

# Multivariate prediction and matrix Szegő theory

N.H. Bingham

*Mathematics Department, Imperial College London, London SW7 2AZ, UK*  
*e-mail: [n.bingham@ic.ac.uk](mailto:n.bingham@ic.ac.uk); [nick.bingham@btinternet.com](mailto:nick.bingham@btinternet.com)*

**Abstract:** Following the recent survey by the same author of Szegő’s theorem and orthogonal polynomials on the unit circle (OPUC) in the scalar case, we survey the corresponding multivariate prediction theory and matrix OPUC (MOPUC).

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## 1. Introduction

Prediction theory for discrete-time stationary stochastic processes, a topic in time series, has been long studied and has benefited greatly from the recent work on orthogonal polynomials on the unit circle (OPUC); see e.g. Simon’s books [Si1, Si2, Si3] for background and references. Here the partial autocorrelation function (PACF)  $\alpha$  plays an important role, as it provides an unrestricted

parametrization of the relevant spectral measure  $\mu$ ; see e.g. the papers by Inoue [In1, In2, In3], Inoue and Kasahara [InKa1, InKa2] and the survey by Bingham [Bi] for background and references. The PACF is essentially the sequence of Verblunsky coefficients, and the unrestricted parametrization is Verblunsky's theorem, in the language of OPUC; see [Si1], Ch. 1,2.

While this is very satisfactory for univariate time series, one often encounters multivariate time series, particularly in areas such as mathematical finance, where the dimensionality  $\ell$  is the number of risky assets held in a portfolio;  $\ell$  reflects the need to diversify one's portfolio (in the style of Markowitz), and may be large (see e.g. [BiFrKi] for a case in point). Multivariate time series form an important area (see e.g. the books by Hannan [Ha] and Reinsel [Re]). Likewise, multivariate prediction goes back to work by Wiener and Masani [WiMa1, WiMa2, WiMa3], Helson and Lowdenslager [HeLo] of 1957-61 (also Masani [Mas1] – [Mas5]). Just as in the univariate case with OPUC, in the multivariate case the matrix theory of OPUC – MOPUC for short below – is crucially relevant. Following the great stimulus to OPUC provided by Simon's books, MOPUC has recently been extensively developed. Our purpose here is to survey these developments with a view to multivariate prediction theory, providing a matrix sequel to [Bi].

Our focus here is on the finite-dimensional case. For a survey of the infinite-dimensional case (as of 1987), see Makagon and Salehi [MakSa].

## 2. The Kolmogorov Isomorphism Theorem

Throughout, we deal with an  $\ell$ -dimensional vector-valued zero-mean stationary process  $X = (X_n)$  in discrete time. We restrict to the  $L_2$  case, so the components have finite variance, which we may take to be 1 for simplicity. The basic result is as follows.

**Kolmogorov Isomorphism Theorem.** *There is a process  $Y$  on  $T$  with orthogonal increments and a probability measure  $\mu$  on  $T$  with*

- (i)  $X_n = \int e^{in\theta} dY(\theta)$ ;
- (ii)  $E[dY(\theta)^2] = d\mu(\theta)$ .
- (iii) *The autocorrelation function  $\gamma$  then has the spectral representation  $\gamma_n = \int e^{-in\theta} d\mu(\theta)$ .*
- (iv) *One has the Kolmogorov isomorphism between  $\mathcal{H}$  (the time domain) and  $L_2(\mu)$  (the frequency domain) given by*

$$X_t \leftrightarrow e^{it}, \quad (KIT)$$

*for integer  $t$  (as time is discrete).*

*Proof.* Parts (i) and (ii) are Cramér representation of 1942 (Doob X.4; Cramér and Leadbetter 7.5). Part (iii) (Herglotz, 1911) follows from (i) and (ii) (Doob; Brockwell & Davis 4.3). Part (iv) is due to Kolmogorov, 1941. This rests on Stone's theorem of 1932 ([Sto], [RiNa] §137, extended to semigroups in [HiPh]

