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# SINGULAR TRAJECTORIES AND SUBANALYTICITY IN OPTIMAL CONTROL AND HAMILTON-JACOBI THEORY

**Abstract.** This survey paper presents several works of the author in optimal control theory, mainly [20, 8, 22]. Under some general assumptions on an analytic optimal control problem, and assuming the absence of singular minimizing trajectories, the value function associated to this problem happens to be subanalytic. In the case of multi-inputs control-affine systems, generically there does not exist any singular minimizer. An application to the Hamilton-Jacobi theory is then presented, where the Hamiltonian is associated to an optimal control problem; namely, if the data are analytic then the unique viscosity solution is subanalytic.

## 1. Introduction

Consider a control-affine system in $\mathbb{R}^n$

$$\dot{x}_u(t) = f_0(x_u(t)) + \sum_{i=1}^{m} u_i(t) f_i(x_u(t)), \tag{1}$$

where $f_0, \ldots, f_m$ are smooth vector fields in $\mathbb{R}^n$, together with a cost of the form

$$C(T,u) = \int_0^T \sum_{i=1}^{m} u_i(t)^2 dt + g(x_u(T)), \tag{2}$$

where $T > 0$, and $g : \mathbb{R}^n \to \mathbb{R}$ is smooth. Denote by $\mathcal{U}_T$ the set of admissible controls on $[0, T]$, i.e. the set of controls such that the associated trajectory $x_u(.)$ is well-defined on $[0, T]$. It is an open subset of $L^2([0, T], \mathbb{R}^m)$.

All definitions given next hold for general optimal control problems. However in this paper we restrict to control-affine systems (1) with a cost of the form (2).

DEFINITION 1. *For all $x_0 \in \mathbb{R}^n$, the mapping $E_{x_0,T} : u \mapsto x_u(T)$ defined on $\mathcal{U}_T$, where $x_u(.)$ denotes the solution of (1) associated to the control $u \in \mathcal{U}_T$ and starting from $x_0$ at time $t = 0$, is called the end-point mapping.*

The end-point mapping is clearly smooth on $\mathcal{U}_T$.

DEFINITION 2. *A trajectory $x_u(.)$ is said to be singular on $[0, T]$ if $u$ is a singular point of the end-point mapping $E_{x_0,T}$, where $x_0 = x_u(0)$. It is said of corank one if the codimension of the range of $dE_{x_0,T}(u)$ is equal to one.*

Let now $M_1$ be a submanifold of $\mathbb{R}^n$, and consider the optimal control problem of determining, among all trajectories solutions of system (1) joining $x_0$ to $M_1$, a trajectory minimizing the cost function $C(t,u)$. If a control $u$, associated to a trajectory $x_u(.)$, is optimal on $[0,T]$, then there exists a nontrivial *Lagrange multiplier* $(\psi,\psi^0)\in\mathbb{R}^n\times\mathbb{R}$ such that

$$\psi.dE_{x_0,T}(u) = -2\psi^0 u,$$

where $x_0 = x_u(0)$. Moreover $\psi - \psi^0\nabla g(x_u(T)) \perp T_{x_u(T)}M_1$. This is a first-order necessary condition for optimality. The well-known *Pontryagin Maximum Principle* (see [17]) parametrizes this condition, and asserts that the trajectory $x_u(.)$ corresponding to this control is the projection of an *extremal*, i.e. a 4-tuple $(x_u(\cdot), p_u(\cdot), p^0(\cdot), u(\cdot))$ solution of the Hamiltonian system

$$\dot{x}_u = \frac{\partial H}{\partial p}(x_u, p_u, p_u^0, u),\ \ \dot{p}_u = -\frac{\partial H}{\partial x}(x_u, p_u, p_u^0, u),\ \ \frac{\partial H}{\partial u}(x_u, p_u, p_u^0, u) = 0,$$

where

$$H(x,p,p^0,u) = \langle p, f_0(x)\rangle + \sum_{i=1}^m u_i\langle p, f_i(x)\rangle + p^0\sum_{i=1}^m u_i^2$$

is the Hamiltonian of the system, $p_u(.) : [0,T] \to \mathbb{R}^n$ is an absolutely continuous mapping called *adjoint vector*, and $p_u^0$ is a real nonpositive number. Moreover there holds at the final time, up to a multiplying scalar,

$$(p_u(T), p_u^0) = (\psi, \psi^0).$$

If $p_u^0 \neq 0$ the extremal is said to be *normal*, otherwise it is said to be *abnormal*.

