

Interpolation of toric varieties

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ABSTRACT. Let $X \subset \mathbb{P}^d$ be an m -dimensional variety in d -dimensional complex projective space. Let k be a positive integer such that the combinatorial number $\binom{m+k}{k}$ is smaller than or equal to d . Consider the following *interpolation problem*: does there exist a variety $Y \subset \mathbb{P}^d$ of dimension strictly smaller than $\binom{m+k}{k}$, with $X \subset Y$, such that the tangent space to Y at a point $p \in X$ is equal to the k th osculating space to X at p , for almost all points $p \in X$? In this paper we consider this question in the *toric* setting. We prove that if X is toric, then there is a unique toric variety Y solving the above interpolation problem. We identify Y in the general case and we explicitly compute some of its invariants when X is a toric curve.

CONTENTS

1. Introduction	1498
2. Toric interpolation	1501
3. Toric curves	1507
4. Normalizations and dual varieties	1511
References	1515

1. Introduction

When a problem can be modeled by polynomial equations, where their solutions correspond to an algebraic variety, it often becomes important to determine the optimal variety that satisfies given constraints. The classical *interpolation problem* in algebraic geometry is the following: find all plane curves of a prescribed degree passing through a given set of points. More generally, one can consider a class of varieties together with a collection of incidence conditions, involving linear subspaces and possibly tangency or higher order osculating

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conditions, and ask for those varieties in the class that satisfy the given conditions. This is both a theoretically and computationally challenging problem in algebraic geometry and related fields.

A classical example in real differential geometry is the following [3, p. 56]: Consider a space curve C . Find a space curve such that its osculating planes are equal to the normal planes of C . The solution to this question is the *evolute* of the curve C , namely the locus of its spherical curvature centers. The study of osculating spaces to a space curve goes back at least to Monge and others in the 18th century. For the case of curves in higher dimensional spaces, see [16].

The k th *osculating space* to an m -dimensional variety $X \subset \mathbb{P}^d$ at a point $p \in X$ is a linear space of dimension $\leq \binom{m+k}{k} - 1$ in \mathbb{P}^d that is tangent to X at p to the order k . The osculating space at $p \in X$ of order $k = 1$ is the embedded tangent space, denoted by $T_{X,p}$. Osculating spaces have been extensively studied in the context of higher order dual varieties, see [17, 18] for the general case and [5, 6] for the case of toric varieties. We refer to 2.2 for more details.

The kind of interpolation problem we consider in this article is the following. Fix a variety $X \subset \mathbb{P}^d$ of dimension m in complex projective space of dimension d . Let k be a positive integer satisfying $\binom{m+k}{k} \leq d$. Consider the set of all varieties $Y \subset \mathbb{P}^d$ of dimension $\leq \binom{m+k}{k} - 1$ such that $X \subset Y$. We say that Y satisfies the k th *interpolation condition* with respect to X – or that Y is a k th *interpolant of X* – if the embedded tangent space to Y at almost all points of X is equal to the k th osculating space to X at that point. A natural question is then:

Determine the existence and uniqueness of a k th interpolant, and, if it exists, explore methods for its construction.

In general, answers are expected to be challenging, particularly with regard to uniqueness, as we are seeking a unique object that satisfies specific local conditions. However, in the case of toric varieties, the rigidity imposed by the torus action on the geometry allows us to provide a complete solution to the problem.

If X and Y are toric varieties and Y satisfies the k th interpolation condition with respect to X , we call Y a k th *toric interpolant of X* .

A toric variety is a (not necessarily normal) algebraic variety containing an algebraic torus as a Zariski open set and such that the multiplicative self-action of the torus extends to the whole variety. Projective spaces are toric varieties and the torus of \mathbb{P}^d is the open subset T^d of projective points with all nonzero coordinates. The action of T^d on \mathbb{P}^d is given by coordinatewise multiplication, that is, multiplication by a point $(t_0 : \dots : t_d) \in T^d$ is given by the diagonal matrix with diagonal entries (t_0, \dots, t_d) . Equivariantly embedded projective toric varieties $X \subset \mathbb{P}^d$ of dimension m not contained in a coordinate hyperplane are rational varieties parameterized by monomials with exponents in the columns of a matrix $A \in \mathbb{Z}^{(m+1) \times (d+1)}$ of rank m , which give the weights of the torus action in (3) below [9, Ch. 5, Prop. 1.5]. In this case, we denote as usual

$X = X_A$. This configuration of column vectors lies on a hyperplane off the origin and so we will assume without loss of generality that the first coordinate of each of these column vectors is equal to 1. In fact, X_A is associated to the affine equivalence class of A by [9, Ch. 5, Prop. 1.2]. We refer the reader to Section 2, where we recall with more detail this characterization, and we show in Theorem 2.5 that the variety X_A is completely determined by its embedded tangent space $T_{X_A, (1:\dots:1)}$.

Given $A \in \mathbb{Z}^{(m+1) \times (d+1)}$ and a positive integer k , consider the matrix $A^{(k)}$ in $\mathbb{Z}^{\binom{m+k}{k} \times (d+1)}$ given in Definition 2.2, originally introduced in [5] in connection with the study of higher order dual varieties of toric varieties. In Theorem 2.6, we prove existence and unicity of toric interpolants, while providing an explicit construction:

$Y := X_{A^{(k)}} \subset \mathbb{P}^d$ is the unique k th toric interpolant of X_A .

We explain that, thanks to the torus action, for Y to be a k th toric interpolant of X_A , it suffices that the tangent space to Y at one point $p \in X_A \subset Y$ with all nonzero coordinates, equals the k th osculating space to X_A at p .

In Section 3, we analyze in more detail the case of toric curves. Let us first recall the concept of cyclic polytopes associated to curves. The m -moment curve in \mathbb{R}^{m+1} is defined by the image of the map $\alpha_m : \mathbb{R} \rightarrow \mathbb{R}^{m+1}$ sending t to the vector $(1, t, t^2, \dots, t^m)$. An m -dimensional *cyclic polytope* is defined as the convex hull of the image, under the moment curve, of a finite number (at least two), of ordered distinct points. It is known that all the images of these points by α_m are vertices of the cyclic polytope, which has dimension m . Its combinatorial structure is independent of the points chosen. Toric curves are associated to matrices $A \in \mathbb{Z}^{2 \times (d+1)}$ of the form

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ \ell_0 & \ell_1 & \dots & \ell_{d-1} & \ell_d \end{pmatrix},$$

where we will always assume, without loss of generality, that $\ell_0 < \ell_1 < \dots < \ell_d$. We describe the degree and number of lattice points of the second interpolants of toric curves. All matrices $A^{(k)}$ arising from toric curves are *positroids* (i.e., all their maximal minors are non-negative) and thus their convex hulls define *positive geometries* in the sense of [1]. The columns of the matrices $A^{(k)}$ correspond to the vertices of a cyclic lattice polytope of dimension k . In the case $k = 2$ we consider the associated polygon and we compute its canonical form. This is an example of a generalized *tree amplituhedron* of type $\mathcal{A}_{d+1,1,2}(Z)$ [1, 2] in the positive Grassmann variety $\mathbb{G}_{1,3}^{\geq 0} = (\mathbb{P}^2)^{\geq 0}$. Indeed, amplituhedra are geometric objects studied in particle physics in relation to integral representations of scattering amplitudes and they are interesting mathematical objects that naturally generalize cyclic polytopes. Our computations can be extended to k th interpolants for any k , based on the description of the facets of combinatorial cyclic polytopes [4]. In the standard case of the toric curves with $\ell_i = i$ for $i = 0, \dots, d$, we compute the degree of the second and the $(d - 1)$ th interpolant.

In Section 4, we describe the normalization of the second interpolant of a toric curve. By computing the degree of its dual variety and the degree of the dual variety of its normalization, we conclude that though the second interpolant is a linear projection of its normalization, it is not a *general* linear projection.

2. Toric interpolation

In this section, we recall some known facts about projective toric varieties X_A associated to a matrix A (2.1) and the notion of higher osculating spaces of a projective variety (2.2). For toric varieties, we prove that hyperosculation of order k can be characterized by an associated matrix $A^{(k)}$ naturally built from A . In Definition 2.4, we introduce the notion of k th (toric) interpolant, and in Theorem 2.6, we prove that the k th toric interpolant of the equivariantly embedded projective toric variety X_A is unique and is equal to the toric variety $X_{A^{(k)}}$. Uniqueness is based on the basic result in Theorem 2.5.

