Twisted analogue of
the Kummer-Leopoldt constant

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Abstract. Let $F$ be a number field and let $p$ be an odd prime. Denote by $S$ the set of $p$-adic and infinite places of $F$. We study a generalization to $K$-theory of the Kummer-Leopoldt constant for the $S$-units introduced in [7, Section 4]. We express in particular its value as the exponent of some Galois module. As an application, we give a new characterization of $(p, i)$-regular quadratic number fields.

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1. Introduction

Let $p$ be an odd prime and let $F$ be a number field. The Kummer-Leopoldt constant [7, Definition 1] $\kappa(F)$ is the smallest integer $c$ satisfying the following property: if $n$ is sufficiently large and $u$ is a unit of $F$ that is a $p^{n+c}$-th power locally at all primes dividing $p$, then $u$ is a global $p^n$-th power. This constant exists when the couple $(F, p)$ satisfies Leopoldt’s conjecture. Given this definition, Kummer’s lemma states that if $p$ is a regular prime number and $F$ is the $p$-th cyclotomic field then $\kappa(F)$ is zero. Kummer’s lemma has been generalized by several authors to $p^n$-th cyclotomic fields, $n \geq 1$ [33], [32], or to totally real number fields [27]. In [33, 32, 27], the authors give an upper bound for the Kummer-Leopoldt constant in terms of special values of the associated $p$-adic $L$-function.

More generally, for an arbitrary number field $F$, the quantity $p^{\kappa(F)}$ is the exponent of the Galois group $\text{Gal}(F^{BP}/\bar{F}_F)$ [7, Théorème 1], where $F^{BP}$ is the

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Bertrandias-Payan field of $F$ [9, 25], $\bar{F}$ is the composite of all $\mathbb{Z}_p$-extensions of $F$ and $L_\bar{F}$ is the maximal abelian unramified $p$-extension of $F$.

The Bertrandias-Payan field $F_{\text{BP}}$ is contained in $\bar{F}$, the maximal abelian pro-$p$-extension of $F$ which is unramified outside the $p$-adic primes. In particular, the Kummer-Leopoldt constant $\kappa(F)$ is trivial if $\bar{F} = \hat{F}$. Number fields with $\hat{F} = \bar{F}$ and satisfying Leopoldt’s conjecture are called $p$-rational fields [20]. Obviously, $\kappa(F)$ is trivial if the field $F$ is $p$-rational. This can be considered as a generalization of Kummer’s lemma since the field $\mathbb{Q}(\mu_p)$ is $p$-rational precisely when $p$ is regular, $\mu_p$ being the group of $p$-th roots of unity.

Let $S$ be the set of $p$-adic and infinite places of $F$ and let $U$ be the group of $S$-units of $F$. In [7, Section 4], the authors define also a Kummer-Leopoldt constant for the $S$-units as the smallest integer $c$ having the following property:

$$\forall n \gg 0, \forall u \in U, (u \in F_v^{p^{c+n}}, \forall v \mid p) \implies u \in U^{p^n},$$

where for $v \mid p$, $F_v$ is the completion of $F$ at $v$.

Denote by $\hat{U}$ and $\hat{F}_v$, respectively, the pro-$p$-completion of $U$ and $F_v$. Let $G_S(F)$ be the Galois group over $F$ of the maximal algebraic extension which is unramified outside $S$. Then

$$\hat{U} \cong H^1(G_S(F), \mathbb{Z}_p(1)) \text{ and } \hat{F}_v \cong H^1(F_v, \mathbb{Z}_p(1)).$$

For an integer $i$, we have a natural localization map

$$\alpha^{(i)} = \bigoplus_{v \mid p} \alpha_v^{(i)} : H^1(G_S(F), \mathbb{Z}_p(i)) \longrightarrow \bigoplus_{v \mid p} H^1(F_v, \mathbb{Z}_p(i))$$

$$x \mapsto (\alpha_v^{(i)}(x))_v$$

where, for each prime $v$ above $p$, $\alpha_v^{(i)} : H^1(G_S(F), \mathbb{Z}_p(i)) \longrightarrow H^1(F_v, \mathbb{Z}_p(i))$ is the restriction homomorphism. For simplicity, if $x \in H^1(G_S(F), \mathbb{Z}_p(i))$, we keep the notation $x := \alpha_v^{(i)}(x) \in H^1(F_v, \mathbb{Z}_p(i))$. Then, we ask the following natural question: Is there a positive integer $c_i$ such that for all $n \gg 0, x \in H^1(G_S(F), \mathbb{Z}_p(i))$

$$(x \in H^1(F_v, \mathbb{Z}_p(i))^{p^{-c_i+n}}, \forall v \mid p) \implies x \in H^1(G_S(F), \mathbb{Z}_p(i))^{p^n}?$$

In this article, we show that such an integer exists when the field $F$ satisfies a twisted Leopoldt’s conjecture (Conjecture 2.1), and we define the twisted analogue of the Kummer-Leopoldt constant $\kappa_i(F)$ to be the smallest value of $c_i$ satisfying this property. The study of the twisted Kummer-Leopoldt constant leads us to define some Galois extensions, in particular we construct a twisted analogue of the Bertrandias-Payan field and an étale analogue of the Hilbert class field (see §2). Using these definitions we express the twisted Kummer-Leopoldt constant as the exponent of a certain Galois group inside the twisted Bertrandias-Payan module (Theorem 3.8).

By the Quillen-Lichtenbaum conjecture, which is now a theorem thanks to the work of Voevodsky and Rost on the Bloch-Kato conjecture, the $p$-adic cohomology group $H^1(G_S(F), \mathbb{Z}_p(i))$ is isomorphic to the pro-$p$-completion of the
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\( \text{K}-\text{theory group } K_{2i-1} F \) [17, Theorem 5.6.8]. Hence for \( i \geq 2 \), the constant \( \kappa_i(F) \) can be considered as a generalization to \( \text{K}-\text{theory of the Kummer-Leopoldt constant.} \)

In the last section of this paper, we study the vanishing of the twisted Kummer-Leopolodt constant. We show, in particular, that \( \kappa_{1-i}(F) = 0 \) if \( F \) is a \((p, i)\)-regular number field in the sense of [2]. Furthermore, we give a new characterization of \((p, i)\)-regular number fields in terms of the triviality of \( \kappa_{1-i}(F) \). More precisely, we prove the following theorem:

**Theorem.** Let \( i \neq 0, 1 \) be an integer and let \( F \) be a number field satisfying the twisted Leopoldt’s conjecture. Then \( F \) is \((p, i)\)-regular if and only if the following three conditions hold:

1. \( \kappa_{1-i}(F) = 0; \)
2. The natural injective map
   
   \[
   H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \rightarrow \bigoplus_{v|p} H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))
   \]

   is an isomorphism;
3. \( H^{(i)} \subset \hat{F}^{(i)} \), where the fields \( \hat{F}^{(i)} \) and \( H^{(i)} \) are defined in Definitions 2.5 and 2.9, respectively.

As an application we get a characterization of \((p, i)\)-regular quadratic number fields in the spirit of [12, §4.1], (see Propositions 4.6 and 4.7 below).

**Notation:** For a number field \( F \), and an odd prime number \( p \), we adopt the following notation throughout this paper:

- \( O_F \) the ring of integers of \( F \);
- \( \mu_p \) the group of \( p \)-th roots of the unity;
- \( E \) the composite of \( F \) and the \( p \)-th cyclotomic field i.e., \( E = F(\mu_p) \);
- \( S \) the set of \( p \)-adic and infinite places;
- \( U \) the group of \( S \)-units in \( F \);
- \( \hat{U} \) the pro-\( p \)-completion of \( U \);
- \( F_v \) the completion of \( F \) at a prime \( v \) of \( F \);
- \( U_v \) the group of local units of \( F \) at a prime \( v \) of \( F \);
- \( \hat{F}_v \) the pro-\( p \)-completion of \( F_v \);
- \( F_\infty \) the cyclotomic \( \mathbb{Z}_p \)-extension of \( F \);
- \( \Gamma \) the Galois group \( \text{Gal}(F_\infty/F) \);
- \( F_n \) the unique subfield of \( F_\infty \) such that \( [F_n : F] = p^n \);
- \( \Gamma_n \) the Galois group \( \text{Gal}(F_\infty/F_n) \);
- \( \Lambda = \mathbb{Z}_p[[\Gamma]] \) the Iwasawa algebra associated to \( \Gamma \);
- \( E_\infty \) the cyclotomic \( \mathbb{Z}_p \)-extension of \( E \);
- \( G_\infty \) the Galois group \( \text{Gal}(E_\infty/F) \);
- \( F_S \) the maximal algebraic extension of \( F \) which is unramified outside \( S \);
- \( \hat{F} \) the maximal abelian pro-\( p \)-extension of \( F \) which is
unramified outside $S$;
$E^\text{ab}_\infty$ the maximal abelian pro-$p$-extension of $E_\infty$ which is unramified outside $S$;
$L^\infty_\infty$ the maximal abelian unramified pro-$p$-extension of $E_\infty$ which splits completely at $p$-adic primes of $E_\infty$;
$X^\infty_\infty$ the Galois group $\text{Gal}(L^\infty_\infty/E_\infty)$;
$G_S(K)$ the Galois group $\text{Gal}(F_S/K)$, for an arbitrary field $K$
inside $F_S/F$;
$M(i)$ the $i$-th Tate twist of a $G_S(F)$-module $M$ ($i \in \mathbb{Z}$);
$M[p^n]$ the kernel of the multiplication by $p^n$;
$M/p^n$ the co-kernel of the multiplication by $p^n$;
$H^n(G_S(F), M)$ the $n$-th continuous cohomology group of $G_S(F)$ with coefficients in $M$;
$I^n(G_S(F), M)$ the localization kernel $\ker(H^n(G_S(F), M) \to \bigoplus_{v \in S} H^n(F_v, M))$;
$M^\vee = \text{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$, the Pontryagin dual of $M$;

For a group $G$ and a commutative ring $R$, let $I_G$ be the augmentation ideal of the group ring $R[G]$; it is the ideal generated by $\{\sigma - 1, \sigma \in G\}$. Unless otherwise stated, $R = \mathbb{Z}_p$.