REMARK 1. Since we did not set any constraint on the control, any singular trajectory is the projection of an abnormal extremal, and conversely. The version of the maximum principle provided here is the weak form of a more general statement, where the control is constrained, and the condition $\frac{\partial H}{\partial u} = 0$ is replaced by a maximum condition, see [17].

REMARK 2. As a consequence of the Maximum Principle, if a control $u$ is singular on $[0,T]$ then it is singular on $[0,t]$, for all $t \in ]0,T]$.

A trajectory is said to be *strictly singular* if it does not admit any normal extremal lift.

REMARK 3. A singular trajectory is of corank one if and only if it admits a unique abnormal extremal lift. It is strict and of corank one if and only if it admits a unique extremal lift which is abnormal.

The paper surveys several results obtained in [20, 8, 22]. Consider an analytic control-affine system of the form (1), together with a cost (2); if there is no singular minimizing trajectory, then the value function associated to this problem is subanalytic (Theorem 1). This situation happens to hold generically provided $m \geq 2$ (Corollary 2).

As a consequence, we obtain that, for certain Hamilton-Jacobi equations, if the data are analytic then the unique viscosity solution, which happens to be the value function associated to an optimal control problem of the latter form, is subanalytic (Theorem 3).

## 2. Subanalyticity of the value function

### 2.1. Subanalytic functions

Let us first define subanalytic sets, see [12], [13].

DEFINITION 3. *Let $M$ be a real analytic finite dimensional manifold. A subset $A$ of $M$ is said to be semi-analytic if and only if, for any $x \in M$, there exists a neighborhood $U$ of $x$ in $M$ and $2pq$ analytic functions $g_{ij}, h_{ij}$ ($1 \leq i \leq p$ and $1 \leq j \leq q$), such that*

$$A \cap U = \bigcup_{i=1}^{p} \{y \in U \mid g_{ij}(y) = 0 \text{ and } \mathrm{h_{ij}(y)} > 0,\ \mathrm{j} = 1 \ldots \mathrm{q}\}.$$

*Let SEM($M$) denote the set of semi-analytic subsets of $M$.*

The image of a semi-analytic subset by a proper analytic mapping is not in general semi-analytic, and thus this class has to be enlarged.

DEFINITION 4. *A subset $A$ of $M$ is said to be subanalytic if and only if, for any $x \in M$, there exists a neighborhood $U$ of $x$ in $M$ and $2p$ couples $(\Phi_i^\delta, A_i^\delta)$ ($1 \leq i \leq p$ et $\delta = 1, 2$), where $A_i^\delta \in SEM(M_i^\delta)$, and for real analytic manifolds $M_i^\delta$, the mappings $\Phi_i^\delta : M_i^\delta \to M$ are proper analytic, such that*

$$A \cap U = \bigcup_{i=1}^{p} (\Phi_i^1(A_i^1) \backslash \Phi_i^2(A_i^2)).$$

*Let SUB($M$) denote the set of subanalytic subsets of $M$.*

The subanalytic class is closed by union, intersection, complementary, inverse image by an analytic mapping, image by a proper analytic mapping. In brief, the subanalytic class is *o-minimal* (see [10]). Moreover subanalytic sets are *stratifiable* in the following sense.

DEFINITION 5. *Let $M$ be a differentiable manifold. A stratum in $M$ is a locally closed sub-manifold of $M$. A locally finite partition $\mathcal{S}$ of $M$ is a stratification of $M$ if any $S \in \mathcal{S}$ is a stratum such that*

$$\forall T \in \mathcal{S} \quad T \cap \mathrm{Fr\,S} \neq \emptyset \Rightarrow \mathrm{T} \subset \mathrm{Fr\,S} \text{ and } \dim \mathrm{T} < \dim \mathrm{S}.$$

Finally, a mapping $f : M \to N$ between two analytic manifolds is said to be *subanalytic* if its graph is a subanalytic subset of $M \times N$.

A basic property of subanalytic functions, which makes them very useful in calculus of variations, and more generally in optimal control theory, is the following, see [19].