2.1. Projective toric varieties. Let $A = \{(1, \mathbf{a}_0), \dots, (1, \mathbf{a}_d)\} \subset \mathbb{Z}^{m+1}$ be a finite set of lattice points in an affine hyperplane off the origin. We denote also by A the $(m + 1) \times (d + 1)$ -matrix whose columns are given by the $(1, \mathbf{a}_j)$'s. Consider the map

$$\iota_A : (\mathbb{C}^*)^m \rightarrow \mathbb{P}^d \quad \text{defined by} \quad \mathbf{t} := (t_1, \dots, t_m) \mapsto (\mathbf{t}^{\mathbf{a}_0} : \dots : \mathbf{t}^{\mathbf{a}_d}), \quad (1)$$

where $\mathbf{t}^{\mathbf{a}_j} = \prod_i t_i^{a_{i,j}}$. Note that if we consider instead the map

$$\iota'_A : (\mathbb{C}^*)^{m+1} \rightarrow \mathbb{P}^d \quad \text{defined by} \quad (t_0, \mathbf{t}) \mapsto (t_0 \mathbf{t}^{\mathbf{a}_0} : \dots : t_0 \mathbf{t}^{\mathbf{a}_d}), \quad (2)$$

we have that $\iota_A(\mathbf{t}) = \iota'_A(t_0, \mathbf{t})$.

The projectively embedded toric variety $X_A \subset \mathbb{P}^d$ associated with A is defined to be the Zariski closure of the image of ι_A . This image is the torus $X_A \cap \{x \in \mathbb{P}^d \mid x_i \neq 0, i = 0, \dots, d\}$ of X_A and it always contains the point

$$\mathbf{1} := (1 : \dots : 1) \in \mathbb{P}^d.$$

Also, $X_A = \overline{\text{Orb}(\mathbf{1})}$ is the closure of the orbit of the point $\mathbf{1}$ by the diagonal action

$$\mathbf{t} *_A (x_0 : \dots : x_d) = (\mathbf{t}^{\mathbf{a}_0} x_0 : \dots : \mathbf{t}^{\mathbf{a}_d} x_d). \quad (3)$$

The variety X_A is an affine invariant of the configuration A and its dimension equals the affine dimension of A [9, Prop. 1.2, Ch. 5]. As we mentioned in the introduction, we will assume without loss of generality that all points in A have first coordinate equal to 1, which implies that $\dim(A) = \text{rank}(A) - 1$. We will moreover assume, without loss of generality, that the matrix A has maximal rank $m + 1$, or equivalently, that the convex hull of the points $\mathbf{a}_0, \dots, \mathbf{a}_d$ is of maximal dimension m . As we also mentioned, Proposition 1.5 in Chapter 5 of the book [9] by Gelfand, Kapranov and Zelevinsky shows that any projective toric variety with an equivariant embedding (that is, with a diagonal torus action) and not contained in a coordinate hyperplane, is of the form X_A . The

degree of the projective variety X_A equals $\text{Vol}(A)$, the lattice volume of A (cf. Theorem 4.16 in [19]). Subtracting multiples of the first row from the other rows, we can also assume without loss of generality that $\mathbf{a}_0 = \mathbf{0}$. When the subgroup $\mathbb{Z}A$ generated by $\mathbf{a}_0 = 0, \dots, \mathbf{a}_d$ equals \mathbb{Z}^m , the lattice volume $\text{Vol}(A)$ of A is defined as $m!$ times the Euclidean volume of the convex hull of the lattice configuration A . Otherwise, it equals this quantity divided by the rank of the quotient $\mathbb{Z}^m/\mathbb{Z}A$.

We present a very simple example:

Example 2.1. Consider the matrices

$$A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix},$$

and $A_4 = (0 \ 1 \ 2 \ 3)$. The closure of the images by the corresponding maps ι_{A_j} , $j = 1, 3, 4$, in (1), equals the same *projective* toric variety: the rational normal curve of degree 3 in \mathbb{P}^3 cut out by the following equations:

$$\{(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3 \mid x_1^2 - x_0x_2 = 0, x_2^2 - x_1x_3 = 0, x_0x_3 - x_1x_2 = 0\}.$$

See also Example 2.3 below.

Note that the columns of A_1 are the injective image of the columns of A_4 by the affine map $m \mapsto (1, m)$. The matrix A_3 has rank 2 and its columns are the injective image of the columns of A_1 by the linear map $(m_1, m_2) \mapsto (m_1, m_2, m_1 + m_2)$.

The columns of the matrix A_2 equal the image of the columns of A_1 via the linear map $(m_1, m_2) \mapsto (3m_1 - m_2, m_2)$ and $A_2 = MA_1$, where $M \in \text{GL}(2, \mathbb{Q})$ is the matrix

$$M = \begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix}.$$

Indeed, the first row of A_2 is not the all 1 vector, but $(1, \dots, 1)$ is in the rowspan of the matrix A_2 . This is clear since $M^{-1}A_2 = A_1$. It is easy to check, as we remarked in (2), that the map j_{A_2} defined by $j_{A_2}(t_0, t_1) = (t_0^3 : t_0^2t_1 : t_0t_1^2 : t_1^3)$ verifies $j_{A_2}(t_0, t_1) = (1 : s : s^2 : s^3) = \iota_{A_1}(s)$ for $s = t_1/t_0$. Thus, A_2 also gives a rational parameterization of the rational normal curve of degree 3.

The vectors $(1, -2, 1, 0)$, $(0, 1, -2, 1)$, $(1, -1, -1, 1)$ that we can read from the exponents of the equations generate the kernel of the matrices A_1, A_2, A_3 , and the space of affine relations among the columns of A_1 (that is, elements of the kernel that add up to 0). Note that it is not enough to select a basis of the kernel; for instance if we omit the equation $x_0x_3 - x_1x_2 = 0$, the points in the line $\{x_1 = x_2 = 0\}$ also satisfy the first two equations. Instead, any two of the equations describe the variety outside the coordinate hyperplanes.

2.2. Osculating spaces. Let $X \subset \mathbb{P}^d$ be a projective algebraic variety of dimension m . Consider the sheaf $\mathcal{L} := \mathcal{O}_{\mathbb{P}^d}(1)|_X$, and let $\mathcal{P}_X^k(\mathcal{L})$ denote the sheaf of k th order principal parts of \mathcal{L} [16, § 6, p. 492]. Recall that the rank of $\mathcal{P}_X^k(\mathcal{L})$

at a smooth point $p \in X$ is $\binom{m+k}{k}$. Indeed, the fiber $\mathcal{P}_X^k(\mathcal{L})_p$ at a point $p \in X$ is isomorphic to the vector space $\mathcal{O}_{X,p}/\mathfrak{m}_{X,p}^{k+1}$, where $\mathfrak{m}_{X,p} \subset \mathcal{O}_{X,p}$ is the maximal ideal in the local ring of X at p . Principal parts bundles play a crucial role in the study of projective duality and differential properties of projective embeddings, see [10, IV.A., pp. 341–346; IV.D., pp. 359–365].

Assume that $U \subset X$ has a parameterization

$$\mathbf{t} := (t_1, t_2, \dots, t_m) \mapsto (x_0(\mathbf{t}) : \dots : x_d(\mathbf{t})).$$

Then, the k th jet map

$$j_k : \mathcal{O}_X^{d+1} \rightarrow \mathcal{P}_X^k(\mathcal{L})$$

restricted to U is given by the matrix

$$A^{(k)}(\mathbf{t}) := \begin{pmatrix} x_0(\mathbf{t}) & x_1(\mathbf{t}) & \cdots & x_d(\mathbf{t}) \\ \frac{\partial x_0(\mathbf{t})}{\partial t_1} & \frac{\partial x_1(\mathbf{t})}{\partial t_1} & \cdots & \frac{\partial x_d(\mathbf{t})}{\partial t_1} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x_0(\mathbf{t})}{\partial t_m} & \frac{\partial x_1(\mathbf{t})}{\partial t_m} & \cdots & \frac{\partial x_d(\mathbf{t})}{\partial t_m} \\ \frac{\partial^2 x_0(\mathbf{t})}{\partial t_1^2} & \frac{\partial^2 x_1(\mathbf{t})}{\partial t_1^2} & \cdots & \frac{\partial^2 x_d(\mathbf{t})}{\partial t_1^2} \\ \frac{\partial^2 x_0(\mathbf{t})}{\partial t_1 \partial t_2} & \frac{\partial^2 x_1(\mathbf{t})}{\partial t_1 \partial t_2} & \cdots & \frac{\partial^2 x_d(\mathbf{t})}{\partial t_1 \partial t_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^k x_0(\mathbf{t})}{\partial t_m^k} & \frac{\partial^k x_1(\mathbf{t})}{\partial t_m^k} & \cdots & \frac{\partial^k x_d(\mathbf{t})}{\partial t_m^k} \end{pmatrix}, \tag{4}$$

where $\frac{\partial^j}{\partial t_i^j}$ denotes the Hasse derivative, that is, $\frac{1}{j!}$ times the standard derivative.

