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ON THE KÄHLER ANGLES OF SUBMANIFOLDS

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To the memory of Giorgio Valli

Abstract: We prove that under certain conditions on the mean curvature and on the Kähler angles, a compact submanifold M of real dimension $2n$, immersed into a Kähler–Einstein manifold N of complex dimension $2n$, must be either a complex or a Lagrangian submanifold of N, or have constant Kähler angle, depending on $n = 1$, $n = 2$, or $n \geq 3$, and the sign of the scalar curvature of N. These results generalize to non-minimal submanifolds some known results for minimal submanifolds. Our main tool is a Bochner-type technique involving a formula on the Laplacian of a symmetric function on the Kähler angles and the Weitzenböck formula for the Kähler form of N restricted to M.

1 – Introduction

Let (N, J, g) be a Kähler–Einstein manifold of complex dimension $2n$, complex structure J, Riemannian metric g, and $F\colon M^{2n} \to N^{2n}$ be an immersed submanifold M of real dimension $2n$. We denote by $\omega(X,Y) = g(JX,Y)$ the Kähler form and by R the scalar curvature of N, that is, the Ricci tensor of N is given by $Ricci = Rg$. The cosine of the Kähler angles $\{\theta_\alpha\}_{1\leq\alpha\leq n}$ are the eigenvalues of $F^*\omega$. If the eigenvalues are all equal to 0 (resp. 1), F is a Lagrangian (resp. complex) submanifold. A natural question is to ask if N allows submanifolds with arbitrary given Kähler angles and mean curvature. An answer is that, the Kähler angles and the second fundamental form of F, and the Ricci tensor of N are interrelated. Conditions on some of these geometric objects have implications

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for the other ones. There are obstructions to the existence of minimal Lagrangian submanifolds in a general Kähler manifold, but these obstructions do not occur in a Kähler–Einstein manifold, where such submanifolds exist with abundance ([Br]). This is the reason we choose Kähler–Einstein manifolds as ambient spaces. An example how the sign of the scalar curvature of N determines the Kähler angles is the fact that if F is a totally geodesic immersion and N is not Ricci-flat, then either F has a complex direction, or F is Lagrangian ([S-V,1]). A relation among the θ_α, ∇dF, and R can be described through a formula on the Laplacian of a locally Lipschitz map κ, symmetric on the Kähler angles of F, where the Ricci tensor of N and some components of the second fundamental form of F appear. Such kind of formula was used for minimal immersions in [W,1] for $n = 1$, and in [S-V,1,2] for $n \geq 2$.

A natural condition for $n \geq 2$ is to impose equality on the Kähler angles. Products of surfaces immersed with the same constant Kähler angle θ into Kähler–Einstein surfaces of the same scalar curvature R, give submanifolds immersed with constant equal Kähler angle θ into a Kähler–Einstein manifold of scalar curvature R. The slant submanifolds introduced and exhaustively studied by B-Y Chen (see e.g. [Che,1,2], [Che-M], [Che-T,1,2]) are submanifolds with constant and equal Kähler angles. Examples are given in complex spaces form, some of them via Hopf's fibration [Che-T,1,2]. A minimal 4-dimensional submanifold of a Calabi–Yau manifold of complex dimension 4, calibrated by a Cayley calibration, also called Cayley submanifold, is just the same as a minimal submanifold with equal Kähler angles ([G]). Existence theory of such submanifolds in $\mathbb{C}^4$, with given inicial boundary data, is guaranteed by the theory of calibrations of Harvey and Lawson [H-L].

Submanifolds with equal Kähler angles have a role in 4 and 8 dimensional gauge theories. For example, each of such Cayley submanifolds in $\mathbb{C}^4$ carries a 21-dimensional family of (anti)-self-dual $SU(2)$ Yang–Mills fields [H-L]. Recentely, Tian [T] proved that blow-up loci of complex anti-self-dual instantons on Calabi–Yau 4-folds are Cayley cycles, which are, except for a set of 4-dimensional Hausdorff measure zero, a countable union of C^1 4-dimensional Cayley submanifolds.

If N is an hyper-Kähler manifold of complex dimension 4 and hyper-Kähler structure $(J_x)_{x\in S^2}$, any submanifold of real dimension 4 that is J_x-complex for some $x \in S^2$, is a minimal submanifold with equal Kähler angles of each (N, J_y, g) ([S-V,2]), and the common Kähler angle is given by $\cos\theta(p) = \|(J_yX)^\top\|$, where X is any unit vector of T_pM. A proof of this assertion is simply to remark that, if $\{X, J_xX, Y, J_xY\}$ is an o.n. basis of T_pM, then the matrix of the Kähler form ω_y

w.r.t. J_y, restricted to this basis, is just a multiple of a matrix in $\mathbb{R}^4$ that represents an orthogonal complex structure of $\mathbb{R}^4$, i.e. of the type $aI + bJ + cK$, where I, J, K defines the usual hyper-Kähler structure of $\mathbb{R}^4$, and $a^2 + b^2 + c^2 = 1$. The square of this multiple is given by $\langle x, y\rangle^2 + \langle J_y X, Y\rangle^2 + \langle J_{x\times y} X, Y\rangle^2 = \|(J_y X)^\top\|^2$. This example suggests us a way to build examples of (local) submanifolds with equal Kähler angles. Let (N, I, g) be a Kähler manifold of complex dimension 4, and $U \subset N$ an open set where an orthornormal frame of the form $\{X_1, IX_1, X_2, IX_2, Y_1, IY_1, Y_2, IY_2\}$ is defined. If for each $p \in U$, we identify T_pN with $\mathbb{R}^4 \times \mathbb{R}^4$, through this frame, we are defining a family of local g-orthogonal almost complex structures $J_x = ai \times i + bj \times j + ck \times k$, for $x = (a, b, c) \in S^2$, where i, j, k denotes de canonical hyper-Kähler structure of $\mathbb{R}^4$. Then any almost J_x-complex 4-dimensional submanifold M is a submanifold with equal Kähler angles of the Kähler manifold (N, I, g). It may not be minimal, because J_x may not be a Kähler structure, or not even integrable.

Such a condition on the Kähler angles, turns out to be more restrictive for submanifolds of non Ricci-flat manifolds, or if M is closed, that is, compact and orientable. A combination of the formula of $\triangle\kappa$ for minimal immersions with equal Kähler angles, with the Weitzenböck formula for $F^*\omega$, lead us in [S-V,2] to the conclusion that the Kähler angle must be constant, and in general it is either 0 or $\frac{\pi}{2}$. Namely, we have:

Theorem 1.1. *Let $F\colon M^{2n} \to N^{2n}$ be a minimal immersion with equal Kähler angles.*

- **(i)** (**[W,1]**) *If $n = 1$, M is closed, $R < 0$, and F has no complex points, then F is Lagrangian.*
- **(ii)** (**[S-V,2], [G]**) *If $n = 2$ and $R \neq 0$, then F is either a complex or a Lagrangian submanifold.*
- **(iii)** (**[S-V,2]**) *If $n \geq 3$, M is closed, and $R < 0$, then F is either a complex or a Lagrangian submanifold.*
- **(iv)** (**[S-V,2]**) *If $n \geq 3$, M is closed, $R = 0$, then the common Kähler angle must be constant.*

