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Division Algebras that Ramify Only Along a Singular Plane Cubic Curve

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Abstract. Let K be the field of rational functions in 2 variables over an algebraically closed field k of characteristic 0. Let D be a finite dimensional K-central division algebra whose ramification divisor on the projective plane over k is a singular cubic curve. It is shown that D is cyclic and that the exponent of D is equal to the degree of D.

Let k be an algebraically closed field of characteristic 0. Let $\mathbb{P}^2 = \operatorname{Proj} k[x,y,z]$ denote the projective plane over k and K the function field of $\mathbb{P}^2$. We view K as the set of all rational functions of the form $f/g \in k(x,y,z)$ where f and g are homogeneous forms in $k[x,y,z]$ of the same degree.

The Brauer group of the projective plane, $\mathrm{B}(\mathbb{P}^2)$, is trivial. Therefore a division algebra D that is central and finite dimensional over K necessarily ramifies at some prime divisor of $\mathbb{P}^2$. By [1, Theorem 1] there is a canonical exact sequence

$$0 \longrightarrow \mathrm{B}(K) \xrightarrow{\ a\ } \bigoplus_C \mathrm{H}^1(K(C), \mathbb{Q}/\mathbb{Z}) \ . \tag{1}$$

The map a measures the ramification of a central K-division algebra D along a prime divisor C on $\mathbb{P}^2$. The group $\mathrm{H}^1(K(C), \mathbb{Q}/\mathbb{Z})$ is the first étale cohomology group of the function field $K(C)$ of C, with coefficients in the constant sheaf $\mathbb{Q}/\mathbb{Z}$. By Kummer theory [4, pp. 125–126] $\mathrm{H}^1(K(C), \mathbb{Q}/\mathbb{Z})$ classifies the finite cyclic Galois extensions of $K(C)$. The "ramification of D along C" is a cyclic extension L of $K(C)$ obtained in the following way. Let A be a maximal order for D over the local discrete valuation ring $\mathcal{O}_C$. Then $L = A \otimes K(C)/(\mathit{radical})$ is a cyclic extension of $K(C)$, which represents an element of $\mathrm{H}^1(K(C), \mathbb{Q}/\mathbb{Z})$. Those C for which L is non-trivial make up the *ramification divisor* of D. A division algebra D is completely determined by its ramification data.

In this article we consider the case where D is a finite dimensional K-central division algebra whose ramification divisor is a reduced cubic curve C that is singular. Our main result is Theorem 1 below which states that every such algebra D is a cyclic algebra with exponent(D) = degree(D). By exponent(D) we mean the exponent of the class of D in the Brauer group $\mathrm{B}(K)$. By degree(D) we mean the square root of the dimension of the vector space D over K.

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If D has ramification divisor C, a nonsingular cubic curve on $\mathbb{P}^2$, then it is known that exponent(D) = degree(D). The reader is referred to [3] and its bibliography for a discussion of this case. M. Van den Bergh has recently announced a proof that if D has odd exponent, then D is cyclic.

In our context, each irreducible component of C is a rational curve whose normalization is isomorphic to $\mathbb{P}^1$. Let C be a reduced curve on $\mathbb{P}^2$ each of whose irreducible components is a rational curve. Write $C = C_1 \cup \cdots \cup C_m$ as a union of irreducible curves. Let $\tilde{C}_i$ denote the normalization of C_i. By our assumption $\tilde{C}_i \cong \mathbb{P}^1$. Let $\tilde{C}$ be the disjoint union $\tilde{C}_1 \coprod \cdots \coprod \tilde{C}_m$. Let Z denote the singular locus of C, which is a finite set of points, hence $Z = \{Z_1, \ldots, Z_s\}$. Let $\pi : \tilde{C} \to C$ be the natural projection and $W = \pi^{-1}(Z)$. Then W is a finite set of points, hence $W = \{W_1, \ldots, W_e\}$. The square

$$
\begin{array}{ccc}
W & \longrightarrow & \tilde{C} \\
\Big\downarrow \pi & & \Big\downarrow \pi \\
Z & \longrightarrow & C
\end{array}
\tag{2}
$$

