New York Journal of Mathematics

New York J. Math. **30** (2024) 1293–1303.

Contraction property of Fock type space of log-subharmonic functions in $\mathbb{R}^m$

David Kalaj

ABSTRACT. We prove a contraction property of Fock type spaces $\mathcal{L}_\alpha^p$ of log-subharmonic functions in $\mathbb{R}^n$. To prove the result, we demonstrate a certain monotonic property of measures of the superlevel set of the function $u(x) = |f(x)|^p e^{-\frac{\alpha}{2}p|x|^2}$, provided that f is a certain log-subharmonic function in $\mathbb{R}^m$. The result recover a contraction property of holomorphic functions in the Fock space $\mathcal{F}_\alpha^p$ proved by Carlen in [Car1991].

CONTENTS

1. Introduction

Let $m \geqslant 1$ and let $\mathbb{R}^m$ be the Euclidean space endowed with the Euclidean norm: $|x| = \sqrt{\langle x, x\rangle}$, where $\langle x, y\rangle = \sum_{i=1}^m x_i y_i$, and $x = (x_1, \dots, x_m), y = (y_1, \dots, y_n) \in \mathbb{R}^m$. If $\alpha > 0$ and $p > 0$ and $m = 2n$ is an even integer, we define the Fock space or Segal-Bargmann space $\mathcal{F}_\alpha^p$ (cf. [Bar62, Bar61, KZ2012]) of entire holomorphic functions f in $\mathbb{C}^n = \mathbb{R}^{2n}$ so that:

$$\|f\|_{p,\alpha}^p := c_{p,\alpha} \int_{\mathbb{R}^m} |f(x)|^p e^{-\frac{\alpha}{2}p|x|^2} dA(x) < \infty,$$

where

$$c_{p,\alpha} = \left(\frac{\alpha p}{2\pi}\right)^{\frac{m}{2}}, \tag{1.1}$$

Received August 29, 2024.

2010 *Mathematics Subject Classification.* Primary 30H20.

Key words and phrases. Holomorphic functions, isoperimetric inequality, Fock space, contraction.

ISSN 1076-9803/2024

and $dA(x)$ is Lebesgue measure on $\mathbb{R}^m$. Note that $c_{p,\alpha}e^{-\frac{\alpha}{2}p|x|^2}dA(x)$ is the Gaussian probabily measure in $\mathbb{R}^m$.

Assume now that $m \in \mathbb{N}$ is an arbitrary integer. We say that a real twice differentiable function f defined in a domain $\Omega \subset \mathbb{R}^m$ is subharmonic if $\Delta f(x) \geqslant 0$ for $x \in \Omega$. Here, Δ is the Laplacian. This definition can also be extended to not necessary double differentiable functions, by using the sub-mean value property ([HK1976]). We say that a mapping f is log-subharmonic, if $\log|f(x)|$ is subharmonic in $\Omega \setminus f^{-1}(0)$. We denote by $\mathcal{L}^p_\alpha$ the space of complex-valued, real-analytic functions whose absolute value is a log-subharmonic function, defined in $\mathbb{R}^m$, with a finite $\|f\|_{p,\alpha}$ norm as defined in (1). Here, m is an arbitrary positive integer. Observe that for $m = 2n$ we have $\mathcal{F}^p_\alpha \subset \mathcal{L}^p_\alpha$: If f is holomorphic in Ω, then $|f(z)|$ is log-subharmonic. Indeed

$$\Delta \log|f(z)| = \sum_{k=1}^{n} \Delta_{z_k} \log|f(z)| = 0,$$

where $z = (z_1, \dots, z_n)$, and

$$\Delta_{z_k} = \frac{\partial^2}{(\partial_{x_k})^2} + \frac{\partial^2}{(\partial_{y_k})^2},$$

$z_k = x_k + iy_k$ for $k = 1, \dots, n$ and $z \in \Omega \setminus f^{-1}(0)$.

2. Motivation and main results

Carlen, in his paper [Car1991] proved the following result:

Theorem 2.1. *If $0 < p < q < \infty$, then $\mathcal{F}^p_\alpha(\mathbb{C}^n) \subset \mathcal{F}^q_\alpha(\mathbb{C}^n)$ and the inclusion is proper and continuous. Moreover*

$$\|f\|_{q,\alpha} \leq \|f\|_{p,\alpha}.$$

Theorem 2.1 is applied in [Car1991] to the coherent state transform in a new proof of Wehrl's entropy conjecture [LIEB1978]. In this paper, among other results, we recover Theorem 2.1 and provide a proof that works for a more general class of mappings, namely real analytic complex mappings whose absolute value is a log-subharmonic function in $\mathbb{R}^m$ and belongs to the Fock-type space $\mathcal{L}^p_\alpha$.