If $n = 2$ and $R = 0$ we cannot conclude the Kähler angle is constant. It is easy to find examples of minimal immersions with constant and non-constant equal Kähler angle, for the case of M not compact and N the Euclidean space. Namely, the most simple family of submanifolds with constant equal Kähler angle of $\mathbb{C}^{2n}$ can be given by the vector subspaces defined by a linear map $F\colon \mathbb{R}^{2n}\to\mathbb{C}^{2n}\equiv(\mathbb{R}^{2n}\times\mathbb{R}^{2n},J_0)$, $F(X)=(X,aJ_\omega X)$, where a is any real number and J_ω is a g_0-orthogonal complex structure of $\mathbb{R}^{2n}$, and where g_0 is the Euclidean metric and $J_0(X,Y)=(-Y,X)$. These are totally geodesic submanifolds with constant equal Kähler angle $\cos\theta=\frac{2|a|}{1+a^2}$, and $F^*\omega(X,Y)=\cos\theta\, F^*g_0\,(\pm J_\omega X,Y)$, with F^*g_0 a J_ω-hermitian euclidean metric. In ([D-S]) we have the following example of non-constant Kähler angle well away from 0. The graph of the anti-i-holomorphic map $f\colon\ \mathbb{R}^4\to\mathbb{R}^4$ given by $f(x,y,z,w)=(u,v,-u,-v)$, where

$$\begin{aligned}&u(x,y,z,w)=\phi(x+z)\,\xi'(y+w)\ ,\\ &v(x,y,z,w)=-\phi'(x+z)\,\xi(y+w)\ ,\\ &\phi(t)=\sin t\,,\qquad \xi(t)=\sinh t\ ,\end{aligned}$$

defines a complete minimal submanifold of $\mathbb{C}^4$ with equal Kähler angles satisfying

$$\cos\theta\ =\ \frac{2\sqrt{\cos^2(x+z)+\,\sinh^2(y+w)}}{1+4\Big(\cos^2(x+z)+\,\sinh^2(y+w)\Big)}\ .$$

This graph has no complex points, for $0\le\cos\theta\le\frac{1}{2}$, and the set of Lagrangian points is a infinite discrete union of disjoint 2-planes,

$$\mathcal{L}\ =\bigcup_{-\infty\le k\le+\infty}\mathrm{span}_{\mathbb{R}}\Big\{(1,0,-1,0),\,(0,1,0,-1)\Big\}\ +\ \Big(0,0,(\tfrac{1}{2}+k)\pi,0\Big)\ .$$

In this paper we present a formula for $\triangle\kappa$, but now not assuming minimality of F, obtaining some extra terms involving the mean curvature H of F. We will see that the above conclusions still hold for F not minimal, but under certain weaker condition on the mean curvature of F. These conclusions show how rigid Kähler–Einstein manifolds are with respect to the Kähler angles and the mean curvature of a submanifold, leading to some non-existence of certain types of submanifolds, depending on the sign of the scalar curvature R of N and on the dimension n.

We summarize the main results of this paper:

Theorem 1.2. *Assume $n=2$, and M is closed, N is non Ricci-flat, and $F\colon\ M\to N$ is an immersion with equal Kähler angles, $\theta_\alpha=\theta\ \forall\alpha$. If*

$$R\,F^*\omega\Big((JH)^\top,\nabla\sin^2\theta\Big)\ \le\ 0 \tag{1.1}$$

then F is either a complex or a Lagrangian submanifold. This is the case when F has constant Kähler angle.

Corollary 1.1. *Let $n = 2$, $R < 0$, and $F\colon M \to N$ be a closed submanifold with parallel mean curvature and equal Kähler angles. If $\|H\|^2 \geq -\frac{R}{8}\sin^2\theta$, then F is either a complex or a Lagrangian submanifold.*

Theorem 1.3. *Assume M is closed, $n \geq 3$, and $F\colon M \to N$ is an immersion with equal Kähler angles.*

(**A**) *If $R < 0$, and if $\delta F^*\omega((JH)^\top) \geq 0$, then F is either complex or Lagrangian.*

(**B**) *If $R = 0$, and if $\delta F^*\omega((JH)^\top) \geq 0$, then the Kähler angle is constant.*

(**C**) *If F has constant Kähler angle and $R \neq 0$, then F is either complex or Lagrangian.*

In case $n = 1$ we obtain:

Proposition 1.1. *If M is a closed surface and N is a non Ricci-flat Kähler–Einstein surface, then any immersion $F\colon M \to N$ either has complex or Lagrangian points. In particular, if F has constant Kähler angle, then F is either a complex or a Lagrangian submanifold.*

This generalizes a result in [M-U], for compact surfaces immersed with constant Kähler angle (and so orientable, if not Lagrangian) into $\mathbb{CP}^2$.

For M not necessarily compact we have the following proposition:

Proposition 1.2. *If $F\colon M \to N$ is an immersion with constant equal Kähler angle θ and with parallel mean curvature, that is, $\nabla^\perp H = 0$, then:*

(**1**) *If $R = 0$, F is either Lagrangian or minimal.*

(**2**) *If $R > 0$, F is either Lagrangian or complex.*

(**3**) *If $R < 0$, F is either Lagrangian, or $\|H\|^2 = -\frac{\sin^2\theta}{4n}R$.*

(**4**) *If $H = 0$, then $R = 0$ or F is either Lagrangian or complex.*

Note that (4) of the above proposition is an improvement of Theorem 1.3 of [S-V,2], for, compactness is not required now. We also observe that from Corollary 1.1, if $n = 2$ and M were closed, that later case of (3) implies as well F to be complex or Lagrangian. Compactness of M is a much more restrictive condition. In [K-Z] it is shown that, if $n = 1$ and N is a complex space form of constant holomorphic sectional curvature 4ρ and M is a surface of non-zero parallel mean curvature and constant Kähler angle, then either F is Lagrangian and M is flat, or $\sin\theta = -\sqrt{\frac{8}{9}}$, $\rho = -\frac{3}{4}\|H\|^2$ and M has constant Gauss curvature $K = -\frac{\|H\|^2}{2}$. These values of θ and ρ ($R = 6\rho$) are according to our relation in (3) of Proposition 1.2. In [Che,2] and [Che-T,2] it is shown explicitly all possible examples of such (non-compact) surfaces of the 2-dimensional complex hyperbolic spaces. In [K-Z] it is also shown all examples of surfaces immersed into $\mathbb{CH}^2$ with non-zero parallel mean curvature and non-constant Kähler angle. In case (1), if F is not minimal, then $(JH)^\top$ defines a global nonzero parallel vector field on M (see Proposition 3.6 of section 3).

Theorem 1.4. *Let F be a closed surface immersed with parallel mean curvature into a non Ricci-flat Kähler–Einstein surface. If F has no complex points and if $\frac{F^*\omega}{Vol_M} \geq 0$ (or ≤ 0) on all M, then F is Lagrangian. If F has no Lagrangian points, then F is minimal.*

2 – Some formulas on the Kähler angles

On M we take the induced metric $g_M = F^*g$, that we also denote by $\langle,\rangle$. We denote by ∇ both Levi–Civita connections of M and N, and by $\nabla_X dF(Y) = \nabla dF(X,Y)$ the second fundamental form of F, a symmetric tensor on M with values on the normal bundle $NM = (dF(TM))^\perp$ of F. The mean curvature of F is given by $H = \frac{1}{2n}\,\mathrm{trace}\nabla dF$. At each point $p \in M$, let $\{X_\alpha, Y_\alpha\}_{1\leq\alpha\leq n}$ be a g_M-orthonormal basis of eigenvectors of $F^*\omega$. On that basis, $F^*\omega$ is a $2n\times 2n$ block matrix

$$F^*\omega \;=\; \bigoplus_{0\leq\alpha\leq n} \begin{bmatrix} 0 & -\cos\theta_\alpha \\ \cos\theta_\alpha & 0 \end{bmatrix},$$

where $\cos\theta_1 \geq \cos\theta_2 \geq \ldots \geq \cos\theta_n \geq 0$, are the corresponding eigenvalues ordered in decreasing way. The angles $\{\theta_\alpha\}_{1\leq\alpha\leq n}$ are the Kähler angles of F at p. We identify the two form $F^*\omega$ with the skew-symmetric operator of T_pM, $(F^*\omega)^\sharp\colon T_pM \to T_pM$, using the musical isomorphism with respect to g_M, that is, $g_M((F^*\omega)^\sharp(X), Y) = F^*\omega(X,Y)$, and we take its polar decomposition, $(F^*\omega)^\sharp = |(F^*\omega)^\sharp|\, J_\omega$, where $J_\omega :\, T_pM \to T_pM$ is a partial isometry with the same kernel $\mathcal{K}_\omega$ as of F^*w, and where $|(F^*\omega)^\sharp| = \sqrt{-(F^*\omega)^{\sharp 2}}$. On $\mathcal{K}_\omega^\perp$, the orthogonal complement of $\mathcal{K}_\omega$ in T_pM, $J_\omega\colon \mathcal{K}_\omega^\perp \to \mathcal{K}_\omega^\perp$ defines a g_M-orthogonal complex structure. On a open set without complex directions, that is $\cos\theta_\alpha < 1$ $\forall\alpha$, we consider the locally Lipschitz map

$$\kappa = \sum_{1\leq\alpha\leq n} \log\left(\frac{1+\cos\theta_\alpha}{1-\cos\theta_\alpha}\right).$$