XXII: spectral representation of groups of unitary transformations of linear operators on Hilbert space); see Doob 636-7, Dunford & Schwartz X.5 for spectral theory.  $\square$

Write  $U$  for the shift  $t \mapsto t+1$ . Then the integer powers  $U^t$  of  $U$  have spectral representation

$$U^t = \int e^{it\theta} Z(d\theta)$$

for a (complex, projection-valued) measure  $Z$  ([RiNa] §109). Since we can write  $X_t = U^t X_0$ , this gives us a spectral representation for  $X = (X_t)$ :

$$X_t = U^t X_0 = \int e^{it\theta} Z(d\theta) X_0$$

(see e.g. Rozanov [Ro2], Th. I.4.2) – *time domain* on the left, *frequency domain* on the right. In terms of  $Z$ , (ii) above becomes

$$E[Z(A)\overline{Z(B)}] = \mu(A \cap B).$$

The fundamental role of the shift (the theme of the classic by Nikolskii [Nik1]) is reflected in the role of shift-invariant subspaces. This is due to Beurling in 1948 in the one-dimensional case [Beu], Lax in 1959 in the finite-dimensional case [Lax2], and Halmos in 1961 in general [Hal]; see e.g. [SzNF] V.3, [FrBh] §3.1 for the Beurling-Lax-Halmos theorem, and the description of invariant subspaces by *inner functions*. For the inner-outer factorisation and the Wold decomposition, see §§4, 5 below.

### 3. Verblunsky's theorem

In the univariate case, the Verblunsky coefficients  $a = (a_n)_{n=1}^\infty$  satisfy  $|a_n| < 1$ , and the bijection

$$a \leftrightarrow \mu$$

of Verblunsky's theorem provides the unrestricted parametrization so useful in statistics and prediction theory; see e.g. Pourahmadi [Pou1, Pou2], [Bi] §2 for background here.

In the  $\ell$ -dimensional case, where one uses MOPUC rather than OPUC, the spectral measure  $\mu$  is now  $\ell \times \ell$  matrix-valued, and corresponds to the  $\ell \times \ell$  covariance matrix by Herglotz's theorem ((iii) above). The Verblunsky coefficients have been studied by Damanik, Pushnitski and Simon [DaPuSi], §3. They show (3.10) that the Verblunsky coefficients are now  $\ell \times \ell$  matrices on the unit circle, satisfying

$$\|a_n\| < 1,$$

that any sequence  $a$  of such matrices can arise in this way, and that the map  $a \leftrightarrow \mu$  is again a bijection (Verblunsky's theorem for MOPUC – [DaPuSi], Th. 3.12). Their method is Bernstein-Szegő approximation ([DaPuSi] §3.6; cf. [Si1] Th. 1.7.8 in the scalar case).

Results of this type go back in the statistical literature to Morf, Vieira and Kailath [MoViKa] in 1978. They showed that the  $a_n$  have singular values  $\alpha_n$  with  $|\alpha_n| \leq 1$ .

The Szegő recursion that leads to OPUC is known in the time-series literature as the *Levinson-Durbin algorithm*. The Levinson-Durbin algorithm was extended to the multivariate case by Whittle [Wh] in 1963, and by Wiggins and Robinson [WigRo] in 1965. In [MoViKa], the authors use the results of [Wh] and [WigRo], which they call the LWR algorithm, and give a normalized version of it. They show that unless the Levinson-Durbin algorithm stops (corresponding to some correlation matrix failing to be positive-definite),  $|\alpha_n| < 1$  for all  $n$ .