2. On certain Galois extensions

Let $F$ be a number field and let $p$ be an odd prime number. We denote by $F_S$ the maximal algebraic extension of $F$ which is unramified outside the set $S$ of $p$-adic and infinite places of $F$. For a subfield $K$ of $F_S$ containing $F$, we denote by $G_S(K)$ the Galois group $\text{Gal}(F_S/K)$. The $p$-ramified Iwasawa module $\chi_K$ is the Galois group over $K$ of the maximal abelian pro-$p$-extension which is unramified outside $S$. In terms of homology groups, we have $\chi_K \cong H_1(G_S(K), \mathbb{Z}_p)$. Indeed, using the cohomology-homology duality, we have:

$$H_1(G_S(K), \mathbb{Z}_p) \cong H^1(G_S(K), \mathbb{Q}_p/\mathbb{Z}_p)^\vee \cong \text{Hom}(G_S(K), \mathbb{Q}_p/\mathbb{Z}_p)^\vee \cong \chi_K.$$}

For an integer $i$, denote by $\chi_K^{(i)}$ the first homology group $H_1(G_S(K), \mathbb{Z}_p(-i))$ which can then be considered as a twisted analogue of the $p$-ramified Iwasawa module $\chi_K$. The module $\chi_K^{(i)}$ has been studied by several authors in the case where $K$ is a multiple $\mathbb{Z}_p$-extension of $F$. For example, [14, 11] for $i = 0$ and [4] for $i \neq 0$. Returning to the case $K = F$, the $\mathbb{Z}_p$-rank of the $p$-ramified Iwasawa module $\chi_F$ is conjecturally equal to $r_2 + 1$, where $r_2$ is the number of complex places of $F$ (Leopoldt’s conjecture). There are many equivalent formulations of this conjecture. In terms of cohomology, it is equivalent to the triviality of the second cohomology group $H^2(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p)$ (e.g., [24, Proposition 12]). More generally, we have the following conjecture (Greenberg [10], Schneider [28], ...
Conjecture 2.1 (C(i)). Let $F$ be a number field. Then for every integer $i \neq 1$, the second cohomology group $H^2(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(i))$ is trivial.

Conjecture C(0) is the Leopoldt’s conjecture, it holds for all $F$ that are abelian over $\mathbb{Q}$ or over an imaginary quadratic number field. If $i \geq 2$, Conjecture C(i) holds for any number field $F$, as a consequence of the finiteness of the $K$-theory groups $K_{2i-2}O_F$ [30]. By a well known result on Brauer groups [13] or [28, §4, Lemma 2], there is no Conjecture C(i).

In the next proposition we give two equivalent formulations of the Conjecture C(i) that we will use in the sequel. These formulations are well known, we add here a proof for the reader’s convenience.

Proposition 2.2. Let $F$ be a number field and let $i \neq 1$ be an integer. The following assertions are equivalent:

1) Conjecture $C(i)$ holds for $F$;
2) the $p$-adic cohomology group $H^2(G_S(F), \mathbb{Z}_p(i))$ is finite;
3) the Galois module $X'_\infty(i-1)_G$ is finite.

Proof. For $k \geq 1$, the exact sequence

$$0 \rightarrow \mathbb{Z}_p(i) \rightarrow \mathbb{Z}_p(i) \rightarrow \mathbb{Z}/p^k(i) \rightarrow 0$$

induces in cohomology the exact sequence

$$H^n(G_S(F), \mathbb{Z}_p(i))/p^k \rightarrow H^n(G_S(F), \mathbb{Z}/p^k(i)) \rightarrow H^{n+1}(G_S(F), \mathbb{Z}_p(i))[p^k]$$

Passing to the direct limit on $k$, we obtain the exact sequence [23, (4.3.4.1)]

$$0 \rightarrow H^n(G_S(F), \mathbb{Z}_p(i)) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow H^n(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(i)) \rightarrow \mathrm{tor}_{\mathbb{Z}_p} H^{n+1}(G_S(F), \mathbb{Z}_p(i)) \rightarrow 0.$$ (1)

In fact, by [31, Proposition 2.3], $H^n(G_S(F), \mathbb{Z}_p(i)) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ is the maximal divisible subgroup of $H^n(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(i))$.

Since the cohomological dimension $\mathrm{cd}(G_S(F)) \leq 2$, we have an isomorphism

$$H^2(G_S(F), \mathbb{Z}_p(i)) \otimes \mathbb{Q}_p/\mathbb{Z}_p \cong H^2(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(i))$$ (2)

Since the $\mathbb{Z}_p$-module $H^2(G_S(F), \mathbb{Z}_p(i))$ is finitely generated (see [23, Proposition 4.2.3]), the equivalence between 1) and 2) follows from the isomorphism (2).

Observe that if $i \neq 1$, $\mathrm{III}^2(G_S(F), \mathbb{Z}_p(i)) \cong X'_\infty(i-1)_G$ [28, Section 6, Lemma 1] and by the local duality theorem, we have

$$H^2(F_v, \mathbb{Z}_p(i)) \cong H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^\vee.$$
In particular, the group $\bigoplus_{v \in S} H^2(F_v, \mathbb{Z}_p(i))$ is finite. Then, the equivalence $2) \iff 3)$ follows from the exact sequence

$$0 \to \bigoplus_{v \in S} \Pi^2(G_S(F), \mathbb{Z}_p(i)) \to H^2(G_S(F), \mathbb{Z}_p(i)) \to \bigoplus_{v \in S} H^2(F_v, \mathbb{Z}_p(i)).$$

**Remark 2.3.** For an integer $i$, we denote by $\mathcal{T}_F^{(i)}$ the $\mathbb{Z}_p$-torsion sub-module of $\mathcal{X}_F^{(i)}$. When the field $F$ satisfies Conjecture $C^{(i)}$ ($i \neq 1$), the cohomology group $H^2(G_S(F), \mathbb{Z}_p(i))$ is finite. Hence the exact sequence (1) (for $n = 1$) induces by duality the following well known cohomological description of $\mathcal{T}_F^{(i)}$ [26, Lemme 4.1]

$$\mathcal{T}_F^{(i)} \simeq H^2(G_S(F), \mathbb{Z}_p(i))^\vee.$$

As in the case where $i = 0$, Conjecture $C^{(i)}$ is related to the $\mathbb{Z}_p$-rank of the module $\mathcal{X}_F^{(i)}$. In [28, §4, Satz 6], the co-ranks of the groups $H^1(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(i))$ were computed. By duality,

$$\text{rank}_{\mathbb{Z}_p} H_1(G_S(F), \mathbb{Z}_p(-i)) = \text{corank}_{\mathbb{Z}_p} H^1(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(i)).$$

It follows that if $i \neq 0, 1$, the field $F$ satisfies $C^{(i)}$ if and only if

$$\text{rank}_{\mathbb{Z}_p} \mathcal{X}_F^{(i)} = \begin{cases} r_2 + r_1 & \text{if } i \text{ is odd;} \\ r_2 & \text{if } i \text{ is even,} \end{cases}$$

(3)

here, as usual, $r_1$ (resp. $r_2$) is the number of real (resp. complex) places. In the sequel we will frequently use the following well known lemma:

**Lemma 2.4** (Tate’s lemma). Let $F$ be a number field and let $i$ be a non-zero integer. Then the Galois cohomology groups $H^k(G, \mathbb{Q}_p/\mathbb{Z}_p(i))$ vanish for all $k \geq 1$, where $G$ is either $G_\infty = \text{Gal}(E_\infty/F)$ or $G_{\infty,v} = \text{Gal}(E_{\infty,v}/F_v)$, $v$ being a finite prime of $F$.

As a consequence of Tate’s lemma, we get that

$$H^1(\Gamma, H^0(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p(i))) = 0,$$

where $\Gamma$ is the Galois group $\text{Gal}(F_\infty/F)$. Indeed, let $\Delta$ be the Galois group $\text{Gal}(E_\infty/F_\infty)$. We have

$$H^0(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p(i)) = H^0(\Delta, H^0(G_S(E_\infty), \mathbb{Q}_p/\mathbb{Z}_p(i))) = H^0(\Delta, \mathbb{Q}_p/\mathbb{Z}_p(i)).$$

(4)

Since $\text{cd}(\Gamma) \leq 1$, the Hochschild-Serre spectral sequence associated to the group extension

$$0 \to \Delta \to G_\infty \to \Gamma \to 0$$

yields the following exact sequence

$$0 \to H^1(\Gamma, H^0(\Delta, \mathbb{Q}_p/\mathbb{Z}_p(i))) \to H^1(G_\infty, \mathbb{Q}_p/\mathbb{Z}_p(i)) \to H^1(\Delta, \mathbb{Q}_p/\mathbb{Z}_p(i))^\Gamma \to 0.$$
By Tate’s Lemma, we get
\[ H^1(\Gamma, H^0(\Delta, Q_p/\mathbb{Z}_p(i))) = 0. \]
From the equality (4), it follows that
\[ H^1(\Gamma, H^0(G_S(F_\infty), Q_p/\mathbb{Z}_p(i))) = 0 \]
as required.

Recall that the \( \mathbb{Z}_p \)-module \( X_F^{(i)} \) is isomorphic to \( X_F = \text{Gal}(\hat{F}/F) \), where \( \hat{F} \) is the maximal abelian pro-\( p \)-extension of \( F \) which is unramified outside \( S \). When the integer \( i \) is non-zero, the \( \mathbb{Z}_p \)-module \( X_F^{(i)} \) can also be realized as a Galois group. Indeed, using Tate’s lemma we get that \( H^1(G_\infty, Q_p/\mathbb{Z}_p(i)) = H^2(G_\infty, Q_p/\mathbb{Z}_p(i)) = 0 \), since \( i \neq 0 \). Therefore, the restriction map
\[ H^1(G_S(F), Q_p/\mathbb{Z}_p(i)) \to H^1(G_S(E_\infty), Q_p/\mathbb{Z}_p(i))^{G_\infty} \]
is an isomorphism. Notice that the Galois group \( G_S(E_\infty) \) acts trivially on \( Q_p/\mathbb{Z}_p(i) \), so we have
\[ H^1(G_S(E_\infty), Q_p/\mathbb{Z}_p(i))^{G_\infty} = H^1(G_S(E_\infty), Q_p/\mathbb{Z}_p(i))^{G_\infty} \approx \text{Hom}(G_S(E_\infty), Q_p/\mathbb{Z}_p(i))^{G_\infty}. \]
Then, by duality, the isomorphism (5) induces the following isomorphism:
\[ X_F^{(i)} \simeq X_\infty(-i)_{G_\infty}, \]
where \( X_\infty = H_1(G_S(E_\infty), \mathbb{Z}_p) \) is the Galois group over \( E_\infty \) of \( E_\infty^{\text{ab}} \), the maximal abelian pro-\( p \)-extension which is unramified outside \( S \).

**Definition 2.5.** Let \( i \neq 0 \) be an integer. We define the field \( \hat{F}^{(i)} \) to be the subfield of \( E_\infty^{\text{ab}} \) fixed by \( I_{G_\infty}(X_\infty(-i)) \); hence
\[ \text{Gal}(\hat{F}^{(i)}/E_\infty) = X_\infty(-i)_{G_\infty} \simeq X_F^{(i)}. \]
When \( i = 0 \), we define \( \hat{F}^{(0)} \) as the composite of the fields \( E_\infty \) and \( \hat{F} \) i.e, \( \hat{F}^{(0)} = E_\infty \hat{F} \).