For each $0 \leq k \leq n$, this map is smooth on the largest open set Ω^0_{2k}, where $F^*\omega$ has constant rank $2k$. On a neighbourhood of a point $p_0 \in \Omega^0_{2k}$, we may take $\{X_\alpha, Y_\alpha\}_{1\leq\alpha\leq n}$ a smooth local g_M-orthonormal frame of M, with $Y_\alpha = J_\omega X_\alpha$ for $\alpha \leq k$, and where $\{X_\alpha, Y_\alpha\}_{\alpha\geq k+1}$ is any g_M-orthonormal frame of $\mathcal{K}_\omega$. Moreover, we may assume that this frame diagonalizes $F^*\omega$ at p_0. Following the computations of the appendix in [S-V,2], without requiring now minimality, we see that the components of the mean curvature of F appear three times in the formula for $\triangle\kappa$. Namely, when we compute (5.9) and (5.10) of [S-V,2], we get respectively, the extra terms $ig(\frac{n}{2}\nabla_\mu H, JdF(\bar\mu))$ and $-ig(\frac{n}{2}\nabla_{\bar\mu} H, JdF(\mu))$, and when we sum $\sum_\beta -R^M(\mu,\bar\beta,\beta,\bar\mu) - R^M(\bar\mu,\bar\beta,\beta,\mu)$ we obtain the extra term $ng(H, \nabla_\mu dF(\bar\mu))$. Then, we have to add in the final expression for $\sum_\beta \mathrm{Hess}\,\tilde g_{\mu\bar\mu}(\beta,\bar\beta)$ of Lemma 5.4 of [S-V,2] the expression $\sum_\beta ig(\frac{n}{2}\nabla_\mu H, JdF(\bar\mu)) - ig(\frac{n}{2}\nabla_{\bar\mu} H, JdF(\mu)) + \cos\theta_\mu\, ng(H, \nabla_\mu dF(\bar\mu))$. Introducing these extra terms in the term $\sum_{\beta,\mu}\frac{32}{\sin^2\theta_\mu}\mathrm{Hess}\,\tilde g_{\mu\bar\mu}(\beta,\bar\beta)$ of (5.7) of [S-V,2], we obtain our more general formula for $\triangle\kappa$:

Proposition 2.1. *For any immersion F, at a point p_0 on a open set where $F^*\omega$ has constant rank $2k$ and no complex directions, we have*

$$\begin{aligned}
\triangle\kappa = {} & 4\,i\sum_\beta Ricci^N(JdF(\beta), dF(\bar\beta))\\
& + \sum_{\beta,\mu}\frac{32}{\sin^2\theta_\mu}\,\mathrm{Im}\Big(R^N\Big(dF(\beta), dF(\mu), dF(\bar\beta), JdF(\bar\mu) + i\cos\theta_\mu dF(\bar\mu)\Big)\Big)\\
& - \sum_{\beta,\mu,\rho}\frac{64(\cos\theta_\mu + \cos\theta_\rho)}{\sin^2\theta_\mu\,\sin^2\theta_\rho}\,\mathrm{Re}\Big(g(\nabla_\beta dF(\mu), JdF(\bar\rho))\; g(\nabla_{\bar\beta} dF(\rho), JdF(\bar\mu))\Big)\\
& + \sum_{\beta,\mu,\rho}\frac{32(\cos\theta_\rho - \cos\theta_\mu)}{\sin^2\theta_\mu\,\sin^2\theta_\rho}\Big(|g(\nabla_\beta dF(\mu), JdF(\rho))|^2 + |g(\nabla_{\bar\beta} dF(\mu), JdF(\rho))|^2\Big)\\
& + \sum_{\beta,\mu,\rho}\frac{32(\cos\theta_\mu + \cos\theta_\rho)}{\sin^2\theta_\mu}\Big(|\langle\nabla_\beta\mu,\rho\rangle|^2 + |\langle\nabla_{\bar\beta}\mu,\rho\rangle|^2\Big)\\
& + \sum_\mu\frac{8n}{\sin^2\theta_\mu}\Big(ig(\nabla_\mu H, JdF(\bar\mu)) - ig(\nabla_{\bar\mu} H, JdF(\mu)) + 2\cos\theta_\mu g(H, \nabla_\mu dF(\bar\mu))\Big)
\end{aligned} \tag{2.1}$$

where "α" $= Z_\alpha = \frac{X_\alpha - iY_\alpha}{2}$ and "$\bar\alpha$" $= \overline{Z_\alpha}$.

Projecting JH on $dF(TM)$, we define a vector field $(JH)^\top$ on M, and we denote by $((JH)^\top)^\flat$ the corresponding 1-form, $((JH)^\top)^\flat(X) = g_M((JH)^\top, X) = g(JH, dF(X))$. If F is a Lagrangian immersion, the above formula on $\triangle\kappa$ leads to a well-known result:

Corollary 2.1. **([W,2])** *If F is a Lagrangian immersion, then $((JH)^\top)^\flat$ is a closed 1-form on M.*

A proof of this corollary will be given in section 3. The formula (2.1) is considerably simplified when F is an immersion with equal Kähler angles. Now we recall the Weitzenböck formula for $F^*\omega$, that we used in [S-V,2]

$$\frac{1}{2}\triangle\|F^*\omega\|^2 = -\langle\triangle F^*\omega, F^*\omega\rangle + \|\nabla F^*\omega\|^2 + \langle SF^*\omega, F^*\omega\rangle\ , \tag{2.2}$$

where $\langle,\rangle$ denotes the Hilbert–Schmidt inner product for 2-forms, and S is the Ricci operator of $\bigwedge^2 T^*M$, and $\triangle = d\delta + \delta d$ is the the Laplacian operator on forms. $F^*\omega$ is a closed 2-form. If it is also co-closed, that is $\delta F^*\omega = 0$, then it is harmonic. If M is compact,

$$\int_M \langle\triangle F^*\omega, F^*\omega\rangle\, Vol_M = \int_M \|\delta F^*\omega\|^2\ Vol_M\ . \tag{2.3}$$

We will use this formula when F has equal Kähler angles.

3 – Immersions with equal Kähler angles

In this section we recall some formulas for immersions with equal Kähler angles. F is said to have equal Kähler angles, if all the angles are equal, $\theta_\alpha = \theta\ \forall\alpha$. In this case, $(F^*\omega)^\sharp = \cos\theta J_\omega$, and J_ω is a smooth almost complex structure away from the set of Lagrangian points $\mathcal{L} = \{p \in M\colon\ \cos\theta(p) = 0\}$. Let $\mathcal{L}^0$ denote the largest open set of $\mathcal{L}$, $\mathcal{C} = \{p \in M\colon\ \cos\theta(p) = 1\}$ the set of complex points, and $\mathcal{C}^0$ its largest open set. Recall that $\cos^2\theta$ is smooth on all M, while $\cos\theta$ is only locally Lipschitz on M, but smooth on $\mathcal{L}^0 \cup (M \sim \mathcal{L})$. For immersions with equal Kähler angles, any local frame of the form $\{X_\alpha, Y_\alpha = J_\omega X_\alpha\}_{1\leq\alpha\leq n}$ diagonalizes $F^*\omega$ on the whole set where it is defined. We use the letters $\alpha, \beta, \mu, \ldots$ to range on the set $\{1, \ldots, n\}$ and the letters $j, k, \ldots$ to range on $\{1, \ldots, 2n\}$. As in the previous section, we denote by "α" $= Z_\alpha = \frac{X_\alpha - iY_\alpha}{2}$ and "$\bar\alpha$" $= \overline{Z_\alpha} = \frac{X_\alpha + iY_\alpha}{2}$, defining local frames on the complexifyied tangent space of M.