#### 4. Szegő's theorem

From a mathematical point of view, the univariate Szegő theorem (see e.g. [Bi] §4) has been extended to the vector-valued case; see e.g. [Nik2] §A4.8.8, §A4.9 for details and references. From a probabilistic point of view, the work of §3 above was continued by Derevyagin, Holtz, Khrushchev and Tyaglov [DeHoKhTy], again using Bernstein-Szegő approximation (§§6, 7). They show (§8) that (using † for the adjoint (conjugate transpose),  $\mu'$  for the density of (the absolutely continuous component of)  $\mu$ , denoted  $w$  below)

$$\log \prod_{n=1}^{\infty} \det(1 - a_n^\dagger a_n) = \int \text{tr} \log \mu' d\theta / 2\pi = \int \text{tr} \log w d\theta / 2\pi$$

for any non-trivial (i.e. of infinite support) matrix-valued probability measure on the unit circle. Call those for which the integral on the right is  $> -\infty$  *Szegő measures*, and the finiteness of the integral *Szegő's condition*, ( $Sz$ ). They deduce that  $\mu$  is a Szegő measure iff

$$\sum \|a_n^\dagger a_n\| < \infty$$

(“ $a \in L_2$ ”). This is Szegő's theorem for MOPUC. They continue (§9) with the matrix version of the Helson-Lowdenslager theorem:

$$\exp \int \frac{1}{\ell} \text{tr} \log \mu' d\theta / 2\pi = \inf_{A,P} \int \frac{1}{\ell} \text{tr} [(A+P)^\dagger d\mu (A+P)],$$

where the infimum is over all matrices  $A$  of determinant 1 and all trigonometric polynomials  $P(e^{i\theta}) = \sum_{k>0} A_k e^{ik\theta}$ .

The Wold decomposition extends from the scalar to the vector case. See e.g. Masani [Mas1, Mas4], Hannan [Ha], and in the context of operator theory, Sz.-Nagy and Foias [SzNF], Ch. 1, Nikolskii [Nik2] §A4.8.8. As in the scalar case, ( $Sz$ ) is the condition that the deterministic component in the Wold decomposition should vanish, leaving only the moving-average component. This is the condition of *non-determinism*, ( $ND$ ) (“( $Sz$ ) = ( $ND$ )”).

As in the scalar case, it is often useful to strengthen the Szegő condition by requiring explicitly that the singular component  $\mu_s$  of the spectral measure  $\mu$

should vanish. This is called the condition of *pure non-determinism*, (*PND*):

$$(PND) = (ND) + \{\mu_s = 0\} = (Sz) + \{\mu_s = 0\}.$$

## 5. Matrix spectral factorizations and matrix Szegő functions

Factorizations are already present in the scalar case. For an analytic function in the Hardy space on the disc, identify the boundary values of the function on the unit circle with the function itself, as usual; then the spectral density  $w$  and the Szegő function  $h$  are related by

$$w = h\bar{h} = |h|^2.$$

Here  $h$  is in the Hardy space  $H_2$ , and is an outer function (see e.g. [Bi] for references and details); one can think of  $h$  as the ‘analytic square root’ of  $w$ . Indeed, Nikolskii ([Nik2], Vol. 1, p. 380) uses the notation

$$h = [w]^{1/2}.$$

In the matrix case, one speaks of the *spectral factorization problem*. The two main traditional approaches to multivariate prediction theory are those of Helson and Lowdenslager [HeLo], based on approximation by polynomials, and of Wiener and Masani [WiMa1] – [WiMa3], based on matrix factorization:

$$W = GG^\dagger, \quad (WM)$$

where again  $G$  is an outer function in the sense of matrix-valued Hardy spaces (see e.g. [Pel3], §13.3 and Appendix 2.3, [RosRov], Ch. 4,5, [Mei] for definitions and details; cf. Stein [Ste], Ch. III).<sup>1</sup> As before, one has an inner-outer factorization, and this corresponds to the previous one on taking determinants ([FrBh], §3.4).

We refer to factorizations of the form  $(WM)$  as *Wiener-Masani factorizations*, and to  $G$ , the matrix analogue of the Szegő function  $h$  in the scalar case ( $\Delta$  in the notation of [Si1]), as a (*matrix*) *Szegő function*. Wiener-Masani factorizations are unique up to unitary equivalence ([WiMa2], Th. 8.12).