Given a matrix A we will denote by $\text{rowspan}(A)$ the subspace of \mathbb{C}^{d+1} spanned by the row vectors of A . The k th osculating space to X at a point corresponding to \mathbf{t} is the linear space

$$\mathbb{P}(\text{rowspan}(A^{(k)}(\mathbf{t}))).$$

2.3. The matrices $A^{(k)}$. We now construct matrices $A^{(k)}$ describing the higher osculating spaces of a toric variety X_A at $\mathbf{1} = (1 : \dots : 1)$. For any k , the k th osculating spaces at the points $\iota_A(\mathbf{t}) = \mathbf{t} *_{A} \mathbf{1}$ in the torus of X_A are translated by this action (defined in (3)).

We also construct matrices denoted by $\tilde{A}^{(k)}$; the matrices $A^{(k)}$ and $\tilde{A}^{(k)}$ will define the same projective toric variety.

Definition 2.2. Let $A \in \mathbb{Z}^{(m+1) \times (d+1)}$ with column vectors $(1, \mathbf{a}_j)$, $j = 0, \dots, d$, and $k \in \mathbb{N}$. We define the associated matrix $\tilde{A}^{(k)}$ as follows. We will add $\binom{m+k}{k} - (m+1)$ new rows to A in its lower part. We order the vectors $\{\mathbf{i} = (i_1, \dots, i_m) \in \mathbb{Z}_{\geq 0}^m \mid 2 \leq |\mathbf{i}| \leq k\}$, with lexicographic order with $0 < 1 < \dots < m$ and use them to label these rows. The entry in the matrix $\tilde{A}^{(k)}$ in the j th column and the i th row is the integer $a_{1,j}^{i_1} \cdots a_{m,j}^{i_m}$. When all the coefficients of A are nonnegative, we define another associated matrix $A^{(k)}$ as follows. Again, we add $\binom{m+k}{k} - (m+$

1) new (ordered) rows to A in its lower part labeled by the vectors $\mathbf{i} = (i_1, \dots, i_m)$. The entry in the matrix $A^{(k)}$ in the j th column and the \mathbf{i} th row is the integer $\binom{\mathbf{a}_j}{\mathbf{i}} := \binom{a_{1,j}}{i_1} \dots \binom{a_{m,j}}{i_m}$.

Since $\binom{a}{i} = \frac{a(a-1)\dots(a-i+1)}{i!}$ for any positive integer $a \geq i$, it is straightforward to see that for any k there exists an integer matrix M_k such that $\tilde{A}^{(k)} = M_k \cdot A^{(k)}$ with $\det(M_k) = \prod_{2 \leq |i| \leq k} i_1! \dots i_m!$. By [5, 2.2], the projective linear space

$$\mathbb{P}(\text{rowspan}(A^{(k)})) = \mathbb{P}(\text{rowspan}(\tilde{A}^{(k)}))$$

equals the k th osculating space of X_A at the point $\mathbf{1}$. This is easily seen by considering the parameterization ι_A to construct the matrix $A^{(k)}(\mathbf{t})$ in (4). Evaluating at $\mathbf{t} = \mathbf{1}$, we get the matrix $A^{(k)}$.

Example 2.3. Let $d \in \mathbb{N}$ and let A be the matrix

$$A_d = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 2 & 3 & \dots & d \end{pmatrix}.$$

When $k = 2$, we have

$$A_d^{(2)} = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 2 & 3 & \dots & d \\ 0 & 0 & 1 & 3 & \dots & \frac{d(d-1)}{2} \end{pmatrix}, \quad \tilde{A}_d^{(2)} = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 2 & 3 & \dots & d \\ 0 & 1 & 4 & 9 & \dots & d^2 \end{pmatrix},$$

and $\tilde{A}_d^{(2)} = M_2 \cdot A_d^{(2)}$, where $M_2 \in \text{GL}(3, \mathbb{Q})$ is the matrix

$$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

Consider the rational parameterizations of $X_{A_d^{(2)}} = X_{\tilde{A}_d^{(2)}}$ from $(\mathbb{C}^*)^2$ to \mathbb{P}^d :

$$\begin{aligned} \iota_{A_d^{(2)}} : (t_1, t_2) &\mapsto (1 : t_1 : t_1^2 t_2 : \dots : t_1^d t_2^{\frac{d(d-1)}{2}}), \\ \iota_{\tilde{A}_d^{(2)}} : (s_1, s_2) &\mapsto (1 : s_1 s_2 : s_1^2 s_2^4 : \dots : s_1^d s_2^{d^2}), \end{aligned}$$

and let $\varphi_{M_2} : (\mathbb{C}^*)^2 \rightarrow (\mathbb{C}^*)^2$ be the $2 : 1$ -map $s \mapsto (s_1 s_2, s_2^2)$. Then, it is straightforward to check that $\iota_{\tilde{A}_d^{(2)}} = \iota_{A_d^{(2)}} \circ \varphi_{M_2}$. We could also consider the first rows of the matrices and get a parameterization of the same (projective) variety

from $(\mathbb{C}^*)^3$ to \mathbb{P}^d , given by the map $\iota'_A(t_0, t_1, t_2) = (t_0 : t_0 t_1 : \dots : t_0 t_1^d t_2^{\frac{d(d-1)}{2}})$.

As noted above, the tangent space at $\mathbf{1}$ to the variety $X_{A_d^{(2)}} = X_{\tilde{A}_d^{(2)}}$ equals the second osculating space of X_A at $\mathbf{1}$. The standard cyclic polygon defined by $d + 1$ points $\alpha_d(j)$, $j = 0, \dots, d$, on the moment curve in the plane is in general presented with vertices equal to the points given by the columns of the matrix $\tilde{A}_d^{(2)}$, but $\iota_{A_d^{(2)}}$ is $1 : 1$ while $\iota_{\tilde{A}_d^{(2)}}$ is $2 : 1$.

2.4. The toric interpolant. We define k th toric interpolants for any $k \in \mathbb{N}$ and we show in Theorem 2.6 that a toric interpolant always exists and that it is unique.

Definition 2.4. Let $X_A \subset \mathbb{P}^d$ be a projective toric variety and $k \geq 1$ a natural number.

- (1) We say that a projective toric variety X_B is a k th toric interpolant of X_A at $\mathbf{p} \in X_A$ if $X_A \subset X_B$ and the tangent space to X_B at the point $\mathbf{p} \in X_A$ is equal to the k th osculating space to X_A at \mathbf{p} .
- (2) We say that X_B is a k th toric interpolant of X_A if X_B is a k th toric interpolant of X_A at almost all points $\mathbf{p} \in X_A$.

We start by showing the basic result that a projective toric embedding $X_A \hookrightarrow \mathbb{P}^d$ is completely determined by the tangent space at the point $\mathbf{1} \in X_A \subset \mathbb{P}^d$.

Theorem 2.5. Let $A \in \mathbb{Z}^{(m+1) \times (d+1)}$ be such that $\text{rowspan}(A) \subset \mathbb{R}^{d+1}$ is a subspace of dimension $m + 1$ containing the vector $(1, \dots, 1) \in \mathbb{R}^{d+1}$. Then, the following statements hold:

- (i) Given another matrix $A' \in \mathbb{Z}^{(m+1) \times (d+1)}$ such that $\text{rowspan}(A')$ has dimension $m + 1$ and contains the vector $(1, \dots, 1) \in \mathbb{R}^{d+1}$, then $X_A = X_{A'}$ if and only if $\text{rowspan}(A) = \text{rowspan}(A')$.
- (ii) The embedded tangent space to X_A at $\mathbf{1} \in X_A \subset \mathbb{P}^d$ is the projectivization $\mathbb{P}(\text{rowspan}(A) \otimes_{\mathbb{R}} \mathbb{C})$.