PROPOSITION 1. *Let $M$ and $N$ be real analytic finite dimensional manifolds, $A$ be a subset of $N$, and $\Phi : N \to M$ and $f : N \to \mathbb{R}$ be subanalytic mappings. We define, for any $x \in M$*

$$\psi(x) = \inf\{f(y) \mid y \in \Phi^{-1}(x) \cap A\}.$$

*If $\Phi_{|\bar{A}}$ is proper then $\psi$ is subanalytic.*

### 2.2. Subanalyticity of the value function for control-affine systems

Consider a control-affine system of the form (1), where $f_0, \dots, f_m$ are analytic vector fields in $\mathbb{R}^n$, together with a cost of the form (2), where $g$ is proper analytic in $\mathbb{R}^n$.

DEFINITION 6. *Let $x_0 \in \mathbb{R}^n$ and $T > 0$; the value function $S(T, x_0, \cdot) : \mathbb{R}^n \to \mathbb{R}^+ \cup \{+\infty\}$ is defined as*

$$S(T, x_0, x) = \inf\{C(T, u) \mid u \in E^{-1}_{x_0,T}(x)\},$$

*with the agreement that $S(x) = +\infty$ if there is no trajectory steering $x_0$ to $x$ in time $T$.*

For all $T > 0$ and $r > 0$, we denote $M_r(x_0, T) = \{x \in \mathbb{R}^n \mid S(T, x_0, x) = r\}$, and $M_{\leq r}(x_0, T) = \{x \in \mathbb{R}^n \mid S(T, x_0, x) \leq r\}$.

The following result has been proved in [20].

THEOREM 1. *Let $r, T > 0$ small enough, and let $K$ be a subanalytic compact subset of $M_{\leq r}(x_0, T)$. If there exists no singular minimizing trajectory steering $x_0$ to any point of $K$, then $S(T, x_0, \cdot)$ is continuous and subanalytic on $K$.*

REMARK 4. The real numbers $r$ and $T$ are assumed to be small enough in order to ensure that any trajectory having a cost less than $r$ is well-defined on $[0, T]$.

REMARK 5. This result is a generalization of results of [1] in sub-Riemannian geometry (see also [2, 3, 15]). It may be further extended to more general optimal control problems, see [21].

REMARK 6. If there exist singular minimizing trajectories, the value function may fail to be continuous and/or subanalytic, see [20, 6].

The main idea underlying the proof of this result is to prove the compactness of the set of Lagrange multipliers associated to normal extremals, and then to apply Proposition 1.

On the other part, the main assumption being the absence of singular minimizers, it is natural to study singular trajectories for control-affine systems, as done in the following section, to investigate how to compute them, at least in a generic context, and to ask under which assumptions one can assert that there does not exist any singular minimizing trajectory.

## 3. Singular trajectories of control-affine systems

Results of this section are contained in [7, 8]. Consider the control-affine system (1), where $(f_0, \ldots, f_m)$ is an $(m+1)$-tuple of smooth vector fields on $M$ and the set of admissible controls $u = (u_1, \ldots, u_m)$ is an open subset of $L^\infty([0, T], \mathbb{R}^m)$.

Recall that a singular trajectory $x(.)$ is the projection of an abnormal extremal $(x(\cdot), p(\cdot), 0, u(\cdot))$. We define, for $t \in [0, T]$ and $i, j \in \{0, \ldots, m\}$,

$$h_i(t) = \langle p(t), f_i(x(t))\rangle, \; h_{ij}(t) = \langle p(t), [f_i, f_j](x(t))\rangle,$$

where [ , ] denotes the Lie bracket of vector fields. From the Maximum Principle, we have, along an abnormal extremal,

$$h_0(t) = \text{constant}, \quad h_i(t) = 0, \quad i = 1, \ldots, m, \tag{3}$$

for all $t \in [0, T]$. Differentiating (3), one gets for $i \in \{0, \ldots, m\}$,

$$h_{i0}(t) + \sum_{j=1}^{m} h_{ij}(t) u_j(t) = 0. \tag{4}$$

DEFINITION 7. *Along an abnormal extremal* $(x(\cdot), p(\cdot), 0, u(\cdot))$ *of the system (1), the Goh matrix* $G(t)$ *(resp. the augmented Goh matrix* $\overline{G}(t)$*) at time* $t \in [0, T]$ *is the* $m \times m$ *skew-symmetric matrix given by*

$$G(t) = \big(h_{ij}(t)\big)_{1 \le i, j \le m} \tag{5}$$

*(resp.* $\overline{G}(t) = \big(h_{ij}(t)\big)_{0 \le i, j \le m}$*).*

If moreover $m$ is odd, the determinant of $\overline{G}(t)$ is the square of a polynomial $\overline{P}(t)$ in the $h_{ij}(t)$ with degree $(m+1)/2$, called the *Pfaffian*. Along the extremal, $\overline{P}(t) = 0$, and, after differentiation, one gets

$$\{\overline{P}, h_0\}(t) + \sum_{i=1}^{m} u_j(t) \{\overline{P}, h_j\}(t) = 0. \tag{6}$$

Define the $(m+2) \times (m+1)$ matrix $\widetilde{G}(t)$ as $\overline{G}(t)$ augmented with the row $(\{\overline{P}, h_j\}(t))_{0 \le j \le m}$.