The early work is well summarized in Masani [Mas3]. Here one finds:

(i) the Wold decomposition (§4; [WiMa1], Th. 7.11). In the scalar case, the deterministic (time-independent) component of  $X_t$  corresponds to the singular component  $\mu_s$  of the spectral measure, the moving-average component to the absolutely continuous component  $\mu_{ac}$  in the Lebesgue decomposition. In the matrix case, the terms Wold-Zasuhin decomposition and Lebesgue-Cramér decomposition are often used; the correspondence is exact in the full-rank case (below), but not in general [Mas1];

(ii) the Kolmogorov Isomorphism Theorem, between the time domain and the spectral domain (§§7, 8);

<sup>1</sup>Wiener and Masani [WiMa3], Def. 3.5 use the term optimal in place of outer. Rozanov [Ro2], Th. II.4.2 uses the term maximal.

(iii) Wiener-Masani factorizations ( $WM$ ) (Th. 9.7 – see also Rozanov [Ro1, Ro2]);

(iv) the matrix extension of Kolmogorov’s formula for the one-step prediction error (eq. (10.1), the main result of [WiMa1] (Th. 7.10); see also Whittle [Wh]);

(v) convergence of the finite-past predictor to the infinite-past predictor (§13) – cf. Baxter’s inequality, §7 below).

To these, we also add

(vi) Whittle’s multivariate extension of the Levinson-Durbin algorithm, mentioned in §3.

See also [Mas4] for extensive commentary on Wiener’s work in this area.

In the matrix case, one needs to discriminate between the *full-rank* and *degenerate-rank* cases – where the rank of the spectral density matrix  $W$  is full ( $\ell$ ) or degenerate ( $m < \ell$ ). The degenerate-rank case is considered in [Ro2], [Mas3], §§11,12, [WiMa3] for  $\ell = 2$ , Matveev [Mat] in the general case; we refer there and to [Nik2] §A4.8.8, §B9.1.5, §B9.4 for details. The generic, and easier, case is the full-rank case, where  $\Gamma$  is positive-definite (the contrast between the full- and degenerate-rank cases is similar to that arising in, e.g., regression, where one encounters multi-collinearity; see e.g. [BiFr], §7.4.

This interesting work dates from the 1960s, before the work of Fefferman and Stein on BMO and of Sarason on VMO. Armed with these, Peller [Pel1] in 1990 considered matrix spectral factorizations

$$w = h^*h = h_{\#}^*h_{\#}^*$$

(recall  $h$  is determined to within unitary equivalence). He introduced the *phase function*

$$u = h_{\#}^*h^{-1},$$

the analogue of the phase function  $\bar{h}/h$  of §6 in the scalar case. He showed, among other results, that the process is completely regular (§11 below) iff

$$u \in VMO.$$

Arov and Dym [ArDy3], §3.16] give matrix factorizations of positive definite functions into factors from the Nevanlinna class.

The simplest, and principal, case is that of a purely non-deterministic process of full rank. See e.g. several papers by Ephremidze, Janashia and Lagvilada [EpJaLa, EpLa, JaLaEp].

More general than the matrix-valued case is the operator-valued case. This is important in operator theory, in non-commutative probability theory, non-commutative Hardy-space theory, non-commutative martingale inequalities etc. See e.g. [RosRov], esp. Ch. 6, Curtain and Zwart [CuZw], Barclay [Bar1, Bar2], Peller [Pel1] – [Pel3].

## 6. The strong Szegő theorem

The strong Szegő theorem, as presented in e.g. [Si1] Ch. 6, [Bi] §5, extends in full to the matrix case. For a short proof, see Böttcher [Bo1]; cf. [Bo2, Bo3],

Basor and Widom [BasWi]. A different approach has been given more recently by Chanzy [Cha1, Cha2].