Let f be a real analytic complex-valued function defined in the Euclidean space $\mathbb{R}^m$, such that $v = |f|$ is a log-subharmonic function in $\mathbb{R}^m$ and such that $u(x) = v(x)^p e^{-\alpha p/2|x|^2}$ is bounded and goes to 0 uniformly as $|x| \to \infty$. Then the superlevel sets $A_t = \{x : u(x) > t\}$ for $t > 0$ are compactly embedded in $\mathbb{R}^m$ and thus have finite Lebesgue measure $\mu(t) = |A_t|$.

Those are the main results:

Theorem 2.2. *Let $\alpha > 0$ and $p > 0$ and assume that f is a real analytic complex valued function such that $v = |f| : \mathbb{R}^m \to [0, +\infty)$ is a log-subharmonic function. Assume further that the function $u(x) = |f(x)|^p e^{-\frac{\alpha p}{2}|x|^2}$ is bounded and*

$u(x)$ tends to 0 *uniformly as* $|x| \to \infty$. *Then the function*

$$g(t) = t \exp\left[\frac{\alpha p(\Gamma(m/2))^{2/m}}{2\pi}\mu^{2/m}(t)\right],$$

is decreasing on the interval $(0, t_\circ)$*, where* $t_\circ = \max_{x\in\mathbb{R}^m} u(x)$.

If $f(x) \equiv 1$, the function g turns out to be constant and this is an important property of g.

The proof of this theorem is mostly based on the methods developed by Nicola and Tilli in [NT2022] (see also the subsequent papers where similar methods are used: [KU2022], [KA2024], [RT2023], [KNOT2022], and [Fr2023]).

By using Theorem 2.2, we will prove the following theorem:

Theorem 2.3. *Let $p > 0$ and $\alpha > 0$. Let $G : [0,\infty) \to \mathbb{R}$ be a convex function. Then the maximum value of*

$$\int_{\mathbb{R}^m} G(|f(x)|^p e^{-\frac{\alpha}{2}p|x|^2})dA(x) \tag{2.1}$$

is attained for

$$f_a(x) = e^{\alpha\langle a,x\rangle - \frac{\alpha}{2}|a|^2},$$

where $a \in \mathbb{C}^n$ is arbitrary, subject to the condition that $f \in \mathcal{L}^p_\alpha$ and $\|f\|_{p,\alpha} = 1$.

Applying Theorem 2.3 to the convex and increasing function $G(t) = t^{q/p}$, we get the extension of theorem [Car1991, Theorem 2] by proving:

Theorem 2.4. *For all $0 < p < q < \infty$ and $0 < \alpha$ and for $f \in \mathcal{L}^p_\alpha(\mathbb{R}^m)$, we have $f \in \mathcal{L}^q_\alpha(\mathbb{R}^m)$ and*

$$\|f\|_{q,\alpha} \le \|f\|_{p,\alpha}$$

with equality for $f_a(x) = e^{\alpha\langle a,x\rangle - \frac{\alpha}{2}|a|^2}$, where $a \in \mathbb{R}^m$ is arbitrary.

Proof of Theorem 2.4. For $\|f\|_{p,\alpha} = N$, $\|f/N\|_{p,\alpha} = 1$ and from Theorem 2.3 we have

$$\int_{\mathbb{R}^m} |f(x)/N|^q e^{-\frac{\alpha}{2}q|x|^2} dA(x) \le \int_{\mathbb{R}^m} e^{-\frac{\alpha}{2}q|x|^2} dA(x) = 1/c_{q,\alpha}.$$

Thus,

$$c_{q,\alpha}\int_{\mathbb{R}^m} |f(x)|^q e^{-\frac{\alpha}{2}q|x|^2} dA(x) \le N^q,$$

or what is the same

$$\|f\|_{q,\alpha} \le \|f\|_{p,\alpha}.$$

The equality statement follows from the equality statement of Theorem 2.4, but can be proved by using the same approach as in the monograph of Zhu [KZ2012, Lemma 2.33]. □

Remark 2.5. *The last theorem is an extension of Theorem* 2.1. *Moreover, its proof is different from the proof in* [Car1991] *and seems to be simpler. We refer to the paper* [GKL2010] *for some related inequalities for log-subharmonic functions in* $\mathbb{R}^n$.