If $m$ is even and the Goh matrix $G(t)$ at time $t$ is invertible (resp. if $m$ is odd and $\widetilde{G}(t)$ is of rank $m$), then we can deduce from equations (4) and (6) the singular control $u(t)$. Let us then set the following definition.

DEFINITION 8. *If $m$ is even (resp. odd), a singular trajectory is said to be of minimal order if it admits an abnormal extremal lift along which rank $G(t) = m$ (resp. rank $\widetilde{G}(t) = m$) almost everywhere on $[0, T]$.*

On the opposite, for arbitrary $m$, a singular trajectory is said to be a *Goh trajectory* if it admits an abnormal extremal lift along which the Goh matrix is identically equal to 0.

THEOREM 2. *Let $m$ be a positive integer with $1 \leq m < n$ and $\mathcal{F}_m$ be the set of $(m+1)$-tuples of linearly independent smooth vector fields on $M$, endowed with the $C^\infty$ Whitney topology. There exists an open set $O_m$ dense in $\mathcal{F}_m$ so that, for all $(m+1)$-tuple $(f_0, \ldots, f_m)$ of $O_m$, every singular trajectory of the associated control-affine system*

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)),$$

*is of minimal order and of corank one. In addition, the complementary of $O_m$ in $\mathcal{F}_m$ is of infinite codimension.*

COROLLARY 1. *With the notations of Theorem 2 and if $m \geq 2$, there exists an open set $O_m$ dense in $\mathcal{F}_m$ so that every control-affine system defined with an $(m+1)$-tuple of $O_m$ does not admit Goh singular trajectories.*

Let us now consider the optimal control problem (1), (2).

PROPOSITION 2. *Let $m$ be a positive integer with $m < n$. Then, there exists an open set $O_m$ dense in $\mathcal{F}_m$ so that every nontrivial singular trajectory of a control-affine system defined by an $(m+1)$-tuple of $O_m$ is strict.*

Corollary 1 together with Proposition 2 yield the next corollary.

COROLLARY 2. *Let $m$ be a positive integer with $2 \leq m < n$. There exists an open set $O_m$ dense in $\mathcal{F}_m$ so that every control-affine system defined with an $(m+1)$-tuple of $O_m$ does not admit minimizing singular trajectories.*

*If in addition the vector fields of the $(m+1)$-tuple in $O_m$ are analytic, then the associated value function is continuous and subanalytic on its domain of definition.*

## 4. Application to Hamilton-Jacobi type equations

Results of this section are contained in [22].

### 4.1. Introduction: viscosity solutions

In the 80's Crandall and Lions [9] introduced the concept of viscosity solution in order to ensure uniqueness of solutions of Hamilton-Jacobi equations. Existence of viscosity solutions was also established under similar assumptions. A general definition of a viscosity solution of a first-order Hamilton-Jacobi equation is the following.

Let $\Omega$ be an open set in $\mathbb{R}^n$ and $H$ be a continuous function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$, called *Hamiltonian*, and $g$ be a continuous function on $\partial\Omega$. Consider the first-order Hamilton-Jacobi equation on $\Omega$

$$H(x, \upsilon(x), \nabla\upsilon(x)) = 0. \tag{7}$$

We first recall the notion of sub- and super-differential.

DEFINITION 9. *Let $\upsilon$ be a scalar function on $\Omega$. The super-differential at a point $x \in \Omega$ is defined as*

$$D^+\upsilon(x) = \{p \in \mathbb{R}^n \mid \limsup_{y \to x} \frac{\upsilon(y) - \upsilon(x) - \langle p, y - x\rangle}{\|y - x\|} \leq 0\}.$$

*Similarly, the sub-differential at $x$ is*

$$D^-\upsilon(x) = \{p \in \mathbb{R}^n \mid \liminf_{y \to x} \frac{\upsilon(y) - \upsilon(x) - \langle p, y - x\rangle}{\|y - x\|} \geq 0\}.$$

We can now define the concept of viscosity solution introduced in [9].