## 7. Baxter's inequality and Baxter's theorem

Baxter used OPUC in a series of probabilistic papers of 1961-63 ([Bax1] – [Bax3]). His results concern, among other things, the weak and strong forms of Szegő's limit theorem (for Toeplitz determinants), finite and infinite Wiener-Hopf equations (in discrete time  $n = 0, 1, 2, \dots$ : finite with  $\sum_{k=0}^n$ , infinite with  $\sum_{k=0}^{\infty}$ ), and the convergence of finite-predictor coefficients (in which one is given a finite section of the past of length  $n$ ) to the corresponding infinite-predictor coefficients. This last depends on *Baxter's inequality* [Bax3]. Baxter's inequality was used by Simon [Si1], Ch. 5, in his proof that the Verblunsky coefficients  $a \in \ell_1$  iff the correlation function  $\gamma \in \ell_1$ , the spectral measure  $\mu$  is absolutely continuous, and its density  $\mu' = w$  is continuous and positive. Simon calls this result *Baxter's theorem* (though Baxter did not formulate the result in this form, and 'the Baxter-Simon theorem' might be better here, but we will follow [Si1]). Perhaps because of this rather involved history, and the fact that Simon's book [Si1] is still comparatively recent, there is as yet no matrix extension to Baxter's theorem. We raise here the question of obtaining one, and turn now to what is known.

Baxter's inequality and convergence of finite predictors in the matrix case were considered by Masani in 1966 ([Mas3] §13) and by Cheng and Pourahmadi [ChePou] in 1993. Theoretical progress in the area since then has been extensive, and the question arises of weakening the conditions that they impose. For more recent developments in the scalar case, see [InKa2].

Results on approximation by such finite-section operators have been given in great generality by Seidel and Silbermann [SeSi] (see §2.5.4), using Banach-algebra techniques (as did Baxter and Simon).

## 8. Nehari sequences and the Levinson–McKean condition

Nehari's theorem of 1957 states that a Hankel operator is a bounded map from  $\ell_2$  on the natural numbers to itself iff the sequence generating it is the sequence of negative Fourier coefficients of a bounded function. See e.g. [Si1] Th. 6.2.17 ("The modern theory of Hankel operators started with the following result of Nehari"), Peller [Pel3]. Finding such a generating sequence is thus a type of *moment problem*, and as with other moment problems there may be no solution, a unique solution or more than one solution; the moment problem is then called insoluble, determinate or indeterminate. The indeterminate case is particularly important; the generating sequence is then called a *Nehari sequence*. This Nehari moment (or interpolation) problem was considered by Adamjan, Arov and Krein [AdArKr] in 1968; they described the solution set in terms of Sarason's concept of *rigidity* [Sa1]. Rigidity has previously been studied in this area in connection with the concept of *complete non-determinism* (CND); see Bloomfield, Jewell

and Hayashi [BlJeHa]. It turns out that (CND) is equivalent to the *intersection of past and future* property (IPF) [IK2]. It has very recently been shown that both are equivalent to the *Levinson–McKean property* (the name comes from work of Levinson and McKean, [LeMcK] p. 105, in continuous time; the name is given by analogy in the discrete-time case). Phase functions (§5) play a crucial role here; see Kasahara and Bingham [KasBi] for details.

The question arises of matrix extensions of these results (work in progress). The matrix Nehari problem has been considered in detail by Arov and Dym [ArDy1, ArDy2], [ArDy3] Ch. 4, 7, 10 (‘strong regularity’), Foias and Frazho [FoFr]. See also Nikolskii [Nik2], B Ch. 1-3, and Dym’s review of Peller’s book on Hankel operators [Pel3] (MR1949210 (2004e:47040)). For matrix phase functions, see §5.

Related to this Nehari problem is the Schur (interpolation) problem, the matrix case of which is considered at book length in Dubovoj, Fritzsche and Kirstein [DuFrKi]; cf. [ArDy3], §7.6.

## 9. Pure minimality

In the scalar case, pure minimality is characterized by ( $\mu_s = 0$  and) Kolmogorov’s condition  $1/w \in L_1$ , using  $w$  for the density of the spectral measure  $\mu$  (now absolutely continuous). This result extends to the multivariate case; see Makagon and Weron [MakWe], [Pou3], Th. 8.10. The spectral density is now a matrix  $W$ , and its inverse  $W^{-1}$  is integrable. It is thus natural that the condition

$$W^{-1} \in L_1$$

should be imposed in studying processes subject to stronger regularity conditions than pure minimality. We turn below to two such conditions – positive angle (§10) and complete regularity (§11). We regard the four conditions in §§8–11, which are in increasing order of strength, as *intermediate* conditions, being intermediate between the *weak* conditions (ND), (PND) (Szegő condition +  $\mu_s = 0$ ) and the strong conditions (B), (sSz) (Baxter’s condition and the strong Szegő condition). See e.g. [Bi] for the scalar case.