For every integer \( i \), we denote by \( \hat{F}^{(i)} \) the subfield of \( \hat{F}^{(i)} \) fixed by the \( \mathbb{Z}_p \)-torsion sub-module \( T_F^{(i)} \) of \( X_F^{(i)} \); hence
\[ T_F^{(i)} \simeq \text{Gal}(\hat{F}^{(i)}/\hat{F}^{(i)}). \]

**Remark 2.6.** In the case \( i = 0 \), we don’t have the isomorphism (6) but we do have the following exact sequence:
\[ 0 \to (X_\infty)_{G_\infty} \to X_F \to \Gamma \to 0. \]
It follows that the field \( \hat{F}^{(0)} \) is the maximal subfield of \( E_\infty^{\text{ab}} \), which is abelian over \( F \).
Let $X'_\infty$ be the Galois group $\text{Gal}(L_\infty/E_\infty)$, where $L_\infty$ is the maximal abelian unramified pro-$p$-extension of $E_\infty$ which splits at $p$-adic primes of $E_\infty$. We have a natural surjective map

$$X_\infty(-i)_{G_\infty} \rightarrow X'_\infty(-i)_{G_\infty}.$$ 

For $i \neq 0$, it is well known that $X'_\infty(-i)_{G_\infty}$ is isomorphic to the localization kernel

$$\text{III}^2(G_S(F), \mathbb{Z}_p(1 - i)) := \ker(H^2(G_S(F), \mathbb{Z}_p(1 - i)) \rightarrow \bigoplus_{v \mid p} H^2(F_v, \mathbb{Z}_p(1 - i))),$$

[28, Section 6, Lemma 1]. For $i \geq 2$, the group $\text{III}^2(G_S(F), \mathbb{Z}_p(i))$ is called the étale wild kernel and does not depend on $S$ containing the $p$-adic places.

In the following proposition, we give an exact sequence which expresses the link between the $\mathbb{Z}_p$-torsion module $\mathcal{F}_F^{(i)}$ and the Pontryagin dual of $X'_\infty(i - 1)_{G_\infty}$. Let $W^{(1-i)}$ be the co-kernel of the injective localization morphism

$$H^0(G_S(F), Q_p/\mathbb{Z}_p(1 - i)) \rightarrow \bigoplus_{v \mid p} H^0(F_v, Q_p/\mathbb{Z}_p(1 - i)),$$

so that $W^{(1-i)} \cong \left( \bigoplus_{v \mid p} H^0(F_v, Q_p/\mathbb{Z}_p(1 - i)) \right) / H^0(G_S(F), Q_p/\mathbb{Z}_p(1 - i)).$

**Proposition 2.7.** Let $F$ be a number field and let $i \neq 1$ be an integer such that $F$ satisfies Conjecture $\mathcal{C}^{(i)}$. Then we have the following exact sequence:

$$0 \rightarrow W^{(1-i)} \rightarrow \mathcal{F}_F^{(i)} \rightarrow \text{Hom}(X'_\infty(i - 1)_{G_\infty}, Q_p/\mathbb{Z}_p) \rightarrow 0. \quad (7)$$

**Proof.** We start by recalling the first part of the Poitou-Tate exact sequence:

$$0 \rightarrow H^0(G_S(F), Q_p/\mathbb{Z}_p(1 - i)) \rightarrow \bigoplus_{v \mid p} H^0(F_v, Q_p/\mathbb{Z}_p(1 - i)) \rightarrow H^2(G_S(F), \mathbb{Z}_p(1 - i))^{\vee} \rightarrow \text{III}^1(G_S(F), Q_p/\mathbb{Z}_p(1 - i)) \rightarrow 0.$$

Clearly, for $i \neq 1$, we have

$$\text{III}^1(G_S(F), Q_p/\mathbb{Z}_p(1 - i)) \cong \text{Hom}(X'_\infty(i - 1)_{G_\infty}, Q_p/\mathbb{Z}_p).$$

Furthermore, if the field $F$ satisfies Conjecture $\mathcal{C}^{(i)}$, Remark 2.3 gives an isomorphism

$$\mathcal{F}_F^{(i)} \cong H^2(G_S(F), \mathbb{Z}_p(i))^{\vee}$$

Summarizing, we can rewrite the Poitou-Tate exact sequence as follows:

$$0 \rightarrow W^{(1-i)} \rightarrow \mathcal{F}_F^{(i)} \rightarrow \text{Hom}(X'_\infty(i - 1)_{G_\infty}, Q_p/\mathbb{Z}_p) \rightarrow 0. \quad \square$$

Note that the group $X'_\infty(-i)_{G_\infty}$ is a quotient of the Galois group $X'_\infty$, thus it could be realized as a Galois group of an abelian and totally decomposed extension of $E_\infty$ (this extension is denoted by $L_\infty$ in [3, Section 2, page 653]).
Definition 2.8. The field $L^{(i)}$ is the subfield of $L'_\infty$ fixed by $I_{G,\infty}(X'_{\infty}(-i))$, hence
\[ \text{Gal}(L^{(i)}/E_\infty) = X'_{\infty}(-i)_{G,\infty}. \]

In [3, Proposition 1], it is noticed that the extension $L^{(i)}$ is not in general abelian over $F$ so we can not use the descent process to realize the group $X'_{\infty}(-i)_{G,\infty}$ as a Galois group over $F$. Using the same methods of Jaulent and Soriano [15, Section 3, page 3], one constructs a field $H^{(i)}$ (this field is denoted by $F$ in [3, Section 2, page 653]) which is a Galois extension over $F$ and the group $X'_\infty(-i)_{G,\infty}$ is isomorphic to the Galois group $\text{Gal}(H^{(i)}/E_{n_0})$, where $E_{n_0} = H^{(i)} \cap E_\infty$ [3, Proposition 2]. Mention that in [3, page 653] the author assumes that $\mu_p \subseteq F$ but the generalization is easy. Let us recall the precise definition of the field $H^{(i)}$.

Definition 2.9. The field $H^{(i)}$ is the composite of the fields $F_\gamma$, where $F_\gamma$ is the subfield of $L^{(i)}$ fixed by a lifting of a topological generator $\gamma$ of $\Gamma$.

Remark 2.10. Since the Galois groups $\text{Gal}(L^{(i)}/E_\infty)$ and $\text{Gal}(H^{(i)}/E_{n_0})$ are isomorphic, and $E_{n_0} = H^{(i)} \cap E_\infty$, we have $L^{(i)} = E_\infty H^{(i)}$.

Let $K/F$ be a cyclic $p$-extension of $F$. Following [9], we say that $K$ is an infinitely embeddable extension of $F$ if it is embeddable in a cyclic $p$-extension of $F$ of arbitrary large degree. By class field theory, a $p$-extension $K/F$ is infinitely embeddable if and only if for any place $v$ of $F$, the local extension $K_v/F_v$ is embeddable in a $\mathbb{Z}_p$-extension of $F_v$. We denote by $F^{BP}$ the composite of all infinitely embeddable extensions of $F$. Obviously the field $F^{BP}$ contains the composite $\tilde{F}$ of all $\mathbb{Z}_p$-extensions of $F$. We set $T_F := \text{Gal}(F^{BP}/\tilde{F})$ to be the Bertrandias-Payan module of $F$ i.e., the $\mathbb{Z}_p$-torsion sub-module of $\text{Gal}(F^{BP}/F)$. Let $\tilde{F}$ be the maximal abelian pro-$p$-extension of $F$ which is unramified outside $S$. In view of [25, Theorem 4.2], we can see that $F^{BP}$ is the subfield of $\tilde{F}$ fixed by the image of $W^{(1)} = \bigoplus_{v \mid p} \mu_p(F_v)/\mu_p(F)$ in $\mathcal{T}_F$, the $\mathbb{Z}_p$-torsion sub-module of $\mathcal{X}_F := \text{Gal}(\tilde{F}/F)$.

In a natural way, we define a twisted analogue of the Bertrandias-Payan field as follows:

Definition 2.11. Let $i \neq 1$ be an integer such that $F$ satisfies Conjecture $C^{(i)}$. The twisted Bertrandias-Payan field $F^{BP,(i)}$ is defined as the subfield of $\tilde{F}^{(i)}$ fixed by the image of $W^{(1-i)}$ in $\mathcal{T}_F^{(i)}$ in the exact sequence (7).

Let $T_F^{(i)}$ be the $\mathbb{Z}_p$-torsion of $\text{Gal}(F^{BP,(i)}/E_\infty)$. Assume that $F$ satisfies Conjecture $C^{(i)}$. By the definition of $F^{BP,(i)}$ and the exact sequence (7), we have the following isomorphism:
\[ T_F^{(i)} \simeq \text{Hom}(X'_{\infty}(i-1)_{G,\infty}, Q_p/\mathbb{Z}_p). \]

In particular, if $i = 0$ we obtain the following isomorphism:
\[ T_F \simeq \text{Hom}(X'_{\infty}(-1)_{G,\infty}, Q_p/\mathbb{Z}_p), \]
(c.f. [25, Theorem 4.2]). Hence $T_F^{(i)}$ is a twisted analogue of the Bertrandias-Payan module. In this context, we have the twist analogue of the exact sequence in [25, Theorem 4.2].

**Corollary 2.12.** Let $F$ be a number field and let $i \neq 1$ be an integer such that $F$ satisfies Conjecture $C^{(i)}$. Then, we have the following exact sequence:

$$0 \rightarrow W^{(1-i)} \rightarrow J_F^{(i)} \rightarrow T_F^{(i)} \rightarrow 0. \quad (8)$$

**Proposition 2.13.** For every integer $i \neq 0, 1$ such that $F$ satisfies Conjecture $C^{(i)}$, the twisted Bertrandias-Payan field $F^{BP,(i)}$ contains the field $L^{(i)}$.

**Proof.** Since $i \neq 0$, we have $\Pi^2(G_S(F), \mathbb{Z}_p(1 - i)) \simeq X'_\infty (-i)_{G_\infty}$. Thus, the Poitou-Tate exact sequence [19, page 682]

$$H^1(G_S(F), \mathbb{Z}_p(1 - i)) \xrightarrow{\nu|_p} \bigoplus_{v \mid p} H^1(F_v, \mathbb{Z}_p(1 - i)) \rightarrow X_F^{(i)}$$

induces a surjective homomorphism:

$$X_F^{(i)} \rightarrow X'_\infty (-i)_{G_\infty}.$$

Its kernel $Y^{(i)} := \text{Gal}(F^{(i)}/L^{(i)})$ is isomorphic to the co-kernel of the localization map

$$H^1(G_S(F), \mathbb{Z}_p(1 - i)) \rightarrow \bigoplus_{v \mid p} H^1(F_v, \mathbb{Z}_p(1 - i)).$$

This map is injective since Conjecture $C^{(i)}$ holds. Thus we have an exact sequence:

$$0 \rightarrow H^1(G_S(F), \mathbb{Z}_p(1 - i)) \rightarrow \bigoplus_{v \mid p} H^1(F_v, \mathbb{Z}_p(1 - i)) \rightarrow Y^{(i)} \rightarrow 0.$$

Taking the restriction to the $\mathbb{Z}_p$-torsion sub-modules, we obtain the following exact sequence:

$$\text{tor}_p H^1(G_S(F), \mathbb{Z}_p(1 - i)) \rightarrow \bigoplus_{v \mid p} \text{tor}_p H^1(F_v, \mathbb{Z}_p(1 - i)) \rightarrow \text{tor}_p Y^{(i)}. \quad (9)$$

Moreover, we have the following well known isomorphisms

$$H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1 - i)) \simeq \text{tor}_p H^1(G_S(F), \mathbb{Z}_p(1 - i))$$

and

$$H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1 - i)) \simeq \text{tor}_p H^1(F_v, \mathbb{Z}_p(1 - i))$$