DEFINITION 10. *Let $\upsilon$ be a continuous function on $\Omega$. The function $\upsilon$ is a viscosity sub-solution of equation (7) if*

$$\forall x \in \Omega \quad \forall p \in D^+\upsilon(x) \quad H(x, \upsilon(x), p) \leq 0.$$

*Similarly, $\upsilon$ is a viscosity super-solution of equation (7) si*

$$\forall x \in \Omega \quad \forall p \in D^-\upsilon(x) \quad H(x, \upsilon(x), p) \geq 0.$$

*Finally, $\upsilon$ is a viscosity solution of equation (7) if it is both a sub-solution and a super-solution.*

This concept is adapted to get existence and uniqueness results, in particular for *Dirichlet problems* of the type

$$\begin{aligned} H(x, \upsilon(x), \nabla\upsilon(x)) &= 0 \ \text{ in } \Omega, \\ \upsilon_{|\partial\Omega} &= g, \end{aligned}$$

so as for many other problems (Cauchy problems, second-order equations, ...), see for instance [9, 16, 4, 5, 11]. Literature on this subject is immense.

In the case of analytic Hamilton-Jacobi equations one could however expect these solutions to be more regular. Of course because of possible shocks one cannot expect to get global analytic solutions. For example in the case of the *eikonal equation*

$$\begin{aligned} \|\nabla\upsilon(x)\|^2 &= 1 \ \text{ in } \Omega, \\ \upsilon_{|\partial\Omega} &= 0, \end{aligned}$$

on a bounded analytic open set $\Omega \subset \mathbb{R}^n$, one can easily see that the unique viscosity solution is

$$v(x) = d(x, \partial\Omega).$$

Of course this function $u$ is not analytic on $\Omega$, due to intersection of characteristic curves (concerning the method of characteristics we refer the reader to the previously cited references). Anyway the function $v$ is, in a sense, "analytic by parts". The right concept in order to describe such objects happens to be the concept of *subanalyticity*.

### 4.2. Subanalytic regularity of viscosity solutions of certain Hamilton-Jacobi equations

THEOREM 3. *Let $\Omega$ be a bounded subanalytic open subset of $\mathbb{R}^n$, $c > 0$ be fixed, and $f_0, \dots, f_m$ be analytic vector fields on $\overline{\Omega}$. For all $x \in \Omega$ and $p \in \mathbb{R}^n$ set*

$$H(x, p) = -\langle p, f_0(x)\rangle + \frac{1}{4}\sum_{i=1}^{m}\langle p, f_i(x)\rangle^2 - c.$$

*Let $\Sigma = \partial\Omega$ and $g$ be a subanalytic function on $\Sigma$. For all $x \in \overline{\Omega}$, consider the optimal control problem of steering $x$ to $\Sigma$ for the affine control system*

$$\dot{x}_u(t) = f_0(x_u(t)) + \sum_{i=1}^{m} u_i(t) f_i(x_u(t)), \tag{8}$$

*and the cost*

$$C(u) = \int_0^{t(x,u)} \Big(\sum_{i=1}^{m} u_i(s)^2 + c\Big)\, ds + g(x_u(t(x,u))), \tag{9}$$

*where $t(x, u)$ is the infimum of times $t$ such that the solution $x_u(.)$ of the control system (8) associated to the control $u$ steers the point $x \in \overline{\Omega}$ to $\Sigma$ in time $t$. We make the following assumptions.*

(10) *• The boundary $\Sigma$ is accessible from $\Omega$, i.e. for any $x \in \Omega$ there exists a time $t$ and a control on $[0, t]$ such that the solution of system (8) associated to this control and starting from $x$ at time $0$, joins $\Sigma$ in time $t$.*

(11) *• There exists no singular minimizing trajectory of the control system (8) for the cost (9), steering $\Omega$ to $\Sigma$.*

*Let $S(x)$ denote the value function associated to the optimal control problem (8), (9). Namely, if $\mathcal{S}$ denotes the set of solutions $(u(.), x(.))$ of (8) defined on various intervals $[0, t_1]$, such that $x(0) \in \overline{\Omega}$ and $x(t_1) \in \Sigma$, one has, for all $x \in \overline{\Omega}$,*

$$S(x) = \inf\{C(u) \mid (u(.), x_u(.)) \in \mathcal{S},\ x_u(0) = x\}. \tag{12}$$