Theorem 2.4 is a counterpart of a similar contraction property of Bergman spaces $\mathbf{B}^p_\alpha$ ([HKZ2000, p. 2]), proved by Kulikov in [KU2022] for holomorphic functions in the unit disk and for $\mathcal{M}-$log-subharmonic functions in the unit ball in $\mathbb{R}^n$ by the author in [KA2024]. It is known that

$$\mathbf{B}^p_\alpha \subset \mathbf{B}^q_\beta, \quad \frac{p}{\alpha} = \frac{q}{\beta} = r, \quad p < q.$$

For $n = 2$, it was asked whether these embeddings are contractions; that is, whether the norm $\|f\|_{\mathbf{B}^{r\alpha}_\alpha}$ is decreasing in α. In the case of Bergman spaces, this question was asked by Lieb and Solovej [LiSo2021]. They proved that such contractivity implies their Wehrl-type entropy conjecture for the $SU(1,1)$ group. In the case of contractions from the Hardy spaces to the Bergman spaces, it was asked by Pavlović in [MP2014] and by Brevig, Ortega-Cerdà, Seip, and Zhao [BOSZ2018] concerning the estimates for analytic functions. The mentioned contraction property proved by Kulikov confirmes these conjectures. An interesting application of Kulikov result has been given by Melentijević in [PM2023].

We end this paper with the construction of a new normed Fock type space:

Definition 2.6 (Fock limit space)**.** *Let f be a holomorphic function in $\mathbb{C}^n$. Then for $\alpha > 0$ we say $f \in \mathcal{F}_\alpha$ if $f \in \bigcap_{p>0} \mathcal{F}^p_\alpha$. Then we define*

$$\|f\|_\alpha := \inf_{p>0} \|f\|_{p,\alpha}.$$

For $\alpha > 0$ define as in [KZ2012, eq. 2.2] the following Banach norm

$$\|f\|_{\infty,\alpha} := \operatorname{esssup}\{|f(z)|e^{-\frac{\alpha}{2}|z|^2}, z \in \mathbb{C}^n\}.$$

Then, we prove

Theorem 2.7. *For every $\alpha > 0$ we have*

$$\|f\|_\alpha = \|f\|_{\infty,\alpha}.$$

In particular $(\mathcal{F}_\alpha, \|\cdot\|_\alpha)$ is a normed subspace of Banach space $\mathcal{F}^\infty_\alpha$.

3. Proof of Theorem 2.2

Proof of Theorem 2.2. We start with the formula

$$\mu(t) = |A_t| = \int_{A_t} dx = \int_t^{\max u} \int_{|u(x)|=\kappa} d\mathcal{H}^{m-1}(x) d\kappa.$$

Then we get

$$-\mu'(t) = \int_{u=t} |\nabla u|^{-1} d\mathcal{H}^{m-1}(x) \tag{3.1}$$

along with the claim that $\{x : u(x) = t\} = \partial A_t$ and that this set is a smooth hypersurface for almost all $t \in (0, t_\circ)$. Here, $dS = d\mathcal{H}^{m-1}$ is $m-1$ dimensional Hausdorff measure. These assertions follow the proof of [NT2022, Lemma 3.2]. We point out that, since u is real analytic, then it is a well-known fact from measure theory that the level set $\{x : u(x) = t\}$ has a zero measure ([MI2020]), and this is equivalent to the fact that the μ is continuous.

Following the approach from [NT2022], our next step is to apply the Cauchy–Schwarz inequality to the $m-1$ dimensional measure of ∂A_t:

$$|\partial A_t|^2 = \left(\int_{\partial A_t} dS\right)^2 \leq \int_{\partial A_t} |\nabla u|^{-1} dS \int_{\partial A_t} |\nabla u| dS. \tag{3.2}$$

Let $\nu = \nu(x)$ be the outward unit normal to ∂A_t at a point x. Note that, ∇u is parallel to ν, but directed in the opposite direction. Thus, we have $|\nabla u| = -\langle \nabla u, \nu\rangle$. Also, we note that since for $x \in \partial A_t$ we have $u(x) = t$, we obtain for $x \in \partial A_t$ that

$$\frac{|\nabla u(x)|}{t} = \frac{|\nabla u(x)|}{u} = \left\langle \nabla \log u(x), \nu \right\rangle.$$

