Kakeya-type sets for geometric maximal operators

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Abstract. We establish an estimate for arbitrary geometric maximal operators in the plane: we associate to any family $\mathcal{B}$ composed of rectangles and invariant by translations and central dilations a geometric quantity $\lambda_{\mathcal{B}}$ called its analytic split and satisfying

$$\log(\lambda_{\mathcal{B}}) \lesssim \| M_{\mathcal{B}} \|^p_p$$

for all $1 < p < \infty$, where $M_{\mathcal{B}}$ is the Hardy-Littlewood type maximal operator associated to the family $\mathcal{B}$.

1. Introduction

In [3], Bateman classified the behavior of directional maximal operators in the plane on the $L^p$ scale for $1 < p < \infty$. Here, we study geometric maximal operators which are more general than directional maximal operators: in particular, their study requires to focus on the interactions between the coupling eccentricity/orientation for a given family of rectangles. Our main result is the construction of so-called Kakeya-type sets for an arbitrary geometric maximal operator which gives an a priori bound on their $L^p$ norm in the same spirit than in [3].
We work in the Euclidean plane $\mathbb{R}^2$: if $U$ is a measurable subset we denote by $|U|$ its Lebesgue measure. We also denote by $\mathcal{R}$ the family containing all rectangles of $\mathbb{R}^2$: for $R \in \mathcal{R}$, we define its orientation as the angle $\omega_R \in [0, \pi)$ that its longest side makes with the $x$-axis and its eccentricity as the ratio $\varepsilon_R \in (0, 1]$ of its shortest side by its longest side. We will also denote by $R'$ the rectangle $R$ translated in its own direction by its length.

A family $\mathcal{B}$ contained in $\mathcal{R}$ is said to be geometric if it is invariant by translations and central dilations i.e. if for any $R \in \mathcal{R}$, any $x \in \mathbb{R}^2$ and $\lambda > 0$, we have

$$x + \lambda R \in \mathcal{B}.$$ 

Given any geometric family $\mathcal{B}$, we define the associated geometric maximal operator $M_\mathcal{B}$ as

$$M_\mathcal{B}f(x) := \sup_{x \in R \in \mathcal{B}} \frac{1}{|R|} \int_R |f|$$

for any $f \in L^\infty$ and $x \in \mathbb{R}^2$. We are interested in the relation between the geometry exhibited by the family $\mathcal{B}$ and the regularity of the operator $M_\mathcal{B}$ on the $L^p$ space for $1 < p < \infty$.

A lot of research has been done in the case where $\mathcal{B}$ is equal to

$$\mathcal{R}_\Omega := \{R \in \mathcal{R} : \omega_R \in \Omega\}$$

where $\Omega$ is an arbitrary set of directions in $[0, \pi)$. In other words, $\mathcal{R}_\Omega$ is the set of all rectangles whose orientation belongs to $\Omega$. We say that $\mathcal{R}_\Omega$ is a directional family and to alleviate the notation we denote

$$M_{\mathcal{R}_\Omega} := M_\Omega.$$ 

Naturally, the operator $M_\Omega$ is said to be a directional maximal operator. The study of those operators goes back at least to Cordoba and Fefferman’s article [6] in which they use geometric techniques to show that if $\Omega = \left\{ \frac{\pi}{2^k} \right\}_{k \geq 1}$ then $M_\Omega$ has weak-type $(2, 2)$. A year later, using Fourier analysis techniques, Nagel, Stein and Wainger proved in [8] that $M_\Omega$ is actually bounded on $L^p(\mathbb{R}^2)$ for any $p > 1$. In [1], Alfonseca has proved that if the set of direction $\Omega$ is a lacunary set of finite order then the operator $M_\Omega$ is bounded on $L^p(\mathbb{R}^2)$ for any $p > 1$. Finally in [3], Bateman proved the converse and so characterized the $L^p$ boundedness of directional operators in the plane.

**Theorem 1.1** (Bateman). Fix an arbitrary set of directions $\Omega \subset [0, \pi)$. We have the following alternative:

- if $\Omega$ is finitely lacunary, then $M_\Omega$ is bounded on $L^p$ for any $p > 1$.
- if $\Omega$ is not finitely lacunary, then $M_\Omega$ is not bounded on $L^p$ for any $p < \infty$.