*For all $x, z \in \Sigma$ define*

$$L(x,z) = \inf\left\{ \int_0^t \Big(\sum_{i=1}^m u_i(s)^2 + c\Big)\, ds \mid x_u(.) \in \mathcal{S},\ x_u(0) = x,\ x_u(t) = z,\right. \tag{13}$$
$$\left. \forall s \in [0,t] \quad x_u(s) \in \overline{\Omega}\right\},$$

*and assume that g satisfies the compatibility condition*

$$\forall x, z \in \Sigma \quad g(x) - g(z) \leq L(x,z). \tag{14}$$

*Then S is well defined on $\overline{\Omega}$, is continuous and subanalytic on $\Omega$, and is a viscosity solution of the Dirichlet problem*

$$H(x, \nabla S(x)) = 0 \text{ on } \Omega,\ S_{|\Sigma} = g. \tag{15}$$

REMARK 7. Denote by $\mathcal{F}_m$ the set of $(m+1)$-tuples of linearly independent vector fields $(f_0, \ldots, f_m)$, endowed with the $C^\infty$ Whitney topology. If $2 \leq m < n$, there exists an open dense subset of $\mathcal{F}_m$ such that any affine control system associated to a (m+1)-tuple of this subset admits no singular minimizing trajectory. This is indeed an obvious adaptation of Corollary 2.

REMARK 8. The compatibility condition (14) is a classical condition for the existence of viscosity solutions. It is automatically satisfied if $g = 0$, that is for Dirichlet problems.

Concerning uniqueness and regularity of $S$ on the whole $\overline{\Omega}$, we have the following results.

PROPOSITION 3. *Under the assumptions of Theorem 3, if moreover*

$$\forall x \in \Sigma \quad \mathrm{Lie}\,(f_1(x), \ldots, f_m(x)) = \mathbb{R}^n, \tag{16}$$

*then S is continuous on $\overline{\Omega}$. As a consequence S is the unique viscosity solution of the Dirichlet problem (15).*

For all $x \in \mathbb{R}^n$ set

$$\Delta(x) = \mathrm{Span}\{f_1(x), \ldots, f_m(x)\},$$
$$\Delta^2(x) = \Delta(x) + \mathrm{Span}\{[f_i, f_j](x),\ 1 \leq i < j \leq m\}.$$

The $m$-tuple $(f_1, \ldots, f_m)$ is said to be *medium-fat* at $x$ if for any vector field $X \in \Delta(x) \setminus \{0\}$ there holds

$$\mathbb{R}^n = \Delta^2(x) + \mathrm{Span}\{[X, [f_i, f_j]](x),\ 1 \leq i, j \leq m\}.$$

Proposition 4. *Under the assumptions of Theorem 3, if the m-tuple of vector fields* $(f_1, \ldots, f_m)$ *is moreover medium-fat at all points of* $\Sigma$*, and if the compatibility inequality (14) is strict, then S is subanalytic on* $\overline{\Omega}$.

Example 1. Let $\Omega$ be a subanalytic bounded open subset of $\mathbb{R}^3$ and $\Sigma = \partial\Omega$. For all $c > 0$ there exists a unique viscosity solution $S$ of the Dirichlet problem

$$-\frac{\partial v}{\partial x_1} + \frac{1}{4}\left(\frac{\partial v}{\partial x_1} + x_2\frac{\partial v}{\partial x_3}\right)^2 + \frac{1}{4}\left(\frac{\partial v}{\partial x_2} - x_1\frac{\partial v}{\partial x_3}\right)^2 - c = 0 \text{ in } \Omega,\ \mathrm{v}_{|\Sigma} = 0,$$

which is continuous and subanalytic on $\overline{\Omega}$. It is indeed an application of Theorem 3, Propositions 3 and 4, with the vector fields

$$f_0 = \frac{\partial}{\partial x_1},\ f_1 = \frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_3},\ f_2 = \frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial x_3}.$$

We then investigate Cauchy-Dirichlet problems on a subset $\Omega$ of $\mathbb{R}^n$.