On tensors and forms we use the Hilbert–Schmidt inner product. We denote by δ the divergence operator on (vector valued) forms, and by div_M the divergence operator on vector fields over M. The $(1,1)$-part of ∇dF with respect to J_ω, is given by $(\nabla dF)^{(1,1)}(X,Y) = \frac{1}{2}(\nabla dF(X,Y) + \nabla dF(J_\omega X, J_\omega Y))$. This tensor is defined away from Lagrangian points, and it vanish on $\mathcal{C}^0$, for, on that set, F is a complex submanifold of N, and J_ω is the induced complex structure.

Proposition 3.1. **([S-V,2])** *On* $(M\sim\mathcal{L})\cup\mathcal{L}^0$,

$$\begin{aligned}
\|F^*\omega\|^2 &= n\cos^2\theta\\
\|\nabla F^*\omega\|^2 &= n\|\nabla\cos\theta\|^2 + \frac{1}{2}\cos^2\theta\|\nabla J_\omega\|^2\\
\delta(F^*\omega)^\sharp &= (\delta F^*\omega)^\sharp = (n-2)\,J_\omega(\nabla\cos\theta)\\
\|\delta F^*\omega\|^2 &= (n-2)^2\,\|\nabla\cos\theta\|^2\\
\cos\theta\delta J_\omega &= (n-1)\,J_\omega(\nabla\cos\theta)
\end{aligned}$$

and on $(M\sim(\mathcal{L}\cup\mathcal{C}))\cup\mathcal{L}^0\cup\mathcal{C}^0$,

$$(1-n)\,\nabla\sin^2\theta =$$

$$= 16\cos\theta\,\mathrm{Re}\Big(i\sum_{\beta,\mu}\Big(g(\nabla_{\bar\mu}dF(\mu),JdF(\beta)) - g(\nabla_{\bar\mu}dF(\beta),JdF(\mu))\Big)\,\bar\beta\Big)\,.$$

In particular, for $n\neq 2$, $J_\omega(\nabla\cos\theta)$, $\|\nabla\cos\theta\|^2$, $\cos^2\theta\|\nabla J_\omega\|^2$, *and* $\cos\theta\,\delta J_\omega$ *can be smoothly extended to all* M. *Furthermore, for* $n\geq 2$, *there is a constant* $C>0$ *such that on* M, $\|\nabla\sin^2\theta\|^2\leq C\cos^2\theta\;\sin^2\theta\;\|(\nabla dF)^{(1,1)}\|^2$.

The estimate on $\|\nabla\sin^2\theta\|^2$ given above follows from the expression on $(1-n)\nabla\sin^2\theta$ and the following explanation. From Schwarz inequality, $|g(\nabla_X dF(Y), JdF(Z))| = |g(\nabla_X dF(Y),\Phi(Z))| \leq \|\nabla_X dF(Y)\|\,\|\Phi(Z)\|$, where $\Phi(Z) = (JdF(Z))^\perp$, and $(\;)^\perp$ denotes the orthogonal projection onto the normal bundle. But (cf. [S-V,2]) $JdF(Z) = \Phi(Z) + dF((F^*\omega)^\sharp(Z))$. An elementary computation shows that

$$\begin{aligned}
\|\Phi(Z)\|^2 &= g\Big(JdF(Z)-dF((F^*\omega)^\sharp(Z)),\, JdF(Z)-dF((F^*\omega)^\sharp(Z))\Big)\\
&= \sin^2\theta\,\|Z\|^2\,.
\end{aligned}$$

Obviously the formula on $\nabla\sin^2\theta$ as well the estimate on $\|\nabla\sin^2\theta\|^2$, are still valid on all complex and Lagrangian points, since those points are critical points for $\sin^2\theta$, and at complex points $JdF(TM)\subset dF(TM)$. Also

Corollary 3.1. *If* $n=2$, $F^*\omega$ *is an harmonic 2-form. If* $n\neq 2$, $F^*\omega$ *is co-closed iff* θ *is constant. For any* $n\geq 2$, *if* $(M\sim\mathcal{L}, J_\omega, g_M)$ *is Kähler, then* θ *is constant and* $F^*\omega$ *is parallel.*

Following chapter 4 of [S-V,2] and using the new expression for $\triangle\kappa$ of Proposition 2.1, with the extra terms involving the mean curvature H, and noting that now both (4.4) and (4.7) + (4.5) of [S-V,2] have extra terms involving H, we obtain:

Proposition 3.2. *Away from complex and Lagrangian points,*

$$\begin{aligned}\triangle\kappa &= \\ &= \cos\theta\left(-2nR + \frac{32}{\sin^2\theta}\sum_{\beta,\mu}R^M(\beta,\mu,\bar\beta,\bar\mu) + \frac{1}{\sin^2\theta}\|\nabla J_\omega\|^2 + \frac{8(n-1)}{\sin^4\theta}\|\nabla\cos\theta\|^2\right)\\ &\quad - \frac{16\,n}{\sin^4\theta}\cos\theta\sum_\beta d\cos\theta\Big(ig(H,JdF(\beta))\bar\beta - ig(H,JdF(\bar\beta))\beta\Big)\\ &\quad + \frac{8n}{\sin^2\theta}\sum_\mu\Big(ig(\nabla_\mu H, JdF(\bar\mu)) - ig(\nabla_{\bar\mu}H, JdF(\mu))\Big)\ .\end{aligned}$$

Let us denote by $\nabla^\perp$ the usual connection in the normal bundle, and denote by $(JH)^\top$ the vector field of M given by

$$g_M((JH)^\top, X) = g(JH, dF(X)) \qquad \forall X\in TM\ .$$

Lemma 3.1. $\forall X,Y\in T_pM$,

(i) $$\begin{aligned} g(\nabla_X H, JdF(Y)) &= -\langle\nabla_X(JH)^\top, Y\rangle - g(H, J\nabla_X dF(Y)) && (\text{on } M)\\ &= -g(H, \nabla_X dF((F^*\omega)^\sharp(Y))) + g(\nabla^\perp_X H, JdF(Y)) && (\text{on } M)\end{aligned}$$

(ii) $$(\tfrac{1}{2}J_\omega((JH)^\top) = \sum_\beta ig(H,JdF(\beta))\bar\beta - ig(H,JdF(\bar\beta))\beta \qquad (\text{on } M\sim\mathcal{L})$$

(iii) $$\begin{aligned}&\sum_\mu 2\,ig(\nabla_\mu H, JdF(\bar\mu)) - 2\,ig(\nabla_{\bar\mu}H, JdF(\mu)) =\\ &\quad= \sum_\mu 4\,\mathrm{Im}\langle\nabla_\mu(JH)^\top,\bar\mu\rangle = -\sum_\mu 2\,id(((JH)^\top)^\flat(\mu,\bar\mu) && (\text{on } M)\\ &\quad= -2n\cos\theta\|H\|^2 - 4\sum_\mu \mathrm{Im}\Big(g(\nabla^\perp_\mu H, JdF(\bar\mu))\Big) && (\text{on } M)\\ &\quad= -\,\mathrm{div}_M\Big(J_\omega((JH)^\top)\Big) + \langle (JH)^\top, \delta J_\omega\rangle && (\text{on } M\sim\mathcal{L}).\end{aligned}$$

(iv) $$\mathrm{div}_M((JH)^\top) = \sum_\mu -4\,\mathrm{Re}\Big(g(\nabla^\perp_\mu H, JdF(\bar\mu))\Big) \qquad (\text{on } M).$$

Proof: Assume that $\nabla Y(p)=0$. Then we have at the point p

$$\begin{aligned} g(\nabla_X H, JdF(Y)) &= d\Big(g(H,JdF(Y))\Big)(X) - g(H,\nabla_X(JdF(Y)))\\ &= -d\langle (JH)^\top, Y\rangle(X) - g(H, J\nabla_X dF(Y))\\ &= -\langle \nabla_X (JH)^\top, Y\rangle - g(H, J\nabla_X dF(Y))\ . \end{aligned}$$