[31, Proposition 2.3]. The exact sequence (9) becomes

$$0 \rightarrow H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1 - i)) \rightarrow \bigoplus_{v \mid p} H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1 - i)) \rightarrow \text{tor}_p Y^{(i)}.$$
Then we obtain that the image of $W^{(1-i)}$ in $X_F^{(i)}$ is contained in the $\mathbb{Z}_{p}$-torsion of the kernel $Y^{(i)} := \text{Gal}(\hat{F}^{(i)}/L^{(i)})$. This means that the field $L^{(i)}$ is contained in $F^{\text{BP},(i)}$. □

The following figure is an illustration of the situation in which we work:

Now, let $K/F$ be a Galois $p$-extension of number fields, with Galois group $G$. If the extension $K/F$ is unramified outside $S$, there exists a natural restriction map

$$f_i : H^2(G_S(F),\mathbb{Z}_p(i)) \rightarrow H^2(G_S(K),\mathbb{Z}_p(i))^G.$$

We denote by $\hat{H}((G,\cdot)$ the modified Tate cohomology groups (see [29]). If $i \neq 0, 1$, the kernel and co-kernel of the map $f_i$ are given by

$$\ker(f_i) \cong H^1(G, H^1(G_S(K), \mathbb{Z}_p(i))) \cong \hat{H}^{-1}(G, H^2(G_S(K), \mathbb{Z}_p(i)))$$

and

$$\text{coker}(f_i) \cong H^2(G, H^1(G_S(K), \mathbb{Z}_p(i))) \cong \hat{H}^0(G, H^2(G_S(K), \mathbb{Z}_p(i)))$$

[1, Proposition 3.1, page 41], [18, Theorem 1.2] and [16, Proposition 2.9] (the proof for $i \neq 0, 1$ is the same as for $i \geq 2$). If $K$ satisfies Conjecture C$^{(i)}$, the group $H^2(G_S(K), \mathbb{Z}_p(i))$ is finite and the above descriptions of the kernel and co-kernel of the map $f_i$ show that, if $G$ is cyclic, $\ker(f_i)$ and $\text{coker}(f_i)$ have the same order.

Similarly for a prime $v$ of $F$ dividing $p$ and a prime $w$ of $K$ above $v$, we have a restriction map [1, Chapter 3]

$$f_{i,v} : H^2(F_v, \mathbb{Z}_p(i)) \rightarrow H^2(K_w, \mathbb{Z}_p(i))^G_w$$
where \( G_w = \text{Gal}(K_w/F_n) \) is the decomposition group of \( w \) in the extension \( K/F \).

Then exactly as in the global case, we have [1, Proposition 3.1, page 41]

\[
\ker(f_{i,v}) \cong H^1(G_w, H^1(K_w, \mathbb{Z}_p(i))) \cong \hat{H}^{-1}(G_w, H^2(K_w, \mathbb{Z}_p(i)))
\]

and

\[
\text{coker}(f_{i,v}) \cong H^2(G_w, H^1(K_w, \mathbb{Z}_p(i))) \cong \hat{H}^0(G_w, H^2(K_w, \mathbb{Z}_p(i))).
\]

Consider the commutative diagram

\[
\begin{array}{ccc}
H^2(G_S(K), \mathbb{Z}_p(i))^G & \rightarrow & \bigoplus_{v \in S, u \in u} H^2(K_u, \mathbb{Z}_p(i))^G \\
\downarrow f_i & & \downarrow \bigoplus_{v \in S} f_{i,v} \\
H^2(G_S(F), \mathbb{Z}_p(i)) & \rightarrow & \bigoplus_{v \in S} H^2(F_v, \mathbb{Z}_p(i))
\end{array}
\]

where for each \( v \in S \), the isomorphism \( \bigoplus_{u \in u} H^2(K_u, \mathbb{Z}_p(i))^G \cong H^2(K_w, \mathbb{Z}_p(i))^G \)

is a consequence of Shapiro’s lemma, \( w \) being a prime of \( K \) above \( v \). It follows that there exists a restriction map

\[
j_i(K/F) : \oplus^2(G_S(F), \mathbb{Z}_p(i)) \rightarrow \oplus^2(G_S(K), \mathbb{Z}_p(i))^G.
\]

We are interested in the dual map

\[
j_i^*(K/F) : (T^i_K)_G \rightarrow T^i_F
\]

when \( K \) is contained in the cyclotomic \( \mathbb{Z}_p \)-extension \( F_\infty \) of \( F \).

We need some additional notation. For all positive integer \( n \), we denote by \( F_n \) the unique sub-extension of \( F_\infty \) such that \( G_n := \text{Gal}(F_n/F) \cong \mathbb{Z}/p^n\mathbb{Z} \) and by \( T^{(i)}_n := T^{(i)}_{F_n} \) the twisted Bertrandias-Payan module of \( F_n \). We define the twisted Bertrandias-Payan module of \( F_\infty \) as the projective limit of \( T^{(i)}_n \) i.e.,

\[
T^{(i)}_\infty := \lim_{\rightarrow} T^{(i)}_n,
\]

where the projective limit is taken via the natural maps \( j_{i,n}^* := j_i^*(F_n/F) : (T^{(i)}_n)_G \rightarrow T^{(i)}_m (n \geq m) \). Let \( \Gamma \) be the Galois group \( \text{Gal}(F_\infty/F) \). Then we have a well-defined homomorphism

\[
j_{i,\infty}^* : (T^{(i)}_\infty)_\Gamma \rightarrow T^{(i)}_F.
\]

In the next lemma we show that \( j_{i,\infty}^* \) is injective, or equivalently that the restriction map

\[
j_{i,\infty} : \oplus^2(G_S(F), \mathbb{Z}_p(i)) \rightarrow (\lim \oplus^2(G_S(F_n), \mathbb{Z}_p(i)))^\Gamma
\]

induced by the maps \( j_i(F_n/F) \) is surjective provided that Conjecture \( C^{(i)} \) holds. More precisely,
Lemma 2.14. Suppose that for every \( n \geq 0 \), the field \( F_n \) satisfies Conjecture \( C^{(i)} \), \( i \neq 0, 1 \). Then, we have a commutative diagram with exact lines

\[
\begin{array}{ccc}
\ker(j_{i,\infty}) & \xhookrightarrow{j_{i,\infty}} & \text{III}^2(G_S(F), \mathbb{Z}_p(i)) \\
& \downarrow & \downarrow g \\
H^1(\Gamma, \lim H^1(G_S(F_n), \mathbb{Z}_p(i))) & \xhookrightarrow{f_{i,\infty}} & \text{III}^2(G_S(F_n), \mathbb{Z}_p(i))^\Gamma
\end{array}
\]

where \( f_{i,\infty} \) is induced by the restriction maps

\[
f_{i,n} : H^2(G_S(F), \mathbb{Z}_p(i)) \to H^2(F_n, \mathbb{Z}_p(i))^{G_n}.
\]

Proof. Let \( \nu \) be a \( p \)-adic prime of \( F \) and let \( n \geq 0 \). For commodity of notation, we denote also by \( \nu \) a prime of \( F_n \) above \( \nu \) and by \( G_{n,\nu} = \text{Gal}(F_{n,\nu}/F_\nu) \) its decomposition group in the extension \( F_n/F \). Let us first show that the restriction homomorphism

\[
f_i(F_{n,\nu}/F_\nu) : H^2(F_\nu, \mathbb{Z}_p(i)) \to H^2(F_{n,\nu}, \mathbb{Z}_p(i))^{G_{n,\nu}}
\]

is injective. The local duality theorem gives an isomorphism

\[
H^2(F_{n,\nu}, \mathbb{Z}_p(i)) \cong H^0(F_{n,\nu}, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^{\nu} \cong H^0(F_{n,\nu}, \mathbb{Q}_p/\mathbb{Z}_p(i-1)).
\]

Using Tate’s lemma and the Hochschild-Serre spectral sequence associated to the extension groups

\[
\text{Gal}(E_{\infty,\nu}/F_{n,\nu}) \xhookrightarrow{\alpha} G_{\infty,\nu} \to G_{n,\nu},
\]

we see that the first cohomology group \( H^1(G_{n,\nu}, H^0(F_{n,\nu}, \mathbb{Q}_p/\mathbb{Z}_p(i-1))) = 0 \). Since \( G_{n,\nu} \) is a cyclic group, it follows that

\[
\hat{H}^{-1}(G_{n,\nu}, H^0(F_{n,\nu}, \mathbb{Q}_p/\mathbb{Z}_p(i-1))) = 0.
\]

Summarizing, we obtain

\[
\ker(\bigoplus_{\nu \in S} f_i(F_{n,\nu}/F_\nu)) := \bigoplus_{\nu \in S} \hat{H}^{-1}(G_{n,\nu}, H^2(F_{n,\nu}, \mathbb{Z}_p(i)))
\]

\[
\simeq \bigoplus_{\nu \in S} \hat{H}^{-1}(G_{n,\nu}, H^0(F_{n,\nu}, \mathbb{Q}_p/\mathbb{Z}_p(i-1))) = 0.
\]

Now, the exact sequence

\[
0 \to \text{III}^2(G_S(F_n), \mathbb{Z}_p(i)) \to H^2(G_S(F_n), \mathbb{Z}_p(i)) \to \bigoplus_{\nu \in S} H^2(F_{n,\nu}, \mathbb{Z}_p(i)) \to H^0(G_S(F_n), \mathbb{Q}_p/\mathbb{Z}_p(1-i))^{\nu} \to 0
\]
leads to the following commutative diagram:

\[
\begin{array}{ccl}
\Pi^2(G_S(F_n), \mathbb{Z}_p(i))^{G_n} \hookrightarrow H^2(G_S(F_n), \mathbb{Z}_p(i))^{G_n} \xrightarrow{\oplus} H^2(F_{n,v}, \mathbb{Z}_p(i))^{G_n} \\
\uparrow j_{i,n} & & \uparrow f_{i,v} & \uparrow \oplus f_{(F_{n,v}/F_v)} \\
\Pi^2(G_S(F), \mathbb{Z}_p(i)^c) & \hookrightarrow H^2(G_S(F), \mathbb{Z}_p(i)) & \xrightarrow{\oplus} H^2(F_v, \mathbb{Z}_p(i)),
\end{array}
\]

where

\[
\bigoplus_{v \in S} H^2(F_v, \mathbb{Z}_p(i)) := \ker(\bigoplus_{v \in S} H^2(F_v, \mathbb{Z}_p(i)) \longrightarrow H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1 - i)^\vee)).
\]

