

## Division Algebras that Ramify Only Along a Singular Plane Cubic Curve

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ABSTRACT. Let  $K$  be the field of rational functions in 2 variables over an algebraically closed field  $k$  of characteristic 0. Let  $D$  be a finite dimensional  $K$ -central division algebra whose ramification divisor on the projective plane over  $k$  is a singular cubic curve. It is shown that  $D$  is cyclic and that the exponent of  $D$  is equal to the degree of  $D$ .

Let  $k$  be an algebraically closed field of characteristic 0. Let  $\mathbb{P}^2 = \text{Proj } k[x, y, z]$  denote the projective plane over  $k$  and  $K$  the function field of  $\mathbb{P}^2$ . We view  $K$  as the set of all rational functions of the form  $f/g \in k(x, y, z)$  where  $f$  and  $g$  are homogeneous forms in  $k[x, y, z]$  of the same degree.

The Brauer group of the projective plane,  $B(\mathbb{P}^2)$ , is trivial. Therefore a division algebra  $D$  that is central and finite dimensional over  $K$  necessarily ramifies at some prime divisor of  $\mathbb{P}^2$ . By [1, Theorem 1] there is a canonical exact sequence

$$(1) \quad 0 \longrightarrow B(K) \xrightarrow{a} \bigoplus_C H^1(K(C), \mathbb{Q}/\mathbb{Z}) .$$

The map  $a$  measures the ramification of a central  $K$ -division algebra  $D$  along a prime divisor  $C$  on  $\mathbb{P}^2$ . The group  $H^1(K(C), \mathbb{Q}/\mathbb{Z})$  is the first étale cohomology group of the function field  $K(C)$  of  $C$ , with coefficients in the constant sheaf  $\mathbb{Q}/\mathbb{Z}$ . By Kummer theory [4, pp. 125–126]  $H^1(K(C), \mathbb{Q}/\mathbb{Z})$  classifies the finite cyclic Galois extensions of  $K(C)$ . The “ramification of  $D$  along  $C$ ” is a cyclic extension  $L$  of  $K(C)$  obtained in the following way. Let  $A$  be a maximal order for  $D$  over the local discrete valuation ring  $\mathcal{O}_C$ . Then  $L = A \otimes K(C)/(\text{radical})$  is a cyclic extension of  $K(C)$ , which represents an element of  $H^1(K(C), \mathbb{Q}/\mathbb{Z})$ . Those  $C$  for which  $L$  is non-trivial make up the *ramification divisor* of  $D$ . A division algebra  $D$  is completely determined by its ramification data.

In this article we consider the case where  $D$  is a finite dimensional  $K$ -central division algebra whose ramification divisor is a reduced cubic curve  $C$  that is singular. Our main result is Theorem 1 below which states that every such algebra  $D$  is a cyclic algebra with  $\text{exponent}(D) = \text{degree}(D)$ . By  $\text{exponent}(D)$  we mean the exponent of the class of  $D$  in the Brauer group  $B(K)$ . By  $\text{degree}(D)$  we mean the square root of the dimension of the vector space  $D$  over  $K$ .

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If  $D$  has ramification divisor  $C$ , a nonsingular cubic curve on  $\mathbb{P}^2$ , then it is known that  $\text{exponent}(D) = \text{degree}(D)$ . The reader is referred to [3] and its bibliography for a discussion of this case. M. Van den Bergh has recently announced a proof that if  $D$  has odd exponent, then  $D$  is cyclic.

In our context, each irreducible component of  $C$  is a rational curve whose normalization is isomorphic to  $\mathbb{P}^1$ . Let  $C$  be a reduced curve on  $\mathbb{P}^2$  each of whose irreducible components is a rational curve. Write  $C = C_1 \cup \dots \cup C_m$  as a union of irreducible curves. Let  $\tilde{C}_i$  denote the normalization of  $C_i$ . By our assumption  $\tilde{C}_i \cong \mathbb{P}^1$ . Let  $\tilde{C}$  be the disjoint union  $\tilde{C}_1 \amalg \dots \amalg \tilde{C}_m$ . Let  $Z$  denote the singular locus of  $C$ , which is a finite set of points, hence  $Z = \{Z_1, \dots, Z_s\}$ . Let  $\pi : \tilde{C} \rightarrow C$  be the natural projection and  $W = \pi^{-1}(Z)$ . Then  $W$  is a finite set of points, hence  $W = \{W_1, \dots, W_e\}$ . The square

$$(2) \quad \begin{array}{ccc} W & \longrightarrow & \tilde{C} \\ \downarrow \pi & & \downarrow \pi \\ Z & \longrightarrow & C \end{array}$$