Now the second integral on the right-hand side of (3.2) can be evaluated by Gauss's divergence theorem:

$$\begin{aligned}\int_{\partial A_t} |\nabla u||dS| &= -t \int_{A_t} \operatorname{div}\left(\nabla \log u(x)\right) dA(x) \\ &= -t \int_{A_t} \Delta \log u(x) dA(x).\end{aligned}$$

Now we plug $u = |f(x)|^p e^{-\frac{\alpha}{2}p|x|^2}$, and calculate

$$-t\Delta \log(|f(x)|^p e^{-\frac{\alpha}{2}p|x|^2}) = -(pt\Delta \log v - t\frac{\alpha}{2}p\Delta|x|^2) \leq 0 + mt\alpha p.$$

By using (3.1) and (3.2), we obtain

$$\begin{aligned}|\partial A_t|^2 &\leq (-\mu'(t)) \int_{\partial A_t} |\nabla u| dS. \\ &\leq -mt\alpha p\mu'(t)\mu(t).\end{aligned}$$

Now we use the isoperimetric inequality for the space:

$$|\partial A_t|^2 \geqslant \pi m^2 |A_t|^{\frac{2(m-1)}{m}} \left(\Gamma(m/2)\right)^{-\frac{2}{m}},$$

which implies that

$$mt\alpha p\mu'(t)\mu(t) + m^2\pi\mu(t)^{\frac{2(m-1)}{m}} \left(\Gamma(m/2)\right)^{-\frac{2}{m}} \leq 0 \tag{3.3}$$

with equality in (3.3) if and only if $v(x) = e^{\alpha\langle x,a\rangle - \frac{\alpha}{2}|a|^2}$ because in that case A_t is a ball centered at a. So,

$$M(t) := \alpha p\mu'(t)\mu(t)^{\frac{2-m}{m}} + \frac{m\pi \left(\Gamma(m/2)\right)^{-\frac{2}{m}}}{t} \leq 0. \tag{3.4}$$

Since $\mu(t^\circ)=0$, we obtain that

$$G(t)=\int_{t_\circ}^t M(t)dt = m\pi(\Gamma(m/2))^{-2/m}\log\frac{t}{t_\circ}+\frac{m}{2}\alpha p\mu^{\frac{2}{m}}(t)$$

is a non-increasing function for $0\le t<t_\circ$.

In the case $v(x)\equiv e^{\alpha\langle a,x\rangle-\frac{\alpha}{2}|a|^2}$, $t_\circ=1$ and $\mu(t_\circ)=0$. Moreover,

$$g(t):=\exp(G(t))=t\exp\left[\frac{\alpha p(\Gamma(m/2))^{2/m}}{2\pi}\mu^{2/m}(t)\right]$$

is non-increasing for $0\le t<t_\circ$.

□

Remark 3.1. *Note that for the function $f(x)\equiv 1$ or*

$$f(x)=e^{-\frac{\alpha}{2}|a|^2}e^{\alpha\langle a,x\rangle},$$

for a fixed a, everywhere in the proof above we have equalities for all values of p and α. Moreover in this case the maximum of $u(x)$ is equal to 1 and achieved for $x=a$.

4. Proof of Theorem 2.3

We need the following lemma:

Lemma 4.1. [KA2024] *Assume that Φ,Ψ are positive increasing functions and g positive non-increasing such that*

$$\int_0^{t_\circ}\Phi(g(t)/t)\,dt=\int_0^{t_\circ}\Phi(1/t)\,dt=c.$$

Then

$$\int_0^{t_\circ}\Phi(g(t)/t)\,\Psi(t)dt\le\int_0^{t_\circ}\Phi(1/t)\,\Psi(t)dt.$$

As in [KU2022, KA2024] where is treated Bergman version of this theorem, we restrict ourselves to the only nontrivial case $\lim_{t\to0^+}G(t)=0$. Let $\mu(t)=\mu(\{x: u(x)>t\})$ be the Lebesgue measure in $\mathbb{R}^m$, where $u(x)=|f(x)|^pe^{-\frac{\alpha p}{2}|x|^2}$. Applying Theorem 2.2 to f, we get that the function

$$g(t)=t\exp\left[\frac{\alpha(\Gamma(m/2))^{2/m}}{2\pi}\mu^{2/m}(t)\right],$$

is decreasing on $(0,t_\circ)$ with $t_\circ=\max_{x\in\mathbb{R}^m}u(x)$. Proposition 5.1 below ensures the existence of $t_\circ$.