The map \(\bigoplus_{v \in S} f_v(F_{n,v}/F_v)\) is injective as the restriction of the map \(\bigoplus_{v \in S} f_v(F_{n,v}/F_v)\) to \(\bigoplus_{v \in S} H^2(F_v, \mathbb{Z}_p(i))\). Since the fields \(F_n, n \geq 0\), satisfy Conjecture \(C^{(i)}\), the group

\[
\lim \ker(f_{i,n}) = \lim H^2(G_n, H^1(G_S(F_n), \mathbb{Z}_p(i)))
\]

is trivial (the proof is exactly the same as [18, Proposition 3.2]). Taking the inductive limit in (10), we then obtain the following commutative diagram with exact lines and columns

\[
\begin{array}{ccl}
\lim \Pi^2(G_S(F_n), \mathbb{Z}_p(i))^{G_n} \hookrightarrow \lim H^2(G_S(F_n), \mathbb{Z}_p(i))^{G_n} & \xrightarrow{\oplus} \lim \bigoplus_{v \in S} H^2(F_{n,v}, \mathbb{Z}_p(i))^{G_n} \\
\uparrow j_{i,\infty} & & \uparrow f_{i,\infty} & \uparrow \oplus f_{(F_{\infty,v}/F_v)} \\
\Pi^2(G_S(F), \mathbb{Z}_p(i)^c) & \hookrightarrow H^2(G_S(F), \mathbb{Z}_p(i)) & \xrightarrow{\oplus} H^2(F_v, \mathbb{Z}_p(i)) \\
\uparrow \ker(j_{i,\infty}) & \hookrightarrow \lim H^1(G_S(F), \mathbb{Z}_p(i))
\end{array}
\]

which shows that the map \(\Pi^2(G_S(F), \mathbb{Z}_p(i)) \longrightarrow \lim \Pi^2(G_S(F_n), \mathbb{Z}_p(i))^{G_n}\) is surjective. Therefore, we get the commutative diagram of the lemma. \(\square\)

**Theorem 2.15.** Let \(F\) be a number field and let \(i \neq 0, 1\) be an integer such that Conjecture \(C^{(i)}\) holds for all the fields \(F_n, n \geq 0\). Then the homomorphism

\[
j_{i,\infty}^* : (T^{(i)}_{\infty})_F \longrightarrow T^{(i)}_F
\]

is injective. If we assume further that \(F\) is totally real and \(i\) is even, we get an isomorphism

\[
(T^{(i)}_{\infty})_F \simeq T^{(i)}_F.
\]

**Proof.** The first claim follows from the Pontryagin dual of the top exact sequence in the commutative diagram of Lemma 2.14 and the isomorphisms

\[
T^{(i)}_F \simeq \Pi^2(G_S(F), \mathbb{Z}_p(i)^\vee),
\]

\[
T^{(i)}_{\infty} \simeq \lim \Pi^2(G_S(F_n), \mathbb{Z}_p(i)^\vee) \simeq \text{Hom}(\lim \Pi^2(G_S(F_n), \mathbb{Z}_p(i)), \mathbb{Q}_p/\mathbb{Z}_p).
\]
Suppose now that $F$ is totally real and $i$ is even. Observe that, for every $n \geq 1$, $F_n$ is also totally real. Using the exact sequence (1), we obtain
\[
\text{rank}_p H^1(G_S(F_n), \mathbb{Z}_p(i)) = \text{co-rank}_p H^1(G_S(F_n), \mathbb{Q}_p / \mathbb{Z}_p(i)) = \text{rank}_p H_1(G_S(F_n), \mathbb{Z}_p(-i)).
\]
Thus, the formula (3) shows that for $n \geq 1$, the group $H^1(G_S(F_n), \mathbb{Z}_p(i))$ is a $\mathbb{Z}_p$-torsion module. Note that for all $n \geq 1$, $H^0(G_S(F_n), \mathbb{Z}_p(i)) \otimes \mathbb{Q}_p / \mathbb{Z}_p$ is trivial. From the exact sequence (1), it follows that the connecting homomorphism is an isomorphism
\[
H^0(G_S(F_n), \mathbb{Q}_p / \mathbb{Z}_p(i)) \simeq H^1(G_S(F_n), \mathbb{Z}_p(i)).
\]
Hence we have a commutative diagram
\[
\begin{array}{ccc}
H^0(G_S(F_n), \mathbb{Q}_p / \mathbb{Z}_p(i)) & \overset{\sim}{\longrightarrow} & H^1(G_S(F_n), \mathbb{Z}_p(i)) \\
\uparrow & & \uparrow \\
H^0(G_S(F), \mathbb{Q}_p / \mathbb{Z}_p(i)) & \overset{\sim}{\longrightarrow} & H^1(G_S(F), \mathbb{Z}_p(i))
\end{array}
\]
where the vertical maps are the restriction maps. Taking the inductive limit, we get
\[
\lim H^1(G_S(F_n), \mathbb{Z}_p(i)) \simeq \lim H^0(G_S(F_n), \mathbb{Q}_p / \mathbb{Z}_p(i)) \simeq H^0(G_S(F_\infty), \mathbb{Q}_p / \mathbb{Z}_p(i)).
\]
Therefore,
\[
H^1(\Gamma, \lim H^1(G_S(F_n), \mathbb{Z}_p(i))) \simeq H^1(\Gamma, H^0(G_S(F_\infty), \mathbb{Q}_p / \mathbb{Z}_p(i))).
\]
As explained after Lemma 2.4, $H^1(\Gamma, H^0(G_S(F_\infty), \mathbb{Q}_p / \mathbb{Z}_p(i)))$ is trivial. Thus, the cohomology group $H^1(\Gamma, \lim H^1(G_S(F_n), \mathbb{Z}_p(i)))$ is trivial. Using this fact and Lemma 2.14, we obtain that
\[
\hat{j}_{i,\infty} : \mathcal{H}_i^2(G_S(F), \mathbb{Z}_p(i)) \longrightarrow (\lim \mathcal{H}_i^2(G_S(F_n), \mathbb{Z}_p(i)))^\Gamma
\]
is an isomorphism. Taking the Pontryagin dual we get the desired isomorphism. \qed

3. The twisted Kummer-Leopoldt’s constant

Let $F$ be a number field and let $S$ be the set of $p$-adic and infinite places of $F$. We set by $A_F$ the $p$-primary part of the $(p)$-class group of $F$. We denote by $U$ the group of $S$-units of $F$ and by $\hat{U}$ the pro-$p$-completion of $U$. A description of the Galois group $\mathcal{X}_F$ is given by the class field exact sequence relative to the decomposition
\[
\hat{U} \xrightarrow{\alpha} \bigoplus_{v|p} \hat{P}_v \xrightarrow{\phi} \mathcal{X}_F \longrightarrow A_F \longrightarrow 0,
\] (11)
where $\alpha$ is the natural pro-$p$-diagonal map and $\varphi$ is the product of the local reciprocity homomorphisms which send each $\hat{F}_v$ to the decomposition group in $X_F$.

In Section 2, we noticed some equivalences formulations of Leopoldt's conjecture in terms of the $\mathbb{Z}_p$-rank of the $p$-ramified Iwasawa module and cohomology groups. Another formulation of this conjecture is the injectivity of the natural pro-$p$-diagonal map $\alpha$ or, equivalently, is the validity of the following property: For all integer $s \geq 1$, there exists an integer $t \geq 1$ such that:

$$\forall u \in U, (u \in F_v^{p^t}, \forall v | p) \implies u \in U^{p^t},$$

[7, Section 4]. Using the isomorphism

$$\hat{U} \cong H^1(G_S(F), \mathbb{Z}_p(1)),$$

the map $\alpha$ is nothing but the localization homomorphism:

$$\alpha^{(1)} = \bigoplus_{v | p} \alpha^{(1)}_v : H^1(G_S(F), \mathbb{Z}_p(1)) \longrightarrow \bigoplus_{v | p} H^1(F_v, \mathbb{Z}_p(1))$$

$$x \longmapsto (\alpha^{(1)}_v(x))_v$$

For an integer $i$, we consider the twisted analogue of the map $\alpha$

$$\alpha^{(i)} = \bigoplus_{v | p} \alpha^{(i)}_v : H^1(G_S(F), \mathbb{Z}_p(i)) \longrightarrow \bigoplus_{v | p} H^1(F_v, \mathbb{Z}_p(i))$$

$$x \longmapsto (\alpha^{(i)}_v(x))_v$$

and if $x \in H^1(G_S(F), \mathbb{Z}_p(i))$, we keep (for simplicity) the notation $x := \alpha^{(i)}_v(x) \in H^1(F_v, \mathbb{Z}_p(i))$. Then, we consider the following property:

$$(\mathcal{Q}_i) \quad \text{For all integer } s \geq 1, \text{ there exists an integer } t \geq 1 \text{ such that:}$$

$$x \in H^1(G_S(F), \mathbb{Z}_p(i)) (x \in H^1(F_v, \mathbb{Z}_p(i))^{p^t}, \forall v | p) \implies x \in H^1(G_S(F), \mathbb{Z}_p(i))$$

**Remark 3.1.** Notice that for all $t' \geq t$, we have

$$H^1(F_v, \mathbb{Z}_p(i))^{p^{t'}} \subseteq H^1(F_v, \mathbb{Z}_p(i))^{p^t}.$$ 

Therefore, we can suppose that $t \geq s$ in the property $$(\mathcal{Q}_i).$$

For every integer $i$, the Poitou-Tate exact sequence with coefficients in the modules $\mathbb{Z}/p^n\mathbb{Z}(i)$ induces, by passing to the projective limit, the following exact sequence [19, page 682]

$$H^1(G_S(F), \mathbb{Z}_p(i)) \bigoplus_{v | p} H^1(F_v, \mathbb{Z}_p(i)) \twoheadrightarrow X^{(1-i)}_F \twoheadrightarrow \Pi^2(G_S(F), \mathbb{Z}_p(i))$$ \quad (12)

When $i = 1$, $\Pi^2(G_S(F), \mathbb{Z}_p(i)) \simeq A_F$ and the exact sequence (12) is nothing but the class field theory exact sequence (11). For $i \neq 1$,

$$\Pi^2(G_S(F), \mathbb{Z}_p(i)) \simeq X^{(i-1)}_{ac}$$
[28, Section 6, Lemma 1] and we have a twisted analogue of (11):

$$H^1(G_S(F), \mathbb{Z}_p(i)) \xrightarrow{\alpha(i)} \bigoplus_{v \mid p} H^1(F_v, \mathbb{Z}_p(i)) \longrightarrow X^{(1-i)}_F \longrightarrow X^i(1 - i)_{G_\infty} \longrightarrow 0.$$  

In the next lemma, for $i \neq 0$, we show an equivalence between the validity of Conjecture C$^{(1-i)}$ and the injectivity of the localization map:

$$\alpha(i) : H^1(G_S(F), \mathbb{Z}_p(i)) \longrightarrow \bigoplus_{v \mid p} H^1(F_v, \mathbb{Z}_p(i)).$$

**Lemma 3.2.** Let $i \neq 0$ be an integer. The following assertions are equivalent:

i) The map $\alpha(i)$ is injective.

ii) Conjecture C$^{(1-i)}$ holds for $(F, p)$.

**Proof.** Remark that for every $p$-adic prime $v$ of $F$, the absolute Galois group of $F_v$ acts non-trivially on $\mathbb{Z}_p(i)$ when $i \neq 0$. Hence the cohomology group $H^0(F_v, \mathbb{Z}_p(i))$ is trivial for every $p$-adic prime $v$. Therefore, the Poitou-Tate exact sequence induces the following exact sequence

$$0 \longrightarrow H^2(G_S(F), \mathbb{Q}_p/Z_p(1 - i))^\vee \longrightarrow H^1(G_S(F), \mathbb{Z}_p(i)) \xrightarrow{\alpha(0)} \bigoplus_{v \mid p} H^1(F_v, \mathbb{Z}_p(i)).$$

This shows that

$$\ker(\alpha(i)) \cong H^2(G_S(F), \mathbb{Q}_p/Z_p(1 - i))^\vee.$$  

\[\square\]

**Remark 3.3.** Although there is no Conjecture C$^{(1)}$, the map $\alpha(0)$ is always injective. Indeed, by the global Poitou-Tate duality, we have

$$\ker(\alpha(0)) = \text{lim} \{((G_S(F), Z_p)) \cong \text{lim} \{((G_S(F), Z/p^n Z))^\vee \}

Furthermore,

$$\text{lim} \{((G_S(F), Z/p^n Z))^\vee = \text{lim} \{\text{Cl}_S(F)/p^n \}

$$\text{Cl}_S(F) \otimes \mathbb{Q}_p/Z_p = 0.$$  

In the next theorem we give other equivalences of the twisted Leopoldt’s conjecture. The proof is an adaptation of that of [7, Proposition 1].