We invite the reader to look at [3] for more details and also [4] where Bateman and Katz introduced their method.
2. Results

Our main result is an *a priori* estimate in the same spirit than one of the main result of [3]. Precisely, to any family $\mathcal{B}$ contained in $\mathcal{R}$ we associate a geometric quantity

$$\lambda_\mathcal{B} \in \mathbb{N} \cup \{\infty\}$$

that we call *analytic split* of $\mathcal{B}$. Loosely speaking, the analytic split $\lambda_\mathcal{B}$ indicates if $\mathcal{B}$ contains a lot of rectangles in terms of orientation and eccentricity. We prove then the following Theorem.

**Theorem 2.1.** For any geometric family $\mathcal{B}$ and any $1 < p < \infty$ we have

$$\log(\lambda_\mathcal{B}) \lesssim_p \|M_\mathcal{B}\|_p^p.$$

An important feature of this inequality is that we do not make any assumption on the family $\mathcal{B}$. In regards of the study of geometric maximal operators, Theorem 2.1 gives a concrete and *a priori* lower bound on the $L^p(\mathbb{R}^2)$ norm of $M_\mathcal{B}$. We insist on the fact that this estimate is concrete since the analytic split is not an abstract quantity associated to $\mathcal{B}$ but has a strong geometric interpretation. No such result was previously known for geometric maximal operators and we give an application in order to illustrate it.

**Theorem 2.2.** Fix any set of directions $\Omega \subset [0, \frac{\pi}{4})$ which is not finitely lacunary and let $\mathcal{B} \leq \mathcal{R}_\Omega$ be a geometric family satisfying for any $\omega \in \Omega$

$$\inf_{R \in \mathcal{B}, \omega R = \omega} e_R = 0.$$

In this case, the operator $M_\mathcal{B}$ is not bounded on $L^p$ for any $p < \infty$.

Observe that since we have $\mathcal{B} \subset \mathcal{R}_\Omega$ we have the trivial pointwise estimate

$$M_\mathcal{B} \leq M_\Omega.$$  

Hence, we have $\|M_\mathcal{B}\|_p < \infty$ if $\|M_\Omega\|_p < \infty$. Surprisingly, Theorem 2.2 states that the converse is also true *i.e.* we have $\|M_\mathcal{B}\|_p = \infty$ if $\|M_\Omega\|_p = \infty$.

3. The family $T$

Given a geometric family $\mathcal{B} \leq \mathcal{R}$, we can always suppose, without loss of generality, that it is of the form

$$\mathcal{B} = \{\vec{t} + \lambda R : \vec{t} \in \mathbb{R}^2, \lambda > 0, R \in B\}$$

where the family $B$ is contained in the family $T$ defined as

$$T = \{R_n(k) : n \geq 0, 0 \leq k \leq 2^n - 1\}.$$ 

Here, for $n \geq 1$ and $k \leq 2^n - 1$, $R_n(k)$ is the parallelogram whose vertices are the points $(0, 0), (0, \frac{1}{2^n}), (1, \frac{k}{2^n})$ and $(1, \frac{k+1}{2^n})$. The parallelogram $R_n(k)$ should be thought as a rectangle whose eccentricity and orientation are

$$\left(e_{R_n(k)}, \omega_{R_n(k)}\right) \simeq \left(\frac{1}{2^n}, \frac{k \pi}{2^n \cdot 4}\right).$$
In the rest of the text, we always identify a geometric family
\[ \mathcal{B}, \mathcal{R}_\Omega \text{ or } \mathcal{F} \leq \mathcal{R} \]
with the family that generates it
\[ B, T_\Omega \text{ or } F \subset T. \]

