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The Kuratowski-Mrówka characterization and weak forms of compactness

La caracterización de Kuratowski-Mrówka y formas débiles de compacidad

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Abstract. For cardinals $\kappa > \aleph_0$, characterizations of the Kuratowski-Mrówka type of initial κ-compactness and final κ-compactness are given. Moreover, a categorical characterization of κ-compactness is given in terms of a closure operator depending on an ultrafilter over κ.

Key words and phrases. Kuratowski-Mrówka characterization of compact spaces, closure operator, weak form of compactness.

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Resumen. Se presentan caracterizaciones del tipo Kuratowski-Mrówka de la κ-compacidad inicial y de la κ-compacidad final, donde $\kappa > \aleph_0$ es un cardinal. Además, se presenta una caracterización categórica de la κ-compacidad, en términos de un operador de clausura que depende de un ultrafiltro sobre κ.

Palabras y frases clave. Caracterización de Kuratowski-Mrówka de los espacios compactos, operador de clausura, formas débiles de compacidad.

1. Introduction

The Kuratowski-Mrówka characterization of compact spaces as those spaces X that satisfy the condition that the second projection $\pi_2 : X \times Y \longrightarrow Y$ is a closed map, for each space Y, gave rise to a categorical approach to compactness (cf. [3], [4], [5] and [7], among others).

The generality of this approach leads to a wide range of applications. In particular, by defining different closure operators in the category of topological spaces and continuous functions, alternative notions of compactness are obtained, some of them well known, like sequential compactness or countable compactness. This points out the relevance of this categorical approach.

This paper describes a closure operator that induces $\mathcal{F}$-compactness, where $\mathcal{F}$ is an ultrafilter over a fixed set of indices. As a consequence and by means of results due to X. Caicedo (cf. [2]), a characterization of the Kuratowski-Mrówka type is found, for certain forms of weak compactness.

2. The Kuratowski-Mrówka characterization

The present work revolves around the Kuratowski-Mrówka characterization of compact topological spaces.

Note in the first place that each filter $\mathcal{F}$ over a set X defines a topology over $X \cup \{\varpi\}$, where $\varpi \notin X$, as follows: if $x \neq \varpi$, the neighborhood filter of x is $\mathcal{V}(x) = \{V \subset X \cup \{\varpi\} : x \in V\}$ and the neighborhood filter of ϖ is $\mathcal{V}(\varpi) = \{F \cup \{\varpi\} : F \in \mathcal{F}\}$ (cf. [1]). Denote by $X_{\mathcal{F}}$ the set $X \cup \{\varpi\}$ endowed with this topology.

The spaces $X_{\mathcal{F}}$, where $\mathcal{F}$ is a filter over the topological space X, play a crucial role in the characterization of weak forms of compactness and allow to simplify the Kuratowski-Mrówka characterization, as it will be seen below.

Suppose that $\mathcal{U}$ is a non convergent ultrafilter over a topological space X. Since $\mathcal{U}$ doesn't have any limit point, for each $x \in X$, there exists an open neighborhood V_x of x such that $V_x \notin \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, then $X \smallsetminus V_x \in \mathcal{U}$, for each $x \in X$. Consider the space $X_{\mathcal{U}}$ and the set $\Delta_0 = \{(x, x) \in X \times X_{\mathcal{U}} : x \in X\}$. For each $x \in X$ the set $V_x \times (X \smallsetminus V_x \bigcup \{\varpi\})$ is a neighborhood of (x, ϖ) in $X \times X_{\mathcal{U}}$, in a such way that $V_x \times (X \smallsetminus V_x \bigcup \{\varpi\}) \bigcap \Delta_0 = \emptyset$, then $(x, \varpi) \notin \overline{\Delta_0}$ for each $x \in X$. This implies that $\pi_2(\overline{\Delta_0}) = X$ and since X is not a closed subset of $X_{\mathcal{U}}$, because $\varpi \in \overline{X}$, it follows that $\pi_2 : X \times X_{\mathcal{U}} \longrightarrow X_{\mathcal{U}}$ is not a closed map.

From the above arguments, one obtains:

Proposition 1. *For a topological space X, the following are equivalent:*

(1) X is compact,

(2) The map $\pi_2 : X \times Y \longrightarrow Y$ is closed, for each topological space Y.