For $f\equiv 1$, g is a constant function equal to 1.

Then,

$$\mu(t)=\left(\frac{2\pi}{\alpha(\Gamma(m/2))^{2/m}}\log\frac{g(t)}{t}\right)^{\frac{m}{2}}.$$

We assume that $\|f\|_{p,\alpha} = 1$, that is

$$I_1 = c_{p,\alpha} \int_0^{t_\circ} \mu(t)dt = c_{p,\alpha} \int_0^{t_\circ} \left(\frac{2\pi}{\alpha(\Gamma(m/2))^{2/m}} \log \frac{g(t)}{t} \right)^{m/2} dt = 1.$$

Now the integral in (2.1) can be rewritten as

$$I_2 = c_{p,\alpha} \int_0^{t_\circ} \left(\frac{2\pi}{\alpha(\Gamma(m/2))^{2/m}} \log \frac{g(t)}{t} \right)^{m/2} G'(t)dt.$$

Then, by Lemma 4.1, by taking $\Phi(s) = c_{p,\alpha} \left(\frac{2\pi}{\alpha(\Gamma(m/2))^{2/m}} \log s \right)^{\frac{m}{2}}$ and $\Psi(t) = G'(t)$, the maximum of I_2 under $I_1 = 1$ is attained for $g \equiv 1$.

5. Additional properties of Fock space and proof of Theorem 2.7

Now we prove the following proposition used in the proof of our main result.

Proposition 5.1. *Assume that f is a real-analytic log-subharmonic function in $\mathbb{R}^m$ belonging to the Fock type space. Then for every x,*

$$|f(x)|^p e^{-\frac{\alpha p}{2}|x|^2} \le c_{p,\alpha} \int_{\mathbb{R}^m} |f(y)|^p e^{-\frac{\alpha p}{2}|y|^2} dA(y). \tag{5.1}$$

Moreover,

$$\lim_{|x|\to\infty} |f(x)| e^{-\frac{\alpha}{2}|x|^2} = 0. \tag{5.2}$$

Notice that (5.1) extends [KZ2012, Theorem 2.7] and the relation (5.2) extends corresponding relation in [KZ2012, p. 38].

Proof. Let $g(y) = |f(x+y)|^p e^{-\alpha p \langle (y+x), x \rangle}$. Now use the mean value property to the log-subharmonic function g (it is also subharmonic).

$$|g(0)| \le c_{p,\alpha} \int_{\mathbb{R}^m} |g(y)| e^{-\frac{\alpha p}{2}|y|^2} dA(y).$$

Then, we have

$$g(0) = |f(x)|^p e^{-\alpha p |x|^2} \le c_{p,\alpha} \int_{\mathbb{R}^m} f^p(y+x) e^{-\frac{\alpha}{2} p \langle (x+y), x \rangle} e^{-\frac{\alpha p}{2}|y|^2} dA(y).$$

Therefore,

$$|f(x)|^p e^{-\alpha p |x|^2} \le c_{p,\alpha} \int_{\mathbb{R}^m} f^p(y) e^{-\alpha p \langle y, x \rangle} e^{-\frac{\alpha p}{2}|y-x|^2} dA(y).$$

So,

$$|f(x)|^p e^{-\frac{\alpha p}{2}|x|^2} \le c_{p,\alpha} \int_{\mathbb{R}^m} f^p(y) e^{-\frac{\alpha p}{2}|y|^2} dA(y).$$

Now, to prove (5.2), we use the following inequality, which is also a consequence of the sub-mean value property of subharmonic functions. Let $B_1(x) = \{y \in \mathbb{R}^m : |y - x| < 1\}$. Then for every subharmonic function g, we have

$$|g(0)| \le \frac{n}{\omega_n}\int_{B_1(0)} |g(y)|dA(y).$$

Thus,

$$|g(0)|e^{-\frac{\alpha p}{2}} \le \frac{n}{\omega_n}\int_{B_1(0)} |g(y)|e^{-\frac{\alpha p}{2}|y|^2}dA(y). \tag{5.3}$$