On the other hand, from $JdF(Y) = dF((F^*\omega)^\sharp(Y)) + (JdF(Y))^\perp$, we get the second equality of (i). For $p\in M\sim\mathcal{L}$, since $J_\omega\beta = i\beta$, and $J_\omega\bar\beta = -i\bar\beta$,

$$\begin{aligned} \sum_\beta ig(H,JdF(\beta))\bar\beta - ig(H,JdF(\bar\beta))\beta &= \\ &= \sum_\beta\ g(H,JdF(J_\omega\beta))\bar\beta + g(H,JdF(J_\omega\bar\beta))\beta\\ &= \sum_\beta\ -g(JH,dF(J_\omega\beta))\bar\beta - g(JH,dF(J_\omega\bar\beta))\beta\\ &= \sum_\beta\ -\langle (JH)^\top, J_\omega\beta\rangle\bar\beta - \langle (JH)^\top, J_\omega\bar\beta\rangle\beta\\ &= \sum_\beta\ \langle J_\omega((JH)^\top),\beta\rangle\bar\beta + \langle J_\omega((JH)^\top),\bar\beta\rangle\beta\\ &= \tfrac12 J_\omega((JH)^\top)\ , \end{aligned}$$

and (ii) is proved. From the first equality of (i),

$$\begin{aligned} \sum_\mu ig(\nabla_\mu H, JdF(\bar\mu)) - ig(\nabla_{\bar\mu}H, JdF(\mu)) &= \\ &= \sum_\mu -i\langle\nabla_\mu (JH)^\top,\bar\mu\rangle + i\langle \nabla_{\bar\mu}(JH)^\top,\mu\rangle = \sum_\mu 2\,\mathrm{Im}\Big(\langle\nabla_\mu(JH)^\top,\bar\mu\rangle\Big)\\ &= \sum_\mu -id((JH)^\top)^\flat(\mu,\bar\mu)\ . \end{aligned}$$

On the other hand, from second equality of (i)

$$\begin{aligned} \sum_\mu g(\nabla_\mu H, JdF(\bar\mu)) &= \sum_\mu -g(H,\nabla_\mu dF(\cos\theta J_\omega(\bar\mu))) + g(\nabla^\perp_\mu H, JdF(\bar\mu))\\ &= \frac{ni}{2}\cos\theta\, g(H,H) + \sum_\mu g(\nabla^\perp_\mu H, JdF(\bar\mu))\ . \end{aligned}$$

Hence

$$\begin{aligned} \sum_\mu ig(\nabla_\mu H, JdF(\bar\mu)) - ig(\nabla_{\bar\mu}H, JdF(\mu)) &= \\ &= -n\cos\theta\|H\|^2 - \sum_\mu\ 2\,\mathrm{Im}\Big(g(\nabla^\perp_\mu H, JdF(\bar\mu))\Big)\ . \end{aligned}$$

Similarly, from $\mathrm{div}_M((JH)^\top) = \sum_\mu 2\langle\nabla_\mu(JH)^\top, \bar\mu\rangle + 2\langle\nabla_{\bar\mu}(JH)^\top, \mu\rangle$ and (i) we get (iv).
Finally, using the symmetry of ∇dF and that $\langle\nabla_Z J_\omega(X), Y\rangle = -\langle\nabla_Z J_\omega(Y), X\rangle$ (cf. [S-V,2])

$$
\begin{aligned}
&\sum_\mu ig(\nabla_\mu H, JdF(\bar\mu)) - ig(\nabla_{\bar\mu} H, JdF(\mu)) \;= \\
&= \sum_\mu \langle\nabla_\mu(JH)^\top, J_\omega(\bar\mu)\rangle + \langle\nabla_{\bar\mu}(JH)^\top, J_\omega(\mu)\rangle \\
&= \sum_\mu -\langle J_\omega(\nabla_\mu(JH)^\top), \bar\mu\rangle - \langle J_\omega(\nabla_{\bar\mu}(JH)^\top), \mu\rangle \\
&= \sum_\mu -\Big\langle \nabla_\mu(J_\omega(JH)^\top) - \nabla_\mu J_\omega((JH)^\top), \bar\mu\Big\rangle - \Big\langle \nabla_{\bar\mu}(J_\omega(JH)^\top) - \nabla_{\bar\mu} J_\omega((JH)^\top), \mu\Big\rangle \\
&= -\tfrac{1}{2}\,\mathrm{div}_M(J_\omega(JH)^\top) + \sum_\mu \langle\nabla_\mu J_\omega((JH)^\top), \bar\mu\rangle + \langle\nabla_{\bar\mu} J_\omega((JH)^\top), \mu\rangle \\
&= -\tfrac{1}{2}\,\mathrm{div}_M(J_\omega(JH)^\top) + \sum_\mu -\langle(JH)^\top, \nabla_\mu J_\omega(\bar\mu)\rangle - \langle(JH)^\top, \nabla_{\bar\mu} J_\omega(\mu)\rangle \\
&= -\tfrac{1}{2}\,\mathrm{div}(J_\omega(JH)^\top) + \langle(JH)^\top, \tfrac{1}{2}\,\delta J_\omega\rangle \;. \blacksquare
\end{aligned}
$$

Using $\mathrm{div}(fX) = f\,\mathrm{div}(X) + df(X)$, with $f = \frac{1}{\sin^2\theta}$, and $X = J_\omega((JH)^\top)$, and that $2\cos\theta d\cos\theta = d\cos^2\theta = -d\sin^2\theta$, we obtain applying Lemma 3.1 to Proposition 3.2

Proposition 3.3. *Away from complex and Lagrangian points*

$$
\begin{aligned}
\triangle\kappa &= \\
&= \cos\theta\left(-2nR + \frac{32}{\sin^2\theta}\sum_{\beta,\mu} R^M(\beta,\mu,\bar\beta,\bar\mu) + \frac{1}{\sin^2\theta}\|\nabla J_\omega\|^2 + \frac{8(n-1)}{\sin^4\theta}\|\nabla\cos\theta\|^2\right) \\
&\quad - \mathrm{div}_M\left(J_\omega\Big(\frac{4n(JH)^\top}{\sin^2\theta}\Big)\right) + g_M\Big(\delta J_\omega, \frac{4n(JH)^\top}{\sin^2\theta}\Big)\;.
\end{aligned}
$$

If $n = 1$ then (M, J_ω, g) is a Kähler manifold (away from Lagrangian points), and so, $\delta J_\omega = \nabla J_\omega = 0$. Obviously the curvature term on M in the expression of $\triangle\kappa$ vanishes. Then, $\triangle\kappa$ reduces to:

Corollary 3.2. *If $n = 1$, away from complex and Lagrangian points*

$$
\triangle\kappa \;=\; -2R\cos\theta - 4\,\mathrm{div}_M\left(J_\omega\Big(\frac{(JH)^\top}{\sin^2\theta}\Big)\right). \tag{3.1}
$$

Now we compute $\triangle \cos^2\theta$ from $\triangle\kappa$ of Proposition 3.3 and applying Proposition 3.1, following step by step the proof of Proposition 4.2 of [S-V, 2]. Recall that, if F has equal Kähler angles at p, then, at p (cf. [S-V,2])

$$\langle SF^*\omega, F^*\omega\rangle \;=\; 16\cos^2\theta \sum_{\rho,\mu} R^M(\rho,\mu,\bar{\rho},\bar{\mu})\ ,$$

where $SF^*\omega$ is the Ricci operator applied to $F^*\omega$, appearing in the Weitzenböck formula (2.2). If (M, J_ω, g_M) is Kähler in a neighbourhood of p, then $\langle SF^*\omega, F^*\omega\rangle = 0$ at p.