(3) The map $\pi_2 : X \times X_{\mathcal{F}} \longrightarrow X_{\mathcal{F}}$ is closed, for each filter $\mathcal{F}$ over X.

(4) The map $\pi_2 : X \times X_{\mathcal{U}} \longrightarrow X_{\mathcal{U}}$ is closed, for each ultrafilter $\mathcal{U}$ over X.

3. $[\lambda, \kappa]$-compact spaces

Intermediate forms of compactness, other than countable compactness, has been considerer by many authors. Characterizations of the Kuratowski-Mrówka type for some of these notions are established in this section.

Definition 1. *Let $\lambda \leq \kappa$ be infinite cardinals. A topological space X is said to be $[\lambda, \kappa]$-compact if every cover of X consisting of at most κ open sets, has a subcover whose cardinality is smaller than λ.*

Equivalently, X is $[\lambda, \kappa]$-compact, if and only if, if every intersection consisting of less than λ sets of a family $\{K_\alpha\}_{\alpha<\kappa}$ of closed subsets of X is not empty, then $\bigcap_{\alpha<\kappa} K_\alpha \neq \emptyset$. Countable compactness is an example of $[\lambda, \kappa]$-compactness, for $\lambda = \kappa = \aleph_0$.

For the sake of simplicity, the $[\aleph_0, \kappa]$-compact spaces will be referred as *κ-compact spaces.*

Definition 2. *A topological space X is $[\lambda, \infty]$-compact or* finally λ-compact *if it is $[\lambda, \kappa]$-compact for each cardinal κ, $\lambda \leq \kappa$. In other words, X is finally λ-compact if every open cover of X has a subcover whose cardinal is less than λ.*

Compactness and the Lindelöf property are examples of finally λ-compactness. In the first case, $\lambda = \aleph_0$, and in the second, $\lambda = \aleph_1$.

First we focus on the κ-compact spaces.

Definition 3. *Let κ be an infinite cardinal. A net $(x_\gamma)_{\gamma\in\Gamma}$ over a set X is an* κ-net *if $|\Gamma| \leq \kappa$.*

The next proposition establishes a characterization of the Kuratowski--Mrówka type for the κ-compact spaces.

Proposition 2. *A topological space X is κ-compact, if and only if, for each filter $\mathcal{F}$ associated to a κ-net $(x_\gamma)_{\gamma\in\Gamma}$, the second projection $\pi_2 : X \times X_{\mathcal{F}} \longrightarrow X_{\mathcal{F}}$ is a closed map .*

Proof. First suppose that X is κ-compact and let $\mathcal{F}$ be the filter associated to the κ-net $(x_\gamma)_{\gamma\in\Gamma}$. Let $M \subset X \times X_{\mathcal{F}}$ be a closed set and $y \in X_{\mathcal{F}} \setminus \pi_2(M)$. If $y \neq \varpi$, then $\{y\}$ is a neighborhood of y contained in $X_{\mathcal{F}} \setminus \pi_2(M)$. Now suppose that $y = \varpi$; for each $x \in X$ there exist an open neighborhood V_x of x and $\gamma_x \in \Gamma$, such that $V_x \times \{x_\gamma : \gamma \geq \gamma_x\} \subset (X \times X_{\mathcal{F}}) \setminus M$. For each $\gamma \in \Gamma$ consider the set $V_\gamma = \bigcup\{V_x : V_x \times \{x_\delta : \delta \geq \gamma\} \subset (X \times X_{\mathcal{F}}) \setminus M\}$. The family $\{V_\gamma\}_{\gamma\in\Gamma}$ is an open cover of X and its cardinal is less or equal than κ. The κ-compactness of X implies the existence of a finite subset $\Gamma' = \{\gamma_1, ..., \gamma_n\}$ of Γ, such that $\bigcup_{i=1,...,n} V_{\gamma_i} = X$. Let $\gamma_0 \in \Gamma$ be such that $\gamma_i \leq \gamma_0$, for each $i = 1, .., n$. It follows that $\{\varpi\} \cup \{x_\gamma : \gamma \geq \gamma_0\}$ is a neighborhood of ϖ contained in $X_{\mathcal{F}} \setminus \pi_2(M)$. One concludes that $X_{\mathcal{F}} \setminus \pi_2(M)$ is an open set and thus $\pi_2(M)$ is closed.