By applying the previous inequality for $g(y) = |f(x+y)|^p e^{-\alpha p\langle (y+x),x\rangle}$, we obtain from (5.3) that

$$\begin{aligned}
|f(x)|^p e^{-\alpha p|x|^2}e^{-\frac{\alpha p}{2}} &\le \frac{n}{\omega_n}\int_{B_1(0)} |f(x+y)|^p e^{-\alpha p\langle (y+x),x\rangle}e^{-\frac{\alpha p}{2}|y|^2}dA(y)\\
&= \frac{n}{\omega_n}\int_{B_1(x)} |f(y)|^p e^{-\alpha p\langle y,x\rangle}e^{-\frac{\alpha p}{2}|y-x|^2}dA(y)\\
&= \frac{n}{\omega_n}e^{-\frac{\alpha p}{2}|x|^2}\int_{B_1(x)} |f(y)|^p e^{-\frac{\alpha p}{2}|y|^2}dA(y).
\end{aligned}$$

Thus,

$$|f(x)|^p e^{-\frac{\alpha p}{2}|x|^2}e^{-\frac{\alpha p}{2}} \le \frac{n}{\omega_n}\int_{B_1(x)} |f(y)|^p e^{-\frac{\alpha p}{2}|y|^2}dA(y).$$

Since $f \in \mathcal{L}^p_\alpha$, it follows that

$$\lim_{|x|\to\infty}\frac{n}{\omega_n}\int_{B_1(x)} |f(y)|^p e^{-\frac{\alpha p}{2}|y|^2}dA(y) = 0.$$

This implies (5.2). □

It follows from the following lemma that $\|f\|_\alpha$ is a norm on $\mathcal{F}_\alpha$. Theorem 2.7 is a direct application of the following lemma

Lemma 5.2. *a) If $f, g \in \mathcal{F}_\alpha$, then $\|f+g\|_\alpha \le \|f\|_\alpha + \|g\|_\alpha$.*

b) For every $\alpha > 0$ and $f \in \mathcal{F}_\alpha$ and $x \in \mathbb{C}^m$ we have $|f(x)|e^{-\frac{\alpha}{2}|x|^2} \le \|f\|_\alpha$.

c) For every $\alpha > 0$ and $f \in \mathcal{F}_\alpha$, $\|f\|_\alpha = \sup_{x\in\mathbb{C}^n}\left(|f(x)|e^{-\frac{\alpha}{2}|x|^2}\right)$.

Proof. Let us restrict ourselves to the case $n = 1$. The general case is a trivial modification of this case.

a) Let $f, g \in \mathcal{F}_\alpha$. Then for every $\alpha > 0$, $f, g \in \mathcal{F}^p_\alpha$ and by the triangle inequality for the norm in $\mathcal{F}^p_\alpha$ we obtain

$$\begin{aligned}
\|f+g\|_\alpha &= \lim_{p\to\infty}\|f+g\|_{p,\alpha}\\
&\le \lim_{p\to\infty}\|f\|_{p,\alpha} + \lim_{p\to\infty}\|g\|_{p,\alpha}\\
&= \|f\|_\alpha + \|g\|_\alpha.
\end{aligned}$$

b) This follows from Proposition 5.1.

c) It follows from (5.1) that

$$|f(x)|e^{-\frac{\alpha}{2}|x|^2} \le \|f\|_{p,\alpha}.$$

By letting $p \to \infty$ we obtain

$$|f(x)|e^{-\frac{\alpha}{2}|x|^2} \le \|f\|_{\alpha}.$$

Thus,

$$\text{ess sup}\,|f(x)|e^{-\frac{\alpha}{2}|x|^2} \le \|f\|_{\alpha}.$$

To prove the converse, fix an $R > 0$ and assume first that $f = P$ is a polynomial. Then

$$\|P\|_{p,\alpha}^p = \int_{|x|\le R} |P(x)|^p e^{-\frac{\alpha}{2}p|x|^2} dx + \int_{|x|>R} |P(x)|^p e^{-\frac{\alpha}{2}p|x|^2} dx.$$

Moreover, for sufficiently large R

$$I(R) := \int_{|x|>R} |P(x)|^p e^{-\frac{\alpha}{2}p|x|^2} dx \le c_P \int_{|x|>R} |z|^{n_P p} e^{-\frac{\alpha}{2}p|x|^2} dx$$

and the last expression is smaller than $\|F\|_{\infty,\alpha}^p$. In fact, the last expression tends to zero as $R \to \infty$. Therefore,

$$\|P\|_{p,\alpha} \le (\|P\|_{\infty,\alpha}^p R^n \omega_n + \|P\|_{\infty,\alpha}^p)^{1/p},$$

where ω_n is the meausre of the unit sphere. Thus,

$$\|P\|_{\alpha} = \lim_{p\to\infty} \|P\|_{p,\alpha} \le \|P\|_{\infty,\alpha}.$$