is commutative. Define a graph  $\Gamma = \Gamma(C)$ . The vertex set of  $\Gamma$  is  $\{Z_1, \dots, Z_s, \tilde{C}_1, \dots, \tilde{C}_m\}$  and the edge set is  $\{W_1, \dots, W_e\}$ . The edge  $W_i$  has positive end the  $\tilde{C}_j$  containing  $W_i$  and negative end the  $Z_t$  defined by  $Z_t = \pi(W_i)$ . Let  $M$  be the incidence matrix of  $\Gamma$ . Then  $M$  induces a boundary map, also denoted  $M$ ,

$$(3) \quad M : (\mathbb{Z}/n)^{(e)} \rightarrow (\mathbb{Z}/n)^{(m)} \oplus (\mathbb{Z}/n)^{(s)}$$

for any positive integer  $n$ . The kernel of  $M$  is the combinatorial cycle space  $H_1(\Gamma, \mathbb{Z}/n)$  of  $\Gamma$ . Since we are assuming each  $\tilde{C}_i \cong \mathbb{P}^1$  is simply connected, it follows that  $H^1(C, \mathbb{Z}/n) = 0$ . Since  $\mathbb{P}^2$  is simply connected,  $H^1(\mathbb{P}^2, \mathbb{Z}/n) \cong H^3(\mathbb{P}^2, \mathbb{Z}/n) = 0$ . Combining Lemma 0.1 and Corollary 1.3 of [2], there is an isomorphism  ${}_n B(\mathbb{P}^2 - C) \cong H_1(\Gamma, \mathbb{Z}/n)$ . Therefore the  $K$ -division algebras  $D$  with exponent dividing  $n$  and that ramify only along  $C$  make up a subgroup of  $B(K)$  that is isomorphic to  $H_1(\Gamma, \mathbb{Z}/n)$ .

Let  $\alpha, \beta$  be elements of  $K$ ,  $n \geq 2$  an integer, and  $\zeta$  a fixed  $n$ th root of unity in  $K$ . The symbol algebra  $(\alpha, \beta)_n$  is the associative  $K$ -algebra generated by  $u, v$  subject to the relations  $u^n = \alpha, v^n = \beta, uv = \zeta vu$ . The ramification divisor of the algebra  $(\alpha, \beta)_n$  is contained in the union of the sets of zeros and poles of the functions  $\alpha$  and  $\beta$  on  $\mathbb{P}^2$ .

The main tool used in proving Theorem 1 is [2, Theorem 2.1] which tells us how to map a symbol algebra  $(\alpha, \beta)_n$  over  $K$  to a sum of weighted edges in the graph  $\Gamma$ . This sum of weighted edges is an element in the edge space,  $\mathbb{Z}/n^{(e)}$ , that is in  $\ker M = H_1(\Gamma, \mathbb{Z}/n)$ . According to [2, Theorem 2.1], the weights on the edges of the graph can be computed in terms of the local intersection multiplicities of the various components of  $\alpha$  and  $\beta$ . Suppose the zeros and poles of  $\alpha$  and  $\beta$  are contained in  $C$ . Let  $P \in Z$  be a singular point on  $C$ . Let  $A_1, \dots, A_t$  be the components of  $C$  corresponding to vertices in  $\Gamma$  that are adjacent to  $P$ , as shown in Figure 1. Assume first that the curve  $\tilde{A}_1$  has only one point  $W_1$  lying over  $P$ . Then the weight (as

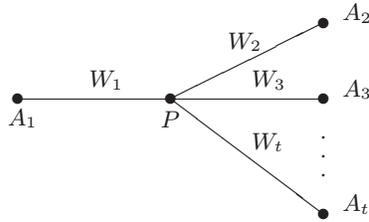


FIGURE 1

an element of  $\mathbb{Z}/n$ ) assigned to the edge  $W_1$  connecting  $P$  to  $A_1$  is

$$(4) \quad \sum_{i=2}^t [v_1(\beta)v_i(\alpha) - v_1(\alpha)v_i(\beta)] (A_1.A_i)_P ,$$

where  $(A_1.A_i)_P$  is the local intersection multiplicity and  $v_i$  is the discrete valuation on  $K$  given by the local ring  $\mathcal{O}_{A_i}$ . If  $A_1$  has multiple tangents at  $P$ , then there will be several edges connecting  $A_1$  to  $P$  in  $\Gamma$ . In this case (4) gives the weight for any one branch  $W_1$  of  $A_1$  at  $P$  where instead of  $(A_1.A_i)_P$  the local intersection multiplicity for the branch that is associated with  $W_1$  is used.