Theorem 4. *Let* $\Omega$ *be a bounded analytic open subset of* $\mathbb{R}^n$*. We consider the Hamiltonian function on* $\Omega \times \mathbb{R}^n$ *defined as*

$$H(x, p) = \langle p, f_0(x)\rangle + \sum_{i=1}^{m}\langle p, f_i(x)\rangle^2,$$

*where* $f_0, \ldots, f_m$ *are analytic vector fields on* $\mathbb{R}^n$*. Let* $\Sigma = \partial\Omega$*,* $T > 0$ *be fixed and g be a subanalytic function on* $[0, T] \times \Sigma$*. Consider the affine control system*

$$x_u'(s) = f_0(x_u(s)) + \sum_{i=1}^{m} u_i(s) f_i(x_u(s)) \tag{17}$$

*and the cost*

$$C(t, u) = \int_0^t \sum_{i=1}^{m} u_i(s)^2 ds + g(t, x_u(t)), \tag{18}$$

*and assume that for all* $t \in ]0, T]$*:*

1. *The boundary* $\Sigma$ *is accessible from* $\Omega$ *in time t, i.e. for all time* $t \in ]0, T]$ *and all* $x \in \Omega$ *there exists a control u on* $[0, t]$ *such that the associated solution* $x_u(.)$ *of (17) satisfies* $x_u(0) = x$ *and* $x_u(t) \in \Sigma$.
2. *There exists no singular minimizing trajectory for the optimal control problem (17), (18), steering* $\Omega$ *to* $\Sigma$ *in time t.*

*For all* $t \in ]0, T]$ *and* $x \in \overline{\Omega}$*, let* $S(t, x)$ *be the value function associated to the optimal control problem of determining a trajectory solution of the control system (17) on* $[0, t]$, *minimizing the cost (18), and such that* $x_u(0) = x$ *and* $x_u(t) \in \Sigma$*. Namely, if* $\mathcal{S}$ *denotes the set of couples* $(u(.), x_u(.))$ *solutions of the control system (17), one has*

$$S(t, x) = \inf \Big\{ C(t, u) \mid (x_u(.), u(.)) \in \mathcal{S},\ x_u(0) = x,\ x_u(t) \in \Sigma \Big\}.$$

*For all* $s, t \in [0, T]$ *such that* $s < t$ *and all* $x \in \Sigma$, $y \in \overline{\Omega}$*, set*

$$S(t, x, s, y) = \inf \left\{ \int_s^t \sum_{i=1}^m u_i(\tau)^2 d\tau \mid (x_u(.), u(.)) \in \mathcal{S},\ x_u(s) = y,\ x_u(t) = x \right\},$$

*and assume that* $g$ *satisfies the compatibility condition*

$$\text{(19)} \qquad \begin{aligned} \forall (s, y) \in \big(\{0\} \times \overline{\Omega}\big) \cup ([0, T[ \times \partial\Omega) \quad \forall (t, x) \in ]s, T] \times \partial\Omega \\ g(t, x) - g(s, y) \leq S(t, x, s, y). \end{aligned}$$

*Then* $S$ *is continuous and subanalytic on* $]0, T] \times \Omega$*, and is a viscosity solution of the Cauchy-Dirichlet problem*

$$\text{(20)} \qquad \begin{aligned} \frac{\partial S}{\partial t} + H(x, \frac{\partial S}{\partial x}) = 0 \quad & \text{in } ]0, \mathrm{T}] \times \Omega, \\ S = g \quad & \text{on } ]0, \mathrm{T}] \times \partial\Omega, \\ \lim_{t \to 0} S(t, x) = g(0, x) \quad & \text{in } \Omega. \end{aligned}$$

REMARK 9. Remark 7 on genericity holds again here.

PROPOSITION 5. *Under the assumptions of Theorem 4, if there holds moreover*

$$\text{(21)} \qquad Lie\ (f_1(x), \ldots, f_m(x)) = \mathbb{R}^n$$

*for all* $x \in \overline{\Omega}$*, then* $S$ *is continuous on* $[0, T] \times \overline{\Omega}$*. As a consequence,* $S$ *is the unique viscosity solution of the Cauchy-Dirichlet problem (20).*

PROPOSITION 6. *Under the assumptions of Theorem 4, if the* $m$*-tuple of vector fields* $(f_1, \ldots, f_m)$ *is moreover medium-fat on* $\overline{\Omega}$*, then* $S$ *is subanalytic on* $[0, T] \times \overline{\Omega}$*.*

REMARK 10. All previous results extend to more general situations. Indeed, under some general conditions on the control system and on the cost, the associated value function is subanalytic; the main assumption is the absence of singular minimizing trajectories. On the other part, it is well-known that, under some general assumptions, the previous value function is a viscosity solution of the Hamilton-Jacobi equation

$$\text{(22)} \qquad \frac{\partial v}{\partial t} + H_1(x, \frac{\partial v}{\partial x}) = 0,$$

where $H_1(x, p) = \max_u H(x, p, u)$. Notice that all comments here also hold in the Dirichlet case where the value function does not depend on $t$. Finally, in [14, 18], the authors prove that under general assumptions on the Hamiltonian $H_1$, there exists an optimal control problem such that the associated value function is exactly the viscosity solution of (22) (*inverse optimal control problem*). Their proof can be quite readily extended to the subanalytic case.