The family \( T \) has a natural structure of binary tree and we develop a vocabulary adapted to this structure: for any \( R \in T \) of scale \( n \geq 1 \), there exist a unique \( R_f \in T \) of scale \( n - 1 \) such that \( R \subset R_f \). We say that \( R_f \) is the parent of \( R \). In the same fashion, observe that there are only two elements \( R_h, R_l \in T \) of scale \( n + 1 \) such that \( R_h, R_l \subset R \). We say that \( R_h \) and \( R_l \) are the children of \( R \). Observe that \( R \in T \) is the child of \( R' \in T \) if and only if \( R \subset R' \) and \( 2|R| = |R'| \) : we will often use those two conditions. We say that a sequence (finite or infinite)
\[ n \in \mathcal{R} \]
with the family that generates it
\[ R \]
isadapted to this structure: for any
\[ i \]
that any
\[ R \]
is the parent of \( R \). If
\[ R \]
is a first element
\[ \mathcal{B} \]
that
\[ \mathcal{F} \]
is a last element
\[ R \]
that
\[ \mathcal{B} \]
is contained in
\[ \mathcal{F} \]
and
\[ \mathcal{B} \]
is called a leaf of \( \mathcal{F} \). Observe that for any
\[ P, R \subset \mathcal{R} \]
we have
\[ \mathcal{P} \]
and
\[ \mathcal{R} \]
for any
\[ i \]
and
\[ i \]
if it satisfies
\[ R_{i+1} \subset R_i \text{ and } 2|R_{i+1}| = |R_i| \text{ for any } i \text{ i.e. if } R_i \text{ is the parent of } R_{i+1} \text{ for any } i. \]
Different situations can occur. A finite path \( P \) has a first element \( R \) and a last element \( R' \) (defined in a obvious fashion) and we will write \( P_{R,R'} := P \). On the other hand, an infinite path \( P \) has no endpoint. For any family \( B \) contained in \( T \), there is a unique parallelogram \( R \in T \) such that any \( R' \in B \) is included in \( R \) and \( |R| \) is minimal. We say that this element \( R_B := R \) is the root of \( B \) and we define the set \([B]\) as
\[ [B] := \{ R \in T : \exists R' \in B, R' \subset R \subset R_B \}. \]
A subset of \( T \) of the form \([B]\) is called a tree generated by \( B \). We define the set \( L_B \) as
\[ L_B = \{ R \in B : \forall R' \in B, R' \subset R \Rightarrow R' = R \}. \]
An element of \( L_B \) is called a leaf of \( B \). Observe that for any \( B \) in \( T \) we have \([B] = [L_B]\) and also \( L_B = L_{[B]} \). The first identity says that the leaves of a tree \([B]\) can be seen as the minimal set that generates \([B]\). The second identity states that \([B]\) is not bigger than \( B \) in the sense that it does not have more leaves. If \( P \) is an infinite path, we have by definition \( L_P = \emptyset \).

4. Analytic split

We associate to any family \( B \) included in \( T \) a natural number \( \lambda_{[B]} \in \mathbb{N} \cup \{ \infty \} \) that we call analytic split. For any tree \([B]\), we define its boundary \( \partial[B] \) as the set of path in \([B]\) that are maximal for the inclusion \( i.e. P \in \partial[B] \) if and only if \( P \) is a path included in \([B]\) such that if \( P' \subset [B] \) is a path that contains \( P \) then \( P = P' \). For any tree \([B]\) and path \( P \in \partial[B] \) we define the splitting number of \( P \) relatively to \([B]\) as
\[ s_{P,[B]} := \# \{ R \in [B] \setminus P : \exists R' \in P, R \subset R', 2|R| = |R'| \}. \]
We say that a tree \([F]\) is a fig tree of scale \( n \) and height \( h \) when
- \([F]\) is finite and \( \# \partial[F] = 2^n \)
- for any \( P \in \partial[F] \) we have \( s_{P,[F]} = n \) and \( \#P = h \).
Observe that by construction we always have \( h \geq n \). We define the analytic split \( \lambda_{[B]} \) of a tree \([B]\) as the integer \( n \) such that \([B]\) contains a fig tree \([F]\) of scale \( n \) and do not contains any fig tree of scale \( n + 1 \). In the case where \([B]\) contains fig trees of arbitrary high scale, we set \( \lambda_{[B]} = \infty \). More generally for any family \( B \) contained in \( T \) (i.e. when \( B \) is not necessarily a tree), we define its analytic split as

\[
\lambda_B := \lambda_{[B]}.
\]

Hence by definition, the analytic split of a family \( B \) is the same as the analytic split of the tree \([B]\). Observe that thanks to Theorem 2.1 this definition is pertinent.