Moreover, let L be a linear subspace in \mathbb{R}^{d+1} of dimension $m+1$ which is defined over \mathbb{Q} and contains the point $(1, \dots, 1)$. Consider the projectivization $\mathbb{P}(L_{\mathbb{C}}) \subset \mathbb{P}^d$ of its extension $L_{\mathbb{C}} := L \otimes_{\mathbb{R}} \mathbb{C}$. There exists a unique equivariantly embedded toric variety $X_A \subset \mathbb{P}^d$ such that $\mathbb{P}(L_{\mathbb{C}})$ is the embedded tangent space $T_{X_A, \mathbf{1}}$ of X_A at $\mathbf{1}$. Indeed, it is enough to take any matrix $A \in \mathbb{Z}^{(m+1) \times (d+1)}$ such that $L = \text{rowspan}(A)$.

Proof. If $\text{rowspan}(A) = \text{rowspan}(A')$ we have that $X_A = X_{A'}$ because the configurations of columns of A and A' are affinely equivalent.

Let ι_A be the rational parameterization of X_A defined in (1) and $U = \iota_A((\mathbb{C}^*)^m)$ the torus of X_A . Taking $k = 1$ in (4) we get that $A^{(1)}(1, \dots, 1) = A$. Therefore, we see that the embedded tangent space at the point $\mathbf{1}$ is the projective linear space spanned by the row vectors of A considered as points in \mathbb{P}^d , which shows item (ii). Then, if $X_A = X_{A'}$, they have the same embedded tangent space at $\mathbf{1}$ and so $\text{rowspan}(A) = \text{rowspan}(A')$

Given such a linear subspace L as in the statement and any choice of matrix A with $L = \text{rowspan}(A)$, then $T_{X_A, \mathbf{1}} = \mathbb{P}(\text{rowspan}(A) \otimes_{\mathbb{R}} \mathbb{C}) = \mathbb{P}(L_{\mathbb{C}})$. \square

We next show the existence and uniqueness of toric interpolants for any $k \in \mathbb{N}$.

Theorem 2.6. Let $A \in \mathbb{Z}^{(m+1) \times (d+1)}$ and $k \in \mathbb{N}$.

- (i) The variety $X_{A^{(k)}}$ is a k th toric interpolant of X_A at all points in the torus of X_A , i.e., all points of X_A in the torus of \mathbb{P}^d .

- (ii) Assume there exists a matrix $B \in \mathbb{Z}^{(m_B+1) \times (d+1)}$ such that X_B is a k th toric interpolant of X_A at one point $\mathbf{p}^* = \iota_A(\mathbf{t}^*)$ in the torus of X_A . Then, $X_B = X_{A^{(k)}}$.

Proof. Assume that the first row of A is given by the vector with all coordinates equal to 1. We have already observed that for any $\mathbf{t} \in (\mathbb{C}^*)^m$, the tangent space to X_A at $\mathbf{p} = \iota_A(\mathbf{t}) = (\mathbf{t}^{\mathbf{a}_0} : \dots : \mathbf{t}^{\mathbf{a}_d})$ is spanned by the row vectors of the matrix $A^{(1)}(\mathbf{t})$ in (4). Note that multiplying the j th row vector of this matrix by t_j for $j = 1, \dots, m$, we get that the rows of the new matrix are the torus translates by the diagonal action $*_A$ in (3) of the rows of the matrix $A^{(1)}(\mathbf{1}) = A$. We can write this as $\mathbf{t} *_A A^{(1)}(\mathbf{1}) = A^{(1)}(\mathbf{t})$. Similarly, the row vectors of $A^{(k)}(\mathbf{t})$ span the k th osculating space to X_A at the point \mathbf{p} . Multiplying its rows by corresponding powers of t_1, \dots, t_m , we see that the rowspan of $A^{(k)}(\mathbf{t})$ equals the rowspan of the matrix $\mathbf{t} *_A A^{(k)}(\mathbf{1}) = \mathbf{t} *_A A^{(k)}$.

Introduce $\binom{m+k}{k} - (m + 1)$ new (ordered) variables $u_{\mathbf{i}}$, where $\mathbf{i} = (i_1, \dots, i_m)$ and $2 \leq |\mathbf{i}| \leq k$. The entry in the matrix $A^{(k)}$ in the j th column and the \mathbf{i} th row is the integer $\binom{\mathbf{a}_j}{\mathbf{i}} := \binom{a_{1,j}}{i_1} \dots \binom{a_{m,j}}{i_m}$. Hence, the toric variety $X_{A^{(k)}}$ has a parameterization

$$\iota_{A^{(k)}} : (\mathbf{t}, \mathbf{u}) := (t_1, \dots, t_m, \dots, u_{\mathbf{i}}, \dots) \mapsto (\dots : \mathbf{t}^{\mathbf{a}_j} \prod_{\mathbf{i}} u_{\mathbf{i}}^{\binom{\mathbf{a}_j}{\mathbf{i}}} : \dots).$$

It follows that the row span of the matrix $A^{(k)}(\mathbf{t}, \mathbf{u})$ is equal to the tangent space to $X_{A^{(k)}}$ at the point $(\dots : \mathbf{t}^{\mathbf{a}_j} \prod_{\mathbf{i}} u_{\mathbf{i}}^{\binom{\mathbf{a}_j}{\mathbf{i}}} : \dots)$. This is a point of X_A if all $u_{\mathbf{i}} = 1$. In particular, we have $X_A \subseteq X_{A^{(k)}}$. The tangent space to $X_{A^{(k)}}$ at the point $\mathbf{p} \in X_A$ is equal to the torus translate by $(\mathbf{t}, \mathbf{1})$ by means of the action $*_{A^{(k)}}$ of the embedded tangent space at the point $(1 : \dots : 1) \in \mathbb{P}^d$, which equals the k th osculating space to X_A at \mathbf{p} . It follows that $X_{A^{(k)}}$ is a k th toric interpolant of X_A at any point \mathbf{p} in the torus of X_A .

Assume a configuration B defines a k th-toric interpolant of X_A . Then, we have the equality of tangent spaces $T_{B, \mathbf{1}} = T_{A^{(k)}, \mathbf{1}}$ since both coincide with the k th-osculating space to X_A at $\mathbf{1}$. Theorem 2.5 implies that $X_B = X_{A^{(k)}}$. \square

Example 2.7. Consider the Del Pezzo surface $X \subset \mathbb{P}^6$ of degree 6 given by the parameterization

$$\iota : (t_1, t_2) \mapsto (1 : t_1 : t_2 : t_1 t_2 : t_1 t_2^2 : t_1^2 t_2 : t_1^2 t_2^2).$$

We get

$$A^{(2)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Consider new variables $u_{(2,0)}$, $u_{(1,1)}$ and $u_{(0,2)}$. Then, we get a parameterization

$$\iota_{A^{(2)}} : (t_1, t_2, u_{(2,0)}, u_{(1,1)}, u_{(0,2)}) \mapsto (1 : t_1 : t_2 : t_1 t_2 u_{(1,1)} : t_1 t_2^2 u_{(1,1)}^2 u_{(0,2)} : t_1^2 t_2 u_{(2,0)} u_{(1,1)}^2 : t_1^2 t_2^2 u_{(2,0)} u_{(1,1)}^4 u_{(0,2)}).$$

We could also subtract twice the sum of the fourth and sixth rows from the fifth row, which gives the vector $(0, 0, 0, 1, 0, 0, 0)$, so that we get the parameterization:

$$\iota_{A^{(2)}} : (t_1, t_2, u_{(2,0)}, u_{(1,1)}, u_{(0,2)}) \mapsto (1 : t_1 : t_2 : t_1 t_2 u_{(1,1)} : t_1 t_2^2 u_{(0,2)} : t_1^2 t_2 u_{(2,0)} : t_1^2 t_2^2 u_{(2,0)} u_{(0,2)}).$$

The corank of $A^{(2)}$ is 1 and the 2nd toric interpolant is the degree 3 hypersurface with equation $x_0 x_4 x_5 - x_1 x_2 x_6 = 0$. This equation does not depend on x_3 because the configuration of columns of $A^{(2)}$ is a pyramid with vertex on its fourth column, as by the previous calculation the fourth column vector lies in the hyperplane $y_4 - 2y_3 - 2y_5 = 1$ in \mathbb{Z}^5 , while all the other column vectors lie in the parallel hyperplane $y_4 - 2y_3 - 2y_5 = 0$.