is commutative. Define a graph $\Gamma = \Gamma(C)$. The vertex set of Γ is $\{Z_1, \ldots, Z_s, \tilde{C}_1, \ldots, \tilde{C}_m\}$ and the edge set is $\{W_1, \ldots, W_e\}$. The edge W_i has positive end the $\tilde{C}_j$ containing W_i and negative end the Z_t defined by $Z_t = \pi(W_i)$. Let M be the incidence matrix of Γ. Then M induces a boundary map, also denoted M,

$$
M : (\mathbb{Z}/n)^{(e)} \to (\mathbb{Z}/n)^{(m)} \oplus (\mathbb{Z}/n)^{(s)} \tag{3}
$$

for any positive integer n. The kernel of M is the combinatorial cycle space $\mathrm{H}_1(\Gamma, \mathbb{Z}/n)$ of Γ. Since we are assuming each $\tilde{C}_i \cong \mathbb{P}^1$ is simply connected, it follows that $\mathrm{H}^1(C, \mathbb{Z}/n) = 0$. Since $\mathbb{P}^2$ is simply connected, $\mathrm{H}^1(\mathbb{P}^2, \mathbb{Z}/n) \cong \mathrm{H}^3(\mathbb{P}^2, \mathbb{Z}/n) = 0$. Combining Lemma 0.1 and Corollary 1.3 of [2], there is an isomorphism ${}_n\mathrm{B}(\mathbb{P}^2 - C) \cong \mathrm{H}_1(\Gamma, \mathbb{Z}/n)$. Therefore the K-division algebras D with exponent dividing n and that ramify only along C make up a subgroup of $\mathrm{B}(K)$ that is isomorphic to $\mathrm{H}_1(\Gamma, \mathbb{Z}/n)$.

Let α, β be elements of K, $n \geq 2$ an integer, and ζ a fixed nth root of unity in K. The symbol algebra $(\alpha, \beta)_n$ is the associative K-algebra generated by u, v subject to the relations $u^n = \alpha$, $v^n = \beta$, $uv = \zeta vu$. The ramification divisor of the algebra $(\alpha, \beta)_n$ is contained in the union of the sets of zeros and poles of the functions α and β on $\mathbb{P}^2$.

The main tool used in proving Theorem 1 is [2, Theorem 2.1] which tells us how to map a symbol algebra $(\alpha, \beta)_n$ over K to a sum of weighted edges in the graph Γ. This sum of weighted edges is an element in the edge space, $\mathbb{Z}/n^{(e)}$, that is in $\ker M = \mathrm{H}_1(\Gamma, \mathbb{Z}/n)$. According to [2, Theorem 2.1], the weights on the edges of the graph can be computed in terms of the local intersection multiplicities of the various components of α and β. Suppose the zeros and poles of α and β are contained in C. Let $P \in Z$ be a singular point on C. Let $A_1, \ldots, A_t$ be the components of C corresponding to vertices in Γ that are adjacent to P, as shown in Figure 1. Assume first that the curve $\tilde{A}_1$ has only one point W_1 lying over P. Then the weight (as

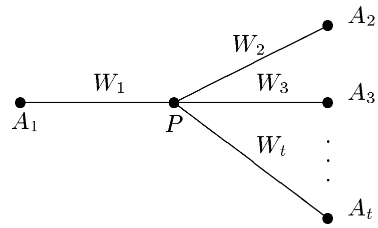

FIGURE 1

an element of $\mathbb{Z}/n$) assigned to the edge W_1 connecting P to A_1 is

$$\sum_{i=2}^{t} [v_1(\beta)v_i(\alpha) - v_1(\alpha)v_i(\beta)]\,(A_1.A_i)_P \ , \tag{4}$$

where $(A_1.A_i)_P$ is the local intersection multiplicity and v_i is the discrete valuation on K given by the local ring $\mathcal{O}_{A_i}$. If A_1 has multiple tangents at P, then there will be several edges connecting A_1 to P in Γ. In this case (4) gives the weight for any one branch W_1 of A_1 at P where instead of $(A_1.A_i)_P$ the local intersection multiplicity for the branch that is associated with W_1 is used.

