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CONVEX FUNCTIONS AND THE HADAMARD INEQUALITY

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Abstract. The Hadamard inequality is proven without resorting to any properties of the derivative. Only the convexity of the function in a closed interval is needed. Furthermore, if the existence of the integral is assumed, then the convexity requirement is weakened to convexity in the sense of Jensen. Both the Hadamard inequality and a corresponding upper bound are generalized for integrals of the Stieljes type.

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§ 1

Definition 1. *The function* $f : [a,b] \longrightarrow \mathbb{R}$ *is convex in* $[a,b]$ *if for any* $\alpha, \beta \in [0,1], \alpha + \beta = 1$, *and any* $x, y \in [a,b]$,

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y) \tag{1}$$

If $-f$ *is convex then* f *is called concave.*

Theorem 1. *If* f *is convex in* $[a,b]$, *then*

(*i*) f is continuous in (a,b), and
(*ii*) f is bounded in $[a,b]$

Proof. (*i*) The proof of this part is adapted from [3], p.110.
Let $I = [x_0 - \delta, x_0 + \delta] \subset (a,b), \delta > 0$ and $M = \max\{f(x_0 - \delta), f(x_0 + \delta)\}$. For any $x \in (x_0, x_0 + \delta)$ there is $t \in (0,1)$ such that $x = (1-t)x_0 + t(x_0 + \delta)$ and therefore

$$x_0 = \frac{1}{1+t}x + \frac{t}{1+t}(x_0 - \delta).$$

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Because of the convexity of f, it follows from these two equalities and from the definition of M that

$$f(x) \leq (1-t)f(x_0) + tf(x_0+\delta) \leq (1-t)f(x_0) + tM \tag{2}$$

and

$$f(x_0) \leq \frac{1}{1+t}f(x) + \frac{t}{1+t}f(x_0-\delta) \leq \frac{f(x)+tM}{1+t} \tag{3}$$

From (2) and (3), successively, we obtain

$$t(M - f(x_0)) \geq f(x) - f(x_0) \geq t(f(x_0) - M)$$

and recalling that $x - x_0 = \delta\, T$, we get

$$|f(x) - f(x_0)| \leq \big|\, t(M - f(x_0))\big| = \frac{\big(M - F(x_0)\big)|x - x_0|}{\delta} \tag{4}$$

The same proof shows that (4) is also valid if $x \in (x_0 - \delta, x_0)$.
Therefore the incremental quotient

$$\frac{|f(x) - f(x_0)|}{|x - x_0|}$$

is bounded for all $x \in I, x \neq x_0$. This implies the continuity of f at any point x_0 of the open interval (a, b).

(Another proof of the continuity can be found in [5], pp. 82–92)

(ii) To prove boundedness, first remark that f is obviously bounded above because of the definition of convexity in a closed interval.

Next, consider any $p < q$ such that $[p, q] \subset (a, b)$. The continuity proven in part (i) implies that f is bounded in $[p, q]$. It remains to prove that it is bounded below in $[a, p]$ and in $[q, b]$. The proof, which is the same in both cases, is as follows. If $x \in [a, p] \subset [a, q]$ then

$$p = \gamma x + (1-\gamma)q, \quad \text{with} \quad \frac{q-p}{q-a} \leq \gamma \leq 1, \tag{5}$$

as it can be easily seen by noting that if $x = a$ then the equality in (5) becomes

$$\gamma = \frac{q-p}{q-a} > 0$$

Now, from the convexity of f it follows that

$$f(p) \leq \gamma f(x) + (1-\gamma)f(q)$$

and

$$f(x) \geq \frac{1}{\gamma}f(p) + \left(1 - \frac{1}{\gamma}\right) f(q) \tag{6}$$

Since $(q-a)/(q-p) \geq 1/\gamma \geq 1$, the right side of (6) gives a lower bound for f over the closed interval $[a, p]$. $\square$

§ 2

Now we prove the main result

Theorem 2. *If f is convex in $[a,b]$ then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}$$

(The left inequality was established by Hadamard using the monotonicity of the derivative, cf. [4], p. 186)

Proof. The existence of the integral follows from Theorem 1. Next we prove the right inequality, although it is intuitively obvious from the geometric meaning of convexity. Let $x = a(1-t)+bt, 0 \leq t \leq 1$. Then

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,dx &= \int_0^1 f(a(1-t)+bt)\,dt \\ &\leq f(a)\int_0^1 (1-t)\,dt + f(b)\int_0^1 t\,dt = \frac{f(a)+f(b)}{2}\end{aligned}$$

Now we prove the left inequality. Writing the integral in the form

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x)\,dx + \int_{\frac{a+b}{2}}^b f(x)\,dx\right] \tag{7}$$

and making the sustitution $x = a + t(b-a)/2$, the first term in the last bracket becomes

$$\int_a^{\frac{a+b}{2}} f(x)\,dx = \frac{b-a}{2}\int_0^1 f\left(a+\frac{t(b-a)}{2}\right)\,dt$$

and the second term, with $x = b - t(b-a)/2$, can be written as

$$\begin{aligned}\int_{\frac{a+b}{2}}^b f(x)\,dx &= -\frac{b-a}{2}\int_1^0 f\left(b-\frac{t(b-a)}{2}\right)\,dt \\ &= \frac{b-a}{2}\int_0^1 f\left(b-\frac{t(b-a)}{2}\right)\,dt\end{aligned}$$

Therefore, replacing these results in (7) and applying the definition of convexity, it follows that