Now suppose that, for each filter $\mathcal{F}$ associated to a κ-net in X, the second projection $\pi_2 : X \times X_{\mathcal{F}} \longrightarrow X_{\mathcal{F}}$ is closed and suppose that $\{V_\lambda\}_{\lambda\in\Lambda}$ is an open cover of X whose cardinal is less than or equal to κ and has no finite subcover. The family Γ of all finite subsets of Λ has a cardinal less or equal than κ and is directed by the relation $\leq$, defined by $\gamma_1 \leq \gamma_2$ if and only if $\gamma_1 \subset \gamma_2$. Now, for each $\gamma \in \Gamma$, we pick $x_\gamma \in X$ such that $x_\gamma \notin \bigcup_{\lambda\in\gamma} V_\lambda$ and denote by $\mathcal{F}$ the filter associated to the κ-net $(x_\gamma)_{\gamma\in\Gamma}$. Consider the subset $M = \{(x_\gamma, x_\gamma) : \gamma \in \Gamma\}$

of $X \times X_{\mathcal{F}}$. If $x \in V_\lambda$ then, for each $\gamma \in \Gamma$ with $\gamma \geq \gamma_0 = \{\lambda\}$, it follows that $x_\gamma \notin V_\lambda$; this means that $V_\lambda \times (\{x_\gamma : \gamma \geq \gamma_0\} \cup \{\varpi\}) \subset (X \times X_{\mathcal{F}}) \smallsetminus \overline{M}$, that is, $(x, \varpi) \notin \overline{M}$, therefore $\varpi \notin \pi_2(\overline{M})$. But, it is apparent that $\varpi \in \overline{\pi_2(\overline{M})}$. This proves that $\pi_2 : X \times X_{\mathcal{F}} \longrightarrow X_{\mathcal{F}}$ is not a closed map, contradicting the hypothesis. We conclude that $\{V_\lambda\}_{\lambda\in\Lambda}$ contains a finite subcover, thus X is κ-compact. ☑

In the particular case of countably compact spaces, one has the next result.

Corollary 1. *A topological space X is countably compact, if and only if, for each elementary filter $\mathcal{F}$ associated to a sequence, the second projection $\pi_2 : X \times X_{\mathcal{F}} \longrightarrow X_{\mathcal{F}}$ is a closed map.*

Now we focus on the finally λ-compact spaces.

Definition 4. *Let κ be an infinite cardinal. A net $(x_\gamma)_{\gamma\in\Gamma}$ over a set X is a* final κ-net *if every subset Γ' of Γ such that $|\Gamma'| < \kappa$ has a upper bound in Γ.*

Arguing in a similar way as in the proof of Proposition 2, one obtains the following result.

Proposition 3. *A topological space X is finally κ-compact, if and only if, for every filter $\mathcal{F}$ associated to a final κ-net $(x_\gamma)_{\gamma\in\Gamma}$, the second projection $\pi_2 : X \times X_{\mathcal{F}} \longrightarrow X_{\mathcal{F}}$ is a closed map.*

Proof. First suppose that X is finally κ-compact and let $\mathcal{F}$ the associated filter to the final κ-net $(x_\gamma)_{\gamma\in\Gamma}$. Let $M \subset X \times X_{\mathcal{F}}$ be a closed set and $y \in X_{\mathcal{F}} \smallsetminus \pi_2(M)$. If $y \neq \varpi$, then $\{y\}$ is a neighborhood of y contained in $X_{\mathcal{F}} \smallsetminus \pi_2(M)$. Suppose now $y = \varpi$. For each $x \in X$ there exist an open neighborhood V_x of x and $\gamma_x \in \Gamma$, such that $V_x \times \{x_\gamma : \gamma \geq \gamma_x\} \subset (X \times X_{\mathcal{F}}) \smallsetminus M$. For each $\gamma \in \Gamma$ consider the set $V_\gamma = \bigcup\{V_x : V_x \times \{x_\delta : \delta \geq \gamma\} \subset (X \times X_{\mathcal{F}}) \smallsetminus M\}$. The family $\{V_\gamma\}_{\gamma\in\Gamma}$ is an open cover of X and the final κ-compactness of X implies the existence of a subset Γ' of Γ, whose cardinal is less than κ and such that $\bigcup_{\alpha\in\Gamma'} V_\alpha = X$. Let $\gamma_0 \in \Gamma$ be such that $\alpha \leq \gamma_0$, for each $\alpha \in \Gamma'$. Then $\{\varpi\} \cup \{x_\gamma : \gamma \geq \gamma_0\}$ is a neighborhood of ϖ contained in $X_{\mathcal{F}} \smallsetminus \pi_2(M)$. It follows that $X_{\mathcal{F}} \smallsetminus \pi_2(M)$ is an open set, therefore $\pi_2(M)$ is closed.