5. Bateman’s construction and Kakeya-type set

In [3], Bateman proves the following Theorem.

**Theorem 5.1** (Bateman’s construction [3]). Suppose that \([F]\) is a fig tree of scale \( n \) and height \( h \): there exists a finite family \( \{R_i : i \in I\} \) included in the geometric family \( \mathcal{F} \) defined as

\[
\mathcal{F} = \{ \vec{t} + \lambda R : \vec{t} \in \mathbb{R}^2, \lambda > 0, R \in [F] \}
\]

such that

\[
\log(n) \left| \bigcup_{i \in I} R_i \right| \lesssim \left| \bigcup_{i \in I} R'_i \right|.
\]

If \( R \) is a rectangle, we denote by \( R' \) the parallelogram \( R \) but shifted of one unit length on the right along its orientation. We fix a \( 2^h \) mutually independent random variables

\[
R_i : (\Omega, \mathbb{P}) \to L_{[F]}
\]

who are uniformly distributed in the set \( L_{[F]} \). We consider also the deterministic vectors

\[
\{ \vec{t}_i = (0, \frac{i - 1}{2^h}) : i \leq 2^h \}
\]

is a deterministic vector. Bateman’s main result in [3] reads as follow

**Theorem 5.2.** We have

\[
\mathbb{P} \left( \log(n) \left| \bigcup_{i \in I} \vec{t}_i + R_i \right| \lesssim \left| \bigcup_{i \in I} T(\vec{t}_i + R_i) \right| \right) > 0.
\]

The proof of this Theorem involves fine geometric estimates, percolation theory and the use of the so-called notion of stickiness of thin tubes of the euclidean plane, see [3] and [4]. Those kind of geometric estimate leads, more generally, to lower bound on maximal operators.
Lemma 5.3. Fix $N > 0$ such that there exists a finite family $\{R_i : i \in I\}$ included in a geometric family $\mathcal{B}$ such that

$$N \left| \bigcup_{i \in I} R_i \right| \leq \left| \bigcup_{i \in J} R_i \right|.$$ 

In this case, for any $p \in (1, \infty)$, we have

$$N \lesssim_p \|M_\mathcal{B}\|_p^p.$$ 

6. Geometric estimates

We need different geometric estimates in order to prove Theorem 2.1. We start with geometric estimates on $\mathbb{R}$ which will help us to prove geometric estimates on $\mathbb{R}^2$. Finally we prove a geometric estimate on $\mathbb{R}^2$ involving geometric maximal operators that is crucial.

If $I$ is a bounded interval on $\mathbb{R}$ and $\tau > 0$ we denote by $\tau I$ the interval that has the same center as $I$ and $\tau$ times its length i.e. $|\tau I| = \tau |I|$. The following lemma can be found in [2].

Lemma 6.1 (Austin’s covering lemma). Let $\{I_\alpha\}_{\alpha \in A}$ a finite family of bounded intervals on $\mathbb{R}$. There is a disjoint subfamily

$$\{I_{\alpha_k}\}_{k \leq N}$$ 

such that

$$\bigcup_{\alpha \in A} I_\alpha \subset \bigcup_{k \leq N} 3I_{\alpha_k}.$$ 

We apply Austin’s covering lemma to prove two geometric estimates on intervals of the real line. The first one concerns union of dilated intervals.