The tangent space to $X_{A^{(2)}}$ at a point

$$\iota_{A^{(2)}}(\mathbf{t}, \mathbf{u}) = \iota_{A^{(2)}}(t_1, t_2, u_{(2,0)}, u_{(1,1)}, u_{(0,2)})$$

is the row span of the matrix $A^{(2)}(\mathbf{t}, \mathbf{u})$ given by

$$\begin{pmatrix} 1 & t_1 & t_2 & t_1 t_2 u_{(1,1)} & t_1 t_2^2 u_{(1,1)}^2 u_{(0,2)} & t_1^2 t_2 u_{(2,0)} u_{(1,1)}^2 & t_1^2 t_2^2 u_{(2,0)} u_{(1,1)}^4 u_{(0,2)} \\ 0 & 1 & 0 & t_2 u_{(1,1)} & t_2^2 u_{(1,1)}^2 u_{(0,2)} & 2t_1 t_2 u_{(2,0)} u_{(1,1)}^2 & 2t_1 t_2^2 u_{(2,0)} u_{(1,1)}^4 u_{(0,2)} \\ 0 & 0 & 1 & t_1 u_{(1,1)} & 2t_1 t_2 u_{(1,1)}^2 u_{(0,2)} & t_1^2 u_{(2,0)} u_{(1,1)}^2 & 2t_1^2 t_2 u_{(2,0)} u_{(1,1)}^4 u_{(0,2)} \\ 0 & 0 & 0 & 0 & 0 & t_1^2 t_2 u_{(1,1)}^2 & t_1^2 t_2^2 u_{(1,1)}^4 u_{(0,2)} \\ 0 & 0 & 0 & t_1 t_2 & 2t_1 t_2^2 u_{(1,1)} u_{(0,2)} & 2t_1^2 t_2 u_{(2,0)} u_{(1,1)} & 4t_1^2 t_2^2 u_{(2,0)} u_{(1,1)}^3 u_{(0,2)} \\ 0 & 0 & 0 & 0 & t_1 t_2^2 u_{(1,1)}^2 & 0 & t_1^2 t_2^2 u_{(2,0)} u_{(1,1)}^4 \end{pmatrix}$$

For a point $\iota_{A^{(2)}}(t_1, t_2, 1, 1, 1) \in X$ with $t_1, t_2 \neq 0$, this is the same as the row span of the matrix

$$\begin{pmatrix} 1 & t_1 & t_2 & t_1 t_2 & t_1 t_2^2 & t_1^2 t_2 & t_1^2 t_2^2 \\ 0 & 1 & 0 & t_2 & t_2^2 & 2t_1 t_2 & 2t_1 t_2^2 \\ 0 & 0 & 1 & t_1 & 2t_1 t_2 & t_1^2 & 2t_1^2 t_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & t_2 \\ 0 & 0 & 0 & 1 & 2t_2 & 2t_1 & 4t_1 t_2 \\ 0 & 0 & 0 & 0 & 1 & 0 & t_1 \end{pmatrix}.$$

which is the same as the second osculating space to X at this point.

3. Toric curves

We shall now study in more detail the case when X is a curve. A toric curve $X \subset \mathbb{P}^d$ which is not contained in a coordinate hyperplane, can be described as

follows. Given integers $\ell_0 = 0 < \ell_1 < \dots < \ell_{d-1} < \ell_d$, set $\ell := \{\ell_0, \dots, \ell_d\}$ and

$$A_{\ell,d} := \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & \ell_1 & \dots & \ell_d \end{pmatrix}.$$

Let $X_{A_{\ell,d}} \subset \mathbb{P}^d$ be the rational curve of degree ℓ_d parameterized by

$$t \mapsto (1 : t^{\ell_1} : \dots : t^{\ell_{d-1}} : t^{\ell_d}).$$

In the special case when $\ell_0 = 1, \ell_1 = 1, \dots, \ell_d = d$, the curve $X_{A_{\ell,d}}$ is a *rational normal curve*.

The k th toric interpolant of $X_{A_{\ell,d}}$ is given by the matrix

$$A_{\ell,d}^{(k)} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & \ell_1 & \ell_2 & \dots & \ell_{d-1} & \ell_d \\ 0 & \binom{\ell_1}{2} & \binom{\ell_2}{2} & \dots & \binom{\ell_{d-1}}{2} & \binom{\ell_d}{2} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & \binom{\ell_1}{k} & \binom{\ell_2}{k} & \dots & \binom{\ell_{d-1}}{k} & \binom{\ell_d}{k} \end{pmatrix}.$$

As we observed in the comment after Definition 2.2 there exists a $(k + 1) \times (k + 1)$ -matrix M_k such that the matrix

$$\tilde{A}_{\ell,d}^{(k)} := \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & \ell_1 & \ell_2 & \dots & \ell_{d-1} & \ell_d \\ 0 & \ell_1^2 & \ell_2^2 & \dots & \ell_{d-1}^2 & \ell_d^2 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & \ell_1^k & \ell_3^k & \dots & \ell_{d-1}^k & \ell_d^k \end{pmatrix}$$

is equal to $M_k \cdot A_{\ell,d}^{(k)}$, with $\det M_k = \prod_{j=1}^k j!$. The column vectors of this second matrix $\tilde{A}_{\ell,d}^{(k)}$ form the vertices of a cyclic polytope $C_{\ell,d}^{(k)}$. They are ordered points on a moment curve. Here, ordered means naively that the second coordinates are increasing, but the main point is that all the maximal minors of the matrix (with the corresponding ordered columns) are positive. Therefore, we can compute the degree of the variety $X_{A_{\ell,d}}^{(k)} = X_{\tilde{A}_{\ell,d}^{(k)}}^{(k)}$ as the lattice volume of the cyclic polytope $C_{\ell,d}^{(k)}$ with respect to the lattice generated by the columns of the matrix $A_{\ell,d}^{(k)}$, which are all consecutive vertices of the cyclic polytope. Note that the matrices $\tilde{A}_{\ell,d}^{(k)}$ are *positroids*: their maximal minors are positive since they are equal to Vandermonde determinants and the ℓ_i satisfy $\ell_i < \ell_j$ for $i < j$. Since $A_{\ell,d}^{(k)} = M_k^{-1} \cdot \tilde{A}_{\ell,d}^{(k)}$, with $\det M_k^{-1} > 0$, also $A_{\ell,d}^{(k)}$ is a positroid.

In general, the *lattice volume* of a k -dimensional lattice polytope is $k!$ times its Euclidean volume and will be denoted by Vol . By decomposing the polytope into simplices, $\text{Vol}(C_{\ell,d}^{(k)})$ can be computed as the sum of the lattice volumes of these simplices, which will all be expressed as Vandermonde determinants in ℓ_1, \dots, ℓ_d . To decompose the polytope, one can use a classical characterization

of its facets, for an account of this, see [8]. Once one knows the volume, one can compute the number of lattice points of the polytope by the formula given by Liu [12, Thm. 1.2, p. 112]:

$$i(C_{\ell,d}^{(k)}) = \sum_{j=0}^k \text{Vol } C_{\ell,d}^{(j)},$$

where $\text{Vol } C_{\ell,d}^{(0)} := 1$.

To illustrate this situation, assume that $k = 2$. Then $C_{\ell,d}^{(2)}$ is a cyclic polygon.