Theorem 1. *Let C be a reduced cubic curve in $\mathbb{P}^2$ and assume C is singular. Let D be a finite dimensional central K-division algebra whose ramification divisor on $\mathbb{P}^2$ is C. Then D is a cyclic algebra and* $\text{exponent}(D) = \text{degree}(D)$.

Proof. Let n be the exponent of the class of D in the Brauer group of K. We use the techniques of [2, Sec. 2] that were mentioned above. Upon desingularization, the singular cubic C consists of one, two or three components each of which is isomorphic to $\mathbb{P}^1$. Therefore the subgroup of $\mathrm{B}(K)$ consisting of classes of division algebras annihilated by n that ramify only along C is isomorphic to $\mathrm{H}_1(\Gamma, \mathbb{Z}/n)$. Here Γ is the graph associated to C and H_1 is simply the combinatorial cycle space of the graph. In each example below, Γ is a planar graph hence the $\mathbb{Z}/n$-rank of $\mathrm{H}_1(\Gamma, \mathbb{Z}/n)$ simply counts the number of regions of Γ.

There are only 6 cases to consider. In each case we show that D is a symbol algebra $(\alpha, \beta)_n$ hence is cyclic.

Case 1: C is irreducible and has a cuspidal singularity. In this case C is simply connected, $\mathrm{H}_1(\Gamma(C), \mathbb{Z}/n) = 0$, hence no non-trivial division algebra can have ramification divisor equal to C.

Case 2: C is irreducible and has a nodal singularity. Let $l_1 = 0$ and $l_2 = 0$ be the equations of the tangent lines to C at the node. The line $l_1 = 0$ intersects the first branch of C with multiplicity 2 and the second branch with multiplicity 1. Similarly, l_2 intersects the first branch of C with multiplicity 1 and the second branch with multiplicity 2. Consider the symbol algebra

$$\Lambda = \left(\frac{l_1}{l_2}, \frac{c}{l_2^3}\right)_n$$

over K. The ramification divisor of Λ must be contained in the curve $l_1 l_2 c = 0$. The graph of $l_1 l_2 c = 0$ is shown in Figure 2. Let W_1 denote the edge of Γ corresponding

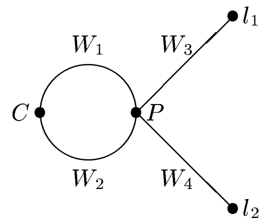

FIGURE 2. The graph for the symbol Λ in Case 2.

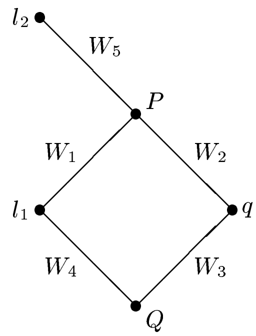

FIGURE 3. The graph for the symbol Λ in Case 3.

to the first branch of C. We apply (4) to determine the weights w_i for the edges W_i of the element in the cycle space corresponding to Λ. In the notation above, we have $\alpha = l_1/l_2$, $\beta = c/l_2^3$, A_1 is the first branch of C, A_2 is the curve $l_1 = 0$, A_3 is the curve $l_2 = 0$, $(A_1.A_2)_P = 2$, $(A_1.A_3)_P = 1$, $v_1(\beta) = 1$, $v_2(\beta) = 0$, $v_3(\beta) = -3$, $v_1(\alpha) = 0$, $v_2(\alpha) = 1$, and $v_3(\alpha) = -1$. From (4) we have

$$w_1 = [(1)(1) - (0)(0)](2) + [(1)(-1) - (0)(-3)](1) = +1 \ .$$

To compute w_2 using (4), we have A_1 is the second branch of C, A_2 is the curve $l_1 = 0$, A_3 is the curve $l_2 = 0$, $(A_1.A_2)_P = 1$, $(A_1.A_3)_P = 2$, and the v_i values are the same as for w_1. From (4) we have

$$w_2 = [(1)(1) - (0)(0)](1) + [(1)(-1) - (0)(-3)](2) = -1 \ .$$

To compute w_3 using (4), we have A_1 is the curve $l_1 = 0$, $A_2 = C$, A_3 is the curve $l_2 = 0$, $(A_1.A_2)_P = 3$, $(A_1.A_3)_P = 1$, $v_1(\beta) = 0$, $v_2(\beta) = 1$, $v_3(\beta) = -3$, $v_1(\alpha) = 1$, $v_2(\alpha) = 0$, and $v_3(\alpha) = -1$. From (4) we have