Proposition 3.4. *Away from complex and Lagrangian points:*

$$\begin{aligned} n\triangle\cos^2\theta \;=\; & -2n\sin^2\theta\cos^2\theta R \;+\; 2\langle SF^*\omega, F^*\omega\rangle \;+\; 2\,\|\nabla F^*\omega\|^2 \\ & + 4(n-2)\,\|\nabla\,|\sin\theta|\,\|^2 - \;4n\,\mathrm{div}_{g_M}((F^*\omega)^\sharp((JH)^\top)) \\ & - \frac{4n\Big(2+(n-4)\sin^2\theta\Big)}{\sin^2\theta}\;\langle\nabla\cos\theta, J_\omega((JH)^\top)\rangle\ . \end{aligned} \tag{3.2}$$

The last term (3.2) *can be written, for* $n=2$ *as*

$$(3.2) \;=\; 8\,F^*\omega((JH)^\top, \nabla\log\sin^2\theta) \tag{3.3}$$

and for $n\geq 3$,

$$(3.2) \;=\; \frac{4n\Big(2+(n-4)\sin^2\theta\Big)}{\sin^2\theta(n-2)}\;\delta F^*\omega((JH)^\top)\ . \tag{3.4}$$

The expressions in (3.3) and (3.4) come from Proposition 3.1 and the fact that $(F^*\omega)^\sharp = \cos\theta J_\omega$.

Remark 1. Let $\omega^\perp = \omega_{|NM}$ be the restriction of the Kähler form ω to the normal vector bundle NM, and $\omega^\perp = |\omega^\perp| J^\perp$ be its polar decomposition, when we identify it with a skew-symmetric operator on the normal bundle, using the musical isomorphism. Let $\cos\sigma_1 \geq \cos\sigma_2 \geq \ldots \geq \cos\sigma_n \geq 0$ be the eigenvalues of $\omega^\perp$. The σ_α are the Kähler angles of NM. If $\{U_\alpha, V_\alpha\}$ is an orthonormal basis of eigenvectors of $\omega^\perp$ at p, then $\omega^\perp = \sum_\beta \cos\sigma_\beta\, U_*^\beta \wedge V_*^\beta$. For each p, $CD(F) = \bigoplus_{\alpha\,\cos\theta_\alpha=1} \mathrm{span}\{X_\alpha, Y_\alpha\}$ defines the vector subspace of complex directions, or equivalently, the largest J-complex vector subspace contained in T_pM. Similarly we define $CD(NM)$, the largest J-complex subspace of NM at p. Then

$$\begin{aligned} F^*\omega \;&=\; \omega_{|CD(F)} \;+ \sum_{\cos\theta_\alpha<1} \cos\theta_\alpha\, X_*^\alpha \wedge Y_*^\alpha\ , \\ \omega^\perp \;&=\; \omega_{|CD(NM)} \;+ \sum_{\cos\sigma_\alpha<1} \cos\sigma_\alpha\, U_*^\alpha \wedge V_*^\alpha\ . \end{aligned}$$

We define the following morphisms between vector bundles of the same dimension $2n$, where $(\ )^\top$ and $(\ )^\perp$ denote the orthogonal projection onto TM and NM respectively,

$$\begin{array}{rcl} \Phi\colon TM & \to & NM \\ X & \to & (JdF(X))^\perp \end{array} \qquad\qquad \begin{array}{rcl} \Xi\colon NM & \to & TM \\ U & \to & (JU)^\top\ . \end{array}$$

Then $\Phi^{-1}(0) = CD(F)$, $\Xi^{-1}(0) = CD(NM)$. Note that $\forall\, X, Y \in TM$ and $\forall\, U, V \in NM$

$$\begin{array}{rclcrcl} (JdF(X))^\top & = & dF((F^*\omega)^\sharp(X)) & \qquad & (JU)^\perp & = & \omega^\perp(U)\ , \\ \Phi(X) & = & JdF(X) - dF((F^*\omega)^\sharp(X)) & & \Xi(U) & = & JU - \omega^\perp(U)\ . \end{array}$$

A simple computation shows that, if $\cos\theta_\alpha \neq 1$, we may take $U_\alpha = \Phi(\frac{Y_\alpha}{\sin\theta_\alpha})$, and $V_\alpha = \Phi(\frac{X_\alpha}{\sin\theta_\alpha})$. Moreover, $CD(NM) = CD(F)^\perp \cap NM$ and $\dim CD(F) = \dim CD(NM)$. Then $\omega^\perp$ and $F^*\omega$ have the same eigenvalues, that is NM and F have the same Kähler angles. We also define $LD(F) = \mathrm{Ker}\, F^*\omega = \mathcal{K}_\omega$, $LD(NM) = \mathrm{Ker}\,\omega^\perp$ the vector subspaces of Lagrangian directions of F and NM respectively. Then we have $J(LD(F)) = LD(NM)$. Furthermore, $J^\perp \circ \Phi = -\Phi \circ J_\omega$, $J_\omega \circ \Xi = -\Xi \circ J^\perp$, $-\Xi\circ\Phi = Id_{TM} + ((F^*\omega)^\sharp)^2$, $-\Phi\circ\Xi = Id_{NM} + (\omega^\perp)^2$. Considering the Hilbert–Schmidt norms, $\|\Phi\|^2 = \|\Xi\|^2 = 2\sum_\alpha \sin^2\theta_\alpha$. If F has equal Kähler angles, $-\Xi\circ\Phi = \sin^2\theta\, Id_{TM}$, $-\Phi\circ\Xi = \sin^2\theta\, Id_{NM}$, and

$$g(\Phi(X), \Phi(Y)) = \sin^2\theta \langle X, Y\rangle \qquad \langle \Xi(U), \Xi(V)\rangle = \sin^2\theta\, g(U, V)\ .$$

If F has equal Kähler angles, since NM and F have the same Kähler angles, we see that, at a point $p \in M$ such that $H \neq 0$, $(JH)^\top = 0$ iff p is a complex point of F. We also note that, from lemma 3.1 (iv), if F has parallel mean curvature, then $(JH)^\top$ is divergence-free, or equivalentely, $((JH)^\top)^\flat$ is co-closed. ▫

In [S-V,2] we have defined non-negative isotropic scalar curvature, as a less restrictive condition than non-negative isotropic sectional curvature of [Mi-Mo]. If such curvature condition on M holds, then $\sum_{\rho,\mu} R^M(\rho, \mu, \bar\rho, \bar\mu) \geq 0$, where $\{\rho, \bar\rho\}_{1\leq\rho\leq n}$ is the complex basis of T^c_pM defined by a basis of eigenvectors of $F^*\omega$. Hence, if F has equal Kähler angles $\langle SF^*\omega, F^*\omega\rangle \geq 0$. A simple application of the Weitzenböck formula (2.2) shows in next proposition, that such curvature condition on M, implies the angle must be constant. No minimality is required.

Proposition 3.5. **([S-V,2])** *Let F be a non-Lagrangian immersion with equal Kähler angles of a compact orientable M with non-negative isotropic scalar curvature into a Kähler manifold N. If $n = 2$, 3 or 4, then θ is constant and*

(M, J_ω, g_M) is a Kähler manifold. For any $n \geq 1$ and θ constant, $F^*\omega$ is parallel, that is, (M, J_ω, g_M) is a Kähler manifold.

Finally, before we prove Corollary 2.1, we state a more general proposition. Let $F\colon M \to N$ be an immersion with equal Kähler angles, and let $M' = \{p \in M : H = 0\}$ be the set of minimal points of F. On $M \sim \mathcal{C}$ a 1-form is defined

$$\sigma \;=\; \frac{2n}{\sin^2\theta}((JH)^\top)^\flat + \frac{\delta F^*\omega}{\sin^2\theta}\;.$$

Following the proof of [G], but now neither requiring $n = 2$ nor $\delta F^*\omega = 0$, we obtain

$$\begin{aligned} \sigma(X) \;&=\; -\operatorname{trace}\tfrac{1}{\sin^2\theta}\; g(\nabla dF(\cdot, X), JdF(\cdot)) \\ d\sigma(X,Y) \;&=\; Ricci^N(JdF(X), dF(Y)) \;=\; RF^*\omega(X,Y)\;. \end{aligned}$$

We note that this form σ is well known (see e.g. [Br], [Che-M], [W,2]). Now we have:

Proposition 3.6. *If $n = 2$, or if $n \geq 2$ and θ is constant, then $\sigma = \frac{2n}{\sin^2\theta}((JH)^\top)^\flat$ and does not vanish on $M \sim (M' \cup \mathcal{C})$. Moreover, if $R = 0$, then $d\sigma = 0$. Thus, if θ is constant $\neq 0$, $\sigma \in H^1(M, \mathbb{R})$, and in particular, if F has non-zero parallel mean curvature, and $R = 0$, then F is Lagrangian and σ is a non-zero parallel 1-form on M.*

For any immersion with constant equal Kähler angles, the following equalities hold

$$\begin{aligned} R\cos\theta\sin^2\theta \;&=\; \sum_\beta 2d((JH)^\top)^\flat(X_\beta, Y_\beta) \\ &=\; -4n\cos\theta\|H\|^2 - \sum_\mu 8\,\mathrm{Im}\Big(g(\nabla^\perp_\mu H, JdF(\bar\mu))\Big)\;, \end{aligned}$$

where $\{X_\alpha, Y_\alpha\}$ is any basis of eigenvectors of $F^*\omega$.