Proposition 3.1. *The lattice volume of and the number of lattice points in $C_{\ell,d}^{(2)}$ are given as*

$$\text{Vol } C_{\ell,d}^{(2)} = \sum_{i=1}^{d-1} \ell_i \ell_{i+1} (\ell_{i+1} - \ell_i)$$

and

$$i(C_{\ell,d}^{(2)}) = \frac{1}{2} \sum_{i=1}^{d-1} \ell_i \ell_{i+1} (\ell_{i+1} - \ell_i) + \ell_d + 1.$$

Proof. We write $C_{\ell,d}^{(2)}$ as the union of the triangles with vertices $(0, 0)$, (ℓ_i, ℓ_i^2) , and $(\ell_{i+1}, \ell_{i+1}^2)$. The lattice area of the triangles are $\frac{1}{2} \ell_i \ell_{i+1} (\ell_{i+1} - \ell_i)$. The lattice length of $C_{\ell,d}^{(1)}$ is ℓ_d . □

Assume $\ell_0 = 0, \ell_1 = 1, \dots, \ell_d = d$ and $k = 2$. If we instead use the matrix

$$A_d^{(2)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & \cdots & d-3 & d-2 & d-1 & d \\ 0 & 0 & 1 & 3 & 6 & \cdots & \binom{d-3}{2} & \binom{d-2}{2} & \binom{d-1}{2} & \binom{d}{2} \end{pmatrix}, \tag{5}$$

we get a different polygon, which we call $P(d)$. It is easy to see that the lattice area of $P(d)$ coincides with its normalized area and thus gives the degree of the associated toric variety $X_{A_d^{(2)}} = X_{\tilde{A}_{\ell,d}^{(2)}}$.

The $(d - 1)$ th toric interpolant of X_{A_d} is given by the matrix

$$A_d^{(d-1)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & \cdots & d-3 & d-2 & d-1 & d \\ 0 & 0 & 1 & 3 & 6 & \cdots & \binom{d-3}{2} & \binom{d-2}{2} & \binom{d-1}{2} & \binom{d}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \binom{d}{d-1} \end{pmatrix}.$$

Hence we get the following parameterization of $X_{A_d^{(d-1)}}$:

$$(t_0, t_1, \dots, t_{d-1}) \mapsto (t_0 : t_0 t_1 : t_0 t_1^2 t_2 : \cdots : t_0 t_1^d t_2^{\binom{d}{2}} \cdots t_{d-1}^d) \in \mathbb{P}^d.$$

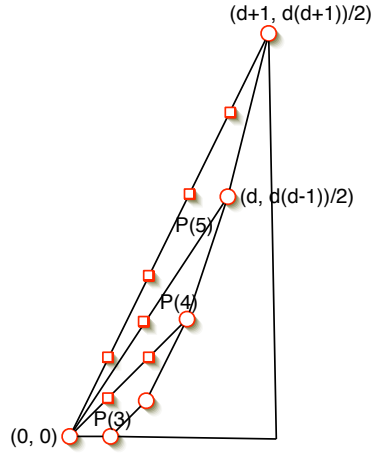


FIGURE 1. The polygons $P(d)$.

Proposition 3.2. $X_{A_d^{(d-1)}}$ is a toric hypersurface of degree 2^{d-1} .

Proof. It is clear that the rank of $A_d^{(d-1)}$ is d as its maximal minor corresponding to its first columns is equal to 1. Then, its kernel has dimension one and so $X_{A_d^{(d-1)}}$ is a hypersurface. It is easy to check that a $(d + 1)$ -vector v lies in the kernel of $A_d^{(d-1)}$ if and only if the polynomial $f_v(t) = \sum_{i=0}^d v_i t^i$ vanishes at $t = 1$ jointly with its derivatives up to order $(d - 1)$. The vector w with coordinates $w_i = (-1)^{d-i} \binom{d}{i}$, satisfies that $f_w(t) = (t - 1)^d$, and so w is a generator of the kernel. Separating its positive from its negative entries, we get that the ideal of $X_{A_d^{(d-1)}}$ is generated by the binomial

$$x_0 x_2^{\binom{d}{2}} \cdots x_{d-2}^{\binom{d}{d-2}} x_d - x_1^{\binom{d}{1}} x_3^{\binom{d}{3}} \cdots x_{d-1}^{\binom{d}{d-1}} = 0$$

if d is even, and

$$x_0 x_2^{\binom{d}{2}} \cdots x_{d-1}^{\binom{d}{d-1}} - x_1^{\binom{d}{1}} x_3^{\binom{d}{3}} \cdots x_{d-2}^{\binom{d}{d-2}} x_d = 0$$

if d is odd. By using $\binom{d}{i} = \binom{d-1}{i-1} + \binom{d-1}{i}$, we see that all four monomials appearing in these two equations have degree $\sum_{i=0}^{d-1} \binom{d-1}{i} = 2^{d-1}$. \square

In [1], the concept of *positive geometries* was introduced in the study of scattering amplitudes in particle physics. A positive geometry is a real semi-algebraic set, together with a rational differential form, with poles along the boundary of the semi-algebraic set, called the *canonical form*. Of special interest were semi-algebraic sets given by certain polytopes, called *generalized tree amplituhedra*, of type $\mathcal{A}_{n,k,m}(Z)$, where Z is a $(k + m) \times n$ -matrix with positive maximal minors – a positroid – in the real positive Grassmann variety $\mathbb{G}_{k,k+m}^{\geq 0}$ [2]. The polygon

$P(d)$ is cyclic and is an example of a generalized tree amplituhedron of type $\mathcal{A}_{d+1,1,2}(Z)$ in $\mathbb{G}_{1,3}^{\geq 0} = (\mathbb{P}^2)^{\geq 0}$, with $Z = A_d^{(2)}$.

Any convex polygon gives rise to a positive geometry, hence has an associated canonical form. In particular, $P(d)$ gives rise to a positive geometry. The following example shows an explicit computation for any d of the procedure outlined in [1, 7.1.1] for computing the canonical form of a cyclic polytope. Using the description of the facets of combinatorial cyclic polytopes [4], our computation could be extended to higher interpolants of toric curves.

Example 3.3. The polygon $P(d)$ can be decomposed as the polygon $P(d - 1)$ union the triangle $\Delta(d)$ with vertices $(0, 0)$, $(d - 1, \binom{d-1}{2})$, and $(d, \binom{d}{2})$. The sides of $\Delta(d)$ have equations $2y - (d - 2)x = 0$, $2y - 2(d - 1)x + d(d - 1) = 0$, and $(d - 1)x - 2y = 0$. Hence, we compute the canonical form (see [1, 11]) of the triangle to be

$$\Omega(\Delta(d)) = \frac{2d(d - 1)}{(2y - (d - 2)x)(2y - 2(d - 1)x + d(d - 1))((d - 1)x - 2y)} dx \wedge dy.$$

The additivity of the canonical form [11, 2.4] then gives

$$\begin{aligned} \Omega(P(d)) &= \frac{2d(d - 1) dx \wedge dy}{(2y - (d - 2)x)(2y - 2(d - 1)x + d(d - 1))((d - 1)x - 2y)} \\ &\quad + \Omega(P(d - 1)). \end{aligned}$$

Hence, the canonical form of the polygon $P(d)$ equals

$$\Omega(P(d)) = \sum_{i=2}^d \frac{2i(i - 1)}{(2y - (i - 2)x)(2y - 2(i - 1)x + i(i - 1))((i - 1)x - 2y)} dx \wedge dy.$$

In particular, we get

$$\Omega(P(2)) = \frac{1}{y(y - x + 1)(x - 2y)} dx \wedge dy$$

and

$$\Omega(P(3)) = \frac{2y - 2x + 3}{y(y - x + 1)(y - 2x + 3)(x - y)} dx \wedge dy.$$

The latter is consistent with the fact that the adjoint curve to $P(3)$ is the line $2y - 2x + 3 = 0$ [11, 2.1].

4. Normalizations and dual varieties

In projective geometry, it is interesting to determine whether a given linear projection is *general*. Indeed, a character (a cycle class, or a number) of a projective variety is said to be *projective* if it is invariant under a *general* linear projection. An example of a projective character of a variety is its degree, and also the degree of its dual variety, i.e., the variety in the dual projective space equal to the (closure of) the set of tangent hyperplanes.

Given a lattice point configuration $A \subset \mathbb{Z}^{m+1}$, let $\bar{A} \subset \mathbb{Z}^{m+1}$ denote the configuration of lattice points in the convex hull of A in \mathbb{R}^{m+1} , and denote by \bar{A} the corresponding matrix. We obtain a *normal* toric variety $X_{\bar{A}} \subset \mathbb{P}^N$, where $N+1$ is the number of lattice points in the configuration \bar{A} . Since the matrix A is obtained from \bar{A} by removing some columns, X_A is equal to a linear projection of its normalization $X_{\bar{A}}$.