Thus, if f is a polynomial, then

$$\|f\|_{\alpha} = \|f\|_{\infty,\alpha}. \tag{5.4}$$

Further, if f is not a polynomial and $\epsilon > 0$ is arbitrary, then for $p = 2$, there exists a polynomial P so that $\|P - f\|_{p,\alpha} < \epsilon$. Moreover,

$$\|f\|_{\alpha} \le \|P\|_{\alpha} + \|f - P\|_{\alpha} = \|P\|_{\infty,\alpha} + \|f - P\|_{\alpha} \le \|P\|_{\alpha} + \epsilon.$$

Since ϵ is arbitrary, we conclude that (5.4) hold for every function $f \in \mathcal{F}_{\alpha}$.

□

Remark 5.3. *One can ask, given a holomorphic function f, when this*

$$\lim_{p\to 0} \|f\|_{\alpha,p}$$

exists. The answer is that limit is infinity except for the case when $f \equiv const$, so we cannot produce a Hardy type space for holomorphic mappings in $\mathbb{C}^n$.

Acknowledgments. After I proved my main results, Kehe Zhu directed me to the reference by Carlen [Car1991], for which I am thankful. I am also grateful to Djordjije Vujadinovic for his valuable insights and discussions on the topic.

References

[Bar62] Bargmann, Valentine. Remarks on a Hilbert space of analytic functions. *Proc. Nat. Acad. Sci. USA* **48** (1962), 199–204. MR0133009, Zbl 0149.09601, doi: 10.1002/cpa.3160200102. 1293

[Bar61] Bargmann, Valentine. On a Hilbert space of analytic functions and an associated integral transform. Part II. A family of related function spaces. Application to distribution theory. *Comm. Pure Appl. Math.* **20** (1967), 1–101. MR0201959, Zbl 0107.09102, doi: 10.1002/cpa.3160140303. 1293

[BOSZ2018] Brevig, Ole Fredrik; Ortega-Cerdà, Joaquim; Seip, Kristian; Zhao, Jing. Contractive inequalities for Hardy spaces. *Funct. Approx. Comment. Math.* **59** (2018), no. 1, 41–56. MR3858278, Zbl 1405.30054, arXiv:1706.00738, doi: 10.7169/facm/1680. 1296

[Car1991] Carlen, Eric A. Some integral identities and inequalities for entire functions and their application to the coherent state transform. *J. Funct. Anal.* **97** (1991), no. 1, 231–249. MR1105661, Zbl 0743.46018, doi: 10.1016/0022-1236(91)90022-W. 1293, 1294, 1295, 1296, 1301

[Fr2023] Frank, Rupert L. Sharp inequalities for coherent states and their optimizers. *Adv. Nonlinear Stud.* **23** (2023), no. 1, Paper No. 20220050, 28 pp. MR4580699, Zbl 1515.81154, arXiv:2210.14798, doi: 10.1515/ans-2022-0050. 1295

[GKL2010] Graczyk, Piotr; Kemp, Todd; Loeb, Jean-Jacques. Hypercontractivity for log-subharmonic functions. *J. Funct. Anal.* **258** (2010), no. 6, 1785–1805. MR2578455, Zbl 1186.60011, arXiv:0802.4260, doi: 10.1016/j.jfa.2009.08.014. 1296

[HK1976] Hayman, Walter K.; Kennedy, Patrick B. Subharmonic functions. I. London Mathematical Society Monographs, 9. *Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York*, 1976. xvii+284 pp. MR0460672, Zbl 0419.31001. 1294

[HKZ2000] Hedenmalm, Haakan; Korenblum, Boris; Zhu, Kehe. Theory of Bergman spaces. Graduate Texts in Mathematics, 199. *Springer-Verlag, New York*, 2000. x+286 pp. ISBN:0-387-98791-6 MR1758653, Zbl 0955.32003, doi: 10.1007/978-1-4612-0497-8. 1296

[KA2024] 1295, 1296, 1298
Kalaj, David. Contraction property of certain classes of $\log -\mathcal{M}$-subharmonic functions in the unit ball. *J. Funct. Anal.* **286**, no. 1, (2024) Paper No. 110203, 29 pp. MR4655699, Zbl 1528.32050, arXiv:2207.02054, doi: 10.1016/j.jfa.2023.110203.