Proof of Proposition 3.6 and Corollary 2.1: We start by proving Corollary 2.1. For a Lagrangian immersion, the formula on $\triangle\kappa$ (valid on Ω^0_0), reduces to

$$\begin{aligned} 0 \;&=\; \triangle\kappa \\ &=\; \sum_{\mu,\beta} 32\,\mathrm{Im}\Big(R^N\Big(dF(\beta), dF(\mu), dF(\bar\beta), JdF(\bar\mu)\Big)\Big) \\ &\quad - \sum_\mu 16\,n\,\mathrm{Im}\Big(g(\nabla_\mu H, JdF(\bar\mu))\Big)\;. \end{aligned}$$

Applying Codazzi equation to the curvature term and noting that $JdF(TM)$ is the orthogonal complement of $dF(TM)$, and that $\sum_\beta \nabla_\mu \nabla dF(\beta,\bar\beta) = \frac{n}{2}\nabla^\perp_\mu H$, we get

$$0 = \sum_{\beta,\mu} \mathrm{Im}\Big(g\Big(\nabla_\beta \nabla dF(\mu,\bar\beta), JdF(\bar\mu)\Big)\Big)\ . \tag{3.5}$$

Note that, since F is Lagrangian, we can choose arbitrarily the orthonormal frame X_α, Y_α. Then we may assume they have zero covariant derivative at a given point p. Since F is a Lagrangian immersion $g(\nabla dF(\beta,\bar\mu), JdF(\mu)) = g(\nabla dF(\bar\mu,\mu), JdF(\beta))$ (see e.g. [S-V,2]). Taking the derivative of this equality at the point p in the direction $\bar\beta$ we obtain

$$\begin{aligned} g\Big(\nabla_{\bar\beta}\nabla dF(\beta,\bar\mu), JdF(\mu)\Big) + g\Big(\nabla dF(\beta,\bar\mu), J\nabla dF(\bar\beta,\mu)\Big) \ = \\ = \ g\Big(\nabla_{\bar\beta}\nabla dF(\bar\mu,\mu), JdF(\beta)\Big) + g\Big(\nabla dF(\bar\mu,\mu), J\nabla dF(\bar\beta,\beta)\Big)\ . \end{aligned}$$

Taking the summation on μ,β and the imaginary part, we obtain from (3.5)

$$\sum_\beta \mathrm{Im}\Big(g(\nabla_{\bar\beta} H, JdF(\beta))\Big) = \sum_\beta \mathrm{Im}\Big(g(\nabla^\perp_{\bar\beta} H, JdF(\beta))\Big) \ = \ 0\ .$$

From Lemma 3.1 we conclude,

$$\begin{aligned} \frac{1}{2}\, i \sum_\beta d((JH)^\top)^\flat(X_\beta, Y_\beta) \ &= \ -\sum_\beta d((JH)^\top)^\flat(\bar\beta,\beta) \\ &= \sum_\beta -2i\,\mathrm{Im}\, g_M(\nabla_{\bar\beta}(JH)^\top,\beta) \ = \ 0\ . \end{aligned}$$

From the arbitrarity of the orthonormal frame, we may interchange X_1 by $-X_1$, obtaining $d((JH)^\top)^\flat(X_1,Y_1) = 0$. Hence $d((JH)^\top)^\flat = 0$.

Now we prove Proposition 3.6. The first part is an immediate conclusion from the expressions for σ, $d\sigma$, and the fact that, under the above assumptions, $\delta F^*\omega = 0$ (see Corollary 3.1), besides the considerations on the zeroes of $(JH)^\top$ in the previous remark. The conclusion that F is Lagrangian and σ is parallel, under the assumption of non-zero parallel mean curvature and $R = 0$, comes from the equalities stated in the proposition, which we prove now, and from Lemma 4.1 of next section. It is obviously true if $\cos\theta = 1$, that is for complex immersions, and it is true for $\cos\theta = 0$, as we have seen above. Now, if $\cos\theta$ is constant and different from 0 or 1, from Proposition 3.3,

$$0 = \triangle\kappa = \cos\theta\Big(-2nR + \frac{32}{\sin^2\theta}\sum_{\beta,\mu} R^M(\beta,\mu,\bar\beta,\bar\mu) + \frac{1}{\sin^2\theta}\,\|\nabla J_\omega\|^2\Big)$$
$$- \frac{4n}{\sin^2\theta}\,\mathrm{div}_M\Big(J_\omega((JH)^\top)\Big) + \frac{4n}{\sin^2\theta}\,g(\delta J_\omega, (JH)^\top)\ .$$

Since $F^*\omega$ is harmonic (see Corollary 3.1), Weitzenböck formula (2.2) with θ constant reduces to

$$16\cos^2\theta\sum_{\beta,\mu} R^M(\beta,\mu,\bar\beta,\bar\mu) = \langle SF^*\omega, F^*\omega\rangle = -\|\nabla F^*\omega\|^2 = -\frac{1}{2}\cos^2\theta\|\nabla J_\omega\|^2\ .$$

Thus, from lemma 3.1

$$\frac{1}{2}R\cos\theta\sin^2\theta = -\,\mathrm{div}_M\Big(J_\omega((JH)^\top)\Big) + g_M(\delta J_\omega, (JH)^\top)$$
$$= -2n\cos\theta\,\|H\|^2 - 4\sum_\mu \mathrm{Im}\Big(g(\nabla^\perp_\mu H, JdF(\bar\mu))\Big)\ . \blacksquare$$

4 – Proofs of the main results

Proof of Proposition 1.1: Assume $\mathcal{C}\cup\mathcal{L} = \emptyset$. Then the formula in Corollary 3.2 is valid on all M with all maps involved smooth everywhere. By applying Stokes we get $\int_M R\cos\theta\; Vol_M = 0$, where $\cos\theta > 0$, which is impossible if $R \neq 0$. ∎

Proof of Proposition 1.2: Follows immediately from Proposition 3.6. ∎

Proof of Theorem 1.4: In case $n = 1$, $F^*\omega$ is a multiple of the volume element of M, that is $F^*\omega = \cos\tilde\theta\, Vol_M$. This $\tilde\theta$ is the genuine definition of Kähler angle given by Chern and Wolfson [Ch-W]. Our is just $\cos\theta = |\cos\tilde\theta|$. While $\cos\tilde\theta$ is smooth on all M, $\cos\theta$ may not be C^1 at Lagrangian points. But we see that the formula (3.1) is also valid on $M \sim \mathcal{L}\cup\mathcal{C}$ replacing $\cos\theta$ by $\cos\tilde\theta$ and the corresponding replacement of κ by $\tilde\kappa$, and $\sin^2\theta$ by $\sin^2\tilde\theta$ and J_ω by J_M, the natural g_M-orthogonal complex structure on M, defining a Kähler structure. We denote this new formula by $(3.1)'$. Note that on $M\sim\mathcal{L}$, $J_\omega = \pm J_M$, the sign being $+$ or $-$ according to the sign of $\cos\tilde\theta$. Hence a change of the sign of $\cos\tilde\theta$ will give a change of sign on $\tilde\kappa$ and on J_ω (w.r.t. J_M). The formula $(3.1)'$ is in fact also valid on $\mathcal{L}^0$. To see this we use the following lemma, as an immediate consequence of Lemma 3.1 (i):