## 10. Positive angle and the matrix Muckenhoupt condition

The *Muckenhoupt condition* ( $A_2$ ) of analysis is important in many areas; see e.g. [Bi] §6.2 for background and references. It occurs in connection with the *positive angle* condition, ( $PA$ ), and the conditions of Helson and Szegő [HeSz] and of Helson and Sarason [HeSa]. Matrix versions of the Muckenhoupt condition are considered at length by Arov and Dym [ArDy1, ArDy3]. For matrix versions of the Helson-Szegő condition, see Pourahmadi [Pou1].

Treil and Volberg [TrVo1] show that the following matrix Muckenhoupt condition is necessary and sufficient for the positive-angle condition ( $PA$ ) in the

multivariate case:

$$\sup_I \left\| \left( \frac{1}{|I|} \int_I W \right)^{1/2} \left( \frac{1}{|I|} \int_I W^{-1} \right)^{1/2} \right\| < \infty, \quad (A_2)$$

where the supremum is taken over all intervals  $I$  of the unit circle. Here the condition that  $W$  be invertible a.e. (which corresponds to the ‘pure’ in pure minimality, see §9 above) has to be imposed explicitly, as noted by Peller in his review of [TrVo2] (MR1428818 (99k;42073)).

As shown in [HeSz, HeSa], the condition  $(PA)$  (and so  $(A_2)$  by above) is equivalent to a condition on the sequence  $\rho(n)$  of regularity coefficients of the form

$$\rho(\cdot) < 1.$$

We note that in his review of earlier work on this problem by Makagon, Miamer and Schröder [MakMiSc], Pourahmadi says “Attempts to obtain a similar result for  $q$ -variate stationary sequences have been unyielding” (MR1443841 (98g;60074)).

## 11. Complete regularity

We turn now to a strengthening of the conditions of §10 above. The process is said to be *completely regular* if  $\rho(n) \rightarrow 0$  as  $n \rightarrow \infty$ ; see [IbRo], Ch. 4, 5. It was shown by Treil and Volberg [TrVo2] that complete regularity is equivalent to the following strengthening of the Muckenhoupt condition  $(A_2)$ :

$$\limsup_{|I| \rightarrow 0} \left\| \left( \frac{1}{|I|} \int_I W \right)^{1/2} \left( \frac{1}{|I|} \int_I W^{-1} \right)^{1/2} \right\| < \infty$$

(cf. Peller [Pel1, Pel3]); here as before  $W^{-1} \in L_1$  needs to be assumed explicitly. Note the form (“ $\rho(\cdot) \rightarrow 0$ ,  $\limsup \dots = 1$ ”) of the strengthenings here of the conditions (“ $\rho(\cdot) < 1$ ,  $\sup \dots < \infty$ ”) of §8 above.

## 12. Hankel operators

Prediction theory has always involved Toeplitz operators (as in the book [GrSz] by Grenander and Szegő), and Toeplitz and Hankel operators have many links in operator theory. So it is natural that Hankel operators are useful in prediction theory. A monograph treatment of Hankel operators is given by Peller ([Pel3]; see also the review by Dym cited above in §8). Connections of Hankel operators with the matrix Muckenhoupt condition and with the matricial Nehari problem are considered by Arov and Dym in [ArDy3], Ch. 10, 11.

## 13. Open questions

We mention two.

*Question 1.* Find the matrix version of Baxter’s theorem.

As mentioned in §7, the matrix version of Baxter’s inequality provides a good starting-point.

*Question 2.* Find the matrix version of [KasBi].

This hinges on solution of the matrix Nehari problem – the step

$$\Gamma \rightarrow H.$$

We hope to return to this elsewhere.

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