i}{2}\begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}.$$

Other $sp(4)$ generators are constructed by (16)–(17). These matrices are orthogonal with respect to the form of trace (7). Norm of matrices s_k is $1/2$, whereas six matrices z_i and h_i have norm $-1/2$.

In this basis the 4×4 Lax matrix for the Kowalevski gyrostat on $so(4)$ reads

$$L^{(4)}(\phi) = \begin{pmatrix} (1-\cos\phi)J_3+\rho & \frac{\sin\phi}{\varkappa}y_- - \frac{\varkappa\sin\phi}{\cos\phi+1} & \cos\phi J_- & \frac{\sin\phi}{\varkappa}y_3 \\ -\frac{\sin\phi}{\varkappa}y_+ + \frac{\varkappa\sin\phi}{\cos\phi+1} & (\cos\phi-1)J_3-\rho & -\frac{\sin\phi}{\varkappa}y_3 & -\cos\phi J_+ \\ \cos\phi J_+ & \frac{\sin\phi}{\varkappa}y_3 & (\cos\phi+1)J_3+\rho & -\frac{\sin\phi}{\varkappa}y_+ + \frac{\varkappa(\cos\phi+1)}{\sin\phi} \\ -\frac{\sin\phi}{\varkappa}y_3 & -\cos\phi J_- & \frac{\sin\phi}{\varkappa}y_- - \frac{\varkappa(\cos\phi+1)}{\sin\phi} & -(\cos\phi+1)J_3-\rho \end{pmatrix}. \tag{23}$$

Here $J_\pm = J_1 \pm iJ_2$ and $y_\pm = y_1 \pm iy_2$.

If we put $\phi = \varkappa\lambda^{-1}$ and take the limit $\varkappa \to 0$ we find the rational Lax matrix for the Kowalevski gyrostat on $e(3)$

$$L_0^{(4)}(\lambda) = \begin{pmatrix} \rho & \frac{y_-}{\lambda} & J_- & \frac{y_3}{\lambda} \\ -\frac{y_+}{\lambda} & -\rho & -\frac{y_3}{\lambda} & -J_+ \\ J_+ & \frac{y_3}{\lambda} & 2J_3+\rho & -\frac{y_+}{\lambda}+2\lambda \\ -\frac{y_3}{\lambda} & -J_- & \frac{y_-}{\lambda}-2\lambda & -2J_3-\rho \end{pmatrix}.$$

According to [3], this matrix has a mysterious property. Namely, it contains the 3×3 Lax matrix $\widehat{L}(\lambda)$ for the Goryachev–Chaplygin gyrostat on $e(3)$ algebra as its $(1,1)$-minor. Remind that the Goryachev–Chaplygin gyrostat with Hamiltonian

$$\hat{H} = J_1^2 + J_2^2 + (2J_3+\rho)^2 + 4x_1$$

is an integrable system at the zero value $b = 0$ of the Casimir element B (3) only. Similar property obeys $(2,2)$-minor of $L_0^{(4)}(\lambda)$. These properties strongly depend on the chosen basis s_i, z_i and h_i.

It is easy to prove that $(1,1)$-minor of the trigonometric Lax matrix (23) cannot be a Lax matrix for any integrable system. It is compatible with the known fact that the Goryachev–Chaplygin gyrostat on $e(3)$ cannot be naturally lifted to $so(4)$ algebra.

4 Conclusion

There are few Lax matrices obtained for deformations of known integrable systems from their undeformed counterpart in the form (10) (see [6, 10] and references within). The important question in construction of these matrices by the Ansatz $L = Y_c \cdot L_0$ (10) is a choice of a proper matrix Y_c for a given rational matrix $L_0(\lambda)$. In all known cases this transformation destroys the original auxiliary algebra, because the corresponding matrices Y_c do not belong it.

In this note we show that if one takes a Lax matrix of the Kowalevski $so(4)$ gyrostat in the symmetric form $L = Y_c^{1/2} \cdot L_0 \cdot Y_c^{1/2}$ and makes a trigonometric change of spectral parameter it restores the original auxiliary $so(3,2)$ algebra and new L respects the trigonometric current involution (21). It means that deformation of the physical space from the orbits of $e(3)$ to that of $so(4)$ algebra is naturally related with transition from rational to trigonometric parametrization of the auxiliary current algebra.

We calculated explicitly the corresponding r-matrices and demonstrated that constant terms in them is due to the Hamiltonian reduction.

The classical r-matrix (22) for the $so(4)$ gyrostat is numeric and the corresponding Lax matrix $L(\phi)$ (19) does not contain ordering problem in quantum mechanics. Hence equation (14) holds true in quantum case both for the Lax matrices (19) and (23).

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