**Theorem 3.4.** Let $F$ be a number field. For all integer $i \neq 0$, the following properties are equivalent:

(i) Conjecture C$^{(1-i)}$ holds for $(F, p)$.

(ii) The property $(B_i)$ is true.

(iii) There exists a positive integer $c_i$ such that for all $n \geq 1$,

$$x \in H^1(G_S(F), \mathbb{Z}_p(i))(x \in H^1(F_v, \mathbb{Z}_p(i))^{p^{c_i+n}}, \forall v \mid p) \Rightarrow x \in H^1(G_S(F), \mathbb{Z}_p(i))^{p^n}.$$
(iv) There exists a positive integer $c_i$ such that for all $n \gg 0$,
$$x \in H^1(G_S(F), Z_p(i))(x \in H^1(F_v, Z_p(i))^{p^{ci+n}}, \forall v \mid p) \Rightarrow x \in H^1(G_S(F), Z_p(i))^{p^n}$$

**Proof.** For a positive integer $t$, the homomorphism $\alpha^{(i)}$ induces the following one
$$\alpha^{(i)}_t : H^1(G_S(F), Z_p(i))/p^t \rightarrow \bigoplus_{v \mid p} H^1(F_v, Z_p(i))/p^t.$$ 

For integers $t \geq s \geq 1$, we consider the following commutative diagram

$$
\begin{array}{ccc}
0 & \rightarrow & \ker(\alpha^{(i)}_t) \rightarrow H^1(G_S(F), Z_p(i))/p^t \rightarrow \bigoplus_{v \mid p} H^1(F_v, Z_p(i))/p^t \\
& & \downarrow a_{st} \quad \downarrow b_{st} \quad \downarrow c_{st} \quad \downarrow \epsilon_{st} \\
0 & \rightarrow & \ker(\alpha^{(i)}_s) \rightarrow H^1(G_S(F), Z_p(i))/p^s \rightarrow \bigoplus_{v \mid p} H^1(F_v, Z_p(i))/p^s
\end{array}
$$

where the vertical maps are the natural ones. Since $\ker \alpha^{(i)} = \lim \ker \alpha^{(i)}_t$, the homomorphism $\alpha^{(i)}$ is injective if and only if the homomorphism $a_{st}$ is trivial for $t \gg s$. According to Lemma 3.2, it follows that the validity of Conjecture $C^{(1-i)}$ is equivalent to the triviality of the homomorphism $a_{st}$, for $t \gg s$. Hence we get the equivalence $(i) \iff (ii)$.

Now we prove the implication $(ii) \implies (iii)$. We suppose that the property $(\mathcal{Q}_i)$ holds and we proceed by induction over $n$. First let $r$ be an integer such that $H^1(F_v, Z_p(i))^{p^r}$ has no $Z_p$-torsion for all prime $v$ above $p$. By $(\mathcal{Q}_i)$ for $s = r + 1$, there is an integer $c_i \geq r$ (see Remark 3.1) such that for all $x \in H^1(G_S(F), Z_p(i))$:

$$\left( x \in H^1(F_v, Z_p(i))^{p^{ci+n}}, \forall v \mid p \right) \Rightarrow x \in H^1(G_S(F), Z_p(i))^{p^{ci+n+1}}. \quad (13)$$

The case $n = 1$ is deduced from $(13)$. Let $n > 1$ and let $x \in H^1(G_S(F), Z_p(i))$ such that $x$ belongs to $H^1(F_v, Z_p(i))^{p^{ci+n}}$ for all $v$ above $p$. According to $(13)$, there is a $y \in H^1(G_S(F), Z_p(i))$ such that $x = y^{p^r+1}$. Since $(y^{p^t})^p = x \in (H^1(F_v, Z_p(i))^{p^{ci+n-1}})^p$, we obtain that $y^{p^t} \in H^1(F_v, Z_p(i))^{p^{ci+n-1}}$, by the choice of $r$. Hence $y^{p^t} \in H^1(G_S(F), Z_p(i))^{p^{ci+n-1}}$, this implies that

$$x = (y^{p^t})^p \in H^1(G_S(F), Z_p(i))^{p^n}.$$ 

The implications $(iii) \implies (ii)$, $(iii) \implies (iv)$ and $(iv) \implies (i)$ are obvious. \qed

**Remark 3.5.**

**i)** From the proof of Theorem 3.4, we see that the truth of $(\mathcal{Q}_i)$ is equivalent to the injectivity of the map $\alpha^{(i)}$ also in the case where $i = 0$. As a consequence of Remark 3.3, the property $(\mathcal{Q}_0)$ is always true.

**ii)** The existence of the constant $c_i$ is trivial in the case of totally real number field $F$ and even integer $i$, since

$$H^1(G_S(F), Z_p(i)) = \text{tor}_{Z_p} H^1(G_S(F), Z_p(i)).$$

In particular, Conjecture $C^{(1-i)}$ holds for $(F, p)$. 

Definition 3.6. We define the twisted Kummer-Leopoldt constant $\kappa_i = \kappa_i(F)$ of the field $F$ to be the minimal integer $c_i$ satisfying the property (iv) of Theorem 3.4.

The aim now is to determine the exact value of the twisted Kummer-Leopoldt constant. We shall express it as the exponent of a certain Galois module.

Lemma 3.7. Let $i \neq 0, 1$ be an integer such that $F$ satisfies Conjecture $C(i)$. The surjective homomorphism $\psi : X_F^{(1-i)} \to X'_\infty(i-1)_{G_\infty}$ factors through a homomorphism $\Psi : T_F^{(1-i)} \to X'_\infty(i-1)_{G_\infty}$ and $\ker(\Psi)$ is isomorphic to the Galois group $\text{Gal}(F^{BP,(1-i)}/\tilde{F}(1-i)L^{(1-i)})$.

Proof. First of all, we recall from the end of the proof of Proposition 2.13 that the image of $W(i)$ in $T_F^{(1-i)}$ is contained in the kernel $Y^{(1-i)} := \ker(\psi : X_F^{(1-i)} \to X'_\infty(i-1)_{G_\infty})$.

Therefore, taking the restriction of the surjective homomorphism $\psi : X_F^{(1-i)} \to X'_\infty(i-1)_{G_\infty}$ to $T_F^{(1-i)}$, we obtain the following commutative diagram with exact lines:

$$
\begin{array}{cccc}
0 & \to & W^{(i)} & \to & T_F^{(1-i)} & \to & 0 \\
& & \downarrow & \& \downarrow \psi & & \\
& & 0 & \to & \text{tor}_{Z_p} Y^{(1-i)} & \to & T_F^{(1-i)}
\end{array}
$$

Thus $\psi$ induces the following homomorphism $\Psi : T_F^{(1-i)} \to X'_\infty(i-1)_{G_\infty}$.

Furthermore, reading the figure in page 379, we see that the kernel $\ker(\Psi)$ is isomorphic to the Galois group $\text{Gal}(F^{BP,(1-i)}/\tilde{F}(1-i)L^{(1-i)})$. □

Theorem 3.8. Let $F$ be a number field and let $i \neq 0, 1$ be an integer such that $F$ satisfies Conjecture $C^{(1-i)}$. Let $\kappa_i$ be the twisted Kummer-Leopoldt constant of $F$. Then $\rho^{\kappa_i}$ is the exponent of the Galois group $\text{Gal}(F^{BP,(1-i)}/\tilde{F}(1-i)L^{(1-i)})$.

Proof. Let us prove that $\rho^{\kappa_i}$ is the exponent of $\ker(\Psi) \simeq \text{Gal}(F^{BP,(1-i)}/\tilde{F}(1-i)L^{(1-i)})$ (Lemma 3.7). Let $j = 1 - i$ and recall that the kernel $Y^{(j)} = \ker(X_F^{(j)} \to X'_\infty(-j)_{G_\infty})$
is equal to the Galois group \( \text{Gal}(\bar{F}(j)/L(j)) \). For \( n \) sufficiently large such that \( p^n \) kills the \( \mathbb{Z}_p \)-torsion \( \mathcal{T}_F(j) \) of \( X_F(j) \), the multiplication by \( p^n \) yields the following exact sequence

\[
0 \longrightarrow Y(j)[p^n] \longrightarrow \mathcal{T}_F(j) \longrightarrow X'_{\infty}(-j)_{G_{\infty}}. 
\]

Comparing with the exact sequence of Corollary 2.12, we get a commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & W(1-j) & \longrightarrow & \mathcal{T}_F(j) & \longrightarrow & T_F & \longrightarrow & 0 \\
& & p^n & & \downarrow g_n & & & \downarrow \Psi & \\
0 & \longrightarrow & Y(j)[p^n] & \longrightarrow & \mathcal{T}_F(j) & \longrightarrow & X'_{\infty}(-j)_{G_{\infty}} & & \\
\end{array}
\]

Using the snake lemma, we obtain that

\[
\ker(\Psi) \simeq \text{coker}(g_n). \tag{14}
\]

Since Conjecture \( C(j) \) holds for \( F \), the map \( \alpha^{(i)} \) is injective (recall that \( j = 1 - i \)). Let us consider the following commutative diagram

\[
0 \longrightarrow H^1(G_S(F), \mathbb{Z}_p(i)) \xrightarrow{\alpha^{(i)}} \bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(i)) \longrightarrow Y(j) \longrightarrow 0 \\
0 \longrightarrow H^1(G_S(F), \mathbb{Z}_p(i))[p^n] \xrightarrow{\alpha^{(i)}} \bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(i))[p^n] \longrightarrow Y(j)[p^n] \\
0 \longrightarrow H^1(G_S(F), p^n) \xrightarrow{\alpha^{(i)}} \bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(i))/p^n \longrightarrow \cdots
\]

It follows that \( \text{coker}(\phi_n) \) is isomorphic to the kernel \( \ker(\alpha^{(i)}_{n}) \). Notice that for \( n \) large enough,

\[
H^1(G_S(F), \mathbb{Z}_p(i))[p^n] \simeq H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(i))
\]

and

\[
H^1(F_v, \mathbb{Z}_p(i))[p^n] \simeq H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(i))
\]

for all \( v \) over \( p \). Hence, we get that \( \text{coker}(\phi_n) \) is isomorphic to \( \text{coker}(g_n) \). Then, by (14)

\[
\text{coker}(g_n) \simeq \ker(\alpha^{(i)}_{n}) \simeq \ker(\Psi).
\]

Since \( p^\nu \) is the exponent of \( \ker(\alpha^{(i)}_{n}) \), for \( n \) large enough, the result follows from Lemma 3.7. \( \square \)
We finish this section with the following proposition in which we consider the case of a CM-field.