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,dx &= \frac{1}{2}\int_0^1 \left[f\left(a+\frac{t(b-a)}{2}\right)+f\left(b-\frac{t(b-a)}{2}\right)\right]\,dt \\ &\geq \int_0^1 f\left(\frac{a}{2}+\frac{b}{2}\right)\,dt = f\left(\frac{a+b}{2}\right) \quad \square\end{aligned}$$

Remark 1. It is now of interest to point out that for the proof of the left inequality, definition 1 of convexity has been only used with $\alpha = \beta = \frac{1}{2}$. Therefore, *if the* *existence of the integral is assumed, then the Hadamard inequality is valid for any convex (J) function in the sense of,* that is, for any function f such that $f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\left[f(x)+f(y)\right]$ for all $x, y \in [a, b]$ (cf. [1], pp. 440–441)

It is possible to prove (see for example [2], p. 164) that ordinary convexity and convexity in the sense of Jensen become equivalent if the function is assumed to be continuous.

§ 3

We now extend some of the previous conclussions to integrals of the Stieljes type. First we prove two lemmas.

Lemma 1. *If h is strictly increasing and convex in $[a, b]$ and h^{-1} has domain $I = [h(a), h(b)]$, then h^{-1} is concave and strictly increasing.*

Proof. That h^{-1} is strictly increasing is clear. Now, for any $y_1, y_2 \in I$ there are unique $x_1, x_2, \in [a, b]$ such that $y_i = h(x_i),\ i = 1, 2$. Therefore, if $\alpha, \beta \in [0, 1]$ and $\alpha + \beta = 1$, then

$$h^{-1}(\alpha y_1 + \beta y_2) = h^{-1}\left(\alpha h(x_1) + \beta h(x_2)\right) \geq h^{-1}\left(h(\alpha x_1 + \beta x_2)\right) =$$

$$\alpha x_1 + \beta x_2 = \alpha h^{-1}(y_1) + \beta h^{-1}(y_2) \quad \square$$

Lemma 2. *If h is convex and increasing in $\varphi([a, b])$, with φ convex in $[a, b]$, then the composition $h(\varphi(x))$ is convex.*

Proof. $h(\varphi(\alpha x_1 + \beta x_2)) \leq h(\alpha\varphi(x_1) + \beta\varphi(x_2)) \leq \alpha h(\varphi(x_1) + \beta h(\varphi(x_2))$ for any $x_1, x_2 \in [a, b]$ $\square$

Theorem 3. *If f is convex in $[a, b]$ and if g is concave and strictly increasing in $[a, b]$ with g^{-1} having domain $[g(a),\ g(b)]$, then*

$$\begin{aligned}&\left(g(b) - g(a)\right) f\left(g^{-1}\left(\frac{g(a) + g(b)}{2}\right)\right)\\ &\leq \int_a^b f(x)\, dg(x) \leq \left(g(b) - g(a)\right)\frac{\left(f(a) + f(b)\right)}{2}\end{aligned} \tag{8}$$

Proof. We first observe that the integral is to be understood in the Stieltjes sense. Equivalently, it can be defined by means of the sustitutions $g(x) = t,\ g(a) = t,\ g(b) = t_2$, as

$$\int_a^b f(x)\, dg(x) = \int_{t_1}^{t_2} f\left(g^{-1}(t)\right)\, dt \tag{9}$$

The proof of the Theorem is now as follows: from (9), Lemma 1 and Theorem 1, we get

$$\begin{aligned}\int_a^b f(x)\, dg(x) = \int_{t_1}^{t_2} f\left(g^{-1}(t)\right) dt &\leq (t_2 - t_1)\frac{\left(f\left(g^{-1}(t_1)\right) + f\left(g^{-1}(t_2)\right)\right)}{2}\\ &= \left(g(b) - g(a)\right)\frac{\left(f(a) + f(b)\right)}{2}\end{aligned}$$

and therefore the right hand side inequality in (8) is proven. The proof of the left inequality follows from the Hadamard inequality and from Lemma 2, since the convexity of $f \circ g^{-1}$ gives

$$\int_{t_1}^{t_2} f\left(g^{-1}(t)\right)\, dt \geq (t_1 - t_1)\, f\left(g^{-1}\left(\frac{t_1+t_2}{2}\right)\right)$$
$$= \big(g(b)-g(a)\big)\, f\left(g^{-1}\left(\frac{g(a)+g(b)}{2}\right)\right). \qquad \square$$

Remark 2. The function $g^{-1}\left(\frac{g(a)+g(b)}{2}\right) : I \times I \longrightarrow \mathbb{R}$, $I \subset \mathbb{R}$, is a direct generalization of the arithmetic mean (consider for example $g(x) = x$). Some of its properties are easily proven, such as:

Proposition 1. *If $g : (p, q) \to \mathbb{R}$ has an inverse and they are both increasing, and if $G(a, b) = g^{-1}\left((g(a) + g(b))/2\right)$ for all a, b such that $p < a < b < q$, then*

(i) $a \leq G(a, b) \leq b$

(ii) If, in addition, g is convex, then

$$\frac{a+b}{2} \leq G(a,b) \leq b$$

Corolary. *If f is increasing, then the conclusion of Theorem 3 can be modified to read*

$$\big(g(b)-g(a)\big)\, f(a) \leq \int_a^b f(x)\, dg(x) \leq \frac{\big(g(b)-g(a)\big)\big(f(a)+f(b)\big)}{2}$$
$$\leq \big(g(b)-g(a)\big)\, f(b)$$

§ 4

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