$$w_3 = [(0)(0) - (1)(1)](3) + [(0)(-1) - (1)(-3)](1) = 0 \ .$$

Similarly, using (4) we find $w_4 = 0$. Therefore Λ has ramification divisor C and exponent n. Since ${}_n\,\mathrm{B}(\mathbb{P}^2 - C) \cong \mathrm{H}_1(\Gamma, \mathbb{Z}/n) \cong \mathbb{Z}/n$ we see that every algebra class of exponent n is some power of the class of Λ, and therefore has degree n.

Case 3: C factors into a line and an irreducible conic and has 2 nodes. Let $q = 0$ be the equation of the conic and $l_1 = 0$ the equation of the line. Let P and

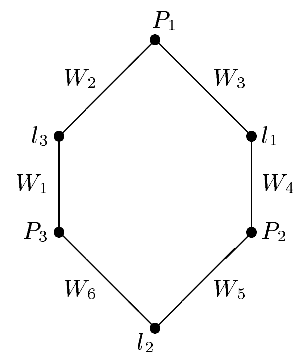

FIGURE 4. The graph for the symbol Λ in Case 5.

Q denote the 2 nodes of C. Let $l_2 = 0$ be the equation of the tangent to $q = 0$ at P. Consider the symbol algebra

$$\Lambda = \left(\frac{l_1}{l_2}, \frac{q}{l_2^2}\right)_n$$

over K. The ramification divisor of Λ is contained in the curve $l_1 l_2 q = 0$. The graph for Λ is shown in Figure 3. We apply (4) to compute the weight w_1 of edge W_1 for the algebra Λ. In the notation above, we have $\alpha = l_1/l_2$, $\beta = q/l_2^2$, A_1 is the curve $l_1 = 0$, A_2 is the curve $l_2 = 0$, A_3 is the curve $q = 0$, $(A_1.A_2)_P = 1$, $(A_1.A_3)_P = 1$, $v_1(\alpha) = 1$, $v_2(\alpha) = -1$, $v_3(\alpha) = 0$, $v_1(\beta) = 0$, $v_2(\beta) = -2$, and $v_3(\beta) = 1$. From (4) we have

$$w_1 = [(0)(-1) - (1)(-2)](1) + [(0)(0) - (1)(1)](1) = +1 .$$

Similarly we compute $w_2 = -1$, $w_3 = +1$, $w_4 = -1$, and $w_5 = 0$. Therefore Λ has ramification divisor C and exponent n. Since ${}_n\mathrm{B}(\mathbb{P}^2 - C) \cong \mathrm{H}_1(\Gamma, \mathbb{Z}/n) \cong \mathbb{Z}/n$ we see that every algebra class of exponent n is some power of the one given, and therefore has degree n.

Case 4: C factors into a line and an irreducible conic and has a cuspidal singularity. In this case C is simply connected, $\mathrm{H}_1(\Gamma, \mathbb{Z}/n) = 0$, hence no division algebra can have ramification divisor equal to C.

Case 5: C factors into 3 lines and has 3 nodes. Let the equation of C be written $l_1 l_2 l_3 = 0$ where each l_i is a linear form. Consider the symbol algebra

$$\Lambda = \left(\frac{l_1}{l_3}, \frac{l_2}{l_3}\right)_n$$

over K. The graph for Λ in this case is the hexagon shown in Figure 4. Using (4) and the same ideas as in the earlier cases, we find that the ramification divisor of Λ is C and exponent$(\Lambda) = n$. Since ${}_n\mathrm{B}(\mathbb{P}^2 - C) \cong \mathrm{H}_1(\Gamma, \mathbb{Z}/n) \cong \mathbb{Z}/n$ we see that every algebra class of exponent n is some power of the one given, and therefore has degree n.

Case 6: C factors into 3 lines and has 1 singular point. In this case C is simply connected, $\mathrm{H}_1(\Gamma, \mathbb{Z}/n) = 0$, hence no division algebra can have ramification divisor equal to C. □

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