**Proposition 3.9.** Let $F$ be a CM-field with totally real subfield $F^+$ and let $i$ be an odd integer. Assume that the field $F^+$ satisfies Conjecture $C^{(1-i)}$. Then the twisted Kummer-Leopoldt constants $\kappa_i : = \kappa_i(F)$ and $\kappa_i^+ : = \kappa_i(F^+)$ are equal.

**Proof.** Let $n$ be an integer such that $p^n$ kills both $\bigoplus_{v|p} H^0(F_v, \mathbb{Q}_p / \mathbb{Z}_p(i))$ and $\mathcal{S}^{(1-i)}_F$. According to the end of the proof of Theorem 3.8, we know that $p^\kappa$ is the exponent of

$$\ker(\alpha_n^{(i)} : H^1(G_S(F), \mathbb{Z}_p(i))/p^n \to \bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(i))/p^n).$$

Let $\tau \in \text{Gal}(F/F^+)$ be the complex conjugation. Consider the decomposition

$$\ker(\alpha_n^{(i)}) = (\ker(\alpha_n^{(i)}))^+ \oplus (\ker(\alpha_n^{(i)}))^-, $$

where $(\ker(\alpha_n^{(i)}))^+ = (1 + \tau) \ker(\alpha_n^{(i)})$. We have to show that $(\ker(\alpha_n^{(i)}))^-$ is trivial and that the exponent of $(\ker(\alpha_n^{(i)}))^+$ is $p^\kappa$. We start by observing that

$$H^1(G_S(F), \mathbb{Z}_p(i))^+ \simeq H^1(G_S(F^+), \mathbb{Z}_p(i)).$$

Since

$$\text{rank}_{\mathbb{Z}_p} H^1(G_S(F), \mathbb{Z}_p(i)) = \text{rank}_{\mathbb{Z}_p} H^1(G_S(F^+), \mathbb{Z}_p(i)),$$

it follows that $H^1(G_S(F), \mathbb{Z}_p(i))^-$ is a $\mathbb{Z}_p$-torsion module. Furthermore, notice that

$$H^0(G_S(F^+), \mathbb{Q}_p / \mathbb{Z}_p(i)) = 0 \iff i \not\equiv 0 \mod [F^+(\mu_p) : F^+].$$

Since $[F^+(\mu_p) : F^+]$ is even and $i$ is odd, we get that $H^0(G_S(F^+), \mathbb{Q}_p / \mathbb{Z}_p(i))$ is trivial. This implies that

$$H^1(G_S(F), \mathbb{Z}_p(i))^- = \text{tor}_{\mathbb{Z}_p} H^1(G_S(F), \mathbb{Z}_p(i)).$$

Using this fact and the choice of $n$, we see that the map

$$(\alpha_n^{(i)}^- : (H^1(G_S(F), \mathbb{Z}_p(i))/p^n)^- \to (\bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(i))/p^n)^-)$$

is nothing but the injection

$$H^0(G_S(F), \mathbb{Q}_p / \mathbb{Z}_p(i)) \to (\bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(i))/p^n)^-. $$

Therefore, $(\ker(\alpha_n^{(i)}))^-$ is trivial for $n$ large enough.

Also, using the isomorphism

$$(H^1(G_S(F), \mathbb{Z}_p(i))/p^n)^+ \simeq H^1(G_S(F^+), \mathbb{Z}_p(i))/p^n$$

we get that $(\ker(\alpha_n^{(i)}))^+$ is the kernel of the map

$$H^1(G_S(F^+), \mathbb{Z}_p(i))/p^n \to (\bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(i))/p^n)^+$$

which is of exponent $p^{\kappa^+}$. ∎
4. On the triviality of the twisted Kummer-Leopoldt constant

Let $i$ be an integer and let $p$ be an odd prime number. The $(p, i)$-regular number fields have been introduced in [2, Definition 1.1] as a generalization of $p$-rational fields [20, 21, 22]. Recall that a number field $F$ is $(p, i)$-regular if the cohomology group $H^2(G_S(F), \mathbb{Z}/p\mathbb{Z}(i))$ is trivial, or equivalently if $F$ satisfies Conjecture $C(i)$ and the $\mathbb{Z}_p$-module $\mathcal{J}^i_F$ is trivial. In particular, this triviality implies that of $\text{Gal}(\mathbb{F}^{Bp,i}(F)/\mathbb{F}^i(F))$, where $\mathbb{F}^i$ is the subfield of $\mathbb{F}^i$ fixed by $\mathcal{J}^i_F$ (Definition 2.5). Hence, by Theorem 3.8, we see that $\kappa_{1-i}$ is trivial for $(p, i)$-regular number fields. In this section, we consider the other implication. Precisely, we give a characterization of the $(p, i)$-regularity in terms of the triviality of $\kappa_{1-i}$.

**Theorem 4.1.** Let $i \neq 0, 1$ be an integer and let $F$ be a number field satisfying Conjecture $C(i)$. Then $F$ is $(p, i)$-regular if and only if the following three conditions hold:

1) $\kappa_{1-i} = 0$;
2) The injective map

$$H^0(G_S(F), Q_p/\mathbb{Z}_p(1 - i)) \longrightarrow \bigoplus_{v | p} H^0(F_v, Q_p/\mathbb{Z}_p(1 - i))$$

is an isomorphism;
3) $H^1(F) \subset \mathbb{F}^i(F)$.

**Proof.** Let us recall that $p^{\kappa_{1-i}}$ is the exponent of $\text{Gal}(\mathbb{F}^{Bp,i}(F)/\mathbb{F}^i(F))$ by Theorem 3.8 and that $\text{Gal}(\mathbb{F}^{Bp,i}(F)/\mathbb{F}^i(F)) \simeq \ker(\Psi : T^i_F \longrightarrow X^i_{\infty}(-i))$ by Lemma 3.7.

Suppose that $F$ is $(p, i)$-regular. Then, the $\mathbb{Z}_p$-torsion module $\mathcal{J}^i_F$ is trivial. Using the exact sequence (8) of Corollary 2.12, we get that the groups $W^{(1-i)}$ and $T^i_F$ are both trivial. Therefore, we obtain Condition 2) from the triviality of $W^{(1-i)}$, and Condition 1) from the triviality of $T^i_F$. Furthermore, the vanishing of $\mathcal{J}^i_F$ shows that $\mathbb{F}^i(F) = \mathbb{F}^i(F)$. Since $L^i$ is contained in $\mathbb{F}^i(F)$, we have $L^i \subset \mathbb{F}^i(F)$. This proves that $H^1(F) \subset \mathbb{F}^i(F)$.

Now assume that the three conditions are satisfied. Using again the exact sequence (8) of Corollary 2.12 we see that $\mathcal{J}^i_F$ and $T^i_F$ are isomorphic, since $W^{(1-i)}$ is trivial by Condition 2). Further, using Remark 2.10 with Condition 3) we obtain that the field $L^i$ is contained in $\mathbb{F}^i(F)$. Hence the morphism $\Psi : T^i_F \longrightarrow X^i_{\infty}(-i)$ is trivial. In particular, the kernel of $\Psi$ equals to $T^i_F$. Therefore, by Theorem 3.8, the Bertrandias-Payan module $T^i_F$ is trivial because of the nullity of $\kappa_{1-i}$. Hence the number field $F$ is $(p, i)$-regular. \qed

**Remark 4.2** (compare with [8, Proposition 2.3]). For the case $i = 0$, using the same arguments in the proof of Theorem 4.1, we can show that $F$ is $p$-rational exactly when the three conditions hold:

1) $\kappa(F) = 0$;
2) The map \( \mu_p(F) \longrightarrow \bigoplus_{v|p} \mu_p(F_v) \) is an isomorphism;

3) \( H_F \subset \overline{F} \).

Here \( \kappa(F) \) is the Kummer-Leopoldt constant for the units [7, Definition 1], \( H_F \) is the Hilbert class field of \( F \) and \( \overline{F} \) is the composite of all \( \mathbb{Z}_p \)-extensions of \( F \).

It is well known that the field of rational numbers \( \mathbb{Q} \) is \( p \)-rational for any prime number \( p \). This is not the case for the \( (p, i) \)-regularity. For example, if the prime \( p \) is irregular, there is at least an integer \( i \) for which \( \mathbb{Q} \) is not \( (p, i) \)-regular (a consequence of [2, (ii, \( \beta \) Proposition 1.3]). It is also well known that all subfields of a \( (p, i) \)-regular number field are \( (p, i) \)-regular. Thus, to study the \( (p, i) \)-regularity of number fields we must suppose that \( \mathbb{Q} \) is \( (p, i) \)-regular. From now on, we assume that \( \mathbb{Q} \) is \( (p, i) \)-regular and we consider the case of quadratic number fields. The aim is to give a characterization of the \( (p, i) \)-regularity of a quadratic number field in the spirit of [12, §4.1].

We start with the following consequence of Theorem 4.1 and Proposition 3.9 that shows the triviality of some twisted Kummer-Leopoldt constants for imaginary quadratic fields.

**Corollary 4.3.** Let \( p \) be an odd prime number and let \( i \) be an even integer such that \( \mathbb{Q} \) is \( (p, i) \)-regular. Then, the Kummer-Leopoldt constant \( \kappa_{1-i}(F) \) is zero for any imaginary quadratic field \( F \). \( \square \)

Now, we prove the following helpful lemma in which we show that the morphism

\[
H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \longrightarrow \bigoplus_{v|p} H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))
\]

is almost always an isomorphism.

**Lemma 4.4.** Let \( F = \mathbb{Q}(\sqrt{d}) \) be a quadratic number field and let \( i \) be an integer. Then, the map \( H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \longrightarrow \bigoplus_{v|p} H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \) is an isomorphism exactly in the following situations:

i) the prime \( p \) splits in \( F/\mathbb{Q} \) and \( i \not\equiv 1 \mod (p-1) \);

ii) the prime \( p \) is inert in \( F/\mathbb{Q} \);

iii) the prime \( p \) ramifies in \( F/\mathbb{Q} \) and \( \frac{2(i-1)}{p-1} \) is even if \( i \equiv 1 \mod \frac{(p-1)}{2} \) and \( (-1)^{\frac{p-1}{2}} \frac{d}{p} \) is a square in \( \mathbb{Q}_p \).