Lemma 4.1. *If $F\colon M^{2n}\to N^{2n}$ is a submanifold with parallel mean curvature, then $(JH)^\top$ is a parallel vector field along $\mathcal{L}$, that is $\nabla (JH)^\top(p)=0\ \forall p\in\mathcal{L}$.*

Now it follows that $\mathrm{div}_M(J_M((JH)^\top))=0$ on $\mathcal{L}$. (In fact we do not need the assumption of parallel mean curvature to prove this equality on $\mathcal{L}^0$.) Hence, the formula $(3.1)'$ on $\triangle\tilde{\kappa}$ is valid on $\mathcal{L}^0$, that is, at interior Lagrangian points. If we assume $\mathcal{C}=\emptyset$, then $(3.1)'$ is valid over all M, because now $\tilde{\kappa}$, $\cos\tilde{\theta}$, J_M, and $\sin^2\tilde{\theta}$ are smooth everywhere and $\mathcal{L}\sim\mathcal{L}^0$ is a set of Lagrangian points with no interior. Integrating and using Stokes, $2R\int_M\cos\tilde{\theta}=0$. Hence if $\cos\tilde{\theta}$ is non-negative or non-positive everywhere, and if $R\neq 0$, then F is Lagrangian. If F has no Lagrangian points, from Lemma 3.1 (iii), since $\delta J_\omega=0$,

$$\mathrm{div}_M\Big(J_\omega(JH)^\top\Big)=2\cos\theta\,\|H\|^2$$

is valid on M. Integration leads to $H=0$. ∎

Proof of Theorem 1.2: If $n=2$, using (3.3) in the expression of $\triangle\cos^2\theta$ in Proposition 3.4, we get an expression that is smooth away from complex points, and valid at interior Lagrangian points, and hence on all $M\sim\mathcal{C}$. Then, following the same steps in the proofs of [S-V,2] chapter 4, combining the formulae for $\triangle\cos^2\theta$ of Proposition 3.4 and the Weitzenböck formula (2.2), and applying Proposition 3.1, we get, away from complex points

$$\sin^2\theta\cos^2\theta R=-2\,\mathrm{div}_M\Big((F^*\omega)^\sharp((JH)^\top)\Big)+2F^*\omega\Big((JH)^\top,\nabla\log\sin^2\theta\Big)\,. \tag{4.1}$$

Set $P=\sin^2\theta\cos^2\theta R+2\,\mathrm{div}_M((F^*\omega)^\sharp((JH)^\top))$. This map is defined and smooth on all M and vanishes on $\mathcal{C}^0$. If $R>0$ (resp. $R<0$), and under the assumption (1.1), we have from (4.1) that $P\leq 0$ (resp. ≥ 0) on $M\sim\mathcal{C}$. Since the remaining set $\mathcal{C}\sim\mathcal{C}^0$ is a set of empty interior, then $P\leq 0$ (resp. ≥ 0) is valid on all M. In fact, from Proposition 3.1, $|F^*\omega((JH)^\top,\nabla\sin^2\theta)|\leq\sqrt{C}\cos^2\theta\,\sin^2\theta\,\|H\|\,\|(\nabla dF)^{(1,1)}\|$. Since $(\nabla dF)^{(1,1)}$ vanishes on $\mathcal{C}^0$, and so also on $\overline{\mathcal{C}^0}$, we can smoothly extend to zero $F^*\omega((JH)^\top,\nabla\log\sin^2\theta)$ on $\overline{\mathcal{C}^0}$. This we can also get from (4.1). Moreover, such equation tells us we can smoothly extend the last term to all complex points, giving exactly the value $2\,\mathrm{div}_M((F^*\omega)^\sharp((JH)^\top))$ at those points. Integration of $P\leq 0$ (respectively ≥ 0) and applying Stokes, we have

$$\int_M\sin^2\theta\cos^2\theta R\,Vol_M\ \leq\ 0\qquad(\text{resp. }\geq 0)$$

and conclude that F is either complex or Lagrangian. ∎

Proof of Corollary 1.1: Instead of using Stokes on the term $\mathrm{div}_M((F^*\omega)^\sharp((JH)^\top)))$, to make it disapear as we did in the proof of Theorem 1.2, we develop it into

$$\begin{aligned}\mathrm{div}_M\Big((F^*\omega)^\sharp((JH)^\top))\Big) &= \mathrm{div}_M\Big(\cos\theta J_\omega((JH)^\top))\Big)\\ &= \cos\theta\,\mathrm{div}_M\Big(J_\omega((JH)^\top))\Big) + d\cos\theta\Big(J_\omega((JH)^\top))\Big)\ ,\end{aligned}$$

and use Lemma 3.1 to give, away from complex and Lagrangian points,

$$\begin{aligned}\sin^2\theta\cos^2\theta R &= -2\cos\theta\,\mathrm{div}_M\Big(J_\omega((JH)^\top)\Big) - 2\Big\langle J_\omega((JH)^\top), \nabla\cos\theta\Big\rangle\\ &\quad + 2F^*\omega\Big((JH)^\top, \nabla\log\sin^2\theta\Big)\\ &= -8\cos^2\theta\|H\|^2 + 2F^*\omega\Big((JH)^\top, \nabla\log\sin^2\theta\Big)\ .\end{aligned}$$

Hence, away from complex and Lagrangian points

$$\sin^4\theta\cos^2\theta R + 8\sin^2\theta\cos^2\theta\|H\|^2 = 2F^*\omega\Big((JH)^\top, \nabla\sin^2\theta\Big)\ .$$

Obviously, this equality also holds at Lagrangian and complex points, for, those points are critical points for $\sin^2\theta$. The corollary now follows immediately from Theorem 1.2. ∎

Proof of Theorem 1.3: If $n \geq 3$ we set

$$\begin{aligned}P &= n\,\triangle\cos^2\theta + 4n\,\mathrm{div}_M\Big((F^*\omega)^\sharp((JH)^\top)\Big) + 2n\sin^2\theta\cos^2\theta R\\ &\quad - 2\|\nabla F^*\omega\|^2 - 2\langle SF^*\omega, F^*\omega\rangle\ .\end{aligned}$$

This map is defined on all M and is smooth. From Proposition (3.4) and using (3.4), on $M \sim \mathcal{C}$

$$P = \frac{4n\Big(2 + (n-4)\sin^2\theta\Big)}{(n-2)\sin^2\theta}\,\delta F^*\omega((JH)^\top) + 4(n-2)\|\nabla|\sin\theta|\|^2\ .$$

In (A) and (B), by assumption, $P \geq 0$ on $M \sim \mathcal{C}$, because for $n \geq 3$, $(2 + (n-4)\sin^2\theta) \geq 0$. But on $\mathcal{C}^0$, $P = 0$, for (M, J_ω, g_M) is a complex submanifold, and so, $(JH)^\top = 0$ and $\langle SF^*\omega, F^*\omega\rangle = 0$. Thus, $P \geq 0$ on all M. Integrating $P \geq 0$ on M we obtain using Stokes, Weitzenböck formula (2.2), and (2.3)

$$\int_M 2nR\sin^2\theta\cos^2\theta\,Vol_M \geq \int_M 2\|\delta F^*\omega\|^2\,Vol_M\ .$$

Thus, if $R < 0$ we conclude F is either complex or Lagrangian, and if $R = 0$ we conclude that $\delta F^*\omega = 0$, which implies, by Corollary 3.1, that θ is constant. This last reasoning proves (C) as well. ∎

Remark 2. In Theorem 1.3 we can replace the condition $\delta F^*\omega((JH)^\top) \geq 0$ by a weaker condition

$$\delta F^*\omega((JH)^\top) \;\geq\; -\frac{(n-2)^2}{4n\Big(2 + (n{-}4)\sin^2\theta\Big)}\;\|\nabla\cos^2\theta\|^2$$

to achieve the same conclusion. This condition is sufficient to obtain $P \geq 0$ in the above proof. Then we can obtain for $n \geq 3$ a corollary similar to Corollary 1.1, by requiring

$$4n^2\cos^2\theta\|H\|^2 + \,n\sin^2\theta\cos^2\theta R - (n{-}2)^2\|\nabla\cos\theta\|^2 \;\geq\; -2n\,\delta F^*\omega((JH)^\top)\;. \;\square$$

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