We now return to the second toric interpolant $X_{A_d^{(2)}}$ of the rational normal curve X_A of degree d studied in the previous section, see (5). We assume $d \geq 3$. We choose this case as an example since it is computable, and we show that $X_{A_d^{(2)}}$ is *not* a general linear projection of its normalization $X_{\bar{A}_d^{(2)}}$. For $d \geq 4$, this follows since the degrees of their dual varieties are different. For $d = 3$, the surface $X_{A_3^{(2)}} \subset \mathbb{P}^3$ has a singular line of multiplicity 3, hence is not a general projection of a smooth surface in \mathbb{P}^5 .

Proposition 4.1. *The degree of $X_{A_d^{(2)}} \subset \mathbb{P}^d$ is $\binom{d+1}{3}$, and it is a toric linear projection of its normalization $X_{\bar{A}_d^{(2)}}$ from $\mathbb{P}^{\frac{1}{12}d(d^2+8)}$ if d is even and from $\mathbb{P}^{\frac{1}{12}d(d^2+11)}$ if d is odd.*

Proof. The lattice polygon $P(d)$ in the plane corresponding to $A_d^{(2)}$ contains the lattice polygon $P(d - 1)$ corresponding to $A_{d-1}^{(2)}$, see Fig. 1. It is easy to see that the difference of the two polygons has lattice area equal to

$$d\binom{d}{2} - (d - 1)\binom{d - 1}{2} - 2\binom{d - 1}{2} - \binom{d}{2} + \binom{d - 1}{2} = \binom{d}{2}.$$

Hence, we get

$$\deg X_{A_d^{(2)}} = \deg X_{A_{d-1}^{(2)}} + \binom{d}{2} = \dots = \sum_{j=2}^d \binom{j}{2} = \binom{d+1}{3}.$$

Let $p(d)$ denote the perimeter of $P(d)$. This polygon has $d + 1$ vertices and $d + 1$ edges. Of the latter, d have lattice length 1. The edge between $(d, \binom{d}{2})$ and $(0, 0)$ is a segment of the line $y = \frac{d-1}{2}x$, which contains $\frac{1}{2}(d - 2)$ lattice points other than the vertices if d is even and $d - 1$ if d is odd. Hence, $p(d) = d + \frac{1}{2}(d - 2) + 1 = \frac{3}{2}d$ if d is even and $p(d) = d + d - 1 + 1 = 2d$ if d is odd. It then follows from Pick’s formula that the number of lattice points in $P(d)$ is $\frac{1}{2}\binom{d+1}{3} + \frac{1}{2} \cdot \frac{3}{2}d + 1 = \frac{1}{12}d(d^2 + 8) + 1$ if d is even and $\frac{1}{2}\binom{d+1}{3} + \frac{1}{2} \cdot 2d + 1 = \frac{1}{12}d(d^2 + 11) + 1$ if d is odd. \square

Proposition 4.2. *The surface $X_{A_3}^{(2)}$ is a quartic nonsingular surface in \mathbb{P}^5 . For $d \geq 4$, the surface $X_{A_d}^{(2)}$ has two singular points, both of multiplicity $d - 1$ if d is even and of multiplicity $\frac{1}{2}(d - 1)$ if d is odd.*

Proof. The vertices of the polygon $P(d)$ are $(0, 0), (1, 0), \dots, (d-1, \binom{d-1}{2}), (d, \binom{d}{2})$. It is easy to check that the vertices $(1, 0), \dots, (d-1, \binom{d-1}{2})$ of $P(d)$ are nonsingular, so it remains to check the vertices $(0, 0)$ and $(d, \binom{d}{2})$ [9, Thm. 3.14, p. 186]:

- If d is even:

$$m_{(0,0)} = \left| \det \begin{pmatrix} 1 & 2 \\ 0 & d-1 \end{pmatrix} \right| = d-1,$$

$$m_{(d, \binom{d}{2})} = \left| \det \begin{pmatrix} 2 & 1 \\ d-1 & d-1 \end{pmatrix} \right| = d-1.$$

- If d is odd:

$$m_{(0,0)} = \left| \det \begin{pmatrix} 1 & 1 \\ 0 & \frac{1}{2}(d-1) \end{pmatrix} \right| = \frac{1}{2}(d-1),$$

$$m_{(d, \binom{d}{2})} = \left| \det \begin{pmatrix} 1 & 1 \\ \frac{1}{2}(d-1) & d-1 \end{pmatrix} \right| = \frac{1}{2}(d-1).$$

This proves the proposition. □

Finally, we derive formulas for the degrees of the dual varieties, which enable us to determine whether the linear projection is generic.

Proposition 4.3. *The degree of the dual variety $(X_{A_d}^{(2)})^\vee$ is equal to $\binom{d-1}{2}(d+3)$ if d is even and $\frac{1}{2}(d-1)(d^2+d-8)$ if d is odd.*

The degree of the dual variety $(X_{A_d}^{(2)})^\vee$ is equal to $\binom{d-1}{2}(d+1)$.

Proof. We shall apply the formula for the degree of the dual variety of a toric surface (see [14, Cor. 1.6, p. 2042], [15, Section 5]). In the first case the surface is normal, and we get

$$\deg(X_{A_d}^{(2)})^\vee = 3 \text{Vol}(P(d)) - 2p(d) + \sum_v \text{Eu}(v),$$

where $\text{Eu}(v)$ denotes the local Euler obstruction at the point of X_A corresponding to the vertex v of $P(d)$. The local Euler obstruction of a variety is a constructible function, introduced by MacPherson in order to define Chern classes of singular varieties [13].

For a normal toric surface, the local Euler obstruction at a vertex v is equal to 2 minus the difference between the lattice area of $P(d)$ and that of the polytope $\text{Conv}(P(d) \setminus \{v\})$ obtained by removing the vertex v and taking the convex hull of all the remaining vertices [14, Cor. 4.4, p. 2052]. It can also be computed as

$1 - c$, where c is the number of interior lattice points of $P(d)$ which are boundary points of $\text{Conv}(P(d) \setminus \{v\})$ [15, Lemma 5.1, p. 514]. We get

$$\text{Eu}((0, 0)) = \text{Eu}((d, \binom{d}{2})) = 2 - \frac{1}{2}d$$

if d is even, and

$$\text{Eu}((0, 0)) = \text{Eu}((d, \binom{d}{2})) = 2 - \frac{1}{2}(d - 1)$$

if d is odd. Hence, we get

$$\text{deg}(X_{A_d^{(2)}}^\vee) = 3\binom{d+1}{3} - 2 \cdot \frac{3}{2}d + d - 1 + 2(2 - \frac{1}{2}d) = \frac{1}{2}(d^3 - 7d + 6)$$

if d is even, and

$$\text{deg}(X_{A_d^{(2)}}^\vee) = 3\binom{d+1}{3} - 2 \cdot 2d + d - 1 + 2(2 - \frac{1}{2}(d - 1)) = \frac{1}{2}(d^3 - 9d + 8)$$

if d is odd.