[KU2022] Kulikov, Aleksei. Functionals with extrema at reproducing kernels. *Geom. Funct. Anal.* **32** (2022), no. 4, 938–949. MR4472587, Zbl 1502.30151, arXiv:2203.12349, doi: 10.1007/s00039-022-00608-5. 1295, 1296, 1298

[KNOT2022] Kulikov, Aleksei; Nicola, Fabio; Ortega-Cerdà, Joaquim; Tilli, Paolo. A monotonicity theorem for subharmonic functions on manifolds. Preprint, 2022. arXiv:2212.14008. 1295

[LIEB1978] Lieb, Elliott H. Proof of an entropy conjecture of Wehrl. *Comm. Math. Phys.* **62** (1978), no. 1, 35–41. MR0506364, Zbl 0385.60089, doi: 10.1007/BF01940328. 1294

[LiSo2021] Lieb, Elliott H.; Solovej, Jan Philip. Wehrl-type coherent state entropy inequalities for $\mathrm{SU}(1,1)$ and its $AX+B$ subgroup. *Partial differential equations, spectral theory, and mathematical physics—the Ari Laptev anniversary volume*, 301–314. EMS Ser. Congr. Rep. *EMS Press, Berlin*, 2021. ISBN:978-3-98547-007-5 MR4331821, Zbl 1480.81064, arXiv:1906.00223, doi: 10.4171/ECR/18-1/18. 1296

[PM2023] Melentijević, Petar. Hypercontractive inequalities for weighted Bergman spaces. *Bull. Lond. Math. Soc.* **55** (2023), no. 6, 2611–2616. MR4689542, Zbl 1532.30016, arXiv:2211.03655, doi: 10.1112/blms.12883. 1296

[MI2020] Mityagin, Boris S. The zero set of a real analytic function. *translated from Mat. Zametki* **107** (2020), no. 3, 473–475; *Math. Notes* **107** (2020), no. 3-4, 529–530.

MR4070868, Zbl 1442.26039, arXiv:1512.07276, doi: 10.1134/S0001434620030189. 1297

[NT2022] NICOLA, FABIO; TILLI, PAOLO. The Faber–Krahn inequality for the short-time Fourier transform. *Invent. Math.* **230** (2022), no. 1, 1–30. MR4480144, Zbl 1497.42024, arXiv:2106.03423, doi: 10.1007/s00222-022-01119-8. 1295, 1297

[RO1978] OSSERMAN, ROBERT. The isoperimetric inequality. *Bull. Amer. Math. Soc.* **84** (1978), no. 6, 1182–1238. MR0500557, Zbl 0411.52006, doi: 10.1090/S0002-9904-1978-14553-4

[MP2014] PAVLOVIĆ, MIROSLAV, Function classes on the unit disc—an introduction. 2nd edition. De Gruyter Stud. Math., 52. *De Gruyter, Berlin*, 2019. xv+553 pp. ISBN:978-3-11-062844-9; ISBN:978-3-11-063085-5; ISBN:978-3-11-062865-4. MR4321142, Zbl 1441.30002, doi: 10.1515/9783110630855. 1296

[RT2023] RAMOS, JOÃO P. G.; TILLI, PAOLO. A Faber–Krahn inequality for wavelet transforms. *Bull. Lond. Math. Soc.* **55** (2023), no. 4, 2018–2034. MR4623699, Zbl 1525.42038, arXiv:2205.07998, doi: 10.1112/blms.12833. 1295

[KZ2012] ZHU, KEHE. Analysis on Fock spaces. Graduate Texts in Mathematics, 263. *Springer, New York*, 2012. x+344 pp. ISBN:978-1-4419-8800-3. MR2934601, Zbl 1262.30003, doi: 10.1007/978-1-4419-8801-0.

(David Kalaj) UNIVERSITY OF MONTENEGRO, FACULTY OF NATURAL SCIENCES AND MATHEMATICS, CETINJSKI PUT B.B. 81000 PODGORICA, MONTENEGRO
davidk@ucg.ac.me

This paper is available via http://nyjm.albany.edu/j/2024/30-57.html.