**Theorem 1.** *Let  $C$  be a reduced cubic curve in  $\mathbb{P}^2$  and assume  $C$  is singular. Let  $D$  be a finite dimensional central  $K$ -division algebra whose ramification divisor on  $\mathbb{P}^2$  is  $C$ . Then  $D$  is a cyclic algebra and  $\text{exponent}(D) = \text{degree}(D)$ .*

**Proof.** Let  $n$  be the exponent of the class of  $D$  in the Brauer group of  $K$ . We use the techniques of [2, Sec. 2] that were mentioned above. Upon desingularization, the singular cubic  $C$  consists of one, two or three components each of which is isomorphic to  $\mathbb{P}^1$ . Therefore the subgroup of  $B(K)$  consisting of classes of division algebras annihilated by  $n$  that ramify only along  $C$  is isomorphic to  $H_1(\Gamma, \mathbb{Z}/n)$ . Here  $\Gamma$  is the graph associated to  $C$  and  $H_1$  is simply the combinatorial cycle space of the graph. In each example below,  $\Gamma$  is a planar graph hence the  $\mathbb{Z}/n$ -rank of  $H_1(\Gamma, \mathbb{Z}/n)$  simply counts the number of regions of  $\Gamma$ .

There are only 6 cases to consider. In each case we show that  $D$  is a symbol algebra  $(\alpha, \beta)_n$  hence is cyclic.

**Case 1:**  $C$  is irreducible and has a cuspidal singularity. In this case  $C$  is simply connected,  $H_1(\Gamma(C), \mathbb{Z}/n) = 0$ , hence no non-trivial division algebra can have ramification divisor equal to  $C$ .

**Case 2:**  $C$  is irreducible and has a nodal singularity. Let  $l_1 = 0$  and  $l_2 = 0$  be the equations of the tangent lines to  $C$  at the node. The line  $l_1 = 0$  intersects the first branch of  $C$  with multiplicity 2 and the second branch with multiplicity 1. Similarly,  $l_2$  intersects the first branch of  $C$  with multiplicity 1 and the second branch with multiplicity 2. Consider the symbol algebra

$$\Lambda = \left( \frac{l_1}{l_2}, \frac{c}{l_2^3} \right)_n$$

over  $K$ . The ramification divisor of  $\Lambda$  must be contained in the curve  $l_1 l_2 c = 0$ . The graph of  $l_1 l_2 c = 0$  is shown in Figure 2. Let  $W_1$  denote the edge of  $\Gamma$  corresponding

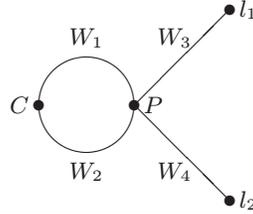


FIGURE 2. The graph for the symbol  $\Lambda$  in Case 2.

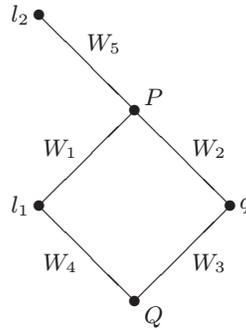


FIGURE 3. The graph for the symbol  $\Lambda$  in Case 3.

to the first branch of  $C$ . We apply (4) to determine the weights  $w_i$  for the edges  $W_i$  of the element in the cycle space corresponding to  $\Lambda$ . In the notation above, we have  $\alpha = l_1/l_2$ ,  $\beta = c/l_2^3$ ,  $A_1$  is the first branch of  $C$ ,  $A_2$  is the curve  $l_1 = 0$ ,  $A_3$  is the curve  $l_2 = 0$ ,  $(A_1.A_2)_P = 2$ ,  $(A_1.A_3)_P = 1$ ,  $v_1(\beta) = 1$ ,  $v_2(\beta) = 0$ ,  $v_3(\beta) = -3$ ,  $v_1(\alpha) = 0$ ,  $v_2(\alpha) = 1$ , and  $v_3(\alpha) = -1$ . From (4) we have

$$w_1 = [(1)(1) - (0)(0)](2) + [(1)(-1) - (0)(-3)](1) = +1 .$$

To compute  $w_2$  using (4), we have  $A_1$  is the second branch of  $C$ ,  $A_2$  is the curve  $l_1 = 0$ ,  $A_3$  is the curve  $l_2 = 0$ ,  $(A_1.A_2)_P = 1$ ,  $(A_1.A_3)_P = 2$ , and the  $v_i$  values are the same as for  $w_1$ . From (4) we have

$$w_2 = [(1)(1) - (0)(0)](1) + [(1)(-1) - (0)(-3)](2) = -1 .$$

To compute  $w_3$  using (4), we have  $A_1$  is the curve  $l_1 = 0$ ,  $A_2 = C$ ,  $A_3$  is the curve  $l_2 = 0$ ,  $(A_1.A_2)_P = 3$ ,  $(A_1.A_3)_P = 1$ ,  $v_1(\beta) = 0$ ,  $v_2(\beta) = 1$ ,  $v_3(\beta) = -3$ ,  $v_1(\alpha) = 1$ ,  $v_2(\alpha) = 0$ , and  $v_3(\alpha) = -1$ . From (4) we have