In the second case, the surface is singular along the curve corresponding to the edge Δ of $P(d)$ joining the vertices $v := (0, 0)$ and $v' := (d, \binom{d}{2})$. In the formula for the degree of the dual surface, also the edge lengths need to be weighted by the Euler obstruction at a general point of the corresponding orbit. This gives [14, Cor. 1.6, p. 2042] $\text{Eu}(\Delta) = \frac{1}{2}d$ if d is even and d if d is odd. The Euler obstruction at each singular vertex is then $\text{Eu}(v) = \text{Eu}(v') = \frac{1}{2}d + 1 - \binom{d}{2}$ if d is even and $\text{Eu}(v) = \text{Eu}(v') = d + 1 - \binom{d}{2}$ if d is odd. We get

$$\begin{aligned} \text{deg}(X_{A_d^{(2)}}^\vee) &= 3\binom{d+1}{3} - 2(\frac{1}{2}d + d) + d - 1 + 2(\frac{1}{2}d + 1 - \binom{d}{2}) \\ &= \frac{1}{2}(d^3 - 2d^2 - d + 2) \end{aligned}$$

if d is even, and

$$\text{deg}(X_{A_d^{(2)}}^\vee) = 3\binom{d+1}{3} - 2(d + d) + d - 1 + 2(d + 1 - \binom{d}{2}) = \frac{1}{2}(d^3 - 2d^2 - d + 2)$$

if d is odd. □

Corollary 4.4. *The surface $X_{A_d^{(2)}}$ is not a general linear projection of its normalization $X_{A_d^{(2)}}$.*

Proof. We have $\text{deg}(X_{A_d^{(2)}}^\vee) \neq \text{deg}(X_{A_d^{(2)}}^\vee)$ for $d \geq 4$. For $d = 3$, we have $\text{deg}(X_{A_3^{(2)}}^\vee) = \text{deg}(X_{A_3^{(2)}}^\vee) = 4$. The nonsingular quartic surface $X_{A_3^{(2)}} \subset \mathbb{P}^5$ is given by the polygon $P(3)$. Its projection $X_{A_3^{(2)}} \subset \mathbb{P}^3$ has a triple line, corresponding to the edge Δ ; indeed, its equation is $x_0x_2^3 - x_1^3x_3 = 0$. Since general

projections to \mathbb{P}^3 of smooth surfaces in \mathbb{P}^5 do not have singular curves of multiplicity > 2 , $X_{A_3}^{(2)}$ is not a general projection of $X_{A_d}^{(2)}$, even though their dual varieties have the same degree. \square

The surfaces $X_{A_4}^{(2)}$ and $X_{A_5}^{(2)}$ have two singular points with local Euler obstruction equal to 0, hence are Gorenstein varieties, whereas for $d \geq 6$, $X_{A_d}^{(2)}$ is not Gorenstein. The surfaces $X_{A_d}^{(2)}$ are non-normal (they have a singular curve corresponding to the edge Δ).

References

- [1] ARKANI-HAMED, NIMA; BAI, YUNTAO; LAM, THOMAS. Positive geometries and canonical forms. *J. High Energy Phys.* (2017), no. 11, 039, 121 pp. MR3747267, Zbl 1383.81273, arXiv:1703.04541, doi: 10.1007/jhep11(2017)039. 1500, 1510, 1511
- [2] ARKANI-HAMED, NIMA; TRNKA, JAROSLAV. The amplituhedron. *J. High Energy Phys.* (2014), no. 10, 030, 33 pp. Zbl 1468.81075, arXiv:1312.2007, doi: 10.1007/jhep10(2014)030. 1500, 1510
- [3] BLASCHKE, WILHELM; LEICHTWEISS, KURT. Elementäre Differentialgeometrie. Die Grundlehren der mathematischen Wissenschaften, Band 1. *Springer-Verlag, Berlin-New York*, 1973. x+369 pp. MR0350630, Zbl 0264.53001, doi: 10.1007/978-3-642-49193-1. 1499
- [4] CORDOUIL, RAUL; DUCHET, PIERRE. Cyclic polytopes and oriented matroids. *European J. Combin.* **21** (2000), no. 1, 49–64. MR1737327(2001g:52020), Zbl 0981.52017, doi: 10.1006/eujc.1999.0317. 1500, 1511
- [5] DICKENSTEIN, ALICIA; DI ROCCO, SANDRA; PIENE, RAGNI. Higher order duality and toric embeddings. *Ann. Inst. Fourier. (Grenoble)* **64** (2014), no. 1, 375–400. MR3330552, Zbl 1329.14098, arXiv:1111.4641, doi: 10.5802/aif.2851. 1499, 1500, 1504
- [6] DICKENSTEIN, ALICIA; PIENE, RAGNI. Higher order selfdual toric varieties. *Ann. Mat. Pura Appl. (4)* **196** (2017), no. 5, 1759–1777. MR3694743, Zbl 1408.14160, arXiv:1609.05189, doi: 10.1007/s10231-017-0637-4. 1499
- [7] EISENBUD, DAVID; STURMFELS, BERND. Binomial ideals. *Duke Math. J.* **84** (1996), no. 1, 1–45. MR1394747(97d:13031), Zbl 0873.13021, doi: 10.1215/S0012-7094-96-08401-X.
- [8] EU, SEN-PENG; FU, TUNG-SHAN; PAN, YEH-JONG. The cyclic sieving phenomenon for faces of cyclic polytopes. *Electron. J. Combin.* **17** (2010), R47, 17 pp. MR2607333(2011b:05013), Zbl 1225.05019, doi: 10.37236/319. 1509
- [9] GEL'FAND, IZRAIL' M.; KAPRANOV, MIKHAIL M.; ZELEVINSKY, ANDREY V. Discriminants, resultants and multidimensional determinants. *Birkhäuser Boston, Inc., Boston, MA*, 1994. ISBN 978-0-8176-4770-4. MR1264417(95e:14045), Zbl 0827.14036, doi: 10.1007/978-0-8176-4771-1. 1499, 1500, 1501, 1513
- [10] KLEIMAN, STEVEN L. The enumerative theory of singularities. *Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976)*, 297–396. *Sijthoff & Noordhoff, Alphen aan den Rijn*, 1977. ISBN 90-286-0097-3. MR0568897, Zbl 0385.14018. 1503
- [11] KOHN, KATHLÉN; PIENE, RAGNI; RANESTAD, KRISTIAN; RYDELL, FELIX; SHAPIRO, BORIS; SINN, RAINER; SOREA, MIRUNA-STEFANA, TELEN, SIMON. Adjoints and canonical forms of polypols. Preprint, 2021. arXiv:2108.11747. 1511
- [12] LIU, FU. Ehrhart polynomials of cyclic polytopes. *J. Combin. Theory Ser. A.* **111** (2005), no. 1, 111–127. MR2144858(2006a:05012), Zbl 1066.05015, arXiv:math/0409337, doi: 10.1016/j.jcta.2004.11.011. 1509
- [13] MACPHERSON, ROBERT D. Chern classes for singular algebraic varieties. *Ann. of Math. (2)* **100** (1974), 423–432. MR0361141, Zbl 0311.14001, doi: 10.2307/1971080. 1513

- [14] MATSUI, YUTAKA; TAKEUCHI, KIYOSHI. A geometric degree formula for A -discriminants and Euler obstructions of toric varieties. *Adv. Math.* **226** (2011), no. 2, 2040–2064. MR2737807(2012e:14103), Zbl 1205.14062, arXiv:0807.3163, doi: 10.1016/j.aim.2010.08.020. 1513, 1514
- [15] NØDLAND, BERNT IVAR UTSTØL. Local Euler obstructions of toric varieties. *J. Pure Appl. Algebra.* **222** (2018), no. 3, 508–533. MR3710713, Zbl 1402.14067, arXiv:1610.02430, doi: 10.1016/j.jpaa.2017.04.016. 1513, 1514
- [16] PIENE, RAGNI. Numerical characters of a curve in projective n -space. *Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976)*, 475–495. *Sijthoff & Noordhoff, Alphen aan den Rijn, 1977*. ISBN 90-286-0097-3. MR0506323, Zbl 0375.14017. 1499, 1502
- [17] PIENE, RAGNI. A note on higher order dual varieties, with an application to scrolls. *Singularities, Part 2* (Arcata, Calif., 1981), 335–432. Proc. Sympos. Pure Math., 40. *Amer. Math. Soc., Providence, RI.*, 1983. ISBN:0-8218-1466-4. MR0713259(85d:14056), Zbl 0515.14031, doi: 10.1090/pspum/040.2/713259. 1499
- [18] PIENE, RAGNI. Higher order polar and reciprocal polar loci. *Facets of algebraic geometry. Vol. II.*, 238–253. London Math. Soc. Lecture Note Ser., 473. *Cambridge Univ. Press, Cambridge*, 2022. ISBN 978-1-108-79251-6. MR4381916, Zbl 1489.14069, arXiv:2002.03691, doi: 10.1017/9781108877855.008. 1499
- [19] STURMFELS, BERND. Gröbner bases and convex polytopes. University Lecture Series, 8. *American Mathematical Society, Providence, RI*, 1996. xii+162 pp. MR1363949(97b:13034), Zbl 0685.13005, doi: 10.1090/ulect/008. 1502

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