Lemma 6.2. Fix $\tau > 0$ and let $\{I_\alpha\}_{\alpha \in A}$ a finite family of bounded intervals on $\mathbb{R}$. We have

$$\left| \bigcup_{\alpha \in A} I_\alpha \right| \simeq \tau \left| \bigcup_{\alpha \in A} \tau I_\alpha \right|.$$ 

Proof. Suppose that $\tau > 1$. We just need to prove that

$$\left| \bigcup_{\alpha \in A} \tau I_\alpha \right| \leq \tau \left| \bigcup_{\alpha \in A} I_\alpha \right|.$$ 

Simply observe that we have

$$\bigcup_{\alpha \in A} \tau I_\alpha \subset \left\{ M I_{\bigcup_{\alpha \in A} I_\alpha} > \frac{1}{\tau} \right\}$$ 

and apply the one dimensional maximal Theorem. □

Now that we have dealt with union of dilated intervals we consider union of translated intervals.
Lemma 6.3. Let $\mu > 0$ be a positive constant. For any finite family of intervals $\{I_\alpha\}_{\alpha \in A}$ on $\mathbb{R}$ and any finite family of scalars $\{t_\alpha\}_{\alpha \in A} \subset \mathbb{R}$ such that, for all $\alpha \in A$

$$|t_\alpha| < \mu \times |I_\alpha|$$

we have

$$\left| \bigcup_{\alpha \in A} I_\alpha \right| \simeq \mu \left| \bigcup_{\alpha \in A} (t_\alpha + I_\alpha) \right|.$$

Proof. We apply Austin's covering lemma to the family $\{I_\alpha\}_{\alpha \in A}$ which gives a disjoint subfamily $\{I_{\alpha k}\}_{k \leq N}$ such that

$$\bigcup_{\alpha \in A} I_\alpha \subset \bigcup_{k \leq N} 3I_{\alpha k}.$$

In particular we have

$$\left| \bigcup_{k \leq N} I_{\alpha k} \right| \simeq \left| \bigcup_{\alpha \in A} I_\alpha \right|.$$

We consider now the family

$$\{(1 + \mu)I_{\alpha k}\}_{k \leq N}$$

which is a priori not disjoint. We apply again Austin's covering lemma which gives a disjoint subfamily that we will denote $\{(1 + \mu)I_{\alpha k_i}\}_{i \leq M}$ who satisfies

$$\bigcup_{k \leq N} (1 + \mu)I_{\alpha k} \subset \bigcup_{i \leq M} 3(1 + \mu)I_{\alpha k_i}.$$

In particular we have

$$\left| \bigcup_{i \leq M} (1 + \mu)I_{\alpha k_i} \right| \simeq \left| \bigcup_{k \leq N} (1 + \mu)I_{\alpha k} \right|.$$

To conclude, it suffices to observe that for any $\alpha \in A$ we have

$$t_\alpha + I_\alpha \subset (1 + \mu)I_\alpha$$

because $|t_\alpha| \leq \mu \times |I_\alpha|$. Hence the family

$$\{t_{\alpha k_i} + I_{\alpha k_i}\}_{i \leq M}$$

is disjoint and so finally

$$\left| \bigcup_{i \leq M} (t_{\alpha k_i} + I_{\alpha k_i}) \right| = \sum_{i \leq M} |I_{\alpha k_i}| \geq \frac{1}{3(1 + \mu)} \left| \bigcup_{i \leq M} 3(1 + \mu)I_{\alpha k_i} \right| \simeq \mu \left| \bigcup_{\alpha \in A} I_\alpha \right|$$

where we have used lemma 6.2 in the last step. \qed
We denote by $\mathcal{P}$ the family containing all parallelograms $R \subset \mathbb{R}^2$ whose vertices are of the form $(p, a), (p, b), (q, c)$ and $(q, d)$ where $p - q > 0$ and $b - a = d - c > 0$. We say that $L_R := p - q$ is the length of $R$ and that $W_R := b - a$ is the width of $R$. For $R \in \mathcal{P}$ and a positive ratio $0 < \tau < 1$ we denote by $\mathcal{P}_{R, \tau}$ the family defined as

$$\mathcal{P}_{R, \tau} := \{S \in \mathcal{P} : S \subset R, L_S = L_R, |S| \geq \tau |R| \}.$$ 

For $R \in \mathcal{P}$ define the parallelogram $\hat{R} \in \mathcal{P}$ as the parallelogram who has same length, orientation and center than $R$ but is $5$ times wider i.e. $W_{\hat{R}} = 5W_R$.