**Proof.** We start with the following well known isomorphisms

\[
H^0(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \approx \mathbb{Z}/p^{w_i} \mathbb{Z} \quad \text{and} \quad H^0(F_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \approx \mathbb{Z}/p^{w_{v,i}} \mathbb{Z},
\]

where

\[
w_i := \max\{n \mid i \equiv 1 \mod [F(\mu_{p^n}) : F]\}
\]

and

\[
w_{v,i} := \max\{n \mid i \equiv 1 \mod [F_v(\mu_{p^n}) : F_v]\}.
\]
To prove the lemma, we discuss on the ramification of the prime \( p \) in \( F/Q \). We start with the case where \( p \) splits in \( F/Q \). Let \( v \) and \( v' \) be the primes of \( F \) above \( p \). Observe that for all \( n \geq 1 \), we have

\[
[F(\mu_{p^n}) : F] = [F_v(\mu_{p^n}) : F_v] = [F_{v'}(\mu_{p^n}) : F_{v'}] = p^{n-1}(p - 1).
\]

Hence, \( H^0(G_S(F), Q_p/\mathbb{Z}_p(1 - i)) \rightarrow \bigoplus_{v|p} H^0(F_v, Q_p/\mathbb{Z}_p(1 - i)) \) is an isomorphism if and only if the groups \( H^0(G_S(F), Q_p/\mathbb{Z}_p(1 - i)) \), \( H^0(F_v, Q_p/\mathbb{Z}_p(1 - i)) \) and \( H^0(F_{v'}, Q_p/\mathbb{Z}_p(1 - i)) \) are trivial. This is equivalent to \( i \not\equiv 1 \mod (p - 1) \).

Suppose now that \( p \) is inert in \( F/Q \) and let \( v \) be the unique prime of \( F \) above \( p \).

Since \( F \cap Q(\mu_p) = Q \) and \( F_v \cap Q_p(\mu_p) = Q_p \), we have

\[
[F(\mu_{p^n}) : F] = [F_v(\mu_{p^n}) : F_v] = p^{n-1}(p - 1) \quad \text{for all } n \geq 1.
\]

Thus the map \( H^0(G_S(F), Q_p/\mathbb{Z}_p(1 - i)) \rightarrow H^0(F_v, Q_p/\mathbb{Z}_p(1 - i)) \) is always an isomorphism.

The remainder case is when \( p \) ramifies in \( F/Q \). Let \( v \) be the unique prime of \( F \) above \( p \). Suppose further that \( d \not\equiv (-1)^{\frac{p-1}{2}} p \) and \( (-1)^{\frac{p-1}{2}} \frac{p-1}{p} \) is a square in \( Q_p \). On the one hand, since \( Q(\sqrt{(-1)^{\frac{p-1}{2}} p}) \) is the unique quadratic subfield of \( Q(\mu_p) \), we can see that \( F \cap Q(\mu_p) = Q \). On the other hand, the condition \( (-1)^{\frac{p-1}{2}} \frac{p-1}{p} \) is a square in \( Q_p \) means that \( F_v \cap Q_p(\mu_p) = F_v \). Therefore for all \( n \geq 1 \), we have

\[
[F(\mu_{p^n}) : F] = p^{n-1}(p - 1) \quad \text{and} \quad [F_v(\mu_{p^n}) : F_v] = p^{n-1}\frac{(p - 1)}{2}.
\]

Comparing the integers \( w_i \) and \( w_{v,i} \), we get that \( w_i = w_{v,i} \) if and only if either \( i \not\equiv 1 \mod \frac{(p-1)}{2} \) or \( i \equiv 1 \mod \frac{(p-1)}{2} \) and \( (\frac{p-1}{p}) \) is even.

To finish the proof we have to show that

\[
H^0(G_S(F), Q_p/\mathbb{Z}_p(1 - i)) \rightarrow H^0(F_v, Q_p/\mathbb{Z}_p(1 - i))
\]

is an isomorphism when either \( d = (-1)^{\frac{p-1}{2}} p \) or \( d \not\equiv (-1)^{\frac{p-1}{2}} p \) and \( (-1)^{\frac{p-1}{2}} \frac{p-1}{p} \) is not a square in \( Q_p \). This is deduced from the fact that in both cases we have

\[
[F(\mu_{p^n}) : F] = [F_v(\mu_{p^n}) : F_v] \quad \text{for all } n \geq 1.
\]

\[ \square \]

**Remark 4.5.** a) When the integer \( i \) satisfies \( i \not\equiv 1 \mod \frac{(p-1)}{2} \), the localization map

\[
H^0(G_S(F), Q_p/\mathbb{Z}_p(1 - i)) \rightarrow \bigoplus_{v|p} H^0(F_v, Q_p/\mathbb{Z}_p(1 - i))
\]

is always an isomorphism.
b) If the integer \( i \) is even, then

\[
H^0(G_S(F), \mathbb{Q}_p / \mathbb{Z}_p(1 - i)) \to \bigoplus_{v \mid p} H^0(F_v, \mathbb{Q}_p / \mathbb{Z}_p(1 - i))
\]

is not an isomorphism exactly when \( p \) ramifies in \( F / \mathbb{Q} \) and the following two conditions hold:

- \( p \equiv 3 \mod (4), d \neq -p \) and \( -d \) is a square in \( \mathbb{Q}_p \);
- \( i \equiv 1 \mod \frac{(p-1)}{2} \) and \( \frac{2(i-1)}{(p-1)} \) is odd.

According to Corollary 4.3 and \( ii) \) of Remark 3.5, we see that the twisted Kummer-Leopoldt constant \( \kappa_{1-i}(F) \) is always zero when \( F \) is an imaginary quadratic field and \( i \) is even or \( F \) is a real quadratic field and \( i \) is odd. Using Theorem 4.1 and Lemma 4.4, we get the following characterizations of the \((p, i)\)-regularity for quadratic fields.

**Proposition 4.6.** Let \( i \geq 2 \) be an integer such that \( \mathbb{Q} \) is \((p, i)\)-regular. For a square free integer \( d > 0 \), let \( F = \mathbb{Q}(\sqrt{(-1)^i d}) \). Suppose that \( F \) satisfies one of the three conditions in Lemma 4.4. Then, \( F \) is \((p, i)\)-regular if and only if \( H^{(i)} \) is contained in \( \overline{F}^{(i)} \). In particular, \( F \) is \((p, i)\)-regular when \( X'_{\infty}(-i)_{\mathbb{Q}_\infty} \) is trivial.

**Proof.** Let \( j = 1 - i \). For simplicity we suppose that \( H^0(G_S(F), \mathbb{Q}_p / \mathbb{Z}_p(j)) = 0 \). Following Theorem 4.1 and Lemma 4.4, we have to prove the equivalence between the triviality of \( \kappa_j \) and the injectivity of the map

\[
\alpha_{j_1}^{(j)} : H^1(G_S(F), \mathbb{Z}_p(j)) / p \to \bigoplus_{v \mid p} H^1(F_v, \mathbb{Z}_p(j)) / p.
\]

Recall that \( \kappa_j \) is trivial precisely when

\[
\alpha_{j_n}^{(j)} : H^1(G_S(F), \mathbb{Q}_p / \mathbb{Z}_p(j)) / p^n \to \bigoplus_{v \mid p} H^1(F_v, \mathbb{Q}_p / \mathbb{Z}_p(j)) / p^n
\]
is injective for \( n \) large. Let’s consider the commutative diagram:

\[
\begin{array}{ccc}
0 \rightarrow (H^1(G_S(F), \mathbb{Z}_p(j)))^p & \rightarrow & H^1(G_S(F), \mathbb{Z}_p(j)) \rightarrow H^1(G_S(F), \mathbb{Z}_p(j))/p \rightarrow 0 \\
\downarrow p^{n-1} & & \downarrow p^{n-1} \\
0 \rightarrow (H^1(G_S(F), \mathbb{Z}_p(j)))^p & \rightarrow & H^1(G_S(F), \mathbb{Z}_p(j)) \rightarrow H^1(G_S(F), \mathbb{Z}_p(j))/p^n \rightarrow 0
\end{array}
\]

where the right vertical map is defined by

\[
x \mod H^1(G_S(F), \mathbb{Z}_p(j))^p \mapsto x^{p^{n-1}} \mod H^1(G_S(F), \mathbb{Z}_p(j))^p^n
\]

and is clearly injective. Hence we have

\[
H^1(G_S(F), \mathbb{Z}_p(j))/p^{n-1} \cong \mathrm{coker}(H^1(G_S(F), \mathbb{Z}_p(j))/p \rightarrow H^1(G_S(F), \mathbb{Z}_p(j))/p^n).
\]

Likewise, we see that for every \( p \)-adic place \( v \)

\[
H^1(F_v, \mathbb{Z}_p(j))/p^{n-1} \cong \mathrm{coker}(H^1(F_v, \mathbb{Z}_p(j))/p \rightarrow H^1(F_v, \mathbb{Z}_p(j))/p^n).
\]

Therefore, the commutative diagram

\[
\begin{array}{ccc}
0 \rightarrow \ker(\alpha_1^{(j)}) & \rightarrow & H^1(G_S(F), \mathbb{Z}_p(j))/p \rightarrow \bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(j))/p \\
\downarrow & & \downarrow \bigoplus_{v|p} \\
0 \rightarrow \ker(\alpha_n^{(j)}) & \rightarrow & H^1(G_S(F), \mathbb{Z}_p(j))/p^n \rightarrow \bigoplus_{v|p} H^1(F_v, \mathbb{Z}_p(j))/p^n
\end{array}
\]

and the snake lemma induce the following exact sequence:

\[
0 \rightarrow \ker(\alpha_1^{(j)}) \rightarrow \ker(\alpha_n^{(j)}) \rightarrow \ker(\alpha_n^{(j)}) \rightarrow \ker(\alpha_{n+1}^{(j)}) \rightarrow 0. \quad (15)
\]

An inductive process and the exact sequence (15) show that \( \ker(\alpha_n^{(j)}) \) is trivial for all \( n \geq 2 \) when \( \ker(\alpha_1^{(j)}) \) is. This means that \( \kappa_j = 0 \) when \( \ker(\alpha_1^{(j)}) \) is trivial. Conversely, if \( \kappa_j = 0 \), the exact sequence (15) shows that \( \ker(\alpha_1^{(j)}) \) is trivial. \( \square \)

The main results of this section can be compared with [8, Proposition 2.3] and [12, Proposition 4.1]. In fact, Condition a) in the above proposition can be interpreted using Kummer theory. Indeed, it is well known that there is a subgroup \( D_F^{(1-i)} \) of \( E^*: = E \setminus \{0\}, E = F(\mu_p) \), such that

\[
H^1(G_S(F), \mathbb{Z}_p(1-i))/p \cong D_F^{(1-i)}/E^*p(-i)
\]

and, for each prime \( v \) of \( F \) above \( p \), a subgroup \( D_v^{(1-i)} \) of \( E_w^* \), \( w \) being a prime of \( E \) above \( v \), such that

\[
H^1(F_v, \mathbb{Z}_p(1-i))/p \cong D_v^{(1-i)}/E_w^*p(-i),
\]
So, Condition a) asserts that the natural map

\[ D_F^{(1-i)} / E^p \to \bigoplus_{v|p} D_v^{(1-i)} / E^w, \]

where for each \( v \) above \( p, w \) is a place of \( E \) dividing \( v \), is injective.

**Example.** The quadratic number field \( F = \mathbb{Q}(\sqrt{\pm p}) \) is \((p, i)\)-regular for every integer \( i \equiv 1 \mod (p - 1) \). In fact, note that \( F \) has a unique \( p \)-adic prime and its class number is less than \( p \) e.g., [8, page 14]. Hence according to [2, (ii, \( \alpha \)), Proposition 1.3], \( F \) is \((p, i)\)-regular. Then the quadratic number field \( F \) satisfies Conjecture \( C^{(i)} \) and \( \kappa_{1-i} = 0 \) for all \( i \equiv 1 \mod (p - 1) \).

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