Gathering these facts leads to a general statement ensuring that the unique viscosity solution of an Hamilton-Jacobi equation is subanalytic, provided that the associated optimal control problem do not admit any singular minimizing trajectory. However the proof of the inverse optimal control problem, mainly based on Kakutani Fixed Point Theorem, is not constructive. Hence in general it may be difficult to check whether or not an underlying optimal control problem admits some singular minimizing trajectories.

**References**

[1] AGRACHEV A., *Compactness for sub-Riemannian length minimizers and subanalyticity*, in: "Control Theory and its Applications", Rend. Sem. Mat. Univ. Polit. Torino **56** (4) (1999), 1–12.

[2] AGRACHEV A. AND GAUTHIER J.P., *On subanalyticity of Carnot-Carathéodory distances*, Ann. Inst. H. Poincaré Anal. Non Linéaire **18** (3) (2001).

[3] AGRACHEV A. AND SARYCHEV A., *Sub-Riemannian metrics: minimality of singular geodesics versus subanalyticity*, ESAIM:COCV **4** (1999), 377–403.

[4] BARDI M. AND CAPUZZO-DOLCETTA I., *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Birkhäuser, Boston 1997.

[5] BARLES G., *Solutions de viscosité des équations de Hamilton-Jacobi*, Math. & Appl. **17**, Springer-Verlag 1994.

[6] BONNARD B. AND TRÉLAT E., *On the role of singular minimizers in sub-Riemannian geometry*, Ann. Fac. Sci. Toulouse Série 6 **X** (3) (2001), 405–491.

[7] CHITOUR Y., JEAN F. AND TRÉLAT E., *Propriétés génériques des trajectoires singulières*, Comptes Rendus Math. **337** (1) (2003), 49–52.

[8] CHITOUR Y., JEAN F. AND TRÉLAT E., *Genericity properties for singular trajectories*, to appear in Journal of Differential Geometry.

[9] CRANDALL M.G. AND LIONS P.L., *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. **277** (1983), 1–42.

[10] VAN DEN DRIES L. AND MILLER C., *Geometric categories and o-minimal structures*, Duke Math. Journal **84** (2) (1996).

[11] EVANS L.C., *Partial differential equations*, American Math. Soc., Providence R.I. 1998.

[12] HARDT R.M., *Stratifi cation of real analytic mappings and images*, Invent. Math. **28** (1975).

[13] HIRONAKA H., *Subanalytic sets*, in: "Number theory, algebraic geometry and commutative algebra, in honor of Y. Akizuki", Tokyo 1973.

[14] ISHII H., *On representation of solutions of Hamilton-Jacobi equations with convex Hamiltonians*, Lecture Notes in Num. Appl. Anal. **8** (1985), 15–52.

[15] JACQUET S., *Subanalyticity of the sub-Riemannian distance*, J. of Dynamical and Control Systems **5** (3) (1999), 303–328.

[16] LIONS P.L., *Generalized solutions of Hamilton-Jacobi equations*, Pitman, 1982.

[17] PONTRYAGIN L. ET AL., *Théorie mathématique des processus optimaux*, Eds Mir, Moscou 1974.

[18] RAMPAZZO F., *Faithful representations for convex Hamilton-Jacobi equations*, preprint, Univ. di Padova.

[19] TAMM M., *Subanalytic sets in the calculus of variations*, Acta Math. **146** (1981).

[20] TRÉLAT E., *Some properties of the value function and its level sets for affine control systems with quadratic cost*, J. of Dyn. and Cont. Syst. **6** (4) (2000), 511–541.

[21] TRÉLAT E., *Etude asymptotique et transcendance de la fonction valeur en contrôle optimal; catégorie log-exp en géométrie sous-Riemannienne dans le cas Martinet*, Ph.D. Thesis, Univ. de Bourgogne, 2000.

[22] TRÉLAT E., *Global subanalytic solutions of Hamilton-Jacobi type equations*, to appear in Annales de l'Institut Henri Poincaré, Analyse non linéaire.


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