**Proposition 6.4.** Fix $0 < \tau < 1$ and any finite family of parallelograms $\{R_i\}_{i \in I} \subset \mathcal{P}$. For each $i \in I$, select an element $S_i \in \mathcal{P}_{R_i, \tau}$. The following estimate holds

$$\left| \bigcup_{i \in I} S_i \right| \geq \frac{\tau}{54} \left| \bigcup_{i \in I} R_i \right|.$$ 

**Proof.** Fix $x \in \mathbb{R}$ and for $i \in I$, denote by $R_i^x$ and $S_i^x$ the segments $R_i \cap \{x \times \mathbb{R}\}$ and $S_i \cap \{x \times \mathbb{R}\}$. For any $i \in I$, observe that there is a scalar $t_i$ satisfying $|t_i| \leq \mu \times |R_i|$ with

$$\mu = 5$$

such that

$$t_i + \tau R_i^x \subset S_i^x.$$ 

Applying lemma 6.3, we then have (since $9 \times (1 + \mu) = 54$)

$$\left| \bigcup_{i \in I} S_i^x \right| \geq \left| \bigcup_{i \in I} (t_i + \tau R_i^x) \right| \geq \frac{1}{54} \left| \bigcup_{i \in I} \tau R_i^x \right|.$$ 

We conclude using lemma 6.2

$$\frac{1}{54} \left| \bigcup_{i \in I} \tau R_i^x \right| \geq \frac{\tau}{54} \left| \bigcup_{i \in I} R_i \right|,$$

and integrating on $x$. \qed

We state a last geometric estimate involving maximal operator: we fix an arbitrary element $R \in \mathcal{P}$ and an element $V \in \mathcal{P}$ included in $R$ such that $L_V = L_u$ and $|V| \leq \frac{1}{2} |R|$. Recall that we denote by $R'$ the parallelogram $R$ translated in its direction by its length.

**Proposition 6.5.** There is a parallelogram $S \in \mathcal{P}_{R', \frac{1}{2}}$ depending on $V$ such that the following inclusion holds

$$S \subset \left\{ M_{V, \frac{1}{2}} > \frac{1}{16} \right\}.$$
Proof. Without loss of generality, we can consider that we have
\[ R := [0, 1]^2. \]
and that the lower left corner of \( V \) is \( O \). The upper left corner of \( V \) is the point \( (0, W_V) \) and we denote by \( (d, 1) \) and \( (d + W_V, 1) \) its lower right and upper right corners. Since \( V \subset R \) we have
\[ d + W_V \leq 1. \]
The upper right corner of \( \frac{1}{2} V \) is the point \( \left( \frac{1}{2}(d + W_V), \frac{1}{2} \right) \) and so for any \( 0 \leq y \leq 1 - \frac{1}{2}(d + W_V) \) we have
\[ (0, y) + \frac{1}{2} V \subset R. \]
This yields our inclusion as follow. Let \( \vec{t} \in \mathbb{R}^2 \) be a vector such that the center of the parallelogram \( \tilde{V} = \vec{t} + 2V \) is the point \((1,0)\). By construction we directly have
\[ |\tilde{V} \cap R'| \geq \frac{1}{16} \]
but moreover for any \( 0 \leq y \leq \frac{1}{2} \) we have
\[ \left| \{(0,y) + \tilde{V} \} \cap R' \right| \geq \frac{1}{16} \]
since the upper right quarter of \( \tilde{V} \) is relatively to \( R' \) in the same position than \( V \) relatively to \( R \). Finally, denoting by \( V^* \) the parallelogram \( \tilde{V} \cap [0,1] \times \mathbb{R} \), the parallelogram \( S \) defined as
\[ S := \bigcup_{0 \leq y \leq \frac{1}{2}} ((0,y) + V^*) \]
satisfies the condition claimed. This concludes the proof. \( \square \)

7. Proof of Theorem 2.1

We fix an arbitrary family \( B \) contained in \( T \) and we prove the following Theorem: combined with Lemma 5.3 it yields Theorem 2.1.