Now suppose that for each filter $\mathcal{F}$ associated to a final κ-net in X, the second projection $\pi_2 : X \times X_{\mathcal{F}} \longrightarrow X_{\mathcal{F}}$ is closed and suppose that $\{V_\lambda\}_{\lambda\in\Lambda}$ is an open cover of X with no subcover with cardinal less than κ. The family Γ of all subsets of Λ with cardinal less than κ is directed by the relation $\gamma_1 \leq \gamma_2$ if $\gamma_1 \subset \gamma_2$, furthermore if $\Gamma' \subset \Gamma$ has a cardinal less than κ, then $\bigcup_{\alpha\in\Gamma'} \alpha$ is a upper bound of Γ' in Γ. For each $\gamma \in \Gamma$ pick $x_\gamma \in X$ such that $x_\gamma \notin \bigcup_{\lambda\in\gamma} V_\lambda$ and denote by $\mathcal{F}$ the filter associated to the κ-net $(x_\gamma)_{\gamma\in\Gamma}$. Consider the subset $M = \{(x_\gamma, x_\gamma) : \gamma \in \Gamma\}$ of $X \times X_{\mathcal{F}}$. If $x \in V_\lambda$, then, for each $\gamma \in \Gamma$ satisfying $\gamma \geq \gamma_0 = \{\lambda\}$, one has that $x_\gamma \notin V_\lambda$; this means that $V_\lambda \times (\{x_\gamma : \gamma \geq \gamma_0\} \cup$

$\{\varpi\}) \subset (X \times X_{\mathcal{F}}) \smallsetminus M$, that is, $(x, \varpi) \notin \overline{M}$, therefore $\varpi \notin \pi_2(\overline{M})$. On the other hand, it is clear that $\varpi \in \overline{\pi_2(\overline{M})}$. This proves that $\pi_2 : X \times X_{\mathcal{F}} \longrightarrow X_{\mathcal{F}}$ is not a closed map, contradicting our hypothesis. We conclude that $\{V_\lambda\}_{\lambda\in\Lambda}$ contains a subcover whose cardinal is less than κ, thus X is finally κ-compact. ☑

4. Closure operators and compactness: the categorical approach

The notions of closure operator and compactness in a category with a proper system of factorization has been studied by some authors like E. G. Manes in [7] and M. M. Clementino, E. Giuli and W. Tholen in [3], [4]. We consider here these notions restricted to the category of topological spaces and continuous functions.

Definition 5. *A closure operator c in the category $\mathcal{T}op$ of topological spaces and continuous functions is given by a family of functions $c_X : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ $(X \in \mathcal{T}op)$ such that:*

(1) c is extensive, that is, $A \subset c_X(A)$, for every $A \subset X$.

(2) c is monotone, in the sense that if $A \subset B$, then $c_X(A) \subset c_X(B)$, for every $A, B \subset X$.

(3) Every continuous map is c-continuous. That is, if $f : X \longrightarrow Y$ is a continuous function then $f(c_X(A)) \subset c_Y(f(A))$, for each $A \subset X$.

Let c be a closure operator in $\mathcal{T}op$, X and Y be topological spaces. A function $f : X \longrightarrow Y$ is c-preserving, if and only if, $c_Y(f(A)) \subset f(c_X(A))$, for each $A \subset X$.