$$w_3 = [(0)(0) - (1)(1)](3) + [(0)(-1) - (1)(-3)](1) = 0 .$$

Similarly, using (4) we find  $w_4 = 0$ . Therefore  $\Lambda$  has ramification divisor  $C$  and exponent  $n$ . Since  ${}_n B(\mathbb{P}^2 - C) \cong H_1(\Gamma, \mathbb{Z}/n) \cong \mathbb{Z}/n$  we see that every algebra class of exponent  $n$  is some power of the class of  $\Lambda$ , and therefore has degree  $n$ .

**Case 3:**  $C$  factors into a line and an irreducible conic and has 2 nodes. Let  $q = 0$  be the equation of the conic and  $l_1 = 0$  the equation of the line. Let  $P$  and

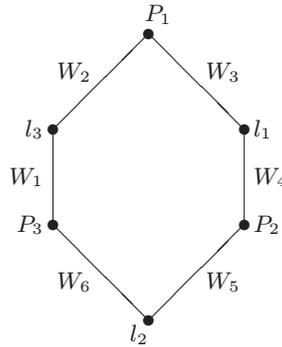


FIGURE 4. The graph for the symbol  $\Lambda$  in Case 5.

$Q$  denote the 2 nodes of  $C$ . Let  $l_2 = 0$  be the equation of the tangent to  $q = 0$  at  $P$ . Consider the symbol algebra

$$\Lambda = \left( \frac{l_1}{l_2}, \frac{q}{l_2^2} \right)_n$$

over  $K$ . The ramification divisor of  $\Lambda$  is contained in the curve  $l_1 l_2 q = 0$ . The graph for  $\Lambda$  is shown in Figure 3. We apply (4) to compute the weight  $w_1$  of edge  $W_1$  for the algebra  $\Lambda$ . In the notation above, we have  $\alpha = l_1/l_2$ ,  $\beta = q/l_2^2$ ,  $A_1$  is the curve  $l_1 = 0$ ,  $A_2$  is the curve  $l_2 = 0$ ,  $A_3$  is the curve  $q = 0$ ,  $(A_1.A_2)_P = 1$ ,  $(A_1.A_3)_P = 1$ ,  $v_1(\alpha) = 1$ ,  $v_2(\alpha) = -1$ ,  $v_3(\alpha) = 0$ ,  $v_1(\beta) = 0$ ,  $v_2(\beta) = -2$ , and  $v_3(\beta) = 1$ . From (4) we have

$$w_1 = [(0)(-1) - (1)(-2)](1) + [(0)(0) - (1)(1)](1) = +1 .$$

Similarly we compute  $w_2 = -1$ ,  $w_3 = +1$ ,  $w_4 = -1$ , and  $w_5 = 0$ . Therefore  $\Lambda$  has ramification divisor  $C$  and exponent  $n$ . Since  ${}_n B(\mathbb{P}^2 - C) \cong H_1(\Gamma, \mathbb{Z}/n) \cong \mathbb{Z}/n$  we see that every algebra class of exponent  $n$  is some power of the one given, and therefore has degree  $n$ .

**Case 4:**  $C$  factors into a line and an irreducible conic and has a cuspidal singularity. In this case  $C$  is simply connected,  $H_1(\Gamma, \mathbb{Z}/n) = 0$ , hence no division algebra can have ramification divisor equal to  $C$ .

**Case 5:**  $C$  factors into 3 lines and has 3 nodes. Let the equation of  $C$  be written  $l_1 l_2 l_3 = 0$  where each  $l_i$  is a linear form. Consider the symbol algebra

$$\Lambda = \left( \frac{l_1}{l_3}, \frac{l_2}{l_3} \right)_n$$

over  $K$ . The graph for  $\Lambda$  in this case is the hexagon shown in Figure 4. Using (4) and the same ideas as in the earlier cases, we find that the ramification divisor of  $\Lambda$  is  $C$  and  $\text{exponent}(\Lambda) = n$ . Since  ${}_n B(\mathbb{P}^2 - C) \cong H_1(\Gamma, \mathbb{Z}/n) \cong \mathbb{Z}/n$  we see that every algebra class of exponent  $n$  is some power of the one given, and therefore has degree  $n$ .

**Case 6:**  $C$  factors into 3 lines and has 1 singular point. In this case  $C$  is simply connected,  $H_1(\Gamma, \mathbb{Z}/n) = 0$ , hence no division algebra can have ramification divisor equal to  $C$ . □

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