Theorem 7.1. There exists a finite family \( \{R_i : i \in I\} \) included in the geometric family \( B \) defined as
\[ B = \{ \vec{t} + \lambda R : \vec{t} \in \mathbb{R}^2, \lambda > 0, R \in B \} \]
which satisfies
\[ \log(n) \left| \bigcup_{i \in I} R_i \right| \lesssim \left| \bigcup_{i \in I} R^t_i \right| \]
where \( n = \lambda_B \).
The family $B$ generates a tree $[B]$; we fix a fig tree $[F] \subset [B]$ of scale $\lambda_B$ and we denote by $h \in \mathbb{N}$ its height. We apply Bateman’s Theorem to obtain a finite family $\{t_i + R_i : i \leq 2^h\}$ included in

$$\mathcal{F} = \left\{ \vec{t} + \lambda R : \vec{t} \in \mathbb{R}^2, \lambda > 0, R \in [F] \right\}$$

which satisfies

$$\log(n) \left| \bigcup_{i \in I} R_i \right| \lesssim \left| \bigcup_{i \in I} R_i' \right|.$$ 

We take advantage of those elements but this time using elements of $B$ and not elements of $[F]$. Let us define $A_1$ as

$$A_1 := \bigcup_{i \in I} R_i$$

and similarly let us define $A_2$ as

$$A_2 := \bigcup_{i \in I} R_i'$$

Figure 1. Theorem 2.1 shows that we can virtually use the tree $[F]$ for the operator $M_B$ even if $B$ has no structure. On the illustration, $B$ is composed of the red dots which represent rectangles who have very different scale and yet they interact at the level of $[F]$.

We apply apply Proposition 6.5: for any $U \in L[F]$ we fix an element $V_U$ of $B$ such that $V_U \subset U$. To each pair $(U, V_U)$ we apply Proposition 6.5 and this gives a parallelogram $S_U \in S_U$ such that

$$S_U \subset \left\{ M_{V_U} \frac{1}{16} > \frac{1}{16} \right\}.$$
We define then the set $B_2$ as
$$B_2 := \bigcup_{i \leq 2^h} \vec{t}_i + T S_{R_i}$$
Because $V_U \in B$, we obviously have
$$M_{V_U} \leq M_B$$
and so $S_U \subset \{M_B \geq \frac{1}{16}\}$. We take the union over $i \leq 2^h$ and we obtain
$$B_2 := \bigcup_{i \leq 2^h} \vec{t}_i + T S_{R_i} \subset \{M_B \geq \frac{1}{16}\}$$
and so finally $|B_2| \leq \left| \left\{M_B \geq \frac{1}{16}\right\} \right|$.
Let us compute $|B_2|$: to do so, we observe that we can use Proposition 6.4 with the families $[\vec{t}_i + R_i : i \leq 2^h]$ and $[\vec{t}_i + T S_{R_i} : i \leq 2^h]$. This yields
$$|B_2| \geq \frac{1}{21 \times 4} |A_2|$$
and so we finally have
$$|A_1| \leq \frac{1}{\log(n)} \left| \left\{M_B \geq \frac{1}{16}\right\} \right|.$$ 
This inequality concludes the proof of Theorem 2.1.

8. Proof of Theorem 2.2

Let $\Omega$ be a directions in $[0, \frac{\pi}{4})$ which is not finitely lacunary and let $B$ be a geometric family such that we have $B \subset R_\Omega$ and also
$$\sup_{\omega \in \Omega} \inf_{R \in B, \omega R = \omega} e_R = 0.$$ 
Let us denote by $T_\Omega$ the family included in $T$ such that
$$R_\Omega = [\vec{t} + \lambda R : \vec{t} \in \mathbb{R}^2, \lambda > 0, R \in T_\Omega].$$
Denote also by $B$ the family included in $T$ that generates $B$ and observe that our hypothesis implies that we have
$$[B] = T_\Omega$$
and so in particular we have
$$\lambda_B = \lambda_{T_\Omega}.$$ 
The following claim will concludes the proof.

Claim. The set of direction $\Omega$ is not finitely lacunary if and only if $\lambda_{T_\Omega} = \infty$.

Applying Theorem 2.1, we obtain for any $1 < p < \infty$
$$\infty = \lambda_{T_\Omega} = \lambda_{[B]} \lesssim \|M_B\|_p^p.$$
References


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