Finally, a topological space X is c-compact if the second projection $p_Y : X \times Y \longrightarrow Y$ is c-preserving for each space Y.

5. $\mathcal{F}$-compact spaces

In this section a closure operator inducing the $\mathcal{F}$-compactness is described, $\mathcal{F}$ being an ultrafilter over a fixed set of indices. In terms of these operators we will find a characterization of the Kuratowski-Mrówka type for some weak forms of compactness.

Definition 6. *Let $\mathcal{F}$ be an ultrafilter over a set I. A family $\{x_i\}_{i\in I}$ of elements of a topological space X is said to $\mathcal{F}$-converge to a point x in X if, for each open neighborhood V of x, one has that $\{i \in I : x_i \in V\} \in \mathcal{F}$.*

Proposition 4. *Let $\mathcal{F}$ be an ultrafilter over a set I, $\{x_i\}_{i\in I}$ be a family of elements of a topological space X and $\Lambda : I \longrightarrow X$ the function defined by $\Lambda(i) = x_i$. The family $\{x_i\}_{i\in I}$ $\mathcal{F}$-converges to a point $x \in X$, if and only if, the ultrafilter $\Lambda(\mathcal{F})$ over X generated by the family $\{\Lambda(F) : F \in \mathcal{F}\}$ converges to x.*

Proof. Suppose that $\{x_i\}_{i\in I}$ $\mathcal{F}$-converges to $x \in X$ and let V be an open neighborhood of x. Since $\{i \in I : x_i \in V\} \in \mathcal{F}$, it follows that $\Lambda(\{i \in I : x_i \in V\}) \in \Lambda(\mathcal{F})$ and, since $\Lambda(\{i \in I : x_i \in V\}) \subset V$, it follows that $V \in \Lambda(\mathcal{F})$. Then $\Lambda(\mathcal{F})$ converges to x.

Conversely, if V is an open neighborhood of x, there exists $F \in \mathcal{F}$ such that $\Lambda(F) \subset V$. But $F \subset \{i \in I : x_i \in V\}$, thus $\{i \in I : x_i \in V\} \in \mathcal{F}$. This means that the family $\{x_i\}_{i\in I}$ $\mathcal{F}$-converges to x. ☑

Definition 7. *Let $\mathcal{F}$ be an ultrafilter over a set I. A topological space X is said to be $\mathcal{F}$-compact if every family $\{x_i\}_{i\in I}$ of elements of X is $\mathcal{F}$-convergent.*

Every ultrafilter $\mathcal{F}$ over a set I gives rise to a closure operator $c^{\mathcal{F}}$ in the category of topological spaces and continuous functions as follows.

Definition 8. *Let X be a topological space and $A \subset X$. An element x is said to be an element of $c_X^{\mathcal{F}}(A)$, if and only if, there exists an I-family $\{a_i\}_{i\in I}$ in A such that $\mathcal{F}$-converges to x.*

Proposition 5. *The family of functions of the form $c_X^{\mathcal{F}} : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$, $A \longmapsto c_X^{\mathcal{F}}(A)$, where X is a topological space, determines a closure operator in $\mathcal{T}op$.*

Proof. The first two conditions are straightforward. To prove the third, let X and Y be topological spaces and $f : X \longrightarrow Y$ be a continuous function. For $A \subset X$ and $y \in f\left(c_X^{\mathcal{F}}(A)\right)$, consider $x \in c_X^{\mathcal{F}}(A)$ such that $f(x) = y$. If $\{a_i\}_{i\in I}$ is an I-family in A $\mathcal{F}$-converging to x, then $\{f(a_i)\}_{i\in I}$ is an I-family in $f(A)$ $\mathcal{F}$-converging to y. In fact, if V is an open neighborhood of y, then $\{i \in I : a_i \in f^{-1}(V)\} \in \mathcal{F}$, that is, $\{i \in I : f(a_i) \in V\} \in \mathcal{F}$. This shows that $f\left(c_X^{\mathcal{F}}(A)\right) \subset c_Y^{\mathcal{F}}(f(A))$, for every $A \subset X$. Hence every continuous function is $c^{\mathcal{F}}$-continuous. This completes the proof. ☑

The following two lemmas are required in order to elucidate the relation between $\mathcal{F}$-compactness and $c^{\mathcal{F}}$-compactness.

Lemma 1. *If X is a topological space and $A \subset X$, then $c_X^{\mathcal{F}}(A) \subset \overline{A}$.*

Proof. Let $x \in c_X^{\mathcal{F}}(A)$ and $V \in \mathcal{V}(x)$. There exists an I-family $\{a_i\}_{i\in I}$ in A, such that $\mathcal{F}$-converges to x; then $\{i \in I : a_i \in V\} \in \mathcal{F}$, therefore $V \cap A \neq \emptyset$. ☑

Lemma 2. *If X is a topological space and $\{x_i\}_{i\in I}$ is an I-family in X determined by the function $\Lambda : I \longrightarrow X$, $i \longmapsto x_i$, then the family $\{x_i\}_{i\in I}$ $\mathcal{F}$-converges to ϖ in $X_{\mathcal{U}}$, where $\mathcal{U}$ the ultrafilter over X generated by the base $\{\Lambda(F) : F \in \mathcal{F}\}$.*

Proof. A basic open neighborhood of ϖ in $X_{\mathcal{U}}$ is of the form $V = \Lambda(F) \cup \{\varpi\}$, where $F \in \mathcal{F}$, it follows that $F \subset \{i \in I : x_i \in V\}$, hence $\{i \in I : x_i \in V\} \in \mathcal{F}$. ☑

The following proposition asserts that the concept of $\mathcal{F}$-compactness coincides with that of compactness with respect to the closure operator $c^{\mathcal{F}}$.

Proposition 6. *Let $\mathcal{F}$ be an ultrafilter over a set I. A topological space X is $c^{\mathcal{F}}$-compact, if and only if, it is $\mathcal{F}$-compact.*

Proof. Suppose that X is $c^{\mathcal{F}}$-compact and let $\{x_i\}_{i\in I}$ be an I-family in X defined by the function $\Lambda : I \longrightarrow X$, $i \longmapsto x_i$. Denote by $\mathcal{U}$ the ultrafilter over X generated by the base $\{\Lambda(F) : F \in \mathcal{F}\}$. By Proposition 4, it suffices to prove that $\mathcal{U}$ converges.

Consider the subset $\Delta = \{(x, x) : x \in X\}$ of $X \times X_{\mathcal{U}}$. From Lemma 2, it follows that the family $\{x_i\}_{i\in I}$ in $\pi_2(\Delta)$, $\mathcal{F}$-converges to ϖ in $X_{\mathcal{U}}$, then $\varpi \in c^{\mathcal{F}}_{X_{\mathcal{U}}}(\pi_2(\Delta))$, hence $\varpi \in \pi_2\left(c^{\mathcal{F}}_{X\times X_{\mathcal{U}}}(\Delta)\right)$, thus $(z, \varpi) \in c^{\mathcal{F}}_{X\times X_{\mathcal{U}}}(\Delta)$ for some $z \in X$. From Lemma 1, it follows that $(z, \varpi) \in \overline{\Delta}$, therefore if $V \in \mathcal{V}(z)$ and $F \in \mathcal{F}$, then $V \cap \Lambda(F) \neq \emptyset$. This implies that $V \in \mathcal{U}$, thus $\mathcal{U}$ converges to z. This proves that X is $\mathcal{F}$-compact.

Conversely, suppose that X is $\mathcal{F}$-compact and consider a topological space Y, $K \subset X \times Y$ and $y_0 \in c_Y(\pi_2(K))$. There exists a family $(y_i)_{i\in I}$ in $\pi_2(K)$ $\mathcal{F}$-converging to y_0. For each $i \in I$, let $x_i \in X$ such that $(x_i, y_i) \in K$. From the $\mathcal{F}$-compactness of X, it follows that $(x_i)_{i\in I}$ $\mathcal{F}$-converges to a point $x_0 \in X$. The family $\{(x_i, y_i)\}_{i\in I}$ $\mathcal{F}$-converges to (x_0, y_0); in fact: if V is an open neighborhood of x_0 in X and W is an open neighborhood of y_0 in Y, it follows that $\{i \in I : (x_i, y_i) \in V \times W\} = \{i \in I : x_i \in V\} \cap \{i \in I : y_i \in W\}$, thus $\{i \in I : (x_i, y_i) \in V \times W\} \in \mathcal{F}$; then $(x_0, y_0) \in c^{\mathcal{F}}_{X\times Y}(K)$, that is, $y_0 \in \pi_2\left(c^{\mathcal{F}}_{X\times Y}(K)\right)$. This shows that $c^{\mathcal{F}}_Y(\pi_2(K)) \subset \pi_2\left(c^{\mathcal{F}}_{X\times Y}(K)\right)$, for every $K \subset X \times Y$ and consequently that X is $c^{\mathcal{F}}$-compact. ☑

Remark 1. *From the proof of the preceding proposition it also follows that a space X is $c^{\mathcal{F}}$-compact, if and only if, for each $\Lambda : I \longrightarrow X$, the projection $\pi_2 : X \times X_{\mathcal{U}} \longrightarrow X_{\mathcal{U}}$ is $c^{\mathcal{F}}$-preserving, where $\mathcal{U} = \Lambda(\mathcal{F})$.*

6. A closure operator generating κ-compactness

The following definition was introduced by H. J. Keisler in 1964 (cf. [6]) and has since then been widely used in the study of the $[\lambda, \kappa]$-compact spaces.

Definition 9. *An ultrafilter $\mathcal{F}$ over a set I is called (λ, κ)-regular if there exists a family $\mathcal{A} \subset \mathcal{F}$, with $|\mathcal{A}| = \kappa$ and such that if $\mathcal{B} \subset \mathcal{A}$ and $|\mathcal{B}| = \lambda$, then $\bigcap \mathcal{B} = \emptyset$.*

The following results due to X. Caicedo are indispensable in what follows. Consider $\kappa^{<\lambda} := \sum_{\delta<\lambda} \kappa^{\delta}$ (in particular, $\kappa^{<\aleph_0} = \kappa$). A family of topological spaces $\mathbb{T}$ is said to be productively $[\lambda, \kappa]$-compact if the product of any family of spaces in $\mathbb{T}$ is $[\lambda, \kappa]$-compact.

Lemma 3 (X. Caicedo [2]). *Let X be a topological space*

(1) If X is $\mathcal{F}$-compact for a (λ, κ)-regular ultrafilter $\mathcal{F}$, then X is $[\lambda, \kappa]$-compact.

(2) If X is $[\lambda, \kappa]$-compact, then for each $\kappa^{<\lambda}$-family of X there exists an ultrafilter (λ, κ)-regular $\mathcal{F}$ over $\kappa^{<\lambda}$, such that the family $\mathcal{F}$-converges.

Theorem 1 (X. Caicedo [2]). *The following assertions are equivalent:*

(1) $\mathbb{T}$ is productively $[\lambda, \kappa]$-compact.

(2) There exists a (λ, κ)-regular ultrafilter $\mathcal{F}$ over $\kappa^{<\lambda}$, such that every space in $\mathbb{T}$ is $\mathcal{F}$-compact.

From these two last results and from the fact that the κ-compactness is preserved by products, for every $\kappa > \aleph_0$ (cf. [2]), one obtains the following corollary.

Corollary 2. *Let $\kappa > \aleph_0$. There exists an ultrafilter $(\aleph_0, \kappa)$-regular $\mathcal{F}_\kappa$ over κ such that any topological space X is κ-compact if and only if it is $\mathcal{F}_\kappa$-compact.*

Now one can state the following characterization of the Kuratowski-Mrówka type of the κ-compact space, with $\kappa > \aleph_0$.

Theorem 2. *Let $\kappa > \aleph_0$. A topological space X is κ-compact if and only if it is $c^{\mathcal{F}_\kappa}$-compact.*

This means that X is κ-compact, if and only if, for each $\Lambda : \kappa \longrightarrow X$, the map $\pi_2 : X \times X_{\mathcal{U}} \longrightarrow X_{\mathcal{U}}$, where $\mathcal{U} = \Lambda(\mathcal{F}_\kappa)$, is $c^{\mathcal{F}_\kappa}$-preserving.

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